INTRODUCTION to MARSDEN and SYMMETRY

Second Edition

Jerrold E. Marsden ▪ Tudor S. Ratiu

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"Geometric mechanics"

(-2010) V.I. Arnold, J.J. Duistermaat, J.E. Marsden

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**MR2593352** Kobilarov, Marin; Marsden, Jerrold E.; Sukhatme, Gaurav S.
Geometric discretization of nonholonomic systems with symmetries.

**MR2542960** (2011b:70020) Bloch, Anthony M.; Marsden, Jerrold E.; Zenkov, Dmitry V.
Quasivelocities and symmetries in non-holonomic systems.

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Stability and drift of underwater vehicle dynamics: mechanical systems with rigid motion symmetry. *Phys. D 105* (1997), no. 1-3, 130–162. (Reviewer: George W. Patrick), *70K20* (58F40 70E15 70H05 70H33 70Q05 93D05)

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MR1444343 (98e:49049) Koon, Wang-Sang; Marsden, Jerrold E.
NOTES AND PROBLEMS

This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P. Q.

A THEOREM ON HARMONIC HOMOLOGIES

J. E. Marsden *

Introduction. A collineation is a one-one mapping of a projective plane onto itself, taking points into points, lines into lines and preserving incidence. ([1], p. 247). A perspective collineation (sometimes called a central collineation) is a collineation which leaves invariant all points on a line h called the axis, and all lines through a point H called the centre. The perspective collineation is an elation if H is incident with h; otherwise it is a homology.
The case (c) proceeds in a way similar to (b) with repeated use of (a).

Note that for $n > 2$, the harmonic homologies appearing may not be permuted amongst themselves.

Much of the above carries through for distributions (in the sense of Schwartz) as well as that in [1]. These results will be given in the near future [3].

**REFERENCES**


University of Toronto
A CORRESPONDENCE PRINCIPLE FOR
MOMENTUM OPERATORS

J. E. Marsden

(received October 1, 1966)

1. Introduction. The purpose of this note is to give a
precise formulation of the correspondence principle between
classical and quantum mechanics, for the case of momentum
operators. (See e.g. I. Segal, Journ. Math. Phys. (1960), 475).

We follow Abraham - Marsden [1] for notation.

2. Induced Vectorfields. Let $M$ be a manifold (configuration space) and $T^*M$ the cotangent bundle (phase space)
with the natural symplectic structure, [1, §14]. Classical mechanics deals with Hamiltonian vectorfields on $T^*M$, while quantum
mechanics is concerned with linear operators on $\mathcal{F}(M)$, the
smooth functions on $M$. Notice that if $X$ is a vectorfield on
$M$, then $i L_X$ may be regarded as a self adjoint operator; (see
[1], exercises for §12).
Marsden, J. E. (1967)
A Correspondence Principle for Momentum Operators

Proof. From [1, §14], we have \( \{ P, f^* \} = -L_x f^* = -X^*(f^*) = -X(f)^* \); the latter equality following from 3 above. This proves the result.

Much of the above carries through for distributions (in the sense of Schwartz) as well as that in [1]. These results will be given in the near future [3].

6. Remarks. It is shown in [1] that the usual linear and angular momenta correspond to the linear and angular momentum operators in quantum mechanics, in the sense we have defined here. Hence it is no surprise that both satisfy the same commutation rules. It is easy to see that \( P(\alpha^m) = \alpha^m \cdot X(m) \). See Sternberg [4], p. 146. This is exploited in [3].

REFERENCES


Our purpose is to generalize Hamiltonian mechanics to the case in which the energy function (Hamiltonian), $H$, is a distribution (generalized function) in the sense of Schwartz. We follow the same general program as in the smooth case. Familiarity with the smooth case is helpful, although we have striven to make the exposition self-contained, starting from calculus on manifolds.
Our purpose is to give an exposition of the foundations of non-linear conservative mechanical systems with an infinite number of degrees of freedom. Systems we have in mind are the vibrating string, the electromagnetic field and quantum mechanics. These are all linear. We also outline a non-linear example, the coupled Maxwell and Dirac fields. Perfect fluids will be discussed elsewhere.

-Serious consideration of analytic issues-
Marsden, J. E. (1968)
Generalized Hamiltonian Mechanics
Archive for Rational Mechanics and Analysis, 28 (5) 323–361
Marsden, J. E. (1968)

**Generalized Hamiltonian Mechanics**

*Archive for Rational Mechanics and Analysis, 28 (5) 323–361*

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**Introduction**

Our purpose is to generalize Hamiltonian mechanics to the case in which the energy function (Hamiltonian), $H$, is a distribution (generalized function) in the sense of Schwartz. We follow the same general program as in the smooth case; see Abraham [1]. Familiarity with the smooth case is helpful, although we have striven to make the exposition self-contained, starting from calculus on manifolds.

The physical motivation for generalized mechanics is clear. Many systems in use actually have singular Hamiltonians. Perhaps the most famous example is that of hard spheres in a box, extensively studied by Sinai. The potential energy in this case is a surface delta function on the walls of the box and on each sphere. Since the usual version of Hamilton's equations break down (even as distributional equations), we view the flow as a limit of smooth flows. Unfortunately, the variational theorems usually fail in the non-smooth case as is seen from elementary examples; see Marsden [5].
Offprint from “Archive for Rational Mechanics and Analysis”,
Volume 28, Number 5, 1968, P. 362–396

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Hamiltonian One Parameter Groups

A Mathematical Exposition of Infinite Dimensional Hamiltonian Systems
with Applications in Classical and Quantum Mechanics

J. E. Marsden

Communicated by C. Truesdell
Introduction

Our purpose is to give an exposition of the foundations of non-linear conservative mechanical systems with an infinite number of degrees of freedom. Systems we have in mind are the vibrating string, the electromagnetic field and quantum mechanics. These are all linear. We also outline a non-linear example, the coupled Maxwell and Dirac fields. Perfect fluids will be discussed elsewhere.

The general Hamiltonian formalism is motivated by the finite dimensional case; see Abraham [1], although the usual difficulties with unbounded operators prevent an exact analogy.
Marsden, J. E. (1969)
Hamiltonian Systems with Spin
Canadian Mathematical Bulletin, pp. 03–208

\[ \mathfrak{so}(3), \mathfrak{su}(2), \ldots \text{“momentum map”} \]
Ebin, David G and Marsden, Jerrold (1970)
Groups of diffeomorphisms and the motion of an incompressible fluid
Annals of Mathematics, 92 (1), 102–163

Focus attention on the group itself.

\[ J, I, \text{ etc. for equations of Euler, Navier-Stokes on manifolds} \]
Fischer, A. E. and Marsden, Jerrold E. (1972)
The Einstein Equations of Evolution—A Geometric Approach
Journal of Mathematical Physics, 13 (4) pp. 546–568

(Cauchy) metric on $M^4$  $\leftrightarrow$ evolution in $M^4 = \text{metrics on } \mathbb{V}^3$

Symmetry $\Theta \times \mathbb{T}$  (shift vector, lapse function)
Marsden, Jerrold E. and Ebin, David G and Fischer, Arthur E. (1972)

*Diffeomorphisms, hydrodynamics and relativity*

in the *Proceedings of the 13th Biennial Semina of the Canadian Mathematical Congress*, pp. 135–279

... Synthesis of the two. Central role of diffeomorphism groups.
We discuss completeness for pseudo-riemannian manifolds, and give new examples of non-complete compact manifolds. The former are simply connected, the latter locally homogeneous.
REDUCTION OF SYMPLECTIC MANIFOLDS WITH SYMMETRY

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(Received September 15, 1972)

We give a unified framework for the construction of symplectic manifolds from systems with symmetries. Several physical and mathematical examples are given; for instance, we obtain Kostant’s result on the symplectic structure of the orbits under the coadjoint representation of a Lie group. The framework also allows us to give a simple derivation of Smale’s criterion for relative equilibria. We apply our scheme to various systems, including rotationally invariant systems, the rigid body, fluid flow, and general relativity.
\[ P, G \quad \Psi : P \rightarrow g^*, \quad \text{momentum map} \]

\[ P_\mu = \Psi^{-1}(\mu)/G_\mu \quad \text{reduced manifold} \]

\[ \text{(Smale, Meyer)} \]

\[ \text{e.g.} \quad G_\mu \leq g^* \cong (T^*G)_\mu. \]

\[ \text{Jacobi elimination of the nodes} \]

Fluids, relativity

Stability of relative equilibria

\[ \text{(Poincaré, Arnold)} \]
Symmetry and Bifurcations of Momentum Mappings*

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Abstract. The zero set of a momentum mapping is shown to have a singularity at each point with symmetry. The zero set is diffeomorphic to the product of a manifold and the zero set of a homogeneous quadratic function. The proof uses the Kuranishi theory of deformations. Among the applications, it is shown that the set of all solutions of the Yang-Mills equations on a Lorentz manifold has a singularity at any solution with symmetry, in the sense of a pure gauge symmetry. Similarly, the set of solutions of Einstein’s equations has a singularity at any solution that has spacelike Killing fields, provided the spacetime has a compact Cauchy surface.
The Hamiltonian structure of the Maxwell-Vlasov equations
*Physica D, 4 (3)*, pp. 394–406

Morrison [25] has observed that the Maxwell-Vlasov and Poisson-Vlasov equations for a collisionless plasma can be written in Hamiltonian form relative to a certain Poisson bracket. We derive another Poisson structure for these equations by using general methods of symplectic geometry. The main ingredients in our construction are the symplectic structure on the co-adjoint orbits for the group of canonical transformations, and the symplectic structure for the phase space of the electromagnetic field regarded as a gauge theory. Our Poisson bracket satisfies the Jacobi identity, whereas Morrison's does not [37]. Our construction also shows where canonical variables can be found and can be applied to the Yang-Mills-Vlasov equations and to electromagnetic fluid
Hamiltonian structures and stability for rigid bodies
with flexible attachments
Archive for Rational Mechanics and Analysis, 98 (1), pp. 71–93

Stability Analysis of a Rigid Body with a Flexible Attachment Using
the Energy–Casimir Method
In Differential Geometry: the interface between pure and applied mathematics
American Mathematical Society, Providence, RI San Antonio, TX (1987),
Contemporary Mathematics 68, pp. 71–93
Abstract

We consider a system consisting of a rigid body to which a linear extensible shear beam is attached. For such a system the Energy–Casimir method can be used to investigate the stability of the equilibria. In the case we consider, it can be shown that a test for (formal) stability reduces to checking the positive definiteness of two matrices which depend on the parameters of the system and the particular equilibrium about which the stability is to be ascertained.

1 Introduction

We consider a rigid body to which a long, flexible appendage is attached. A coordinate reference frame is fixed in the rigid body with the origin at the center of mass of the rigid body. The flexible attachment is assumed to lie along the second coordinate axis when the configuration is at rest. (see Figure 1.) The equations of motion for such a configuration, under suitable assumptions and with the appendage modeled as a linear extensible shear beam, are derived by Krishnaprasad and Marsden in [2]. In deriving the equations of motion they use Hamiltonian methods in the context of Poisson manifolds and reduction. (see [2] for the explicit formula for the Poisson brackets involved.) If we assume that the momentum of the system which arises from the appendage rotating with the

Stability Analysis of a Rigid Body with a Flexible Attachment Using
the Energy–Casimir Method

In Differential Geometry: the interface between pure and applied mathematics
American Mathematical Society, Providence, RI San Antonio, TX (1987),
Contemporary Mathematics 68, pp. 71–93

T. Posbergh, P.S. Krishnaprasad and J. Marsden

Figure 1: The Geometry of the Configuration

rigid body is negligible, then our Hamiltonian is of the form

\[ H = \frac{1}{2} \mathbf{J}^{-1} \mathbf{p} \cdot \mathbf{p} + \frac{1}{2} \int_0^\varepsilon \left| \frac{\mathbf{m}(s)}{\rho_0} \right|^2 ds + \frac{1}{2} \int_0^\varepsilon \mathbf{K} \frac{\partial \mathbf{r}}{\partial s} \cdot \frac{\partial \mathbf{r}}{\partial s} ds. \] (1)

We assume that \( \mathbf{J} \) is the inertia matrix of the rigid body and that \( \rho_0 \) is the
uniform mass per unit length of the attached appendage of length \( L \). The reduced
phase space is coordinated at any time by \( \omega \), the convected angular velocity
vector of the rigid body; \( \mathbf{r}(s) \), the convected displacement of the shear beam at
a point \( s \), \( 0 \leq s \leq \ell \); and \( \mathbf{m}(s) \) the momentum density of the shear beam at
the point \( s \). The vector \( \mathbf{p} \) is the body angular momentum vector of the rigid body,
thus \( \mathbf{p} = J \omega \). Finally, \( \mathbf{K} \) is the diagonal matrix of elastic coefficients.

In our investigation we are interested in the stability of the system about
equilibria points. These equilibria will satisfy,
Symmetry, Stability, Geometric Phases, and Mechanical Integrators (Part II)

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J. E. Marsden*, O. M. O'Reilly†, F. J. Wicklin‡, B. W. Zombro§

[Part I of this paper appeared in *NLS* 1:1, pp. 4–11.]

**Geometric Phases**

The application of the methods described in Part I is still in its infancy, but the previous example clearly indicates the power of reduction and suggests that the REMM will be applied to dynamic problems in many fields, including chemistry, quantum and classical physics, and engineering. Apart from the computational simplification afforded by reduction, reduction also permits us to put into a mechanical context a concept known as the geometric phase, or *holonomy*.

A well-known example of holonomy is the Foucault pendulum. During a single rotation of the earth, the plane of the pendulum’s oscillations is shifted by an angle which depends only on the latitude of the pendulum’s location. Specifically, if a pendulum located at latitude $\alpha$ is swinging in a plane, then after twenty-four hours, the plane of its oscillations will have shifted by an angle of $-2\pi \sin \alpha$. This holonomy is a result of parallel translation: if an orthonormal coordinate frame undergoes parallel transport along a line of latitude $\alpha$, then after one revolution the frame will have rotated by an amount equal to the phase shift of the Foucault pendulum. (See Figure 6.)

Geometrically, the holonomy of the Foucault pendulum is equal to the solid angle swept out by the pendulum’s axis during one rotation of the earth. Thus a pendulum at the north pole of the earth will experience a holonomy of $-2\pi$, whereas a pendulum on the earth’s equator experiences no holonomy. Both of these results are with respect to the laboratory frame.

A less familiar example of holonomy was presented by Hannay [1985] and discussed further by Berry [1985, 1988]. Consider a frictionless, non-circular, planar hoop of wire on which is placed a small bead. The bead is set in motion and allowed to slide along

the wire at a constant speed. Clearly the bead will return to its initial position after, say, $T$ seconds, and will continue to return every $T$ seconds after that. Suppose however, that the wire hoop is slowly rotated in its plane by 360 degrees while the bead is in motion. At the end of the rotation, the bead is not in the location where we might expect it, but instead will be found at a shifted position which is completely determined by the shape of the hoop. In fact, the shift in position depends only on the length of the hoop, $L$, and on the area it encloses, $A$. The shift is approximately given by $8\pi^2 A/L^2$ as an angle, or by $4\pi A/L$ as length. (See Hannay [1985] or Marsden, Montgomery, and Ratiu [1990] for a derivation of these formulas.) To be completely concrete, if the bead’s initial position is marked with a tick and if the time of rotation is a multiple of the bead’s period, then at the end of rotation the bead is found $4\pi A/L$ units from its initial position. This is shown in Figure 7. We remark that if the hoop is circular then the angular shift is $2\pi$ and so the holonomy is not observable.

There is a similar explicit formula for the freely rotating rigid body. Suppose that a rigid body has spatial angular momentum given by the vector $\mu$ and has total energy $E$ given by (RBH). If the (reduced) trajectory on the angular momentum sphere (Figure 4) is periodic with period $T$ then the trajectory must enclose some surface area, $S$ on this sphere. A formula of Montgomery’s (see,

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**Figure 6:** The parallel transport of a coordinate frame along a curved surface (after Arnold [1978]).
known as the dynamic phase and depends explicitly on the system's energy and the period of the reduced trajectory. It is possible to observe the holonomy of a rigid body with a simple experiment. Put a rubber band around a book so that the cover will not open. (A "tall," thin book works best.) With the front cover pointing up, gently toss the book in the air so that it rotates about its middle axis, as shown in Figure 8. Catch the book after a single rotation and you will find that it has also rotated by 180 degrees about its long axis—that is, the front cover is now facing the floor! (Cushman and others have given a careful analysis of this problem.)

There are further examples of familiar everyday occurrences which demonstrate holonomy. We have already mentioned the fact that a falling cat often manages to land upright, and can even accomplish this feat if released while upside down with total angular momentum zero. Montgomery [1990] treated the cat as a deformable body and characterized the deformations which allow a cat to reorient itself without violating conservation of angular momentum. In showing that such deformations are possible, Montgomery casts the falling cat problem into geometric language. Let the shape of a cat refer to the location of the cat's body parts relative to each other, but without regard to the cat's orientation in space. Let the configuration of a cat refer both to the cat's shape and to its orientation with respect to some fixed reference frame. More precisely, if $Q$ is the configuration space and $G$ is the group of rigid motions, then $Q/G$ is the shape space.

For example, if the cat is completely rigid then it will always have the same shape, but we can give it a different configuration by rotating it through, say, 180 degrees about some axis. If we require that the cat have the same shape at the end of its fall as it had at the beginning, then the cat problem may be formulated as follows: Given an initial configuration, what is the most efficient way for a cat to achieve a desired final configuration if the final shape is required to be the same as the initial shape? It turns out that the solution of the falling cat problem is closely related to Wong's equations, which describe the motion of a particle in a Yang-Mills field (Montgomery [1990], Wilczek [1988], and Shapere [1989]).

Geometrically, the picture for the falling cat problem is analogous to that presented earlier for a rigid body. We think of the cat as tracing out some path in configuration space during its fall. The projection of this path onto the shape space results in a trajectory in the shape space, and the requirement that the cat's initial and final shapes are the same means that the trajectory is a closed loop. Furthermore, if we want to know the most efficient configuration path which satisfies the initial and final conditions, then we want to find the shortest path with respect to a metric induced by the function we wish to minimize.

Intuitively, we may define holonomy as a difference between the initial and final configuration of a system which results from a cyclic change of the system's shape. A simple example (due to Cherry [1989] and shown in Figure 9) is to stand with your arms at your side, your palm facing forward, and your thumb facing out. Keeping your arm straight, lift your arm sideways until it is

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**Figure 9:** An example of holonomy. Although the arm completes a cycle in its shape space, there is a 180 degree rotation in the configuration space.
This is the first paper of a five part work in which we study the Lagrangian and Hamiltonian structure of classical field theories with constraints. Our goal is to explore some of the connections between initial value constraints and gauge transformations in such theories (either relativistic or not). To do this, in the course of these four papers, we develop and use a number of tools from symplectic and multisymplectic geometry. Of central importance in our analysis is the notion of the "energy-momentum map" associated to the gauge group of a given classical field theory. We hope to demonstrate that many different and apparently unrelated facets of field theories can be thereby tied together and understood in an essentially new way. In Part I we develop some of the basic theory of classical fields from a spacetime covariant viewpoint. We begin with a study of the covariant Lagrangian and Hamiltonian formalisms, on jet bundles and multisymplectic manifolds, respectively. Then we discuss symmetries, conservation laws, and Noether's theorem in terms of "covariant momentum maps."

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Momentum Maps
and
Classical Fields

Part I: Covariant Field Theory

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ABSTRACT. The paper develops discretization schemes for mechanical systems for integration and optimization purposes through a discrete geometric approach. We focus on systems with symmetries, controllable shape (internal variables), and nonholonomic constraints. Motivated by the abundance of important models from science and engineering with such properties, we propose numerical methods specifically designed to account for their special geometric structure. At the core of the formulation lies a discrete variational principle that respects the structure of the state space and provides a framework for constructing accurate and numerically stable integrators. The dynamics of the systems we study is derived by vertical and horizontal splitting of the variational principle with respect to a nonholonomic connection that encodes the kinematic constraints and symmetries. We formulate a discrete analog of this principle by evaluating the Lagrangian and the connection at selected points along a discretized trajectory and derive discrete momentum equation and discrete reduced Euler-Lagrange equations resulting from the splitting of this principle. A family of nonholonomic integrators that are general, yet simple and easy to implement, are then obtained and applied to two examples—the steered robotic car and the snakeboard. Their numerical advantages are confirmed through comparisons with standard methods.
1. Introduction. The goal of this paper is to develop integrators for mechanical systems subject to nonintegrable constraints on the velocities, i.e., nonholonomic constraints. We study systems that evolve on a configuration manifold $Q = M \times G$ constructed from a Lie group $G$ whose action leaves the kinetic energy invariant (and so $G$ is a group of symmetries) and a vector space $M$ that describes the system internal shape. This general configuration space applies to systems from several domains, e.g., locomotion systems found in nature [23, 28, 22], vehicles used in robotics and aerospace [34, 35, 2, 5], systems in molecular dynamics [20, 37].

Their dynamics is derived by explicitly factoring out the group invariance through reduction by symmetry, and consequently splitting the equations of motion into vertical—corresponding to symmetries aligned with the constraints and defining the

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Key words and phrases. Nonholonomic systems, discrete mechanics, variational integrators, symmetries, reduction.

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5. Examples.

5.1. Car with simple dynamics. We study the kinematic car model defined in [23] with added simple dynamics (Fig. 3). The configuration space is $Q = S^1 \times S^1 \times SE(2)$ with coordinates $q = (\psi, \sigma, \theta, x, y)$, where $(\theta, x, y)$ are the orientation and position of the car, $\psi$ is the rolling angle of the rear wheels, and $\sigma$ is defined by $\sigma = \tan(\phi)$ where $\phi$ is the steering angle. The car has mass $m$, rear wheel inertia $I$, rotational inertia $K$, and we assume that the steering inertia is negligible. The car is controlled by rear wheels torque $f^\psi$ and steering velocity $u^\sigma$. The Lagrangian is then expressed as:

$$L(q, \dot{q}) = \frac{1}{2} \left( I \dot{\psi}^2 + K \dot{\theta}^2 + m(\dot{x}^2 + \dot{y}^2) \right),$$

and the constraints (see [23]) are

$$\cos \theta dx + \sin \theta dy = \rho d\psi,$$

$$-\sin \theta dx + \cos \theta dy = 0,$$

$$d\theta = \frac{\rho}{I} \sigma d\psi,$$
where $l$ is the distance between front and rear wheel axles, and $\rho$ is the radius of the wheels. These constraints simply encode the fact that the car must turn in the direction in which the front wheels are pointing, that the car cannot slide sideways, and that the change in orientation is proportional to the steering angle and turning rate of the wheels.

Note now that for any element $g = (\alpha, a, b)$ of $\text{SE}(2)$, the action $\Phi_g(q) = (\phi, \phi_L, \theta + \alpha, a + \cos(\alpha)x - \sin(\alpha)y, b + \sin(\alpha)x + \cos(\alpha)y)$ leaves the Lagrangian and constraints invariant. As the shape coordinates are $r = (\psi, \sigma)$, the reduced Lagrangian thus becomes

$$\ell(r, u, \xi) = \frac{1}{2} \left( u^T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} u + \xi^T \begin{bmatrix} K & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \xi \right),$$

where $\xi$ is used as a vector in $\mathbb{R}^3$ of coordinates with respect to the standard Lie algebra basis (see App.B).

The matrix representation of the connection $A$ dependent on $r$ becomes:

$$[A(r)] = \begin{bmatrix} -\xi \sigma & 0 \\ -\rho & 0 \\ 0 & 0 \end{bmatrix}$$

(26)
6. **Conclusion.** This paper was concerned with the discretization of nonholonomic mechanical systems through a discrete variational approach. Our main contribution was the derivation of reduced integrators for systems with Lie group structure and internal controllable shape dynamics. The geometric nature of the integrators enabled us to identify a discrete nonholonomic momentum map and discrete constraint forces which in certain cases have properties similar to their continuous analogs. Numerical comparisons with standard integration methods as well as other nonholonomic integrators revealed that the variational and reduced nature of the proposed algorithms contributes to stable and accurate integration even at larger time steps. It would be useful to investigate the nature of these numerical results further through backward error analysis and to also establish a notion of optimality of the chosen discretization. Further insight is still necessary to precisely define the notions of a discrete curvature of a connection as well as discrete constraint forces. It is interesting to determine how the proposed integrators fit in the more general framework of Lie groupoids [18] and whether some of the raised issues can be explained through this more general viewpoint.