Waltzing Solitons

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Abstract

Goal: explain the cooperative “Waltzing” behaviour seen in space-time plots for the solutions of a coupled set of PDEs

\[
\frac{\partial t m}{=}- (vm)_x - mv_x \quad \text{with} \quad m = u - u_{xx}, \\
\frac{\partial t n}{=}- (un)_x - nu_x \quad \text{with} \quad n = v - v_{xx}.
\]
Abstract (cont)

The explanation of the “Waltzing” behaviour will come in the form of particle-like singular solutions.
Summary

Main message:

1. The particle-like solutions of these nonlocal nonlinear PDEs may be associated to *singular momentum maps*.
2. The waltzing is a type of *cooperative dynamics* of a two-species Hamiltonian particle system.

We follow previous work with Jerry Marsden (1942 - 2010).
How do transformations of manifolds by Lie groups lead to momentum maps?

A Lie group $G$ acts on a manifold $M$ by flows of vector fields.
Momentum Map & coadjoint motion on a Lie group

\[ T^*G \xrightarrow{\Phi_g(t)} T^*G \]

Equivariant Momentum Map

\[ J(0) \xrightarrow{\text{Ad}^*_g(t)^{-1}} J(t) \]

\[ g^* \xrightarrow{\text{Ad}^*_g(t)^{-1}} g^* \simeq T^*G/G \]

\[ \frac{dJ}{dt} = \pm \text{ad}_\xi J \text{ with } \xi \in g \] (EP eqn)


A Momentum Map $J : T^*M \mapsto \mathfrak{g}^*$ is defined for the Lie algebra action $\xi_M(q)$ of $\xi \in \mathfrak{g}$ on $q$ in manifold $M$ by the pairings

$$J^\xi(p, q) := \langle J(p_q), \xi \rangle_{\mathfrak{g}^* \times \mathfrak{g}}$$

$$= \langle p_q, \xi_M(q) \rangle_{T^*M \times TM}$$

where $p_q \in T_q^*M$ is the momentum at position $q \in M$ and $\xi_M(q)$ is the vector field tangent to the flow of $g(t) \in G$ at $q$.

$J^\xi(p, q)$ is the Hamiltonian for action $\xi_M(q)$ & its cotangent lift,

$$\frac{dq}{dt} = \{ q, J^\xi \} = \xi_M(q) \quad \text{and} \quad \frac{dp}{dt} = \{ p, J^\xi \} = -\frac{d\xi_M}{dq}^T \cdot p$$
Momentum maps evolve by the Euler-Poincaré eqn

The Euler-Poincaré (EP) equation for $J(Q, P)$ is

$$\frac{dJ}{dt} = \pm \text{ad}^*_{\xi} J, \quad J \in g^*, \quad \frac{dQ}{dt} = \xi(Q, t) \in g$$

The EP equation governs the evolution of the momentum map $J \in g^*$ derived from the cotangent lift of the action $\xi_M(Q)$ of the Lie algebra $\xi \in g$ on the manifold $M$ at point $Q \in M$.

The EP eqn is Hamiltonian, with a Lie-Poisson bracket,

$$\frac{dJ}{dt} = \{ J, J^\xi \}_\text{LP} \quad \text{for} \quad \{ F, H \}_\text{LP} = \mp \left\langle J, \left[ \frac{dF}{dJ}, \frac{dH}{dJ} \right] \right\rangle_{g^* \times g}$$

EP equation for Rigid bodies, Fluids, Solitons, Shapes

- EP eqn: \[ \frac{d}{dt} \frac{\delta l}{\delta u} + \text{ad}^* u \frac{\delta l}{\delta u} = 0 \] with \( l(u) = \frac{1}{2} \| u \|_g^2 \) & \( u \in \mathfrak{g} \)
- Rigid body: \( \Pi_t + \Omega \times \Pi = 0 \) with \( l(\Omega) = \frac{1}{2} \Omega \cdot I \Omega, \Omega \in \mathfrak{so}(3) \)
- Ideal Fluids: \( u_t + u \cdot \nabla u + \nabla p = 0 \) & \( \text{div} u = 0 \) (EPDiff vol)
- As a PDE: \( m_t + u \cdot \nabla m + (\nabla u)^T \cdot m + m \text{div} u = 0, m := \frac{\delta l}{\delta u} \)
- As an optimal control problem (boundary value problem):
  \[ \min_{Q(t), U(t)} \int_0^T \left[ \ell(Q, U) + \langle P, \dot{Q} - F(Q, U) \rangle \right] dt \]
Momentum map $J_{\text{Sing}} : T^* \text{Emb}(S, M) \to \mathfrak{x}(M)^*$

Let $u \in \mathfrak{x}(M)$ for left action of vector fields $\mathfrak{x}$ on $M$. Compute:

$$\left\langle J_{\text{Sing}}(Q, P), u \right\rangle_{\mathfrak{x}^* \times \mathfrak{x}} = \left\langle (Q, P), u \circ Q \right\rangle = \int_S P_i(s) u^i(Q(s)) d^k s$$

$$= \int_M J_{\text{Sing}} \cdot u(x) d^n x = \int_M \int_S \left( P_i(s) \delta(x - Q(s)) d^k s \right) u^i(x) d^n x$$

Thus, $J_{\text{Sing}} : T^* \text{Emb}(S, M) \to \mathfrak{x}(M)^*$ is given by

$$J_{\text{Sing}}(Q(t), P(t)) = \int_S P(s, t) \delta(x - Q(s, t)) d^k s =: m(x, t)$$

with EPDiff: $\frac{dm}{dt} = -\text{ad}^*_u m = -u \cdot \nabla m - (\nabla u)^T \cdot m - m \text{ div } u$
The CH Lagrangian is \( L(u) = \frac{1}{2} \int_{\mathbb{R}} u^2 + u_x^2 \, dx = \frac{1}{2} \|u\|_{H^1}^2 \).

For this Lagrangian, the TW velocity profile is the Peakon:

\[
m = \frac{\delta L}{\delta u} = u - u_{xx}, \text{ or } u = K \ast m \text{ with } K(x, y) = \frac{1}{2} e^{-|x-y|}
\]

\[
m_t = -(mu)_x - mu_x, \quad \text{(CH/EPDiff in 1D)}
\]

\[
\frac{d}{dt} \frac{\delta L}{\delta u} = -\text{ad}_u^* \frac{\delta L}{\delta u} \quad \text{(EPDiff in 2D)}
\]
Cross coupled CH solutions – Waltzing peakons

The Lagrangian for cross coupled CH (CCCH) is

\[ l(u, v) = \int_{\mathbb{R}} uv + u_x v_x \, dx \]

The corresponding two EPDiff equations in 1D are

\[ \partial_t m = - \text{ad}^* v \, m = - (vm)_x - mv_x \quad \text{with} \quad m := \frac{\delta l}{\delta v} = u - u_{xx} , \]

\[ \partial_t n = - \text{ad}^* u \, n = - (un)_x - nu_x \quad \text{with} \quad n := \frac{\delta l}{\delta u} = v - v_{xx} . \]

Cross coupled CH: Localized solutions

Figure: Coherent excitations in $u$ showing the collision of a propagating peakon pair with a stationary single peakon (left) and the overtaking collision of two propagating peakon pairs. Figures courtesy of James Percival.
CCCH momentum maps for Waltzing Peakons

The CCCH Hamiltonian with \( K(x, y) = \frac{1}{2} e^{-|x-y|} \) is

\[
h(n, m) = \int_{\mathbb{R}} n K \ast m \, dx = \int_{\mathbb{R}} m K \ast n \, dx
\]

The corresponding two EPDiff equations in 1D on \( \mathbb{R} \) are

\[
\partial_t m = - \text{ad}_{\delta h/\delta m}^* m = - (vm)_x - mv_x \quad \text{with} \quad v := \frac{\delta h}{\delta m} = K \ast n,
\]

\[
\partial_t n = - \text{ad}_{\delta h/\delta n}^* n = - (un)_x - nu_x \quad \text{with} \quad u := \frac{\delta h}{\delta n} = K \ast m.
\]

This Hamiltonian system has two singular momentum maps

\[
m(x, t) = \sum_{a=1}^{M} m_a(t) \delta(x - q_a(t)), \quad n(x, t) = \sum_{b=1}^{N} n_b(t) \delta(x - r_b(t))
\]
Hamiltonian dynamics of Waltzing Peakons

Hamilton’s equations for canonical pairs \((q_a, m_a)\) and \((r_a, n_a)\)

\[
H = \frac{1}{2} \sum_{a,b=1}^{M,N} m_a(t)n_b(t)e^{-|q_a(t)-r_b(t)|}
\]

govern the dynamics of the two singular momentum maps

\[
m(x, t) = \sum_{a=1}^{M} m_a(t) \delta(x - q_a(t))
\]

and

\[
n(x, t) = \sum_{b=1}^{N} n_b(t) \delta(x - r_b(t))
\]

The singular momentum-map solutions explain Hamiltonian cooperative dynamics (waltzing!) of localised coherent collective degrees of freedom for the coupled set of PDEs:

\[
\begin{align*}
\partial_t m &= - (vm)_x - mv_x \quad \text{with} \quad m = u - u_{xx}, \\
\partial_t n &= - (un)_x - nu_x \quad \text{with} \quad n = v - v_{xx}.
\end{align*}
\]
Cross coupled CH collisions of singular solutions

Figure: Singular solution behaviour for interactions of three cross coupled EP parabolic dipole compactons and one single compacton. (Movie)
Cross coupled CH collisions of 12 singular solutions

Figure: London Underground Map: Singular solution behaviour for interactions of 12 cross coupled EP parabolic dipole compactons.
Three key points of the lecture

Point #1:
EPDiff admits a singular momentum map
\[ J_{\text{Sing}} : T^* \text{Emb}(S, M) \to \mathfrak{X}(M)^*. \]

Point #2:
In the problem discussed here, \( J_{\text{sing}} \) for multiple species describes Hamiltonian cooperative dynamics (waltzing!) of the coherent collective degrees of freedom encoded in \( J_{\text{sing}} \).

Point #3:
The waltzing behavior is seen as a braiding between the space-time trajectories of two different species.
Thanks for listening!
Metamorphosis & integrable CH(2,1)

Metamorphosis for 1D images yields the CH(2,1) system for momentum map \( m = u - u_{xx} \) and template intensity \( \rho \),

\[
m_t + (mu)_x + mu_x = -\rho \rho_x \quad \text{with} \quad \rho_t + (\rho u)_x = 0.
\]

This is a completely integrable Lie-Poisson Hamiltonian system that admits soliton solutions similar to shallow water waves. **CH(2,1) has a Lax pair.** That is, it follows from compatibility \( \psi_{xxt} = \psi_{txx} \) of the isospectral \( d\lambda/dt = 0 \) linear system

\[
\psi_{xx} = \left( \frac{1}{4} - m\lambda - \rho^2 \lambda^2 \right) \psi,
\]

\[
\psi_t = -\left( u + \frac{1}{2\lambda} \right) \psi_x + \frac{1}{2} u_x \psi.
\]

BiHamiltonian, infinite integrable hierarchy, etc., but no singular solutions.
The link to the CH(n,k) sequence of soliton equations

The compatibility condition $\psi_{xxt} = \psi_{txx}$ for

$$\psi_{xx} = Q(x, \lambda)\psi, \quad \text{and} \quad \psi_t = -U(x, \lambda)\psi_x + \frac{1}{2} U_x(x, \lambda)\psi,$$

gives the EP equation on the Bott-Virasoro group in 1D,

$$Q_t + (QU)_x + QU_x = \frac{1}{2} U_{xxx}.$$

Then setting

$$Q(x, \lambda) = \frac{1}{4} + \sum_{i=1}^{n} \lambda^i q_i(x, t) \quad \text{and} \quad U(x, \lambda) = \sum_{j=1}^{k} \lambda^{-j} u_j(x, t)$$

generates CH(n,k) sequence of integrable equations, n,k$\in\mathbb{Z}$.

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Figure: Dam-break results for the CH(2,1) system show evolution of the density $\rho$ (left panel) and velocity $u$ (right panel) in a periodic domain. Figures are courtesy of L Ó Náraigh.
**CH(2,-1) soliton solutions**

**Figure:** Dam-break results for the CH(2,-1) system for $\rho$ (left) and $m$ (right), arising in a periodic domain. Soliton solutions are seen to emerge and propagate in both directions. Figures are courtesy of V Putkaradze.
Thanks again!