Transport and resonance transitions from time-frequency analysis

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Outline

- Frequency map analysis:
  - Use of Fourier transform to characterize the trajectories.
  - It works for quasiperiodic trajectories.
  - Something else is needed for chaotic trajectories.

- Extension of the frequency map analysis for systems that are not nearly integrable, or that are not in action-angle variables:
  - New method of time-frequency analysis based on wavelets.
  - Definition of the instantaneous frequency.
  - Extraction of the instantaneous frequency using wavelet transform.

Outline (Cont.)

- Transport mechanism in the spatial circular restricted three body problem: resonance transitions.
  - Calculation of time-varying frequencies associated with numerical solutions.
  - Determination of resonance islands, temporary resonance trappings, transport between exterior and interior resonances.

Frequency Map

Completely integrable system expressed in action-angle variables,

$$H(I,\theta) = H_0(I).$$

The motion is confined to a torus:

$$\begin{aligned}\dot{I}_k &= 0\\ \dot{\theta}_k &= \frac{\partial H_0}{\partial I_k}(I^0) =: \omega_k(I^0)\end{aligned}$$

$$\begin{aligned}&\Rightarrow I_k(t) = I_k^0,\\ &\Rightarrow \theta_k(t) = \omega_k(I^0)t + \theta_k^0, \qquad k = 1, \ldots, n.\end{aligned}$$

- **Frequency Map**:

$$I \to \omega(I).$$

The dynamics on the torus is completely characterized by the frequencies:

- $\omega_1, \omega_2, \ldots, \omega_n$ satisfy a ***resonance relation*** if for some integer numbers m_k:

$$m_1\omega_1 + m_2\omega_2 + \cdots + m_n\omega_n = 0.$$

Torus is filled with lower dimensional tori.

- ***Nonresonant*** torus:

$$m \cdot \omega \neq 0, \quad \text{for all integer vector } m.$$

Trajectories are dense in the torus: quasiperiodic.

- For an integrable Hamiltonian $H_0(I)$:

  Frequency Map:

$$I \to \omega(I) = \partial H_0 / \partial I.$$

  The frequency map can be extracted from the solutions in complex coordinates:

$$z_k(t) = \sqrt{2I_k^0}\, e^{i\, \omega_k(I^0) t + \theta_k^0}.$$

$$\Longrightarrow \textit{Fourier Analysis}$$

- For nearly integrable systems, most of the tori remain: KAM Theorem.

  We can still associate frequencies with quasiperiodic trajectories.
- If we can't associate frequencies, then the trajectory is chaotic.
- What if the system is not in action-angle variables?
- What if the system is not nearly integrable?

Frequency Analysis

- Laskar (1990, 1993) proposed an efficient algorithm based on Fourier analysis to find the frequencies of the time series obtained as solutions of nearly integrable Hamiltonian systems.
- Assuming that chaotic trajectories seem regular in short intervals of time, he divided the trajectory in several subintervals, and use Fourier analysis to extract the frequency in each of them.
- We extend frequency map analysis to systems that
  - are not nearly integrable, or
  - that are not in action angle variables.

Instantaneous frequency

Representation of a function f in the form

$$f(t) = a(t)\, e^{i\,\phi(t)} \Rightarrow^{?} \omega(t) = \frac{1}{2\pi}\phi'(t).$$

Instantaneous frequency is uniquely defined for ***analytic signals***.

- $f = u + i\, v$ is an analytic signal if

$$H\left[u(t)\right] = v(t).$$

where the Hilbert transform is $H\,u(t) = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{u(\eta)}{\eta - t}d\eta$.

The ***Instantaneous frequency*** of an analytic signal $f = u + i\,v$ is defined by the time derivative of the instantaneous phase ϕ:

$$\phi(t) = \mathrm{Arctan}\frac{v(t)}{u(t)} \Rightarrow \omega(t) = \frac{1}{2\pi}\phi'(t).$$

Extraction of the instantaneous frequency

- Wavelet transform: $L_\psi f(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \overline{\psi} \left(\frac{t-b}{a}\right) dt$,

  $\psi \in L^2(\mathbb{R})$ is a Wavelet,

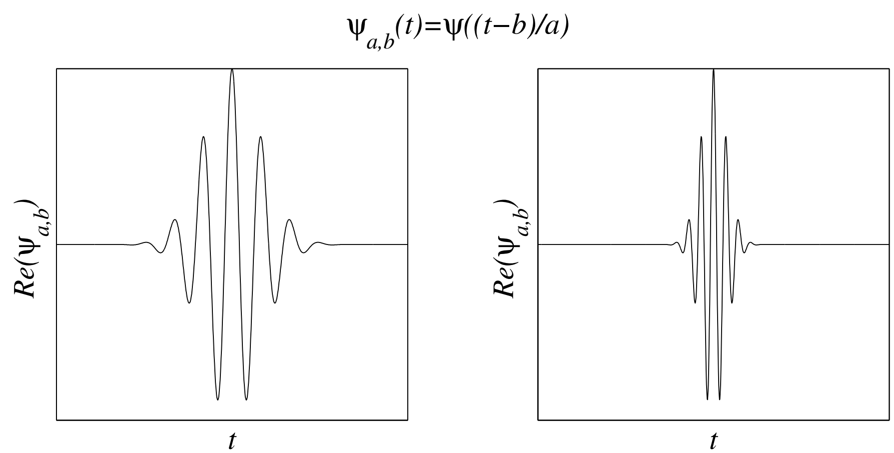
$$\psi_{a,b}(t) = a^{-1/2} \psi \left(\frac{t-b}{a}\right), \qquad a > 0, b \in \mathbb{R}.$$

Wavelet transform:

$$L_\psi f(a,b) =< f, \psi_{a,b} >= \int_{-\infty}^{\infty} f(t) \overline{\psi}_{a,b} dt.$$

Morlet-Grossman Wavelet: $\psi(t) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{2\pi i\lambda t}\, e^{-t^2/2\sigma^2}.$

$$L_\psi f(a,b) = \frac{1}{\sigma\sqrt{2\pi a}} \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i\lambda\left(\frac{t-b}{a}\right)}\, e^{-\left(\frac{t-b}{a}\right)^2/2\sigma^2} dt,$$



Low frequency | High frequency

Large a | Small a

$$frequency = \frac{\text{constant}}{a}, \qquad time = b.$$

CWT of an analytic signal

$f(t) = A_f(t)\exp[i\,\phi_f(t)]$ analytic signal. If the wavelet ψ is an analytic signal,

$$\psi(t) = A_\psi(t)\exp[i\phi_\psi(t)],$$

then

$$L_\psi f(b,a) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} M_{ab}(t)\exp[i\Phi_{ab}(t)]dt,$$

where

$$M_{ab} = A_f(t)\overline{A}_\psi\left(\frac{t-b}{a}\right),$$

$$\Phi_{ab}(t) = \phi_f(t) - \phi_\psi\left(\frac{t-b}{a}\right).$$

Let t_0 such that $\Phi'_{ab}(t_0) = 0$

$$L_\psi f(b,a) \approx \frac{1}{\sqrt{a}} f(t_0) \overline{\psi}\left(\frac{t_0 - b}{a}\right) \sqrt{\frac{2\pi}{|\Phi''_{ab}(t_0)|}}\, e^{i \operatorname{sgn} \Phi''_{ab}(t_0)\pi/4}.$$

(Method of stationary phase).

Note: $t_0 = t_0(b,a)$.

Definition 1 *Ridge of the wavelet transform: $t_0(b,a) = b$.*

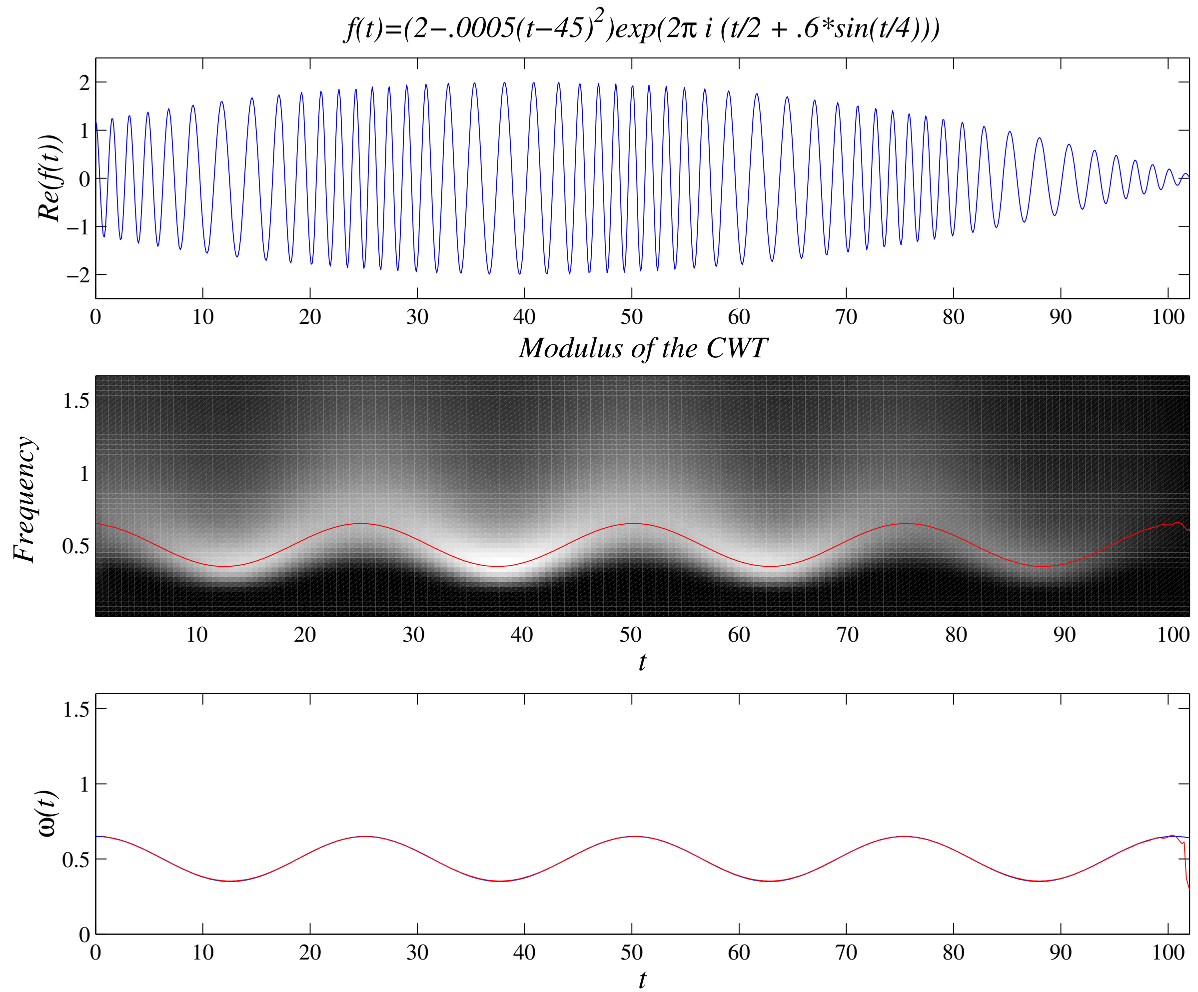
Since $\Phi'_{ab}(t_0) = 0$, this is,

$$\Phi'_{ab}(t_0) = \phi'_f(t_0) - \frac{1}{a}\phi'_\psi\left(\frac{t_0 - b}{a}\right) = 0$$

$$a = a_r(b) = \frac{\phi'_\psi(0)}{\phi'_f(b)}.$$

Example

- Variable phase and amplitude



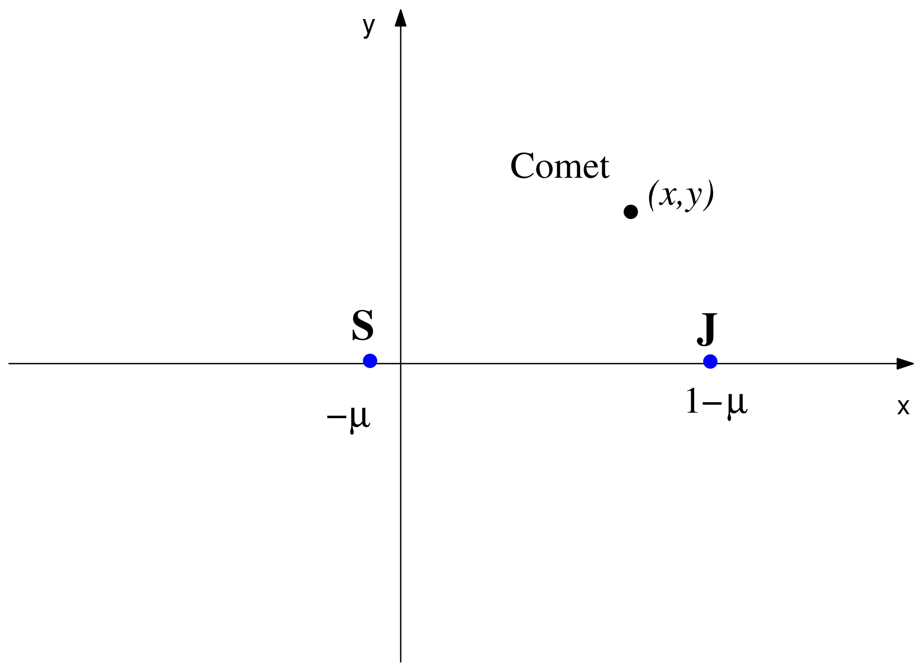
The CRTBP

- The ***Circular Restricted Three Body Problem*** describes the motion of a body in the gravitational field of two ***primaries*** that are moving in circles.
- Our case: the two primaries are the Sun and Jupiter, and the system describes the motion of a comet.
- μ is the ratio between the mass of the Jupiter and the total mass:

$$\mu = \frac{m_J}{m_J + m_S},$$

for the Sun-Jupiter system, $\mu = .0009537$.

- The primaries rotate with angular velocity normalized to 1.
- The system is expressed in a rotating coordinate system with origin at the primaries center of mass. In the rotating coordinates, the equations of motion are autonomous.



- Energy is conserved:

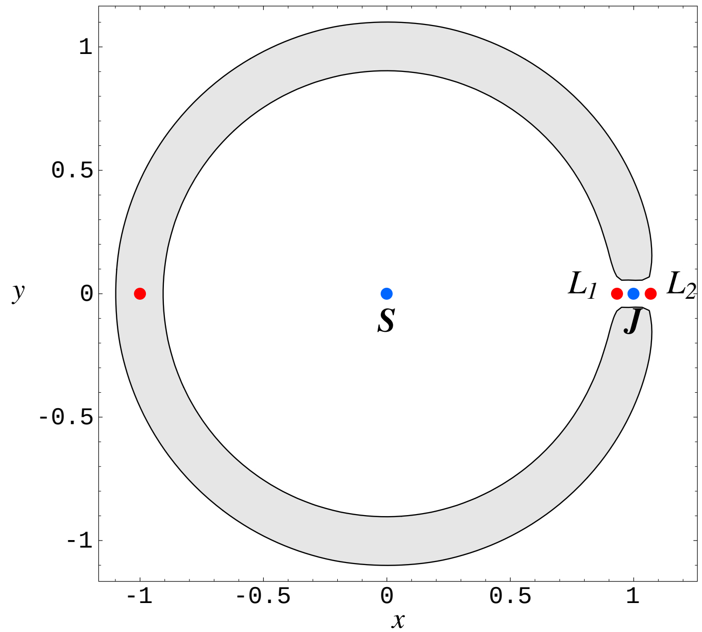
$$
\begin{aligned}
E &= \frac{1}{2}(v_x^2 + v_y^2 + v_z^2) - \Omega, \\
\Omega &= \frac{1}{2}(x^2 + y^2 + \mu(1-\mu)) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \\
r_1 &= \sqrt{(x+\mu)^2 + y^2 + z^2}, \\
r_2 &= \sqrt{(x-1+\mu)^2 + y^2 + z^2}
\end{aligned}
$$

Our case: $E = -1.515$, energy of comet Oterma.

Hill's Regions: $v_x = v_y = v_z = 0$,

$$E + \Omega \geq 0.$$

Hill's region for $E = -1.515$,



- It has 5 equilibrium points: 3 on the x-axis are called Libration points, and 2 Lagrangian points. In the inertial frame these equilibrium points are ***periodic orbits***.

Regularization

Kustaanheimo and Stiefel transformation. p. 582 Szebehely

$$A_4 = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix}$$

$$\begin{aligned} (x, y, z) &= A_4\mathbf{u} \\ (p_x, p_y, p_z, 0) &= \frac{1}{2}({A_4}^T)^{-1}\mathbf{p_u} = \frac{1}{2r}(A_4)\mathbf{p_u} \end{aligned}$$

$$\begin{aligned} x &= u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ y &= 2u_1u_2 - 2u_3u_4 \\ z &= 2u_1u_3 + 2u_2u_4 \end{aligned}$$

Defining:

$$\Omega = \frac{1}{2}(\mu - 1) - E, \qquad \frac{dt}{d\tau} = \frac{r}{2\Omega^2}$$

Hamiltonian function:

$$\begin{aligned}
K &= \frac{1}{4}(p_{u_1}^2 + p_{u_2}^2 + p_{u_3}^2 + p_{u_4}^2) + \frac{1}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2) \\
&+ \frac{2}{\Omega^2}(p_{u_1}u_1 - p_{u_2}u_2 - p_{u_3}u_3 + p_{u_4}u_4)(u_1u_2 - u_3u_4) \\
&+ \frac{1}{\Omega^2}(-p_{u_2}u_1 - p_{u_1}u_2 + p_{u_4}u_3 + p_{u_3}u_4)(u_1^2 - u_2^2 - u_3^2 + u_4^2) \\
&+ \frac{2(\mu - 1)}{\Omega^3}(u_1^2 - u_2^2 - u_3^2 + u_4^2)(u_1^2 + u_2^2 + u_3^2 + u_4^2) \\
&+ \frac{2(\mu - 1)}{\Omega r_1}(u_1^2 + u_2^2 + u_3^2 + u_4^2) \\
r_1^2 &= 4(u_1u_3 + u_2u_4)^2 + 4(u_1u_2 - u_3u_4)^2 \\
&\quad + (\Omega + u_1^2 - u_2^2 - u_3^2 + u_4^2)^2
\end{aligned}$$

Time-Frequency Analysis

- Choose a slice of initial conditions: (x^0, v_x^0), with $y^0 = 0$, $z^0 = .05$, $v_z^0 = 0$.
  Energy is fixed $E = -1.515$ ($C = 3.03$ Oterma's energy).
- Integrate the trajectory in regularized coordinates. Transform to rotational coordinates $(x, y, z, v_x, v_y, v_z)(t)$, $t \in [0, T]$, $T = 318$ Jupiter years.
- Express the orbit in inertial coordinates:

$$
\begin{aligned}
f_1(t) &= (X + i\,Y)(t) = e^{-it}(x + iy)(t) \\
f_2(t) &= (Z(t) + i\,HZ(t)) = z(t) + i\,Hz(t)
\end{aligned}
$$

- Extract the instantaneous frequencies $\omega_1(t)$, $\omega_2(t)$, $t \in [0, T]$.

Frequency Map:

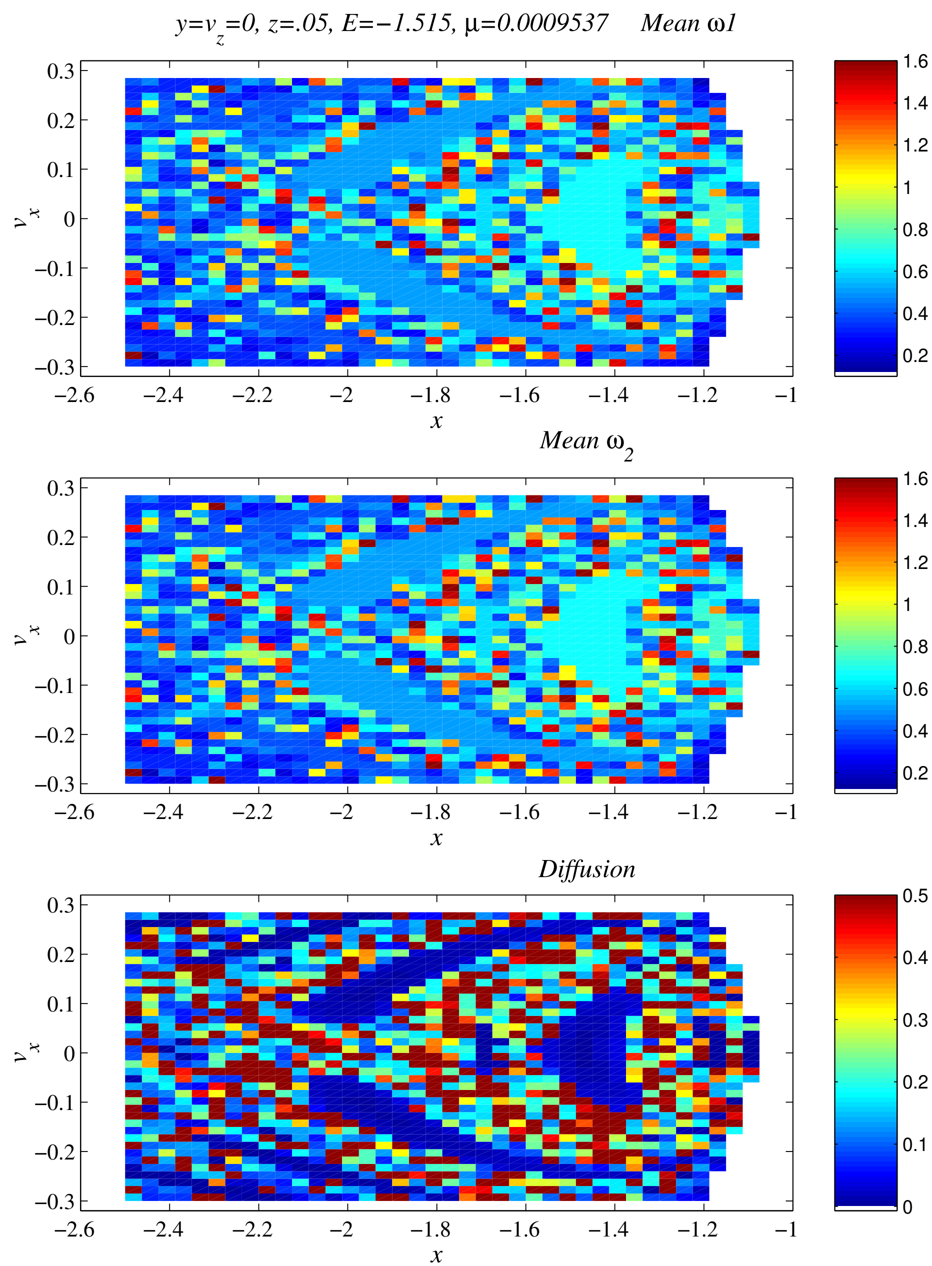
$$(x^0, v_x^0) \longmapsto (\omega_1(t), \omega_2(t)), \qquad t \in [0, T].$$

For each initial condition (x^0, v_x^0) we compute

- Mean Frequencies $\widetilde{\omega_1}$, $\widetilde{\omega_2}$.
- Mean deviation:

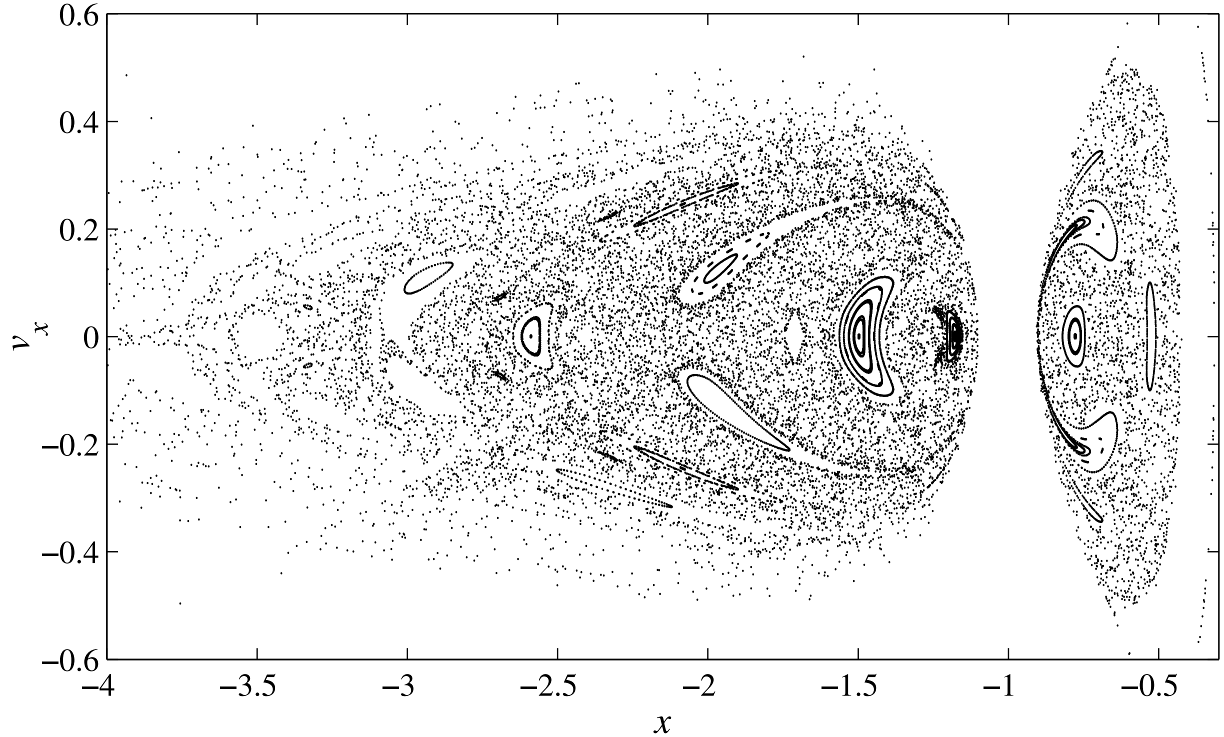
$$\text{Diffusion} = \frac{1}{T}\int_0^T |\omega_1(t) - \widetilde{\omega_1}|dt + \frac{1}{T}\int_0^T |\omega_2(t) - \widetilde{\omega_2}|dt.$$

Results



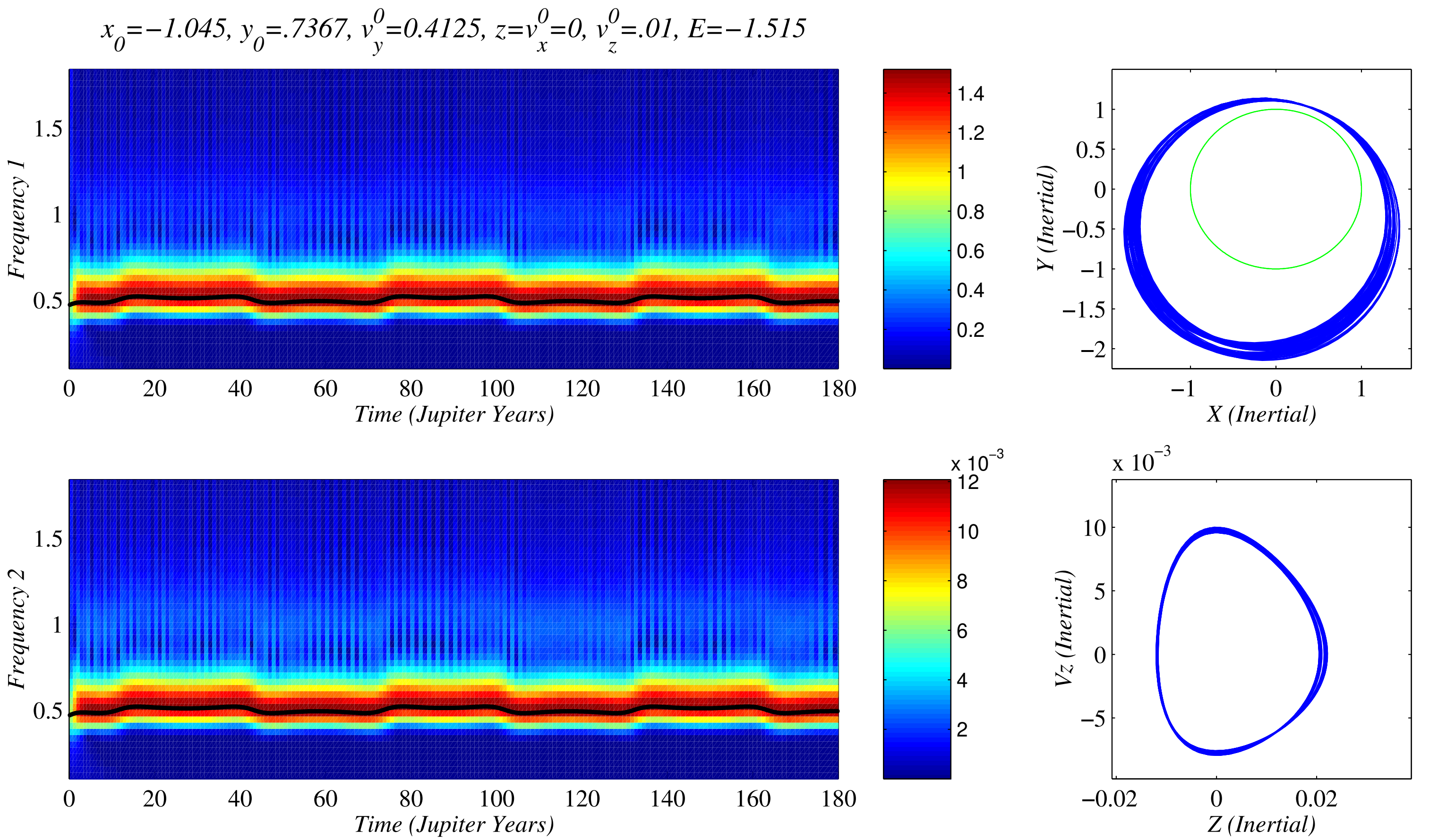
y=v_z=0, z=.05, E=−1.515, μ=0.0009537 Mean ω1
Mean ω_2
Diffusion
v_x
x

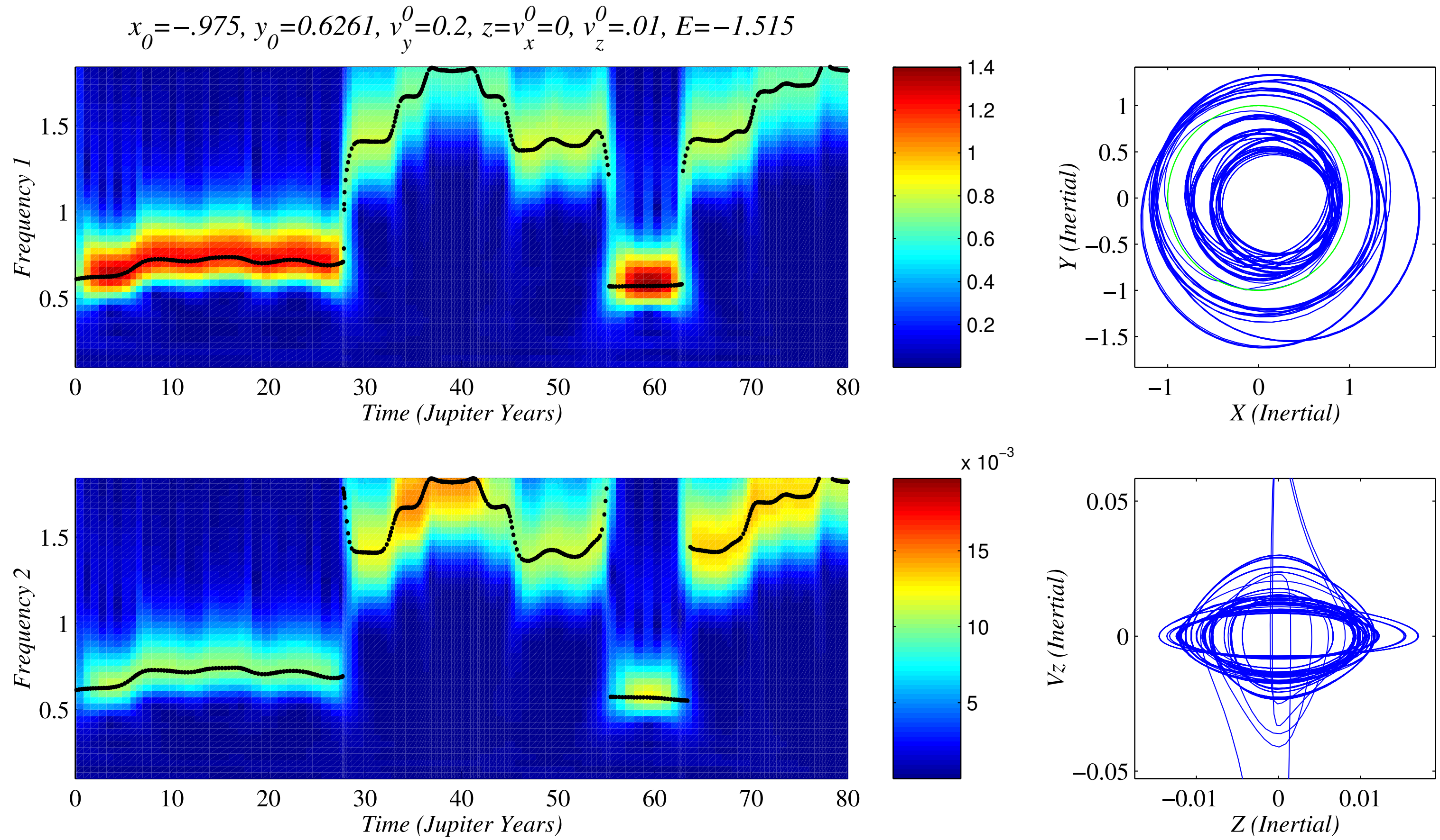
Dynamics in the plane

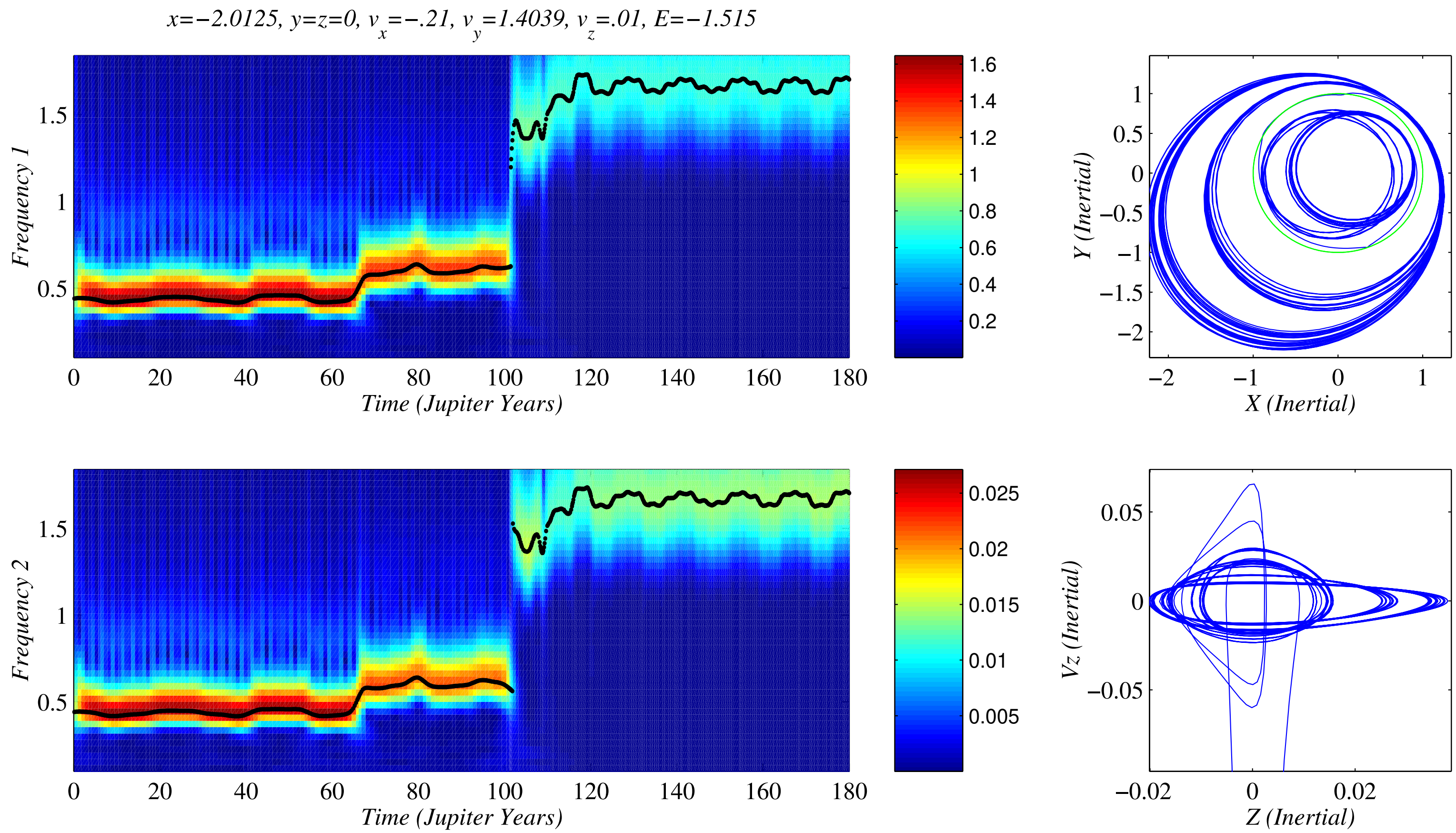


- In the inner region, there is a torus in 3:2 resonance with Jupiter. In the outer region, there is a torus in 2:3 resonance with Jupiter.

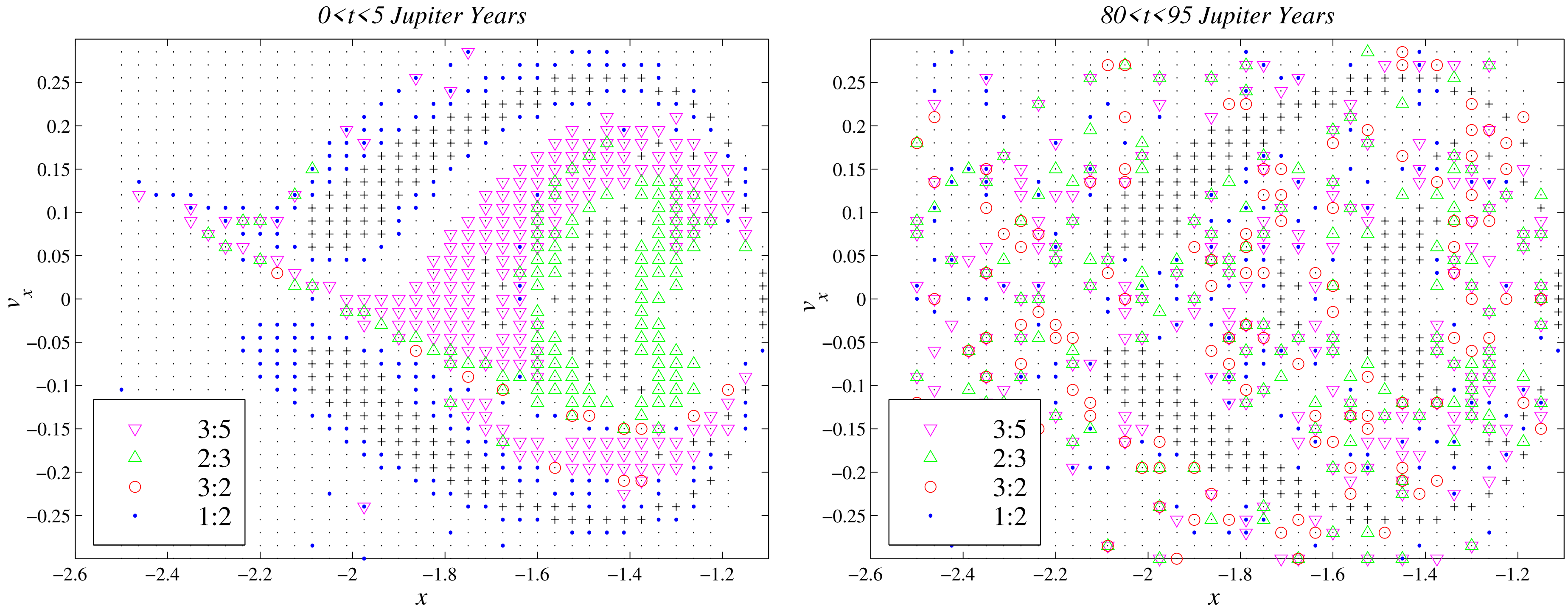
Examples


x0=−1.045, y0=.7367, v0y=0.4125, z=v0x=0, v0z=.01, E=−1.515
Frequency 1
Frequency 2
Time (Jupiter Years)
Y (Inertial)
X (Inertial)
Vz (Inertial)
Z (Inertial)
x 10−3


x_0=−.975, y_0=0.6261, v_y^0=0.2, z=v_x^0=0, v_z^0=.01, E=−1.515
Frequency 1
Frequency 2
Time (Jupiter Years)
0
10
20
30
40
50
60
70
80
0.5
1
1.5
1.4
1.2
1
0.8
0.6
0.4
0.2
x 10^−3
15
10
5
Y (Inertial)
X (Inertial)
Vz (Inertial)
Z (Inertial)
−1.5
−1
−0.5
0
0.5
1
0.05
−0.05
−0.01
0.01

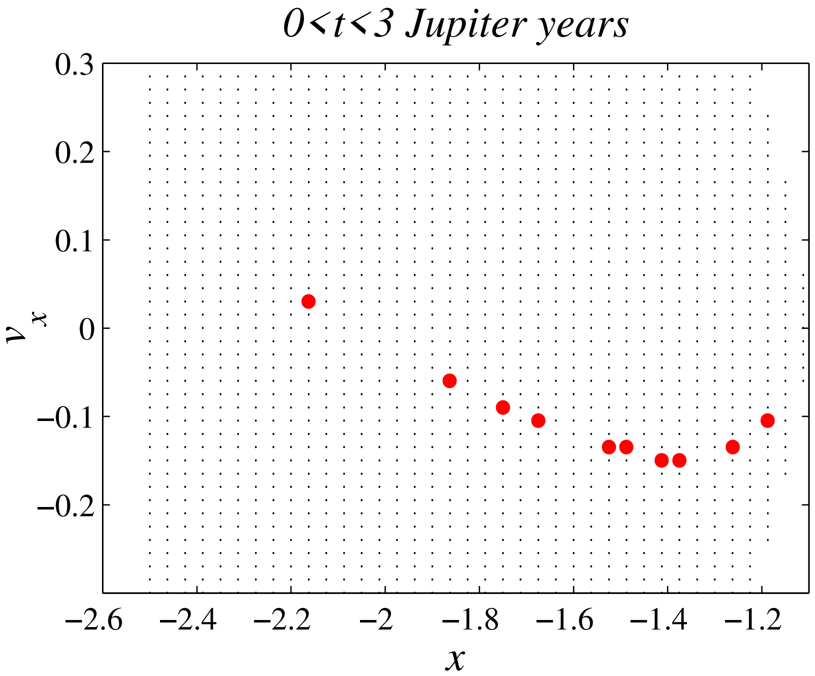

x=−2.0125, y=z=0, v_x=−.21, v_y=1.4039, v_z=.01, E=−1.515
Frequency 1
Time (Jupiter Years)
Y (Inertial)
X (Inertial)
Frequency 2
Time (Jupiter Years)
Vz (Inertial)
Z (Inertial)

Temporary resonance trappings

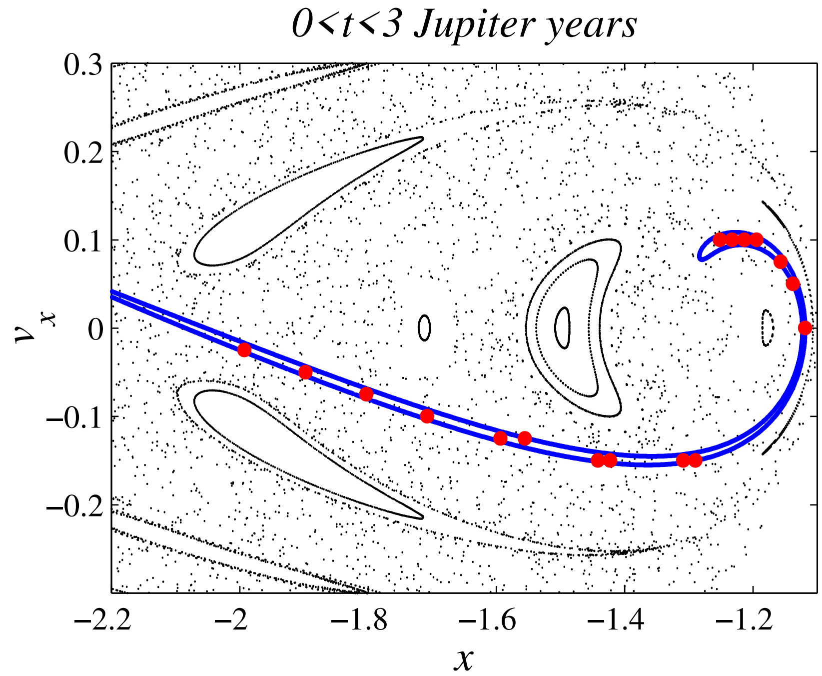


Rapid Transitions

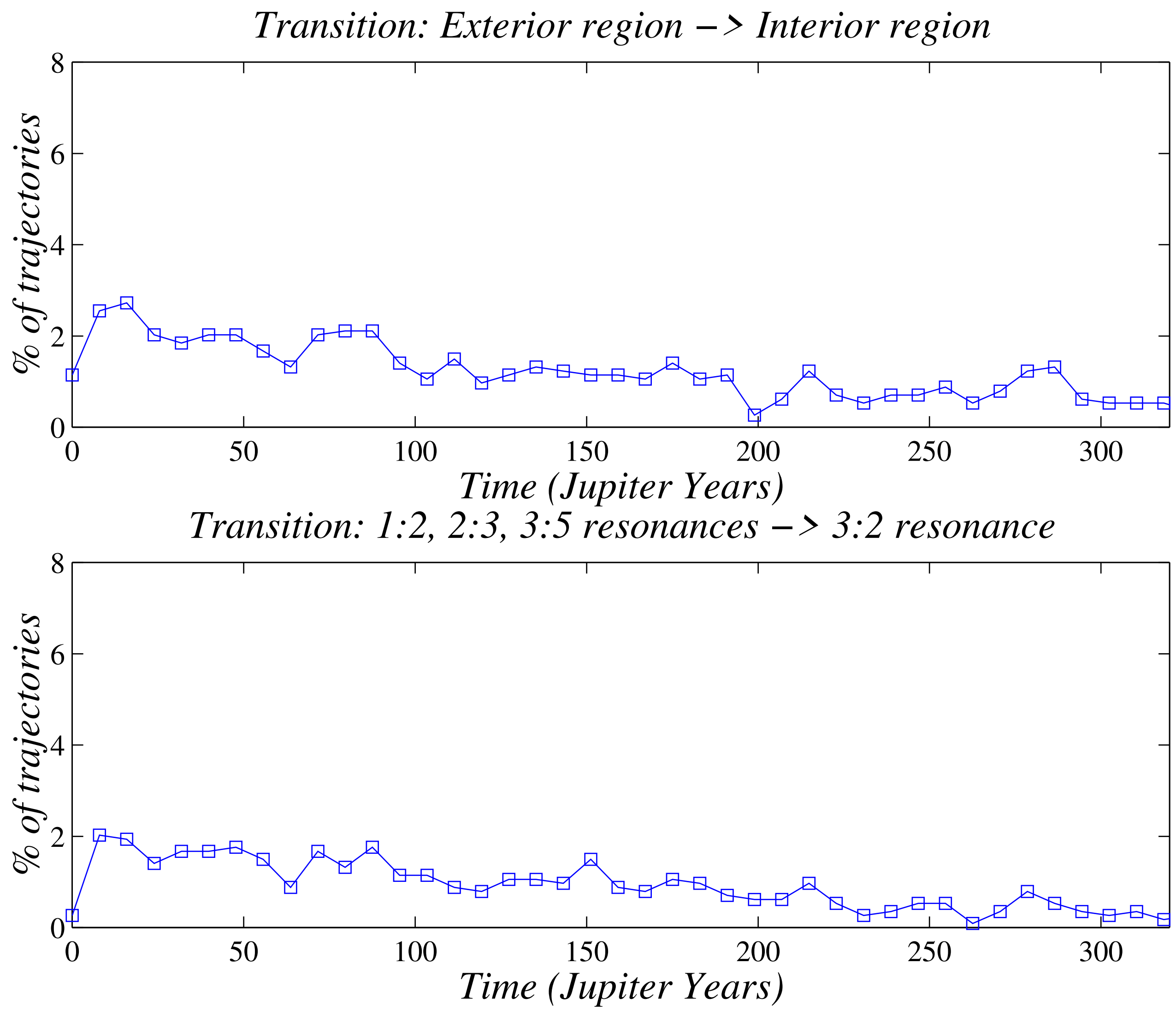
$z^0 = .05$



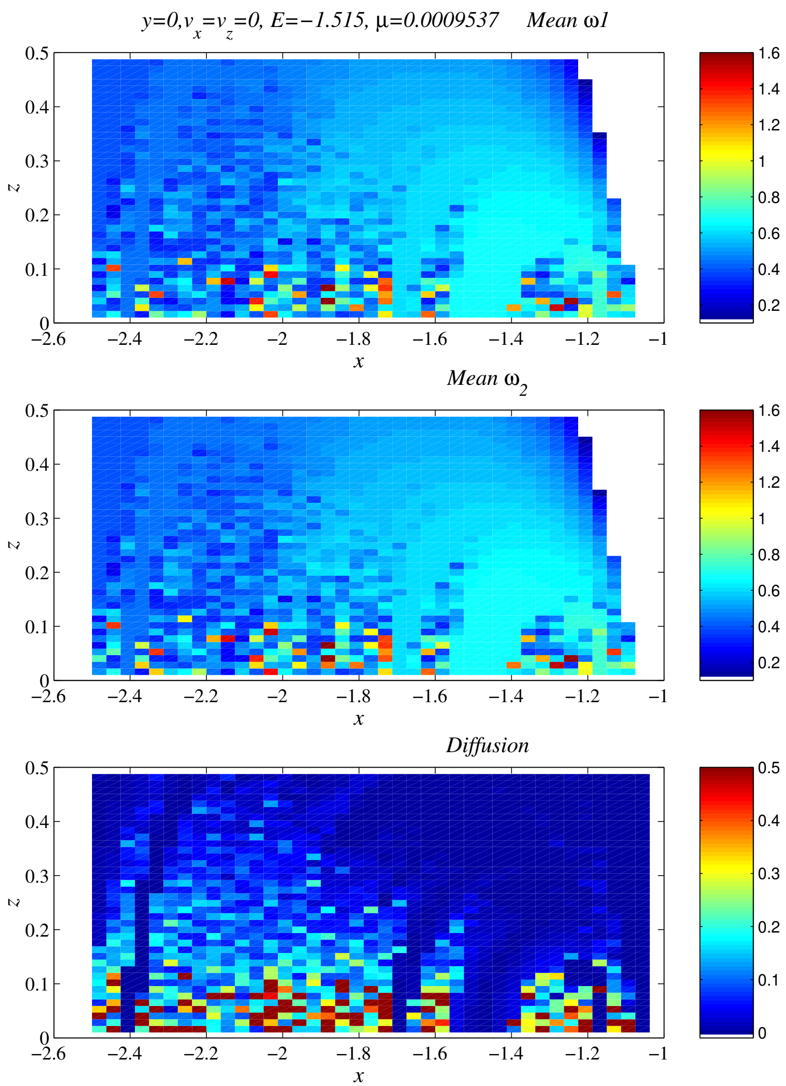
$z^0 = 0$



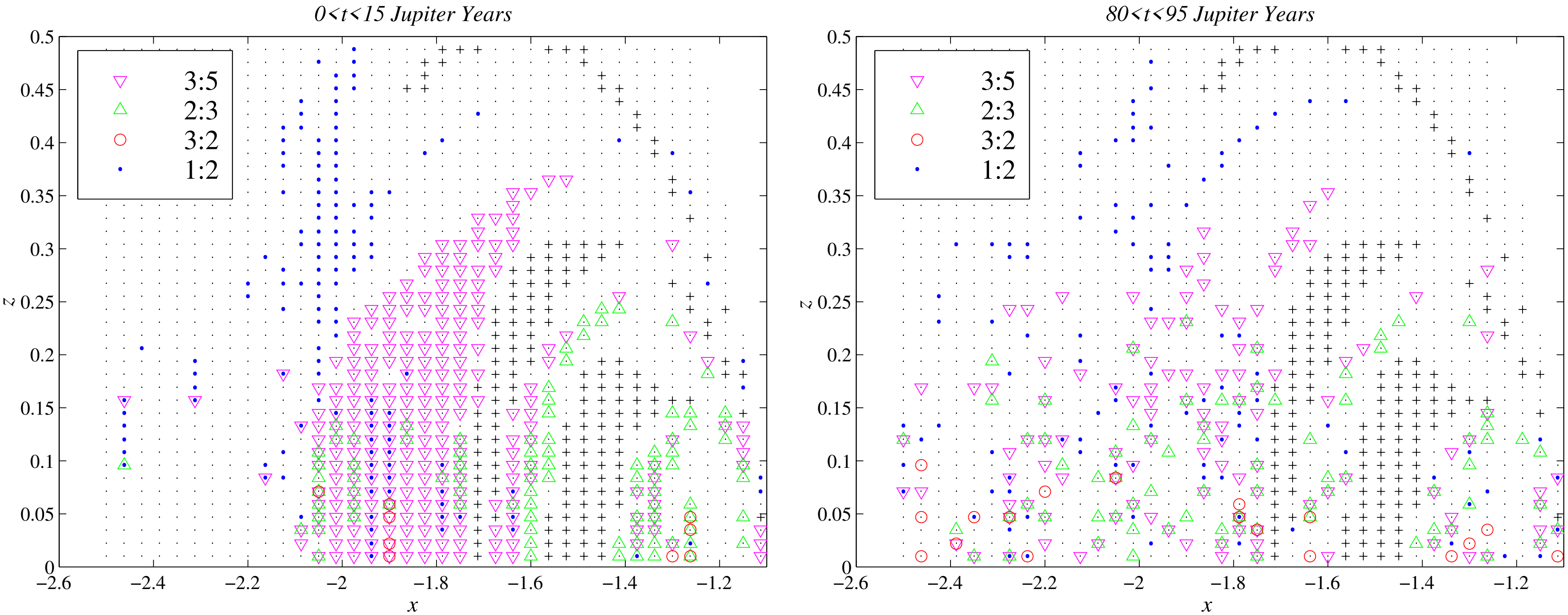
Histograms of transitions



Slice: $y = v_x = v_z = 0$



Temporary resonance trappings



Conclusions

- The phase space structure of the restricted three body problem can be described with the frequencies associated with the numerical solutions, and characterized in terms of
  - resonance islands,
  - chaotic zones with low and high diffusion.
- Detection of transport along resonances: showed transition between 1:2, 2:3, 3:5 and 3:2 resonances.
- For this particular energy level, most of the transport in the restricted three body problem can be explained with the planar model.