Geometry of the Full 2-Body Problem

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Full Body Workshop

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Example of a FBP: asteroid pairs.
This talk will be about the geometry and reduction for such problems and is based on work with Hernan Cendra, with input from various others in the group.
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Others will speak about dynamical systems aspects of these problems, including transport rates, etc.
For mechanical systems with symmetry, some of the tools are:

- *Momentum maps*, ie conserved quantities
- *Reduction*, shape space
- *Stability and the energy-momentum method*
- *Geometric phases*
**Restricted Problems**

- *Restricted* means that one part of the system evolves in the field of another part of the system; Examples are
  - Spherical pendulum on a Merry-Go-Round (the pendulum dynamics does not affect the rotation of the Merry-Go-Round)
  - Fluid flow on a rotating earth (the fluid does not affect the Earth’s rotation)
  - Restricted 3-body problem
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  - Restricted 3-body problem

- We typically handle restricted problems by the theory of *moving systems*. 
**Restricted 3-body Problem**

- consider the *planar case*—the *spatial case* is similar

- **Kinetic energy** (wrt inertial frame) in rotating coordinates:
  \[
  K(x, y, \dot{x}, \dot{y}) = \frac{1}{2} \left[ (\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 \right]
  \]

- **Lagrangian** is K.E. − P.E., given by
  \[
  L(x, y, \dot{x}, \dot{y}) = K(x, y, \dot{x}, \dot{y}) - V(x, y);
  \quad V(x, y) = -\frac{1 - \mu}{r_1} - \frac{\mu}{r_2}
  \]

- **Euler-Lagrange equations**:
  \[
  \ddot{x} - 2\omega \dot{y} = -\frac{\partial V_\omega}{\partial x}, \quad \ddot{y} + 2\omega \dot{x} = -\frac{\partial V_\omega}{\partial y}
  \]

  where the *effective potential* is
  \[
  V_\omega = V - \frac{\omega^2 (x^2 + y^2)}{2}
  \]
Equations for the third body are those of a particle moving in an effective potential plus a magnetic field (Jacobi, Hill, etc.)
Geometric mechanics provides a general theory for (usually) nonrestricted problems: mechanical systems with symmetry. Eg, notions of amended potential, relative equilibria, stability by the energy-momentum method, variational integration algorithms (symplectic integrators), etc.
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- **Configuration Manifold**: $Q = \text{SE}(3) \times \text{SE}(3)$

- The *shape space* $Q/G$ gives the *system shape* and plays an important role in reduction theory.

- Lots of work by many people, as in Dan Scheeres talk
**Material points** in a reference configuration $X_i; i = 1$ for body 1 and $i = 2$ for body 2

Points in the current configuration $x_i$.

Given a configuration

$((A_1, r_1), (A_2, r_2)) \in \text{SE}(3) \times \text{SE}(3)$,

the *material* and *spatial* points are related by

$$x_1 = A_1 X_1 + r_1 \quad \text{and} \quad x_2 = A_2 X_2 + r_2$$
Reduction for the FBP
Reduction for the FBP

□ **Lagrangian** equals kinetic minus potential energy:

\[
L(A_1, r_1, A_2, r_2, \dot{A}_1, \dot{r}_1, \dot{A}_2, \dot{r}_2) \\
= \frac{1}{2} \int_{B_1} \|\dot{x}_1\|^2 d\mu_1(X_1) + \frac{1}{2} \int_{B_2} \|\dot{x}_2\|^2 d\mu_2(X_2) + \int_{B_1} \int_{B_2} \frac{G d\mu_1(X_1) d\mu_2(X_2)}{\|x_1 - x_2\|} \\
= \frac{m_1}{2} \|\dot{r}_1\|^2 + \frac{1}{2} \langle \Omega_1, I_1 \Omega_1 \rangle + \frac{m_2}{2} \|\dot{r}_2\|^2 + \frac{1}{2} \langle \Omega_2, I_2 \Omega_2 \rangle \\
+ \int_{B_1} \int_{B_2} \frac{G d\mu_1(X_1) d\mu_2(X_2)}{\|A_1X_1 - A_2X_2 + r_1 - r_2\|}.
\]

□ Here, for instance, \( \Omega_1 = A_1^{-1} \dot{A}_1 \) is the body angular velocity of the first body (with the usual identification of \( 3 \times 3 \) skew matrices with vectors.)
Goal: Reduce by overall translations and rotations and bring the machinery of geometric mechanics to bear.

SE(3) acts by the diagonal left action on $Q$:

$$(A, r) \cdot (A_1, r_1, A_2, r_2) = (AA_1, Ar_1 + r, AA_2, Ar_2 + r).$$

Momentum map

$$J : TQ \to \mathfrak{se}(3)^*$$

is the total linear and angular momentum.

Shape space $Q/G$: one copy of SE(3); coordinatized by the relative attitude $A = A_1^{-1}A_2$ and relative position $R = A_2^T(r_1 - r_2)$. 
General reduction theory says that the variational principle and the equations of motion drop (in the appropriate sense) to the quotient space $(TQ)/G$. (Similarly to $(T^*Q)/G$ for Hamiltonian mechanics.)
Some Reduction Theory

General reduction theory says that the variational principle and the equations of motion drop (in the appropriate sense) to the quotient space \((TQ)/G\). (Similarly to \((T^*Q)/G\) for Hamiltonian mechanics.

To get a nice realization of \((TQ)/G\), one chooses a connection \(A : TQ \rightarrow g\) on the bundle \(Q \rightarrow Q/G\) (assume that this is a principle bundle—i.e., there are no singularities.)
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In this case, one gets a natural identification

\[
\alpha_A : (TQ)/G \rightarrow T(Q/G) \times \tilde{\mathfrak{g}}
\]

where \(\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G\) is the \textit{associated bundle}. Similarly for the Hamiltonian side of the story.
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First of all, shape space is given by

$$X = (G \times G)/G \cong G,$$

where the map $\pi : Q = G \times G \rightarrow G$ is given by

$$x = \pi(g_1, g_2) = g_1^{-1}g_2.$$

A natural connection on the bundle $Q \rightarrow Q/G$ is given by

$$A(g_1\xi_1, g_2\xi_2) = \text{Ad}_{g_2}\xi_2$$

This connection has zero curvature.
The FBP Case

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- This has picked out one of the bodies as special.
This gives rise to the identification

\[(T(G \times G)) \mod G \cong G \times g \times g\]

where we map the class of \((g_1, \dot{g}_1, g_2, \dot{g}_2)\) to \((x, w, \xi_2)\), where \(x = g_1^{-1}g_2\) and \(\xi_2 = g_2^{-1}\dot{g}_2\) as above and where \(w = \dot{x}x^{-1}\).
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Thus, by general theory, the equations of motion will reduce to equations for the variables \((x, w, \xi_2)\).
In general, the equations of motion for a given invariant Lagrangian $L : TQ \to \mathbb{R}$ drop to equations on $(TQ)/G \cong T(Q/G) \times \tilde{\mathfrak{g}}$. 
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The equations correspond to breaking up the variational principle into two parts: one for horizontal variations (Lagrange part of the equations) and one for vertical variations (Poincaré part of the equations).
One can work this all out quite explicitly for the general case of $Q = G \times G$, where, say, $G = \text{SE}(3)$ and for Lagrangians of the form

$$L(g_1\xi_1, g_2\xi_2) = \frac{1}{2} \left[ \text{Tr}(K_1\xi_1^T\xi_1) + \text{Tr}(K_2\xi_2^T\xi_2) \right] - V(g_1^{-1}g_2).$$
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The reduced Lagrangian is given by $\mathcal{l}(x, w, \xi_2)$ which equals the above expression, where $\xi_1 = -w + x\xi_2x^{-1}$. 
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Similarly, one gets the Hamiltonian version of the equations, the reduced Poission structure, etc.
For numerics as well as analysis of stability of relative equilibria (analog of the libration points), the variational and Hamiltonian structures are useful.

Previous works guessed these structures and missed the variational structure altogether. Using reduction, one derives them in a simple and natural way, one gets the Jacobi integrals naturally, etc.

Extra symmetries give extra conserved quantities and further reductions.
Restricted Simpler Case

- Restricted (as in restricted 3-body problem) simple case already exhibits the basic ejection and collision dynamics

- Point mass moving in the $xy$-plane under the gravitational field of a uniformly rotating elliptical body, without affecting its uniform rotation.
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- **Equations of motion** relative to a rotating Cartesian coordinate frame and appropriately normalized:
  \[ \ddot{x} - 2\dot{y} = \frac{\partial V}{\partial x} \quad \text{and} \quad \ddot{y} + 2\dot{x} = \frac{\partial V}{\partial y}, \]
where

\[ V(x, y) = \frac{1}{\sqrt{x^2 + y^2}} + \frac{1}{2}(x^2 + y^2) + U_{22}; \]

and where

\[ U_{22} = \frac{3C_{22} (x^2 - y^2)}{(x^2 + y^2)^{5/2}} \]

- The coefficient \( C_{22} \) is the **ellipticity**.
- **Jacobi integral**: \( J = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V. \)
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□ *Jacobi integral:* \( J = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V. \)

□ *Moving systems approach* gives, as in the RCTBP, the Lagrangian and Hamiltonian structure and Jacobi integral.
Lagrangian (kinetic minus potential energy) written in the rotating system and with angular velocity normalized to unity, is

\[ L = \frac{1}{2}[(\dot{x} - y)^2 + (x + \dot{y})^2 + \dot{z}^2] - U(x, y, z). \]

where

\[ U(x, y, z) = \frac{1}{r} - U_{22}. \]

Euler–Lagrange equations produce the previous equations and the Legendre transformation gives the Hamiltonian structure, the Jacobi integral, etc.
The Jacobi integral (energy) is an indicator of the type of global dynamics possible.

For energies above a threshold, $E > E_S$, corresponding to symmetric saddle points, movement between the *realm* near the asteroid (interior realm) and away from the asteroid (exterior realm) is possible. For energies $E \leq E_S$, no such movement is possible.
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As in the CRTBP, motion between realms is mediated by phase space tubes.

General theory allows us to transition what we learned in the CRTBP to this case.
Phase space in each realm organized further into different \textit{resonance regions}, connected via \textit{lobes}.

Zero velocity curves: locus of $(x, y)$ where $0 = J + V(x, y)$

(a) (b)
F2BP: Phase Space Structure

Poincaré sections in the different realms, $U_1$ and $U_2$ are linked by tubes in the phase space. Under the Poincaré map $f_1$ on $U_1$, a trajectory reaches an exit, the last Poincaré cut of a tube before it enters another realm. The map $f_{12}$ takes points in the exit of $U_1$ to the entrance of $U_2$. The trajectory then evolves under the action of the Poincaré map $f_2$ on $U_2$.

See Shane’s talk and the material on the FBP website for further information.
Selected References


For papers, movies, etc., visit http://www.cds.caltech.edu/~marsden and http://www.cds.caltech.edu/~shane
The End

Typesetting Software: \TeX, Textures, \LaTeX, hyperref, texpower, Adobe Acrobat 4.05
Graphics Software: Adobe Illustrator 10.0.1
\LaTeX\ Slide Macro Packages: Wendy McKay, Ross Moore