Geometry of phase space transport in dynamical systems

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MultiSTEPS: MultiScale Transport in Environmental & Physiological Systems, www.multisteps.esm.vt.edu
The tale of a confused comet

- comet Oterma from 1910 to 1980
- Rapid transition: outside to inside Jupiter’s orbit; temporarily captured.
The tale of a confused comet

- Oterma’s orbit in rotating frame with special nearby orbits (green)
Natural Pathways for Fuel Efficiency
Natural Pathways for Fuel Efficiency

Orbiting Jupiter’s moons

zero fuel trajectory

Fuel-efficient tours of Jupiter’s moons
Interplanetary transport network

Natural pathways winding through the solar system
Oceanic transport network

Ocean currents: natural pathways on Earth
Atmospheric transport network
Atmospheric transport network
Transport networks: overview

- Main objective: geometric description of transport
  - insight into phase space mixing and regions of further interest
  - efficient control schemes

- Motivating principle: structures guiding transport
  — especially systems with symmetry, e.g., Hamiltonian

- celestial mechanics example

- geophysical flow example
Interplanetary transport: main ideas

- Break $N$-body problem into several 3-body problems
- Invariant manifolds of unstable bound orbits act as **separatrices** (codimension 1 surfaces)
- Determine **transport**, e.g., collisions, transitions
3-Body Problem

Restricted 3-body approximation

- \( P \) in field of two massive bodies, \( m_1 \) and \( m_2 \)
- \( x-y \) frame rotates w.r.t. \( X-Y \) inertial frame
3-Body Problem

Equations of motion in **rotating frame** describe $P$ moving in effective potential plus a coriolis force (goes back to work of Jacobi, Hill, etc)

$$\bar{U}(x,y)$$

Effective Potential
Hamiltonian system

Hamiltonian function (2 d.o.f.) — time-independent

\[
H(x, y, p_x, p_y) = \frac{1}{2}((p_x + y)^2 + (p_y - x)^2) + \bar{U}(x, y),
\]

where \( p_x \) and \( p_y \) are the conjugate momenta, and

\[
\bar{U}(x, y) = -\frac{1}{2}(x^2 + y^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}
\]

where \( r_1 \) & \( r_2 \) are the distances of \( P \) from \( m_1 \) & \( m_2 \) and

\[
\mu = \frac{m_2}{m_1 + m_2} \in (0, 0.5]
\]

For systems of interest, \( \mu \approx 10^{-6} - 10^{-2} \)
Motion in energy surface

- **Energy surface** of energy $E$ is codim-1 surface

\[ \mathcal{M}(E) = \{(q, p) \mid H(q, p) = E\}. \]

- In 2 d.o.f., 3D surfaces foliating the 4D phase space (in 3 d.o.f., 5D energy surfaces)
Realms of possible motion

\( M_\mu(E) \) partitioned into three **realms**

e.g., Earth realm = phase space around Earth

- Energy \( E \) determines their connectivity
Realms of possible motion

Case 1: $E<E_1$

Case 2: $E_1<E<E_2$

Case 3: $E_2<E<E_3$

Case 4: $E_3<E<E_4$

Case 5: $E>E_4$
Orbits in neck regions between realms

- Orbits exist around $L_1$ & $L_2$; periodic & quasi-periodic
  - Unstable bound orbits: Lyapunov, halo and Lissajous orbits
  - their stable/unstable invariant manifolds are tubes, play a key role

The location of all the equilibria for $\mu = 0.3$
Realms and tubes

- Realms connected by **tubes** in phase space $\simeq S^k \times \mathbb{R}$
  — Conley & McGehee, 1960s, found these locally for planar case, speculated on use for **“low energy transfers”**
Motion near saddles

- Near $L_1$ or $L_2$, linearized vector field has eigenvalues $\pm \lambda$ and $\pm i\omega_j$, $j = 2, \ldots, N$

- Under local change of coordinates $H(q, p) = \lambda q_1 p_1 + \sum_{i=2}^{N} \frac{\omega_i}{2} (p_i^2 + q_i^2)$
Equilibrium point is of type
saddle $\times$ center $\times \cdots \times$ center ($N - 1$ centers)
i.e., rank 1 saddle

the $N$ canonical planes
Motion near saddles

□ For energy $\hbar$ just above saddle pt, $(q_1, p_1) = (0, 0)$ is normally hyperbolic invariant manifold of bound orbits

$$\mathcal{M}_h = \sum_{i=2}^{N} \frac{\omega_i}{2} (p_i^2 + q_i^2) = h > 0.$$
Motion near saddles

□ Note that $\mathcal{M}_h \simeq S^{2N-3}$

- $N = 2$, the circle $S^1$, a single periodic orbit
- $N = 3$, the 3-sphere $S^3$, a set of periodic and quasi-periodic orbits

the $N$ canonical planes
Motion near saddles

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- $N = 2$, the circle $S^1$, a single periodic orbit
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Four “cylinders” or **tubes** of asymptotic orbits: stable, unstable manifolds, $W^s_\pm(\mathcal{M}_h), W^u_\pm(\mathcal{M}_h), \simeq S^1 \times \mathbb{R}$ for $N = 2$
Motion near saddles

- **B**: bounded orbits (periodic/quasi-periodic): $S^3$
- **A**: asymptotic orbits to 3-sphere: $S^3 \times \mathbb{R}$ (*tubes*)
- **T**: transit and **NT**: non-transit orbits.
Motion near saddles: 3-body problem

- **B**: bounded orbits (periodic/quasi-periodic): $S^3$
- **A**: asymptotic orbits to 3-sphere: $S^3 \times \mathbb{R}$ (tubes)
- **T**: transit and **NT**: non-transit orbits.

![Diagram](image)

Projection to configuration space.
Tube dynamics: inter-realm transport

- **Tube dynamics**: All motion between realms connected by necks around saddles must occur through the interior of tubes\(^1\)

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\(^1\)Koon, Lo, Marsden, Ross [2000,2001,2002], Gómez, Koon, Lo, Marsden, Masdemont, Ross [2004]
Some remarks on tube dynamics

- Tubes are general; consequence of rank 1 saddle – e.g., ubiquitous in chemistry

- Tubes persist
  - in presence of additional massive body
  - when primary bodies’ orbit is eccentric

- Observed in the solar system (e.g., Oterma)
- Even on galactic and atomic scales!

Tube dynamics

- Motion between Poincaré sections on $\mathcal{M}(E)$
- System reduced to $k$-map dynamics between the $k \ U_i$
Tube dynamics

- Motion between Poincaré sections on $\mathcal{M}(E)$
- System reduced to $k$-map dynamics between the $k U_i$
Identifying orbits by itinerary

Regions of common orbits labeled using itineraries

- by looking at intersections of labeled tubes → tube hopping

Itineraries for multiple 3-body systems possible too.
Identifying orbits by itinerary

- itinerary \((X, J, S)\), same as Oterma
- search for an initial condition with this itinerary
- first in 2 d.o.f., then in 3 d.o.f.
Identifying orbits by itinerary — 2 d.o.f.

Consider how tubes connect Poincaré sections $U_i$
Identifying orbits by itinerary — 2 d.o.f.

\[ ([J], S) = (T_{[J], S} \cap U_3) \]

\[ (X, [J]) = (T_{X,[J]} \cap U_3) \]

Poincare Section \( U_3 \)
\{ \( x = 1 - \mu, y > 0, \dot{x} < 0 \) \}

\( J \) realm
Identifying orbits by itinerary — 2 d.o.f.

**Tile with label** \((X, [J], S)\)

- Denote the intersection \((X, [J]) \cap ([J], S)\) by \((X, [J], S)\)

\[
(X, [J], S) = (X, [J]) \cap ([J], S)
\]
Identifying orbits by itinerary — 2 d.o.f.

- Forward and backward numerical integration
Identifying orbits by itinerary — 2 d.o.f.

$(X, [J], S)$

Longer itineraries...
Identifying orbits by itinerary — 2 d.o.f.

... correspond to smaller pieces of phase space
Identifying orbits by itinerary — 2 d.o.f.

Orbit with \((X, J, S, J, X)\)
Theorem of global orbit structure

- says we can construct an orbit with any itinerary, eg (... J, X, J, S, J, S, ...), where X, J and S denote the different realms (symbolic dynamics)

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2Main theorem of Koon, Lo, Marsden, and Ross [2000] Chaos
Identifying orbits by itinerary — 3 d.o.f.

- **Similar for 3 d.o.f.:** Invariant manifold tubes $S^3 \times \mathbb{R}$
- **Poincaré section of energy surface**
  - at $x = \text{constant}$, $(y, \dot{y}, z, \dot{z}) \subset \mathbb{R}^4$
Identifying orbits by itinerary — 3 d.o.f.

- **Similar for 3 d.o.f.**: Invariant manifold tubes $S^3 \times \mathbb{R}$
- **Poincaré section of energy surface**
  - at $x = \text{constant}$, $(y, \dot{y}, z, \dot{z}) \subset \mathbb{R}^4$
- **Tube cross-section** is a topological **3-sphere** $S^3$ of radius $r$
  - $S^3$ projection: disk $\times$ disk
Determining interior of $S^3$

$S^3$ projection: disk $\times$ disk

$$y^2 + \dot{y}^2 + z^2 + \dot{z}^2 = r^2$$

$$r_y^2 + r_z^2 = r^2$$

(y, $\dot{y}$) Plane $\times$ (z, $\dot{z}$) Plane
For fixed \((z, \dot{z})\), projection onto \((y, \dot{y})\) is a closed curve

\[
y^2 + \dot{y}^2 = r^2 - (z^2 + \dot{z}^2)
\]

\[
y^2 = r^2 - \frac{r_z^2}{r_y}
\]
Determining interior of $S^3$

For different $(z, \dot{z})$, a different closed curve in $(y, \dot{y})$

\[ y^2 + \dot{y}^2 = r^2 - (z^2 + \dot{z}^2) \]
\[ \frac{r^2}{y} = r^2 - \frac{r^2}{z} \]

(y, \dot{y}) Plane \hspace{1cm} \times \hspace{1cm} (z, \dot{z}) Plane
Determining interior of $S^3$

Cross-section of tube effectively reduced to a two-parameter family of closed curves

$$y^2 + \dot{y}^2 = r^2 - (z^2 + \dot{z}^2)$$

(y, \dot{y}) Plane  \quad \times \quad (z, \dot{z}) Plane
Determining interior of $S^3$

- Can be demonstrated numerically: \( \{ \text{int}(\gamma z\dot{z}) \} (z, \dot{z}) \)

- Provides nice way to calculate interior of tube, intersections of tubes, etc.
Intersection of phase volumes

Find \((X,J,S)\) orbit via tube intersection
Intersection of phase volumes

Find \((X,J,S)\) orbit via tube intersection
All orbits in intersection correspond to transition

3D view  xy-plane projection

Gómez, Koon, Lo, Marsden, Masdemont, Ross, Nonlinearity [2004]
Other orbits obtained this way

Another example
On the tubes, rather than in the tubes

An $L_1$-$L_2$ heteroclinic connection
Example: Comet transport between outside and inside of Jupiter (i.e., Oterma-like transitions)
Phase volume ratio gives the relative probability to pass from outside to inside Jupiter’s orbit.
Transition probabilities

□ Jupiter family comet transitions: X → S, S → X
Capture time determined by tube dynamics

Temporary capture time profiles are structured
Related systems

- Results apply to similar problems in chemistry, biomechanics, ship capsize
Tubes leading to capsize

- Ship motion is Hamiltonian,

\[ H = \frac{p_x^2}{2} + \frac{R^2 p_y^2}{4} + V(x, y), \]

\[ V(x, y) \]

\[ E < E_c \quad E > E_c \]
Tubes leading to capsize

Poincare section

transition state
Tubes leading to capsize

- Wedge of trajectories leading to imminent capsize

- Tubes are a useful paradigm for predicting capsize even in the presence of random forcing, e.g., random waves

- Could inform control schemes to avoid capsize in rough seas
Some other transport activities inspired by Jerry
FTLE for Riemannian manifolds

- We can define the FTLE for Riemannian manifolds\(^3\)

\[
\sigma^T_t(x) = \frac{1}{|T|} \ln \left\| D\phi^{t+T} \right\| = \frac{1}{|T|} \log \left( \max_{y \neq 0} \frac{\left\| D\phi^{t+T}_t(y) \right\|}{\|y\|} \right)
\]

with \(y\) a small perturbation in the tangent space at \(x\).

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\(^3\)Lekien & Ross [2010] Chaos
Atmospheric flows: Antarctic polar vortex

ozone data
Atmospheric flows: Antarctic polar vortex

ozone data + LCSs (red = repelling, blue = attracting)
Atmospheric flows and lobe dynamics

orange = repelling LCSs, blue = attracting LCSs

Hurricane Andrea, 2007

Atmospheric flows and lobe dynamics

Hurricane Andrea at one snapshot; LCS shown (orange = repelling, blue = attracting)
Atmospheric flows and lobe dynamics

orange = repelling (stable manifold), blue = attracting (unstable manifold)
Atmospheric flows and lobe dynamics

orange = repelling (stable manifold), blue = attracting (unstable manifold)
Atmospheric flows and lobe dynamics

Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out
Atmospheric flows and lobe dynamics

Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out
Atmospheric flows and lobe dynamics

Sets behave as lobe dynamics dictates
Coherent sets and set-based definition of FTLE

- FTLE from covariance during 24 hours starting 09:00 1 May 2007
Coherent sets and set-based definition of FTLE

- Coherent sets during 24 hours starting 09:00 1 May 2007
Navigation in an aperiodic setting

- Selectively jumping between large air masses using control
- Moving between mobile subregions of different finite-time itineraries
Biological adaptation

Long range transport of plant pathogen spores

- Might organisms which travel via the atmosphere have adaptations to best take advantage of the “atmospheric superhighway”?
Final words on geometry of transport

- **Invariant manifold** and invariant manifold-like structures are related to transport; form template or skeleton

- In Hamiltonian systems with rank-1 saddles:
  - **Tube dynamics**: the interior of tube manifolds
    - related to capture, escape, transition, collision
    - applications to orbital mechanics, ship capsize, ...

- In the atmosphere:
  - **Lagrangian coherent structures**
    - the skeleton of air
    - boundaries between air masses
    - link with set-oriented and topological methods
Some Papers:

FREE Book

Book available:

Dynamical systems, the three-body problem, and space mission design
Koon, Lo, Marsden, Ross

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