Variational Principles, Dirac Structures, and Reduction

Dedicated to Alan@60

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Joint work with Hiroaki Yoshimura (and others)

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Alanfest, Vienna, August 7, 2003
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What Dirac did; what we do

- Dirac (who knew mechanics!), studied Lagrangian systems with constraints, including those arising from degeneracies.
- See his article on “Generalized Hamiltonian Mechanics” in the *Canadian J. of Math.*, circa 1950.
- He also worked on the program of going to the Hamiltonian side via the Legendre transformation and computing the associated Poisson brackets.
- **Lesson** learned from examples and applications: In many if not most cases, one does *not start* on the Hamiltonian side, but rather on the Lagrangian side with a variational principle.
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Dirac understood this very clearly and it is how his papers are written; but this seems to have been a forgotten lesson!

While it is quite appropriate that *Dirac structures* are named after him, it seems that workers in the field have so far left out Lagrangian mechanics from the story! *Our goal is to fill this gap (or canyon).*

You are wrong if you believe that this gap can be trivially filled by simply waving a Legendre transformation wand.
Examples

• Standard nondegenerate Lagrangian and Hamiltonian systems, possibly with symmetry, possibly reduced.

• Specific case: *the dynamics of asteroid pairs*, such as Ida and Dactyl:
Examples

- **Nonholonomic mechanics.** Specific example is the rolling penny. Have a look at one of my favorite books:
• **Electrical Networks.** Specific example; how to analyze the dynamics associated with this network:
Theoretical Developments

• General development of Dirac structures (Courant, Weinstein, Dorfman) and reduction theory of Dirac structures (Van der Schaft, Blankenstein, Ratiu).

• Application to nonholonomic systems and circuits, but on the Hamiltonian side (horrors!) by Van der Schaft, Maschke, Bloch, Crouch and others.
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- Lagrangian and Hamiltonian reduction theory (see article of JM and Alan on the history of the subject).

- General symplectic and Poisson reduction are fine, but one wants more detail for the case of tangent and cotangent bundles. Why? Well, that is how one does examples!
Theoretical Developments

- Hamiltonian reduction of cotangent bundles\(^1\) Start with \(H : T^* Q \to \mathbb{R}\) and a Lie group \(G\) acting (free and proper for simplicity) on \(Q\). Choose a principal connection \(A\) on the shape space bundle \(Q \to Q/G\).

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• Mechanics and examples tell you that this global perspective is the right one—that is, one should really choose a connection at this point. Then,

\[
T^*Q/G \cong_A T^*(Q/G) \times \tilde{\mathfrak{g}}^*
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with its natural Poisson structure (containing curvature terms from \(A\)), etc.

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• Reduced equations on this space are called the \textit{Hamilton–Poincaré equations}. When \(Q = G\), you get the \textit{Lie–Poisson equations} on \(\mathfrak{g}^*\).

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If you do symplectic reduction in this context by imposing a momentum map constraint $J = \mu$ then one gets an **associated coadjoint orbit bundle**

$$J^{-1}(\mu)/G_\mu = T^*(Q/G) \times \tilde{O}_\mu$$

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• This is actually useful in examples; eg, the asteroid pair problem, in the dynamics of a fluid with a free surface. (The Hodge decomposition provides the connection).
Theoretical Developments

- **Lagrangian reduction of tangent bundles**: Start with $L : TQ \to \mathbb{R}$ and a Lie group $G$ acting (free and proper for simplicity) on $Q$. Choose a principal connection $A$ on the shape space bundle $Q \to Q/G$, so that

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• *Don’t you dare* choose a metric and identify $TQ$ and $T^*Q$ or a Killing form and identify $\mathfrak{g}$ and $\mathfrak{g}^*$!!

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Theoretical Developments

• Many other variations on the theme: For example, semi-direct product reduction theory\(^3\) and its recent generalization to group extensions (including Bott Virasoro, Camassa-Holm, Dym, etc).

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Theoretical Developments

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• **Stability theory**\(^4\) for relative equilibria; energy-momentum method (Arnold method, energy-Casimir, block diagonalization, ...).

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\(^4\)Arnold, JEM, Simo, Lewis, ...
Nonholonomic Mechanical Systems

- Constraints such as rolling constraints. Dynamics governed by the *Lagrange–d’Alembert principle*: start with a distribution $\Delta \subset TQ$ and ask that

$$\delta \int L \, dt = 0$$

for all $\delta q \in \Delta$. One also impose the condition $\dot{q} \in \Delta$. 
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![Diagram showing varied curve $q(t)$ and points $q(a)$ and $q(b)$ with $\delta q(t) \in \Delta$.](image-url)
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• In general, symmetries need not lead to conservation laws; \textit{Momentum equation} discovered by BKMM. This, together with holonomy (geometric phases) plays a crucial role in locomotion (eg, how snakes move).

• The equations on the Hamiltonian side (assuming the Lagrangian is regular) are governed by an \textit{almost Poisson structure} and, in a sense, by an almost symplectic structure. In fact, the Jacobiator is measured by the curvature of $\Delta^5$.

\footnote{Van der Schaft and Maschke, Bates and Sniatycki, Koon and JEM, Marle, ...}
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Despite this, the system *is* described by a Dirac structure, as I will explain below.

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\(^5\)Van der Schaft and Maschke, Bates and Sniatycki, Koon and JEM, Marle, ...
Despite the above, there is also an energy-momentum method for stability (applies to the classical examples: rat-tleback, the unicycle, ... ); there is also a picture similar to the bundle picture above with a beautiful intrinsic geometric description.\textsuperscript{6}

\textbf{Resulting equations:} the \textit{Lagrange-d’Alembert-Poincaré} equations.

\textbf{(Not quite as bad as hemi-quasi-twisted-algebroids, ...)}

\textbf{Circuits} are typically not only nonholonomic (because of the Kirchhoff current laws), they are also often degenerate (giving \textit{primary} constraints in the sense of Dirac).

\textsuperscript{6}Obtained by Cendra, Marsden and Ratiu in 2001.
Variational methods are useful!

- Shell collisions: thin shell models using *multisymplectic variational methods, AVI + subdivision + collision methods* (JEM, Ortiz, Cirac–West)
A Question for you.

- The Euler–Lagrange equations come from Hamilton’s principle.
- Hamilton’s equations come from Hamilton’s phase space variational principle.
- The Euler–Poincaré equations come from the Euler–Poincaré variational principle.
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- **Question:** what is it for the Lie–Poisson equations? (I can make the question harder and ask this for the Hamilton–Poincaré equations or the Lagrange–Poincaré equations.)
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- Euler–Poincaré equations on $g$

\[
\frac{d}{dt} \frac{\delta l}{\delta \nu} = \text{ad}^*_\nu \frac{\delta l}{\delta \nu},
\]
A Question for you.

• Answer:\(^7\)

\[ \delta \int_a^b l(\nu(t)) \, dt = 0 \]

for \textit{constrained} variations of the form

\[ \delta \nu(t) = \dot{\eta}(t) + [\nu(t), \eta(t)], \]

where \( \eta(t) \) has fixed endpoints.

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- **Answer:**
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  for constrained variations of the form
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  where \( \eta(t) \) has fixed endpoints.

- **Method of proof:** reduce Hamilton’s principle on \( TG \)

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• Back to our question: how are the Lie–Poisson equations

$$\dot{\mu} = \text{ad}_{\delta h/\delta \mu}^* \mu$$

on $\mathfrak{g}^*$ variational?
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Answer: A Pontryagin type principle:

\[ \delta \int_a^b (\langle \mu(t), v(t) \rangle - h(\mu(t))) \, dt = 0 \]

for variations of the form

\[ \delta v(t) = \dot{\eta}(t) + [v(t), \eta(t)], \]

where \( \eta(t) \) has fixed endpoints, but \( \delta \mu \) are arbitrary.
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where \( \eta(t) \) has fixed endpoints, but \( \delta \mu \) are arbitrary.

- The Legendre transformation \( v = \delta h/\delta \mu \) is part of the
  variational principle! Neat! We will need this sort of thing.

- **Method:** reduce Hamilton’s phase space principle.
Recall: a **Dirac structure** on a manifold $R$ is: a subbundle $D \subset TR \times T^*R$ such that $D = D^\perp$, where the perp is with respect to the natural pairing

$$\langle\langle (u, \alpha), (v, \beta) \rangle\rangle = \langle \beta, u \rangle + \langle \alpha, v \rangle$$
Implicit Hamiltonian Systems

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- **Standard examples**: graph of an (almost) symplectic or (almost) Poisson structure. [Not enough for nonholonomic systems—one needs to put in the constraints].
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- **Standard examples**: graph of an (almost) symplectic or (almost) Poisson structure. [Not enough for nonholonomic systems—one needs to put in the constraints].

- For a symplectic or Poisson manifold, a **Hamiltonian vector field** $X_H$ on $R$ associated to a function $H$ satisfies

$$(X, dH) \in D$$

at each point of $R$. For good reason, Van der Schaft calls these things **implicit Hamiltonian systems**; they are an important part of his theory of “interconnected” and “port controlled” systems.
Before defining an implicit Lagrangian system, we will need some more terminology and a “big diagram”.\footnote{Some parts of this picture are due to Tulczyjew; the full diagram was formulated by Cendra, JM and Ratiu.}

Namely, we need two natural diffeomorphisms; first there is the diffeomorphism

\[ TT^*Q \rightarrow T^*T^*Q \]

associated with the canonical symplectic form on \( T^*Q \).

Second, there is the natural diffeomorphism between

\[ TT^*Q \rightarrow T^*TQ \]

given in coordinates by \((q,p,\delta q,\delta p) \rightarrow (q,\delta q,\delta p,p)\) and determined intrinsically by the commutativity of the following “big diagram”.
The Big Diagram

\[ T^*T^*Q \xrightarrow{\Omega_b} TT^*Q \xrightarrow{\kappa_Q} T^*TQ \]

\[ T^*Q \xrightarrow{\pi^2} TT^*Q \xrightarrow{T_{\pi_Q}} TQ \]

\[ T^*Q \xrightarrow{\tau_{T^*Q}} TQ \]

\[ TQ \xrightarrow{\pi_T} TQ \]

\[ TQ \oplus T^*Q \text{ Pontryagin Space} \]

\[ TQ \xrightarrow{pr_1} Q \]

\[ T^*TQ \xrightarrow{\pi^{1}} TQ \xrightarrow{\tau_Q} Q \]

\[ T^*TQ \xrightarrow{\pi^{TQ}} TQ \xrightarrow{pr_3} Q \]

\[ T^*TQ \xrightarrow{\pi_T} TQ \xrightarrow{pr_2} TQ \]

\[ T^*TQ \xrightarrow{\Omega^b} TT^*Q \xrightarrow{T_{\pi_Q}} TQ \]
Let $D \subset TT^*Q \times T^*T^*Q$ be a given Dirac structure on $T^*Q$. Let $L : TQ \to \mathbb{R}$ be a given Lagrangian and let

$$DL \in T^*T^*Q$$

be $dL \in T^*TQ$ transferred over to $T^*T^*Q$ by the canonical diffeomorphisms.
Implicit Lagrangian Systems

Let $D \subset TT^*Q \times T^*T^*Q$ be a given Dirac structure on $T^*Q$. Let $L : TQ \to \mathbb{R}$ be a given Lagrangian and let $\mathcal{D}L \in T^*T^*Q$

be $dL \in T^*TQ$ transferred over to $T^*T^*Q$ by the canonical diffeomorphisms.

An implicit Lagrangian system relative to the given Dirac structure $D$ is a vector field $X$ on $T^*Q$ satisfying

$$(X, \mathcal{D}L) \in D$$

at each point of $T^*Q$. 
• Hamilton’s principle may be rewritten so that it fits very well with the above definition.

• Write Hamilton’s principle this way:

\[
0 = \delta \int_a^b L(q(t), \dot{q}(t)) \, dt \\
= \int_a^b \left( \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \, dt \\
= \int_a^b \left( \dot{p} \delta q - \dot{p} \delta q + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \, dt \\
= \int_a^b \left( -\dot{p} \delta q - p \delta \dot{q} + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \, dt + p \delta q \bigg|_a^b \\
= -\int_a^b \left\{ \left( \dot{p} - \frac{\partial L}{\partial q} \right) \delta q + \left( p - \frac{\partial L}{\partial \dot{q}} \right) \delta \dot{q} \right\} \, dt + p \delta q \bigg|_a^b.
\]
This derivation can be given entirely intrinsically using objects in the big diagram.
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• Notice that the final equations include the Legendre transformation \( p = \partial L / \partial \dot{q} \) as part of the equations. The boundary term, as is (now) standard, gives the canonical one-form.
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• Notice that the final equations include the Legendre transformation $p = \partial L / \partial \dot{q}$ as part of the equations. The boundary term, as is (now) standard, gives the canonical one-form.

• The equations (where $X(q, p) = (q, p, \dot{q}, \dot{p})$) written this way are exactly the Dirac structure equations, namely, $(X, D_L) \in D$, including the Legendre transform and also the correct identification of $\dot{q}$. 
Now we are ready to state the nonholonomic equations in terms of Dirac structures.

Given the constraint distribution $\Delta \subset TQ$, we will now define an associated Dirac structure $D_\Delta$ on $T^*Q$.

It will save time if we define $D_\Delta$ when $Q = V$, a vector space. Then at each point $q \in V$, $\Delta \subset V$ and

$$D_\Delta \subset TT^*Q \times T^*T^*Q$$

becomes, at each fiber point $(q, p) \in V \times V^*$,

$$D_\Delta \subset (V \times V^*) \times (V^* \times V)$$

Let

$$D_\Delta = \{(v, \beta), (y, -v) | v \in \Delta, y - \beta \in \Delta^o\}$$

where $\Delta^o \subset V^*$ is the polar of $\Delta$. 
It is easy to check that $D_\Delta$ is a Dirac structure.
Nonholonomic Example

- It is easy to check that $D_\Delta$ is a Dirac structure.
- Then we have

Theorem. The statement that $(X,\mathcal{D}L) \in D_\Delta$ is equivalent to the Lagrange–d’Alembert equations, including the dynamic equations, the condition $\dot{q} \in \Delta$, and the constraints.
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Theorem. The statement that $(X, \mathcal{D}L) \in D_\Delta$ is equivalent to the Lagrange-d’Alembert equations, including the dynamic equations, the condition $\dot{q} \in \Delta$, and the constraints.

- Of course this is also directly tied to the Lagrange-d’Alembert variational structure of the equations.
• You are seeing the tip of a big iceberg. Much is left to uncover.
Concluding Remarks

• You are seeing the tip of a big iceberg. Much is left to uncover.

• Conjectures are easy to make: Reduction of Dirac structures should produce, for instance, a Dirac structure for the Lagrange-d’Alembert-Poincaré equations as a special case; that is, Lagrange-d’Alembert reduction should be consistent with Dirac reduction. Ditto for the Lie-Poisson variational principle described in the “question to you”. A metaprinciple emerges.
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- In particular, the reduction of the standard Dirac structure on $T^*Q$ is not the standard Dirac structure on $g^*$, but rather should be one on $g \times g^*$ associated with the reduced “Pontryagin” space.
Concluding Remarks

In doing PDE, relativistic field theories or even quantum mechanics, multisymplectic and multipoisson structures are the way to do. What is a **multi-Dirac** structure? (Think of a rolling ball of jello, or squishy tires on a road to motivate nonholonomic PDE’s).
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  ◦ Go to my home page for free templates for making TeX slides like these ones. Click on “Marslides”.


The End