Minisymposium:
Applications of numerical geometric dynamics and control
to the memory of

Jerry Marsden (1942-2010)
Applications of Discrete Geometric Control

Marin Kobilarov
California Institute of Technology

July 18, 2011
Caltech Small Satellite Testbed

Autonomous Assembly of Reconfigurable Space Telescope (AaREST) (collaboration between Caltech and University of Surrey, UK)

Reconfiguration Testbed: autonomous navigation and docking

prototype hardware (Univ. of Surrey) 
redocking in orbit
Computational-Theoretical Focus

Approach: a consistent approach to Modeling, Mechanics, Control, and Optimization exploiting the geometric structure of nonlinear dynamics
Computational-Theoretical Focus

Approach: a consistent approach to Modeling, Mechanics, Control, and Optimization exploiting the geometric structure of nonlinear dynamics
Goal: a general framework for computational dynamics and control
Computational-Theoretical Focus

Approach: a consistent approach to Modeling, Mechanics, Control, and Optimization exploiting the geometric structure of nonlinear dynamics

Goal: a general framework for computational dynamics and control

▶ Geometric Integration
▶ Discrete Geometric Control
▶ Implementing Fast Algorithms for Real-time Applications
▶ Basis for Global Optimization for Mechanical Systems
Discrete Mechanics

How to integrate the systems dynamics properly?

- Standard integrators dissipate energy and accumulate errors
- Alternative approach: discretize the variational principle of mechanics, obtaining a *variational integrator*
Discrete Mechanics

How to integrate the systems dynamics properly?

- Standard integrators dissipate energy and accumulate errors
- Alternative approach: discretize the variational principle of mechanics, obtaining a variational integrator

Continuous vs. Discrete Mechanics

- state space structure preservation
- increased numerical accuracy and stability
- suitable numerical framework for optimal control

Mechanical System Modeling

- systems described by $L = K - V$.
- configuration space $Q$ with configuration $q \in Q$
  - typically $Q = G \times M$
  - $G$ – Lie group, e.g. $SE(3)$ denoting the vehicle pose
  - $M$ – shape space, e.g. joint angles
- nonholonomic constraints $\dot{q} \in D$, distribution $D_q \subset T_q Q$
- symmetries associated with group transformations
- external forces, e.g. gravity, friction
- Primary application area: robotics and aerospace
Example: mechanical systems on Lie groups

Consider mechanical system on $n$-dim. Lie group $G$ (algebra $\mathfrak{g}$) with Lagrangian

$$\ell(g, \xi) = K(\xi) - V(g),$$

where $K : \mathfrak{g} \to \mathbb{R}$ and $V : G \to \mathbb{R}$ are given kinetic and potential energy functions. The system is subject to forcing $f(t) \in \mathfrak{g}^*$. Two types of systems considered

- fully actuated: $f$ can act in any direction
- under-actuated: $f = \sum_{i=1}^{c} f^i(\phi)u^i$, where $f^i \in \mathfrak{g}^*$ are allowed directions with controllable parameters $\phi \in \mathbb{M} \subset \mathbb{R}^m$, and control inputs $u \in \mathbb{U} \subset \mathbb{R}^c$, $c \leq n$
Example: mechanical systems on Lie groups

Consider mechanical system on \( n \)-dim. Lie group \( G \) (algebra \( \mathfrak{g} \)) with Lagrangian

\[
\ell(g, \xi) = K(\xi) - V(g),
\]

where \( K : \mathfrak{g} \to \mathbb{R} \) and \( V : G \to \mathbb{R} \) are given kinetic and potential energy functions. The system is subject to forcing \( f(t) \in \mathfrak{g}^* \). Two types of systems considered

- fully actuated: \( f \) can act in any direction
- under-actuated: \( f = \sum_{i=1}^{c} f^i(\phi)u^i \), where \( f^i \in \mathfrak{g}^* \) are allowed directions with controllable parameters \( \phi \in \mathcal{M} \subset \mathbb{R}^m \), and control inputs \( u \in \mathcal{U} \subset \mathbb{R}^c \), \( c \leq n \)

The system is required to move from a fixed initial state \((g(0), \xi(0))\) to a fixed final state \((g(T), \xi(T))\) during a time interval \([0, T]\). The problem is to find the optimal control \( u^* = \arg\min_u J(u, T) \) where the cost function \( J \) is defined by

\[
J(u, T) = \frac{1}{2} \int_{0}^{T} \|u(t)\|^2 \, dt, \tag{1}
\]

subject to the dynamics and boundary state conditions.
Trajectory Discretization

Trajectory: $g_{0:N} := \{g_0, ..., g_N\}$, where $g_k \approx g(kh) \in G$ and $h = T/N$. Approximate curve b/n $g_k$ and $g_{k+1}$ through a Lie algebra element $\xi_k \in \mathfrak{g}$ such that $\xi_k = \tau^{-1}(g_k^{-1}g_{k+1})/h$, where: the retraction map $\tau : \mathfrak{g} \to G$ is a $C^2$-diffeomorphism around the origin such that $\tau(0) = e$

Given a map $\tau : \mathfrak{g} \to G$, its right-trivialized tangent $d\tau_\xi : \mathfrak{g} \to \mathfrak{g}$ and its inverse $d\tau_\xi^{-1} : \mathfrak{g} \to \mathfrak{g}$ are such that, for a some $g = \tau(\xi) \in G$ and $\eta \in \mathfrak{g}$, the following holds (Hairer et al; Bou-Rabee and Marsden)

$$\partial_\xi \tau(\xi) \cdot \eta = d\tau_\xi \cdot \eta \cdot \tau(\xi), \quad (2)$$
$$\partial_\xi \tau^{-1}(g) \cdot \eta = d\tau_\xi^{-1} \cdot (\eta \cdot \tau(-\xi)). \quad (3)$$
Discrete Variational Formulation

A mechanical system on Lie group $G$ with kinetic energy $K$, potential energy $V$, subject to forces $f$, satisfies the following equivalent conditions:

1. The discrete reduced Lagrange-d’Alembert principle holds

$$\delta \sum_{k=0}^{N-1} \left[ K(\xi_k) - \frac{V(g_k) + V(g_{k+1})}{2} \right] + \sum_{k=0}^{N-1} \frac{1}{2} \left[ \langle f_k, g_k^{-1}\delta g_k \rangle + \langle f_{k+1}, g_{k+1}^{-1}\delta g_{k+1} \rangle \right] = 0,$$

(4)

where $\xi_k = \tau^{-1}(g_k^{-1}g_{k+1})/h$.

2. The discrete reduced Euler-Poincaré equations of motion hold

$$\mu_k - \text{Ad}^*_{\tau(h\xi_k)^{-1}} \mu_{k-1} = h (-g_k^* \partial_g V(g_k) + f_k),$$

(5a)

$$\mu_k = (d\tau^{-1}_{h\xi_k})^* \partial_\xi K(\xi_k),$$

(5b)

$$g_{k+1} = g_k \tau(h\xi_k).$$

(5c)
Numerical Benefits

- essentially coordinate-free: avoids issues with expensive chart switching that cause sudden jumps or singularities, e.g. due to gimbal lock, that prevent convergence in iterative optimization;

- minimum reduced dimension; no required Lagrange multipliers enforcing, e.g. matrix orthogonality constraints or quaternion unit norms;

- symplectic structure and momentum preservation, and energy approximation close to the true energy – leading to an accurate and numerically robust approximation of the dynamics and optimality conditions as a function of the time step

- predictable computation even at lower resolutions allowing increased run-time efficiency.
Discrete Mechanics Algorithm

Input: state \((g_k, \xi_{k-1}, \mu_{k-1}) \in G \times g \times g^*\) and force \(f_k \in g^*\)

1. Compute (explicitly) the next momentum

\[
\mu_k = \text{Ad}_{\tau(h\xi_{k-1})}^* \mu_{k-1} + h \left( -g^*_k \partial_g V(g_k) + f_k \right),
\]

2. Solve the Legendre transform for \(\xi_k\) (implicitly)

\[
\mu_k = (d\tau_{h\xi_k}^{-1})^* \partial_{\xi} K(\xi_k),
\]

3. Update \(g_{k+1} = g_k \tau(h\xi_k)\)

Output: \(g_{k+1}, \xi_k, \mu_k\)

Very easy to implement:

```matlab
function [g, v, p] = lieint_step(g, v, p, f, h, S)
    p = S.Ad(h*v)'*p + h*f;
    v = fsolve(@(v) legt(v, [], p, h, S), g*S.cay(h*v));
    g = g*S.cay(h*v);

function f = legt(v, p, h, S)
    f = p - S.dcayinv(h*v)'*(S.J.*v);
```
Convergence Issues

The critical step is 2). How to guarantee that it succeeds?

- Define $\psi(\xi) = \mu_k - (d \tau_h^{-1})^{*} \partial \xi K(\xi)$ and solve $\psi(\xi) = 0$.

- Solution through Newton iterations $\xi = \xi - D\psi^{-1}(\xi) \cdot \psi(\xi)$

- Consider the error $e(\xi) = \frac{1}{2} \| \psi(\xi) \|^2$. The solution $\psi(\xi^*) = 0$ can be reached if for all iterates $\xi \in g$

  1. The Jacobian $D\psi(\xi)$ is full rank
  2. The Hessian $D^2e(\xi) = D\psi(\xi)^T D\psi(\xi) + \sum_i D^2\psi_i(\xi) \cdot \psi_i(\xi)$ has positive eigenvalues (i.e. local convexity)
Convergence Issues

The critical step is 2). How to guarantee that it succeeds?

- Define $\psi(\xi) = \mu_k - (d\tau_{h\xi}^{-1})^* \partial_\xi K(\xi)$ and solve $\psi(\xi) = 0$.
- Solution through Newton iterations $\xi = \xi - D\psi^{-1}(\xi) \cdot \psi(\xi)$
- Consider the error $e(\xi) = \frac{1}{2}\|\psi(\xi)\|^2$. The solution $\psi(\xi^*) = 0$ can be reached if for all iterates $\xi \in \mathfrak{g}$
  1. The Jacobian $D\psi(\xi)$ is full rank
  2. The Hessian $D^2e(\xi) = D\psi(\xi)^TD\psi(\xi) + \sum_i D^2\psi_i(\xi) \cdot \psi_i(\xi)$ has positive eigenvalues (i.e. local convexity)

Take the smallest eigenvalue of $H$ and plot (for the case $G = SO(3)$):
- blue: if positive (convexity)   red: negative (singularity/saddle)

truncated exponential map $\tau = \exp$  full derivative for $\tau = \text{cay}$
Examples: optimal control trajectories

Satellite
Car in a tunnel
LittleDog on terrain

latched-mode reconfiguration
dockgin maneuver
little dog
**Unified Framework**

Derivation of mechanics and optimal control through a common principle

![Diagram]

Discrete configurations, velocities, and discrete force:

- \( q_k \) and \( q_{k+1} \)
- \( v_k = q_{k+1} - q_k \)
- \( f_k \)
- \( \lambda_{N-1}, \lambda_N \)

- Feasible set of trajectories satisfy the dynamics:
  \[
  \overrightarrow{D}(L)(x_{k-1}) \rightarrow \overrightarrow{D}(L)(x_k) = f_k, \text{ for any forcing } f_k
  \]

- Optimal trajectory is then determined by:
  \[
  \overrightarrow{D}(L)(x_k) - \overrightarrow{D}(L)(x_{k-1}) = \lambda_k,
  \]
  higher-order Lagrangian \( \mathcal{L}(x) = \langle \overrightarrow{D}(L)(x_k), f_k \rangle - \langle \overrightarrow{D}(L)(x_k), f_{k+1} \rangle \)

Dynamics expressed through left and right *differential vector fields*\(^1\)

\[
\overrightarrow{D} : \mathcal{X} \rightarrow T\mathcal{X} \quad \text{and} \quad \overrightarrow{D} : \mathcal{X} \rightarrow T\mathcal{X}, \text{ where } \mathcal{X} \sim TQ
\]

\(^1\)Marrero, Martín de Diego, Martínez, 2006.
Vector Field Expressions

- **Vector Space**: state \( x = (q, v) \in TQ \)
  \[
  \mathbf{D}(q, v) = (\partial_v - \frac{h}{2}\partial_q)(q, v), \quad \mathbf{D}(q, v) = (\partial_v + \frac{h}{2}\partial_q)(q + hv, v).
  \]

- **Lie groups**: state \( x = (g, \xi) \in G \times g \)
  \[
  \mathbf{D}(x) = \left( (d\tau_{h\xi} )^* \frac{\partial}{\partial \xi} - \frac{h}{2}g^* \frac{\partial}{\partial g} \right)(g, \xi),
  \quad \mathbf{D}(x) = \left( (d\tau_{-h\xi} )^* \frac{\partial}{\partial \xi} + \frac{h}{2}g^* \frac{\partial}{\partial g} \right)(g\tau(h\xi), \xi).
  \]

- **Nonholonomic**: state \( x = (g, \xi, r, u) \in G \times g \times TM \) where \( (r, u) \in TM \)
  \[
  \mathbf{D}(x) = \left( w, (d\tau_{h\xi} )^* \eta - \frac{h}{2}g^* \frac{\partial}{\partial g} \right)(g, \xi, r, u),
  \quad \mathbf{D}(x) = \left( w, (d\tau_{-h\xi} )^* \eta + \frac{h}{2}g^* \frac{\partial}{\partial g} \right)(g\tau(h\xi), \xi, r + hu, u),
  \]
  where \( (w, \eta) \in TM \times g \) are regarded as sections of the vector bundle that act independently in two directions: vertical \( (w = 0, \eta \in s_r) \) and horizontal (of the form \( (w, \eta) = (\delta r, -A(r)\delta r) \)) according to the connection.
Vector Field Expressions

▶ Vector Space: state $x = (q, v) \in TQ$

$\overleftarrow{D}(q, v) = (\partial_v - \frac{h}{2} \partial_q)(q, v), \overrightarrow{D}(q, v) = (\partial_v + \frac{h}{2} \partial_q)(q + hv, v)$.

▶ Lie groups: state $x = (g, \xi) \in G \times \mathfrak{g}$

$\overleftarrow{D}(x) = \left( (d\tau_{h\xi}^{-1})^* \frac{\partial}{\partial \xi} - \frac{h}{2} g^* \frac{\partial}{\partial g} \right) (g, \xi),$

$\overrightarrow{D}(x) = \left( (d\tau_{-h\xi}^{-1})^* \frac{\partial}{\partial \xi} + \frac{h}{2} g^* \frac{\partial}{\partial g} \right) (g \tau(h\xi), \xi).$

▶ Nonholonomic: state $x = (g, \xi, r, u) \in G \times \mathfrak{g} \times TM$ where $(r, u) \in TM$

$\overleftarrow{D}(x) = \left( w, (d\tau_{h\xi}^{-1})^* \eta - \frac{h}{2} g^* \frac{\partial}{\partial g} \right) (g, \xi, r, u),$

$\overrightarrow{D}(x) = \left( w, (d\tau_{-h\xi}^{-1})^* \eta + \frac{h}{2} g^* \frac{\partial}{\partial g} \right) (g \tau(h\xi), \xi, r + hu, u),$

where $(w, \eta) \in TM \times \mathfrak{g}$ are regarded as sections of the vector bundle that act independently in two directions: vertical ($w = 0, \eta \in s_r$) and horizontal (of the form $(w, \eta) = (\delta r, -A(r)\delta r)$) according to the connection.

Can be generalized through the groupoid formalism (current work with D.M. de Diego and F. Jimenez)
Open Questions:

▶ Obtaining unified *regularity conditions* of mechanics and control based on the discrete differential vector fields defined in this section, i.e. loosely speaking

regularity of $L(x) \rightarrow \text{regular}$ mechanics solution,
regularity of $L(x) \rightarrow \text{regular}$ optimal control solution.

Regularity is directly related to convergence. A fruitful direction is to establish *a-priori* regions of attraction based on regularity. That will guarantee that a solution can be found with numerical *robustness*.
Open Questions:

- Obtaining unified *regularity conditions* of mechanics and control based on the discrete differential vector fields defined in this section, i.e. loosely speaking

  regularity of \( L(x) \rightarrow \) regular mechanics solution,
  regularity of \( L(x) \rightarrow \) regular optimal control solution.

Regularity is directly related to convergence. A fruitful direction is to establish *a-priori* regions of attraction based on regularity. That will guarantee that a solution can be found with numerical robustness.

- Is it possible to use discrete controllability to answer questions about controllability in general, since

  discrete controllability \( \xrightarrow{N \to \infty} \) continuous controllability
DGC: fast implementation for real-time systems

Demo: DGC Library, lietest
Randomized Global Motion Planning

How to deal with more complicated and constrained systems?
Idea: treat the set of solution trajectories as a probability space and solve optimal control through adaptive importance sampling.
Randomized Global Motion Planning

How to deal with more complicated and constrained systems?
Idea: treat the set of solution trajectories as a probability space and solve optimal control through adaptive importance sampling.

▶ A simple nonholonomic car

Randomized Global Motion Planning

How to deal with more complicated and constrained systems?
Idea: treat the set of solution trajectories as a probability space and solve optimal control through adaptive importance sampling.

- A simple nonholonomic car

- A double integrator

Randomized Global Motion Planning

How to deal with more complicated and constrained systems?
Idea: treat the set of solution trajectories as a probability space and solve optimal control through adaptive importance sampling.

▶ A simple nonholonomic car

▶ A double integrator

▶ A helicopter in an urban terrain

Caltech Small Satellite Testbed

Autonomous Assembly of Reconfigurable Space Telescope (AaREST) (collaboration between Caltech and University of Surrey, UK)

Reconfiguration Testbed: autonomous navigation and docking

prototype hardware (Univ. of Surrey)
BACKUPS
Necessary Conditions

The trajectory of a discrete mechanical system on a Lie group $G$ with algebra $\mathfrak{g}$ and Lagrangian $\ell(\xi) = \frac{1}{2} \langle \mathbb{I} \xi, \xi \rangle$, with fixed initial and final configurations and velocities $g(0) \in G$, $\xi(0) \in \mathfrak{g}$ and $g(T) \in G$, $\xi(T) \in \mathfrak{g}$, minimizes the total control effort only if the discrete body-fixed velocity curve $\xi_{0:N-1}$ satisfies the following

$$
\nu_k - \text{Ad}^*_{\tau(h\xi_{k-1})} \nu_{k-1} = 0, \quad k = 1, \ldots, N - 1 \quad (6a)
$$

$$
\tau^{-1}(\tau(h\xi_0) \cdots \tau(h\xi_{N-1}) \cdot (g(0)^{-1} g(T))^{-1}) = 0, \quad (6b)
$$

where:

$$
\nu_k = (d\tau^{-1}_{h\xi_k})^* \partial_{\xi} \mathcal{K}_{(\lambda_0:N,k)}(\xi_k), \quad (6c)
$$

$$
\mathcal{K}_{(\lambda_0:N,k)}(\xi_k) = \langle (d\tau^{-1}_{h\xi_k})^* \mathbb{I} \xi_k, \lambda_k - \text{Ad}_{\tau(h\xi_k)} \lambda_{k+1} \rangle / h, \quad (6d)
$$

$$
\lambda_0^b = 2 \left( \mu_0 - \mathbb{I} \xi(0) \right) / h, \quad (6e)
$$

$$
\lambda_k^b = \left( \mu_k - \text{Ad}^*_{\tau(h\xi_{k-1})} \mu_{k-1} \right) / h, \quad k = 1, \ldots, N - 1 \quad (6f)
$$

$$
\lambda_N^b = 2 \left( \mathbb{I} \xi(T) - \text{Ad}^*_{\tau(h\xi_{N-1})} \mu_{N-1} \right) / h, \quad (6g)
$$

$$
\mu_k = (d\tau^{-1}_{h\xi_k})^* \mathbb{I} \xi_k. \quad (6h)
$$

Note: The proposition defines $Nn$ equations (6a)-(6b) in the $Nn$ unknowns $\xi_0, \ldots, \xi_{N-1}$. A solution can be found using nonlinear root finding.