JOURNAL OF GEOMETRIC MECHANICS
doi:10.3934/jgm.2009.1.223

Volume **1**, Number **2**, June **2009**
pp. **223–266**

THE GEOMETRY AND DYNAMICS OF INTERACTING RIGID BODIES AND POINT VORTICES

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(Communicated by Manuel de León)

Abstract. We derive the equations of motion for a planar rigid body of circular shape moving in a 2D perfect fluid with point vortices using symplectic reduction by stages. After formulating the theory as a mechanical system on a configuration space which is the product of a space of embeddings and the special Euclidian group in two dimensions, we divide out by the particle relabeling symmetry and then by the residual rotational and translational symmetry. The result of the first stage reduction is that the system is described by a non-standard magnetic symplectic form encoding the effects of the fluid, while at the second stage, a careful analysis of the momentum map shows the existence of two equivalent Poisson structures for this problem. For the solid-fluid system, we hence recover the ad hoc Poisson structures calculated by Shashikanth, Marsden, Burdick and Kelly on the one hand, and Borisov, Mamaev, and Ramodanov on the other hand. As a side result, we obtain a convenient expression for the symplectic leaves of the reduced system and we shed further light on the interplay between curvatures and cocycles in the description of the dynamics.

1. **Introduction.** In this paper, we use symplectic reduction to derive the equations of motion for a rigid body moving in a two-dimensional fluid with point vortices. Despite the fact that this setup is easily described, it may come as a surprise that these equations were derived (using straightforward calculations) only recently by Shashikanth, Marsden, Burdick and Kelly [43] and Borisov, Mamaev

2000 *Mathematics Subject Classification.* Primary: 53D20; Secondary: 76M60.
Key words and phrases. Symplectic reduction, Kaluza-Klein, point vortices, perfect fluids.

and Ramodanov [7]. We also mention here the pioneering work of Koiller [23], where these equations were obtained in the limit case where the cylinder radius goes to zero.

In order to see why this problem remained open for so long, and to set the stage for our approach, let us trace some of the history of fluid-rigid body interaction problems.

1.1. **Rigid bodies in potential flow.** The equations of motion for a rigid body of mass m_b and inertia tensor $\mathbb{I}_b$ in potential flow were first described by Kirchhoff [20] and are given by

$$\dot{\mathbf{L}} = \mathbf{A} \times \mathbf{V} \quad \text{and} \quad \dot{\mathbf{A}} = \mathbf{A} \times \mathbf{\Omega}, \tag{1}$$

where $\mathbf{V}$ and $\mathbf{\Omega}$ are the linear and angular velocity of the body, while $\mathbf{L}$ and $\mathbf{A}$ are the linear and angular momentum, related by $\mathbf{L} = m\mathbf{V}$ and $\mathbf{A} = \mathbb{I}\mathbf{\Omega}$. Here, $m = m_b + m_f$ is the total mass of the rigid body, consisting of the mass m_b and the *virtual mass* m_f induced by the fluid. Similarly, $\mathbb{I} = \mathbb{I}_b + \mathbb{I}_f$, where $\mathbb{I}_f$ is the virtual inertia tensor due to the fluid.

The main difference between these equations and the Euler equations governing the motion of a rigid body in vacuum is the appearance of the non-zero term $\mathbf{A} \times \mathbf{V}$ on the right-hand side of the equation for $\mathbf{L}$. In other words, the center of mass no longer describes a uniform straight trajectory and is a non-trivial degree of freedom. From a geometric point of view, the motion of a rigid body in a potential flow can therefore be considered as a curve on the special Euclidian group SE(3) consisting of all translations and rotations in $\mathbb{R}^3$, where the latter describe the orientation of the body while the former encode the location of the center of mass.

The kinetic energy for the rigid body in a potential flow is a quadratic function and hence determines a metric on SE(3). The physical motions of the rigid body are geodesics with respect to this metric. By noting that the dynamics is invariant under the action of SE(3) on itself, the system can be reduced from the phase space $T^*\,\mathrm{SE}(3)$ down to one on the dual Lie algebra $\mathfrak{se}(3)^*$: in this way, one derives the Kirchhoff equations (1). The situation is similar for a planar rigid body moving in a two-dimensional flow: the motion is then a geodesic flow on SE(2), the Euclidian group of the plane, and reduces to a dynamical system on $\mathfrak{se}(2)^*$. The dynamics of the reduced system is still described by the Kirchhoff equations, but additional simplifications occur: since both $\mathbf{A}$ and $\mathbf{\Omega}$ are directed along the z-axis, the right-hand side of the equation for $\mathbf{A}$ is zero. Throughout this paper, we will deal with the case of planar bodies and two-dimensional flows only.

The procedure of reducing a mechanical system on a Lie group G whose dynamics are invariant under the action of G on itself is known as ***Lie-Poisson reduction***. It was pointed out by Arnold [4] that both the Euler equations for a rigid body as well as the Euler equations for a perfect fluid can be derived using this approach. For more about the history of these and related reduction procedures, we refer to [31]. The geometric outlook on the Kirchhoff equations, developed by Leonard [26], turned out to be crucial in the study of stability results for bottom-heavy underwater vehicles; see also [38].

1.2. **Point vortices.** Another development in fluid dynamics to which the name of Kirchhoff is associated, is the motion of point vortices in an inviscid flow. A ***point vortex*** is defined as a singularity in the vorticity field of a two-dimensional flow: $\omega = \Gamma\delta(\mathbf{x} - \mathbf{x}_0(t))$, where the constant Γ is referred to as the ***strength*** of the point vortex. By substituting a superposition of N point vortices into the Euler equations for a perfect fluid, one obtains the following set of ODEs for the evolution of the vortex locations $\mathbf{x}_i$, $i = 1, \ldots, N$:

$$\Gamma_k \frac{d\mathbf{x}_k}{dt} = J\frac{\partial H}{\partial \mathbf{x}_k}, \tag{2}$$

where $H = -W_G(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ and W_G is the so-called ***Kirchhoff-Routh function***, which is derived using the Green's function for the Laplacian with appropriate boundary conditions. We will return to the explicit form for W_G later. For point vortices moving in an unbounded domain, W_G was derived by Routh and Kirchhoff, whereas Lin [28] studied the case of vortices in a bounded container with fixed boundaries.

It is useful to recall here how the point vortex system is related to the dynamics of an inviscid fluid. In order to do so, we recall the deep insight of Arnold [4], who recognized that the motion of a perfect fluid in a container $\mathcal{F}$ is a geodesic on the group $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{F})$ of volume-preserving diffeomorphisms of $\mathcal{F}$, in much the same way as the motion of a rigid body is a geodesic on the rotation group SO(3). The group $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{F})$ acts on itself from the *right* and leaves the fluid kinetic energy invariant, a result known as ***particle relabeling symmetry***. Hence, by Lie-Poisson reduction, the system can be reduced to one on the dual $\mathfrak{X}^*_{\mathrm{vol}}(\mathcal{F})$ of the Lie algebra of $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{F})$, and the resulting equations are precisely Euler's equations.

Moreover, any group acts on its dual Lie algebra through the co-adjoint action, and a trajectory of the reduced mechanical system starting on one particular orbit of that action is constrained to remain on that particular orbit. Marsden and Weinstein [29] showed that for a perfect fluid, the co-adjoint orbits are labelled by vorticity and, when specified to the co-adjoint orbit corresponding to the vorticity of N point vortices, Euler's equations become precisely the vortex equations (2).

For a detailed overview of the geometric approach to fluid dynamics, we refer to [5]. The dynamics of point vortices is treated by different authors, see [40, 36, 3].

1.3. **The rigid body interacting with point vortices.** Given the interest in both rigid bodies and point vortices, it is natural to study the dynamics of a planar rigid body interacting with N point vortices. After pioneering work by Koiller [23], it wasn't until the recent work of Shashikanth, Marsden, Burdick and Kelly [43] (SMBK) and Borisov, Mamaev and Ramodanov [7] (BMR) that the equations of motion for this dynamical system were established in their full generality. Both groups proceeded through an ad-hoc calculation to derive the equations of motion, and showed that the resulting equations are in fact Hamiltonian (see theorems 2.1 and 2.2 below). However, both sets of equations are different at first sight.

The SMBK equations are formally identical to the Kirchhoff equations (1) together with the point vortex equations (2), but the definitions of the momenta $\mathbf{L}$, $\mathbf{A}$ and the Hamiltonian H are modified to include the effect of the rigid body (through the ambient fluid) on the point vortices, and vice versa. The configuration space is $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$ and the Poisson structure is the sum of the Poisson structures on the individual factors. From the BMR point of view, the dynamic variables are the velocity $\mathbf{V}$ and angular velocity $\mathbf{A}$, together with the locations of the vortices. The Hamiltonian is simply the sum of the kinetic energies for both subsystems, and to account for the interaction between the point vortices and the rigid body, BMR introduce a non-standard Poisson structure involving the stream functions of the fluid.

Somewhat miraculously, the equations of motion obtained by SMBK and BMR turn out to be equivalent: Shashikanth [41] establishes the existence of a Poisson map taking the canonical SMBK Poisson structure into the BMR Poisson structure; this is recalled in theorem 2.3 below. However, a number of questions remain. Most importantly, it is not obvious why one mechanical problem should be governed by two Poisson structures, which are at first sight very different but turn out to be related by a certain Poisson map. Moreover, it is not entirely clear why this system is Hamiltonian in the first place: in the work of SMBK and BMR, the Hamiltonian structure is derived only afterwards by direct inspection.

1.4. **Main contributions and outline of this paper.** We shed more light on the issues addressed above by uncovering the geometric structures that govern this problem. We begin this paper by giving an overview of rigid body dynamics and aspects of fluid mechanics in section 2. This material is mostly well-known and serves to set the tone for the rest of the paper. In section 3 we then introduce the so-called Neumann connection, giving the response of the fluid to an infinitesimal motion of the rigid body. This connection has been described before, but a detailed overview of its properties seems to be lacking. In particular, we derive an expression for the curvature of this connection when the fluid space is an arbitrary Riemannian manifold, generalizing a result of Montgomery [34].

The remainder of the paper is then devoted to using reduction theory to obtain the equations of motion for the fluid-solid system in a systematic way. We will derive the equations of motion for this system by reformulating it first as a geodesic flow on the Cartesian product of the group SE(2) of translations and rotations in 2D, and a space of embeddings describing the fluid configurations. Two symmetry groups act on this space: the particle relabeling symmetry group $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{F}_0)$ acts on the right, and the group SE(2) itself acts on the left, making the system into a ***left-right system***. Here $\mathcal{F}_0$ is the reference configuration for the fluid, *i.e.* it is the exterior of the disc in the reference configuration (see figure 1). Dividing out by the combined action of these symmetries is therefore an example of ***reduction by stages***.

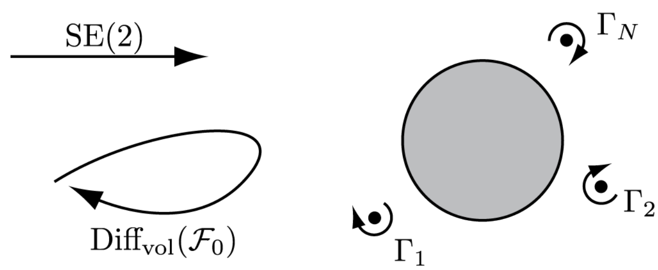

FIGURE 1. The two symmetry groups in the problem of a cylinder interacting with N point vortices are the particle relabeling group $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{F}_0)$ and the Euclidian group $SE(2)$.

After symplectic reduction with respect to $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{F})$ in section 4, we obtain a dynamical system on the product space $T^*\,\mathrm{SE}(2) \times \mathbb{R}^{2N}$, where the first factor is the phase space of the rigid body, while the second factor describes the locations of the point vortices. The dynamics is governed by a ***magnetic symplectic form***:

it is the sum of the canonical symplectic forms on both factors, together with a non-canonical interaction term. From a physical point of view, the interaction term embodies the interaction between the point vortices and the rigid body through the ambient fluid. Mathematically speaking, the map associating to each rigid body motion the corresponding motion of the fluid can be viewed as a connection, and the magnetic term is closely related to its curvature.

Note that it is essential here to do *symplectic* reduction with respect to the particle relabeling symmetry rather than Poisson reduction, even though the latter might be conceptually simpler. Recall that symplectic reduction reduces the system to a co-adjoint orbit; as we shall see below, in the case of fluids interacting with solids these co-adjoint orbits are precisely labelled by the vorticity of the fluid. Symplectic reduction — in particular the selection of one particular level set of the momentum map — therefore amounts to imposing a specific form for the vorticity field. In our case, this is where we introduce the assumption that the vorticity is concentrated in N point vortices.

If instead we had performed Poisson reduction, we would have obtained the Euler equations for the fluid (in spatial representation) coupled to the equations of motion for the rigid body. In two dimensions, one particular feature of this set of equations is that the vorticity of a perfect fluid is advected with the fluid flow but remains otherwise unchanged. From a geometrical point of view, this "advection property" of the classical Euler equations corresponds to the fact that the co-adjoint orbits for the fluid-solid system are precisely the spaces of constant vorticity. This will be made more precise in section 2.1 below.

After factoring out the particle relabeling symmetry, the resulting dynamical system is invariant under translations and rotations in the plane and can then be reduced with respect to the group SE(2). This is the subject of section 5. Physically, this corresponds to rewriting the equations of motion obtained after the first reduction in body coordinates. However, because of the presence of the magnetic term in the symplectic form and the fact that SE(2) acts diagonally on $T^* \mathrm{SE}(2) \times \mathbb{R}^{2N}$, this is not a straightforward task.

First, we derive the reduced Poisson structure on the reduced space $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$. Because of the magnetic contributions to the symplectic form, the reduced Poisson structure is not just the sum of the Poisson structures on the individual factors, but includes certain non-canonical contributions as well. We then show that the momentum map for the SE(2)-symmetry naturally defines a Poisson map taking this Poisson structure into the product Poisson structure, possibly with a cocycle if the momentum map is not equivariant (this happens when the total strength of the point vortices is nonzero). We do the computations for a general product $T^*H \times P$ first, where H is a Lie group, P is a symplectic manifold, and the product is equipped with a magnetic symplectic form. In this way, we generalize the "coupling to a Lie group" scenario (see [24]) to the case where magnetic terms are present.

In this way, the results of SMBK and BMR are put on a firm geometric footing: the BMR Poisson structure is the one obtained through reduction and involves the interaction terms, while the SMBK Poisson structure is simply the product Poisson structure. The Poisson map induced by the SE(2)-momentum map described above turns out to be precisely Shashikanth's Poisson map. As a consequence, we also obtain an explicit prescription for the symplectic leaves of this system.

1.5. **Relation with other approaches.** Our method consists of rederiving the SMBK and BMR equations by reformulating the motion of a rigid body in a fluid as a geodesic problem on the combined fluid-solid configuration space. By imposing the assumption that the vorticity is concentrated in N point vortices, and dividing out the symmetry, we obtain first of all the BMR equations, and secondly (after doing a momentum shift) the SMBK equations. This procedure is worked out in the body of the paper — here we would like to reflect on the similarities with other dynamical systems.

Recall that the dynamics of a particle of charge e in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ can be described in two ways. One is by using canonical variables $(\mathbf{q}, \mathbf{p})$ and the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2m} \left\| \mathbf{p} - e\mathbf{A} \right\|^2, \tag{3}$$

while for the other one we use the kinetic energy Hamiltonian $H_{\mathrm{kin}} = \left\| \mathbf{p} \right\|^2 / 2m$ but now we modify the symplectic form to be

$$\Omega_{\mathbf{B}} = \Omega_{\mathrm{can}} + e\mathbf{B}. \tag{4}$$

A simple calculation shows that both systems ultimately give rise to the familiar Lorentz equations. From a mathematical point of view, this can be seen by noting that the ***momentum shift map*** $\mathcal{S} : \mathbf{p} \mapsto \mathbf{p} - e\mathbf{A}$ maps the dynamics of the former system into that of the latter:

$$\mathcal{S}^* \Omega_{\mathbf{B}} = \Omega_{\mathrm{can}} \quad \text{and} \quad \mathcal{S}^* H_{\mathrm{kin}} = H.$$

In other words, both formulations are related by a symplectic isomorphism, thus making them equivalent.

An over-arching way of looking at the dynamics of a charged particle in a magnetic field is as a geodesic problem (with respect to a certain metric) on a higher-dimensional space: this is part of the famed ***Kaluza-Klein approach***. In this case, spacetime is replaced by the product manifold $\mathbb{R}^4 \times \mathrm{U}(1)$ and charged particles trace out geodesics on this augmented space. The standard, four-dimensional formulation of the dynamics can then be obtained by dividing out by the internal $\mathrm{U}(1)$-symmetry, resulting in the familiar Lorentz equations on $\mathbb{R}^4$. More information about these constructions can be found in [45], [48] and [31].

Our approach to the fluid-structure problem is similar but more involved because of the presence of two different symmetry groups. However, the underlying philosophy is the same: by reformulating the motion of a rigid body and the fluid as a geodesic problem on an infinite-dimensional manifold, we follow the philosophy of Kaluza-Klein of trading in the complexities in the equations of motion for an increase in the number of dimensions of the configuration space. Then, by dividing out the $\mathrm{Diff}_{\mathrm{vol}}$-symmetry, we obtain a reduced dynamical system governed by a magnetic symplectic form (the analogue of (4)), which is mapped after a suitable momentum shift $\mathcal{S}$ into an equivalent dynamical system governed by the canonical symplectic form but with a modified Hamiltonian. After reducing by the residual $SE(2)$-symmetry, the former gives rise to the BMR bracket, while the latter is nothing but the SMBK system.

1.6. **A note on integrability.** The case of a circular cylinder is distinguished because of the existence of an additional conservation law, associated with the symmetry that rotates the cylinder around its axis. When the circular cylinder interacts with one external point vortex, a simple count of dimensions and first integrals suggests that this problem is integrable, a fact first proved by Borisov and Mamaev [6]. Indeed, for a single vortex the phase space is a symplectic leaf of $\mathfrak{se}^*(2) \times \mathbb{R}^2$, which is generically four-dimensional. On the other hand, for the circular cylinder there are two conservation laws: the total energy and the material symmetry associated with rotations around the axis of the cylinder, which hints at Liouville integrability.

In the case of an ellipsoidal cylinder with nonzero eccentricity, Borisov, Mamaev and Ramodanov [8] gather numerical evidence to show that the interaction with one vortex is chaotic. This is to be contrasted with the motion of point vortices in an unbounded domain (see [36]), which is integrable for three vortices or less.

2. **The fluid-solid problem.**

This section is subdivided into two parts. In the first, and longest, part we describe the general setup for a planar rigid body interacting with a 2D flow. The second part is then devoted to discussing a number of simplifying assumptions that will make the subsequent developments clearer. By separating these assumptions from the main problem setting, we hope to convince the reader that the method outlined in this paper does not depend on any specific assumptions on the rigid body or the fluid, and can be generalized to more complex problems. At the end of this paper, we discuss how these assumptions can be relaxed.

2.1. **General geometric setting.**

2.1.1. *Kinematics of a Rigid Body.* Throughout this paper, we consider the motion of a ***planar rigid body*** interacting with a ***2D flow***. We introduce an inertial frame $\mathbf{e}_{1,2,3}$, where $\mathbf{e}_{1,2}$ span the plane of motion and $\mathbf{e}_3$ is perpendicular to it. The configuration of the rigid body is then described by a rotation with angle β around $\mathbf{e}_3$ and a vector $\mathbf{x}_0 = x_0\mathbf{e}_1 + y_0\mathbf{e}_2$ describing the location of a fixed point of the body, which we take to be the center of mass.

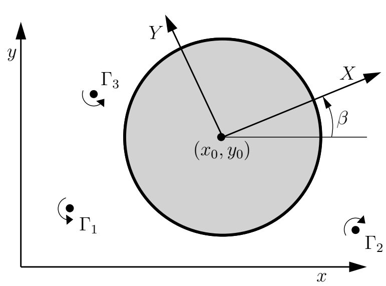

FIGURE 2. The configuration of the rigid body and the point vortices.

The orientation and position $(\beta; x_0, y_0)$ determine an element g of the Euclidian group SE(2) given by

$$g = \begin{pmatrix} R & \mathbf{x}_0 \\ 0 & 1 \end{pmatrix}, \quad \text{where } R = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}. \tag{5}$$

Written in this way, the group composition and inversion in SE(2) are given by matrix multiplication and inversion. The Lie algebra of SE(2) is denoted by $\mathfrak{se}(2)$ and essentially consists of infinitesimal rotations and translations. Its elements are matrices of the form

$$\begin{pmatrix} 0 & -\Omega & V_x \\ \Omega & 0 & V_y \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that $\mathfrak{se}(2)$ can be identified with $\mathbb{R}^3$ by mapping such a matrix to the triple (Ω, V_x, V_y). We define a basis of $\mathfrak{se}(2)$ given by

$$\mathbf{e}_\Omega = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

For future reference, we also introduce a moving frame fixed to the rigid body, denoted by $\mathbf{b}_{1,2,3}$. The transformation from body to inertial frame is given by $\mathbf{x} = R\mathbf{X} + \mathbf{x}_0$, where $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2$ and $\mathbf{X} = X\mathbf{b}_1 + Y\mathbf{b}_2$.

The ***angular*** and ***translational velocities*** of the rigid body relative to the inertial frame are defined as

$$\boldsymbol{\omega} = \dot{\beta}\mathbf{e}_3 \quad \text{and} \quad \mathbf{v} = \dot{x}_\circ \mathbf{e}_1 + \dot{y}_\circ \mathbf{e}_2, \tag{7}$$

where dots denote derivatives with respect to time. These quantities can be expressed in the body frame: the body angular velocity $\boldsymbol{\Omega}$ and the body translational velocity $\mathbf{V}$ are related to the corresponding inertial quantities by

$$\boldsymbol{\Omega} = \boldsymbol{\omega} \quad \text{and} \quad \mathbf{V} = R^T \mathbf{v}. \tag{8}$$

For the case of a planar rigid body, the angular velocity is oriented along the axis perpendicular to the plane and so determines a scalar quantity: $\boldsymbol{\omega} = \omega \mathbf{e}_3$ and $\boldsymbol{\Omega} = \Omega \mathbf{b}_3$. From a group-theoretic point of view, if the motion of a rigid body is given by a curve $t \mapsto g(t)$ in SE(2), then we may define an element $\xi(t) \in \mathfrak{se}(2)$ by putting

$$\xi(t) = \frac{d}{d\epsilon} g(t)^{-1} g(t+\epsilon)\Big|_{\epsilon=0},$$

and it can easily be checked that ξ coincides with the body angular and translational velocities $(\Omega, \mathbf{V})$.

The ***kinetic energy*** of the rigid body is given by the following expression:

$$T_{\text{body}}(\Omega, \mathbf{V}) = \frac{\mathbb{I}}{2}\Omega^2 + \frac{m}{2}\mathbf{V}^2 \tag{9}$$

where $\mathbb{I}$ is the moment of inertia of the body and m is its mass. The kinetic energy defines an inner product on $\mathfrak{se}(2)$, given by

$$\langle\langle (\Omega_1, \mathbf{V}_1), (\Omega_2, \mathbf{V}_2) \rangle\rangle_{\mathfrak{se}(2)} = \begin{pmatrix} \Omega_1 & \mathbf{V}_1 \end{pmatrix} \mathbb{M}_m \begin{pmatrix} \Omega_2 \\ \mathbf{V}_2 \end{pmatrix}, \tag{10}$$

for $(\Omega_1, \mathbf{V}_1)$ and $(\Omega_2, \mathbf{V}_2)$ in $\mathfrak{se}(2)$, and where $\mathbb{M}_m$ is given by

$$\mathbb{M}_m = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & m\mathbf{I} \end{pmatrix}. \tag{11}$$

Here, $\mathbf{I}$ is the 2-by-2 identity matrix. By left extension, this inner product induces a left-invariant metric on the whole of SE(2):

$$\langle\langle (g, \dot{g}_1), (g, \dot{g}_2) \rangle\rangle_{\text{SE}(2)} := \left\langle\left\langle g^{-1}\dot{g}_1, g^{-1}\dot{g}_2 \right\rangle\right\rangle_{\mathfrak{se}(2)}, \tag{12}$$

for $(g, \dot{g}_1)$ and $(g, \dot{g}_2)$ in $T\,\text{SE}(2)$.

2.1.2. *Incompressible Fluid Dynamics.* The geometric description of an incompressible fluid goes back to the work of Arnold [4], who described the motion of an incompressible fluid in a fixed container $\mathcal{F}$ as a geodesic on the diffeomorphism group of $\mathcal{F}$. In the case of a fluid interacting with a rigid body, the fluid container may change over time, reflecting the fact that the rigid body moves.

Arnold's formulation can be extended to cover this case by considering the ***space of embeddings*** of the reference configuration of the fluid (denoted by $\mathcal{F}_0$) into $\mathbb{R}^2$. Recall that an embedding $\varphi : \mathcal{F}_0 \hookrightarrow \mathbb{R}^2$ maps each reference point $X \in \mathcal{F}_0$ to its current configuration $x = \varphi(X)$. In order to reflect the fact that the fluid is taken to be incompressible, we require that any fluid embedding φ is ***volume preserving***: if $d\mathbf{x}$ is the Euclidian area element on $\mathbb{R}^2$ and $d\mathbf{x}_0$ is a fixed area element on $\mathcal{F}_0$, then we require

$$\varphi^*(d\mathbf{x}) = d\mathbf{x}_0. \tag{13}$$

The space of all such volume-preserving embeddings is denoted by $\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2)$. In the sequel, we will specify additional boundary conditions on the fluid configurations, stating for example that the fluid is free to slide along the boundary of the solid, but this does not make any difference for the current expository treatment.

A ***motion*** of a fluid is described by a curve $t \mapsto \varphi(t)$ in $\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2)$. The ***material velocity field*** is the tangent vector field $(\varphi_t, \dot{\varphi}_t)$ along the curve. Here, $\dot{\varphi}_t$ is a map from $\mathcal{F}_0$ to $T\mathbb{R}^2$, whose value at a point $x \in \mathcal{F}_0$ is given by

$$\dot{\varphi}_t(x) = \frac{d}{dt}\varphi_t(x) \in T_{\varphi(x)}\mathbb{R}^2.$$

Note that $\dot{\varphi}_t$ is not a vector field in the traditional sense; rather, it is a vector field along the map φ_t. In contrast, the ***spatial velocity field*** $\mathbf{u}_t$, defined as

$$\mathbf{u}_t = \dot{\varphi}_t \circ \varphi_t^{-1} \tag{14}$$

is a proper vector field, defined on $\varphi_t(\mathcal{F}_0)$.

The motion of a fluid can be described using the kinetic-energy Lagrangian:

$$T_{\mathrm{fluid}}(\varphi, \dot{\varphi}) = \frac{\rho}{2}\int_{\mathcal{F}_t} \|\mathbf{u}\|^2 \, d\mathbf{x}, \tag{15}$$

where $\mathbf{u}$ is the Eulerian velocity field (14) and the integration domain is the spatial domain of the fluid at time t: $\mathcal{F}_t = \varphi_t(\mathcal{F}_0)$. Just as in the case of the rigid body, this kinetic energy induces a metric on the space $\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2)$, given by

$$\langle\!\langle \dot{\varphi}_1, \dot{\varphi}_2 \rangle\!\rangle_{\mathrm{Emb}} = \rho \int_{\mathcal{F}_0} \dot{\varphi}_1 \cdot \dot{\varphi}_2 \, d\mathbf{x}_0. \tag{16}$$

By changing variables, this metric can be rewritten in spatial form as follows:

$$\langle\!\langle \dot{\varphi}_1, \dot{\varphi}_2 \rangle\!\rangle_{\mathrm{Emb}} = \rho \int_{\mathcal{F}_t} \mathbf{u}_1 \cdot \mathbf{u}_2 \, d\mathbf{x},$$

where $\mathbf{u}_i$ is the Eulerian velocity associated to $\dot{\varphi}_i$: $\mathbf{u}_i = \dot{\varphi}_i \circ \varphi_i^{-1}$, for $i = 1, 2$, and the integration is again over the spatial domain $\mathcal{F}_t := \varphi_t(\mathcal{F}_0)$ of the fluid.

2.1.3. *The Configuration Space of the Fluid-Solid System.* The motion of a rigid body in an incompressible fluid combines aspects of both rigid-body and fluid dynamics. We assume that the body occupies a circular region $\mathcal{B}_0$ in the reference configuration, and that the remainder of the domain, denoted by $\mathcal{F}_0$, is taken up by the fluid. The configuration space for the fluid-solid system is made up of pairs $(g, \varphi) \in \mathrm{SE}(2) \times \mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2)$ satisfying the following conditions.

1. The embedding φ represents the configuration of the fluid. In particular, φ is volume-preserving, *i.e.* (13) is satisfied. In addition, we assume that φ approaches the identity at infinity suitably fast.[1]
2. The element $g \in \mathrm{SE}(2)$ describes the configuration of the rigid body.
3. The fluid satisfies a "slip" boundary condition: the normal velocity of the fluid coincides with the normal velocity of the solid, while the tangent velocity can be arbitrary, reflecting the fact that there is no viscosity in the fluid. Mathematically, this boundary condition is imposed by requiring that, as sets, $\varphi(\partial\mathcal{F}_0)$ is equal to $g(\partial\mathcal{B}_0)$, where $g \in \mathrm{SE}(2)$ is interpreted as a affine embedding taking the body reference configuration $\mathcal{B}_0$ into its current location in $\mathbb{R}^2$.

We denote the space of all such pairs as Q; this is a submanifold of $\mathrm{SE}(2) \times \mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2)$. The kinetic energy of the fluid-solid system is given by the sum of the rigid-body energy T_{body} and the kinetic energy T_{fluid} of the fluid:

$$\begin{aligned} T &= T_{\mathrm{fluid}} + T_{\mathrm{body}} \\ &= \frac{\rho}{2}\int_{\mathcal{F}_t} \|\mathbf{u}\|^2\, d\mathbf{x} + \frac{\mathbb{I}}{2}\Omega^2 + \frac{m}{2}\mathbf{V}^2. \end{aligned} \tag{17}$$

Similarly, there exists a metric on Q given by the sum of the metrics (12) and (16):

$$\langle\!\langle (\dot\varphi_1, \dot g_1), (\dot\varphi_2, \dot g_2) \rangle\!\rangle = \langle\!\langle \dot\varphi_1, \dot\varphi_2 \rangle\!\rangle_{\mathrm{Emb}} + \langle\!\langle \dot g_1, \dot g_2 \rangle\!\rangle_{\mathrm{SE}(2)}\,. \tag{18}$$

The dynamics of rigid bodies moving in perfect fluids was studied before by Kelly [18], Radford [39], and Kanso *et al.* [16, 17]. A similar configuration space, but with the SE(2)-factor replaced by a suitable set of smooth manifolds, was studied in [27] for the dynamics of a liquid drop.

2.1.4. *Particle Relabelling Symmetry.* The kinetic energy T_{fluid} of the fluid is invariant if we replace $\dot\varphi$ by $\dot\varphi \circ \phi$, where ϕ is a volume-preserving diffeomorphism from $\mathcal{F}_0$ to itself. This represents the ***particle relabeling symmetry*** referred to in the introduction. Note that these symmetries act on the *right* on Q.

Recall that a diffeomorphism $\phi : \mathcal{F}_0 \to \mathcal{F}_0$ is volume-preserving if $\phi^* d\mathbf{x}_0 = d\mathbf{x}_0$, where $d\mathbf{x}_0$ is the volume element on $\mathcal{F}_0$. The group of all volume-preserving diffeomorphisms is denoted by $\mathrm{Diff}_{\mathrm{vol}}$. This group acts on the right on $\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2)$ by putting $\varphi \cdot \phi = \varphi \circ \phi$, and hence also on Q. The action of $\mathrm{Diff}_{\mathrm{vol}}$ on Q makes Q into the total space of a principal fiber bundle over SE(2). In other words, if we define the projection $\pi : Q \to \mathrm{SE}(2)$ as being the projection onto the first factor: $\pi(g, \varphi) = g$, then the fibers of π coincide precisely with the orbits of $\mathrm{Diff}_{\mathrm{vol}}$ in Q.

2.1.5. *Vorticity and Circulation.* In classical fluid dynamics, the ***vorticity*** $\boldsymbol{\mu}$ is defined as the curl of the velocity field: $\boldsymbol{\mu} = \nabla \times \mathbf{u}$, and the ***circulation*** around the rigid body is the line integral of $\mathbf{u}$ along any curve encircling the rigid body. In two dimensions, $\boldsymbol{\mu}$ can be written as $\boldsymbol{\mu} = \mu\mathbf{e}_3$, where μ is called the scalar vorticity.

According to Noether's theorem, there is a conserved quantity associated to the particle relabeling symmetry. This conserved quantity turns out to be precisely the circulation of the fluid:

$$\frac{d}{dt}\oint_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{l} = 0,$$

[1] We will not be concerned with any functional-analytic issues concerning these infinite-dimensional manifolds of mappings. Instead, the reader is referred to [12] for more information.

and Noether's theorem hence becomes Kelvin's theorem, which states that circulation is materially constant. As a consequence of Green's theorem, the circulation of the fluid is related to the vorticity:

$$\oint_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{l} = \int_{\mathcal{S}} \nabla \times \mathbf{u} \cdot d\mathbf{S},$$

where $\mathcal{S}$ is any surface whose boundary is $\mathcal{C}$. It would lead us too far to explore the geometry of vorticity and circulation in detail (for this, we refer to [5]) but at this stage we just note that *the conservation of vorticity is closely linked to the particle relabeling symmetry.*

2.1.6. *The Helmholtz-Hodge Decomposition.* If (in addition to being incompressible) the fluid is irrotational, meaning that $\nabla \times \mathbf{u} = 0$, and has no circulation, then there exists a velocity potential Φ such that $\mathbf{u} = \nabla\Phi$. In the presence of point vortices, the fluid is not irrotational but the velocity field $\mathbf{u}$ can be uniquely decomposed in an irrotational part and a vector field $\mathbf{u}_\mathrm{v}$ representing the "rotational" contributions (see for instance [40] or [36]). This is the well-known Helmholtz-Hodge decomposition:

$$\mathbf{u} = \nabla\Phi + \mathbf{u}_\mathrm{v}. \tag{19}$$

Here, $\mathbf{u}_\mathrm{v}$ is a divergence-free vector field which is tangent to the boundary of $\mathcal{F}$, while the potential Φ is the solution to Laplace's equation $\nabla^2\Phi = 0$, subject to the boundary conditions that the normal derivative of Φ equals the normal velocity of the rigid body, and that the velocity vanishes at infinity. In other words, one has

$$\nabla^2\Phi = 0 \quad \text{and} \quad \frac{\partial\Phi}{\partial n}\Big|_{\partial\mathcal{F}} = (\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_0) + \mathbf{v}) \cdot \mathbf{n} \quad (\text{for } \mathbf{x} \in \partial\mathcal{F}), \tag{20}$$

and Φ goes to zero as $|\mathbf{x}|$ goes to infinity. Here $\boldsymbol{\omega}$ and $\mathbf{v}$ are the angular and translational velocity of the rigid body, expressed in a spatial frame, while $\mathbf{x}_0$ represents the location of the center of mass.

Since Φ depends on ω and $\mathbf{v}$ in a linear way, we may decompose Φ, following Kirchhoff, as

$$\Phi(g, \dot{g}; \mathbf{x}) = v_x\Phi_x(g, \mathbf{x}) + v_y\Phi_y(g, \mathbf{x}) + \omega\Phi_\omega(g, \mathbf{x}). \tag{21}$$

Here Φ_x, Φ_y, and Φ_ω are elementary velocity potentials corresponding to infinitesimal translations in the x- and y-direction and to a rotation, respectively. They satisfy the Laplace equation with the following boundary conditions:

$$\frac{\partial\Phi_\omega}{\partial n} = (\mathbf{e}_3 \times \mathbf{x}) \cdot \mathbf{n}, \quad \frac{\partial\Phi_x}{\partial n} = \mathbf{e}_1 \cdot \mathbf{n}, \quad \text{and} \quad \frac{\partial\Phi_y}{\partial n} = \mathbf{e}_2 \cdot \mathbf{n}.$$

Note that the elementary potentials depend on the location of the rigid body (encoded by g) only. For the sake of clarity, we will suppress these arguments when no confusion is possible. The elementary velocity potentials for a circular body are given below: the reader can check that they indeed depend on the location of the rigid body.

The Helmholtz-Hodge decomposition defines a connection on the principal fiber bundle $Q \to \mathrm{SE}(2)$. To see this, note that any tangent vector to Q is of the form $(g, \dot{g}; \varphi, \dot{\varphi})$. We define its horizontal and vertical part by applying the Helmholtz-Hodge decomposition to $\mathbf{u} = \dot{\varphi} \circ \varphi^{-1}$, and we put

$$(g, \dot{g}; \varphi, \dot{\varphi})^H = (g, \dot{g}; \varphi, \nabla\Phi \circ \varphi) \quad \text{and} \quad (g, \dot{g}; \varphi, \dot{\varphi})^V = (g, 0; \varphi, \mathbf{u}_\mathrm{v} \circ \varphi)$$

We will verify in section 3 below that this prescription indeed defines a connection, which we term the ***Neumann connection***, since its horizontal subspaces are found by solving the Neumann problem (20). This connection was used in a variety of contexts, ranging from the dynamics of fluid drops [27, 35] to problems in optimal transport [19].

2.1.7. *The Lie Algebra of Divergence-free Vector Fields and its Dual.* At least on a formal level, $\mathrm{Diff}_{\mathrm{vol}}$ is a Lie group with associated to it a Lie algebra, denoted by $\mathfrak{X}_{\mathrm{vol}}$ and consisting of divergence-free vector fields which are parallel to the boundary of $\mathcal{F}_0$. The bracket on $\mathfrak{X}_{\mathrm{vol}}$ is the Jacobi-Lie bracket of vector fields (which is the negative of the usual bracket of vector fields) and its dual, denoted by $\mathfrak{X}^*_{\mathrm{vol}}$, is the set of linear functionals on $\mathfrak{X}_{\mathrm{vol}}$.

In the interest of making the paper self-contained, we now recall briefly the explicit realization of this dual space. Note first of all that each one-form α on $\mathcal{F}_0$ induces a linear functional on $\mathfrak{X}_{\mathrm{vol}}$ by putting

$$\langle \mathbf{u}, \alpha \rangle = \int_{\mathcal{F}_0} \alpha(\mathbf{u})\, d\mathbf{x}_0,$$

with exact one-forms giving rise, through Stokes' theorem, to the zero functional: $\langle \mathbf{u}, \mathbf{d}f \rangle = 0$ for all $\mathbf{u} \in \mathfrak{X}_{\mathrm{vol}}$. In this way, the dual space $\mathfrak{X}^*_{\mathrm{vol}}$ can be identified with the set of one-forms on $\mathcal{F}_0$ modulo exact forms:

$$\mathfrak{X}^*_{\mathrm{vol}}(\mathcal{F}_0) = \Omega^1(\mathcal{F}_0)/\mathbf{d}\Omega^0(\mathcal{F}_0),$$

(see [5]) while the duality pairing between elements of $\mathfrak{X}_{\mathrm{vol}}$ and $\mathfrak{X}^*_{\mathrm{vol}}$ is given by

$$\langle \mathbf{u}, [\alpha] \rangle = \int_{\mathcal{F}_0} \alpha(\mathbf{u})\, d\mathbf{x}_0,$$

where $\mathbf{u} \in \mathfrak{X}_{\mathrm{vol}}$ and $[\alpha] \in \mathfrak{X}^*_{\mathrm{vol}}$. Note that the right-hand side does not depend on the choice of representative α.

A more convenient interpretation of the dual Lie algebra $\mathfrak{X}^*_{\mathrm{vol}}$ is as the set of exact two-forms on $\mathcal{F}_0$. Any class $[\alpha]$ in $\mathfrak{X}^*_{\mathrm{vol}}$ is uniquely determined by the exterior differential $\mathbf{d}\alpha$ and by the value of α on the generators of the first homology of $\mathcal{F}_0$.

Intuitively speaking, this isomorphism can be understood as follows. Let us first consider the case where the domain is simply connected. In this case, we may reconstruct α by integration from $\mathbf{d}\alpha$, and the resulting α is determined up to an exact one-form. Hence, we obtain a one-to-one mapping from $\Omega^1(\mathcal{F}_0)/\mathbf{d}\Omega^1(\mathcal{F}_0)$ to $\mathbf{d}\Omega^1(\mathcal{F}_0)$. If the domain is multiply-connected, as is the case for the solid-fluid problem, the integration used to reconstruct α from $\mathbf{d}\alpha$ will also depend on the homology class of the path along which the integration is performed. In this case, there is a one-to-one mapping from $\Omega^1(\mathcal{F}_0)/\mathbf{d}\Omega^1(\mathcal{F}_0)$ to $\mathbf{d}\Omega^1(\mathcal{F}_0) \times \mathbb{R}^n$, where n is the rank of the first homology of $\mathcal{F}_0$.

In our case, the first homology group of $\mathcal{F}_0$ is generated by any closed curve $\mathcal{C}$ encircling the rigid body, and its pairing with α is given by

$$\Gamma = \int_{\mathcal{C}} \alpha.$$

Under this identification, $\mathbf{d}\alpha$ represents the ***vorticity***, while Γ represents the ***circulation***. Since in our case the circulation is assumed to be zero, it follows that $[\alpha]$ is completely determined by $\mathbf{d}\alpha$. The implications of non-zero circulation have been explored in [47].

From a geometric point of view, the vorticity field can be interpreted as an element of $\mathfrak{X}^*_{\mathrm{vol}}$: if we assume that the fluid is moving on an arbitrary Riemannian manifold, then the vorticity can be defined by

$$\mu = \mathbf{d}(\varphi^* \mathbf{u}^\flat). \tag{22}$$

Here $\flat$ is the flat operator associated to the metric. In the case of Euclidian spaces, this definition reduces to the one involving the curl of $\mathbf{u}$. Indeed, in this case, $\mathbf{u}^\flat$ is given by $u^i dx^i$, where u^i denotes the ith component of the velocity. The exterior derivative $\mathbf{d}\mathbf{u}^\flat$ then corresponds to the z-component of the curl of the vector field $\mathbf{u}$:

$$\mathbf{d}\mathbf{u}^\flat = \partial_i u^j dx^i \wedge dx^j = \epsilon_{0ij} \partial_i u^j dx^1 \wedge dx^2 = \mathbf{e}_z \cdot (\nabla \times \mathbf{u}) dx^1 \wedge dx^2.$$

The definition (22) is slightly different from the one in [5], where vorticity is defined as a two-form on the inertial space $\mathbb{R}^2$, whereas in our interpretation, vorticity lives on the material space $\mathcal{F}_0$. Both definitions are related by push-forward and pull-back by φ, and hence carry the same amount of information. Our definition has the advantage that μ is naturally an element of $\mathfrak{X}^*_{\mathrm{vol}}$, which is preferable from a geometric point of view.

We finish this section by noting that any Lie group acts on its Lie algebra and its dual Lie algebra through the adjoint and the co-adjoint action, respectively. For the group of volume-preserving diffeomorphisms, both are given by push-forward: if ϕ is an element of $\mathrm{Diff}_{\mathrm{vol}}$, and $\mathbf{u}$ and $\mathbf{d}\alpha$ are elements of $\mathfrak{X}_{\mathrm{vol}}$ and $\mathfrak{X}^*_{\mathrm{vol}}$, respectively, then

$$\mathrm{Ad}_\phi(\mathbf{u}) = \phi_* \mathbf{u} \quad \text{and} \quad \mathrm{CoAd}_\phi(\mathbf{d}\alpha) = \mathbf{d}(\phi_* \alpha).$$

2.2. **Point vortices interacting with a circular cylinder.** In this section, we impose some specific assumptions on the rigid body and the fluid. It should be pointed out that while these assumptions greatly simplify the exposition, the general reduction procedure can be carried out under far less stringent assumptions. Later on, we will discuss how some of these assumptions may be removed.

2.2.1. *The Rigid Body.* In order to tie in this work with previous research efforts, we assume the rigid body to be ***circular*** with radius R and neutrally buoyant (*i.e.* its body weight m is balanced by the force of buoyancy). If the density of the fluid is set to $\rho = 1$, this implies that the body has mass $m = \pi R^2$. The moment of inertia of the body around the axis of symmetry is denoted by $\mathbb{I}$.

For a rigid planar body of circular shape, the elementary velocity potentials Φ_x, Φ_y, and Φ_ω occurring in (21) can be calculated analytically (see [25]) and are given by

$$\Phi_x = -R^2 \frac{x - x_0}{(x-x_0)^2 + (y-y_0)^2}, \quad \text{and} \quad \Phi_y = -R^2 \frac{y - y_0}{(x-x_0)^2 + (y-y_0)^2}, \tag{23}$$

while $\Phi_\omega = 0$, reflecting the rotational symmetry of the body. Here (x_0, y_0) are the coordinates of the center of the disc.

In some cases, it will be more convenient to express Φ in body coordinates. In analogy with (21), we may write

$$\Phi(g, \dot{g}, \mathbf{X}) = V_x \Phi_X(\mathbf{X}) + V_y \Phi_Y(\mathbf{X}) + \Omega \Phi_\Omega(\mathbf{X}),$$

where (V_x, V_y) and Ω are the translational and angular velocity in the body frame, respectively. For the circular cylinder, the elementary potentials in body frame are given by

$$\Phi_X = -R^2 \frac{X}{X^2 + Y^2}, \quad \Phi_Y = -R^2 \frac{Y}{X^2 + Y^2} \quad \text{and} \quad \Phi_\Omega = 0. \tag{24}$$

Note that Φ_X, Φ_Y and Φ_Ω do not depend on the location of the rigid body, in contrast to Φ_x, Φ_y and Φ_ω.

2.2.2. *Point Vortices.* As for the fluid, we make the fundamental assumption that the vorticity is concentrated in N ***point vortices*** of strengths Γ_i, $i = 1, \ldots, N$, and that there is no circulation. Considered as a two-form on $\mathcal{F}_0$, the former means that the vorticity is given by

$$\mu = \sum_{i=1}^{N} \Gamma_i \delta(\bar{\mathbf{x}} - \bar{\mathbf{x}}_i) \mathbf{d}\bar{x} \wedge \mathbf{d}\bar{y}, \tag{25}$$

where $(\bar{x}, \bar{y})$ are coordinates on $\mathcal{F}_0$, and $\bar{\mathbf{x}}_i$ is the reference location of the ith vortex, $i = 1, \ldots, N$. As pointed out above, μ is an element of $\mathfrak{X}^*_{\rm vol}$.

2.2.3. *The Kirchhoff-Routh Function.* Shashikanth *et al.* [43] showed that the kinetic energy for the vortex system is the negative of the Kirchhoff-Routh function W_G for a system of N point vortices moving in a domain with moving boundaries:

$$T_{\rm vortex}(\mathbf{X}_1, \ldots, \mathbf{X}_N) = -W_G(\mathbf{X}_1, \ldots, \mathbf{X}_N). \tag{26}$$

The precise form of W_G is given by

$$W_G(\mathbf{X}_1, \ldots, \mathbf{X}_N) = \sum_{i,j(i>j)} \Gamma_i \Gamma_j G(\mathbf{X}_i, \mathbf{X}_j) + \frac{1}{2} \sum_i \Gamma_i g(\mathbf{X}_i, \mathbf{X}_i), \tag{27}$$

where $G(\mathbf{X}_0, \mathbf{X}_1)$ is a Green's function for the Laplace operator of the form

$$G(\mathbf{X}_0, \mathbf{X}_1) = g(\mathbf{X}_0, \mathbf{X}_1) + \frac{1}{4\pi} \log \|\mathbf{X}_0 - \mathbf{X}_1\|^2. \tag{28}$$

The function g is harmonic in the fluid domain and is the stream function of $\mathbf{u}_I$. As pointed out in [43], in the case of a circular cylinder, g can be calculated explicitly using Milne-Thomson's circle theorem [33], and is given by

$$g(\mathbf{X}, \mathbf{Y}) = \frac{1}{4\pi} \log \|\mathbf{X}\|^2 - \frac{1}{4\pi} \log \left\| \mathbf{X} - \frac{R^2}{\|\mathbf{Y}\|^2} \mathbf{Y} \right\|^2.$$

The general form of the Green's function was obtained by Lin [28]; see also [10].

From a geometrical point of view, Marsden and Weinstein [29] showed that the vortex energy can be obtained (up to some "self-energy" terms) by inverting the relation $\mu = \nabla \times \mathbf{u}$, where μ is the scalar part of (25), and substituting the resulting velocity field $\mathbf{u}$ generated by N vortices into the expression for the kinetic energy of the fluid. Although their analysis was for an unbounded fluid domain, their result can easily be extended to the case considered here, yielding again the negative of the Kirchhoff-Routh function as in (26).

In the case of a solid interacting with point vortices, the total kinetic energy will be the sum of the kinetic energy of the vortices, given by the Kirchhoff-Routh function, and that of the rigid body. The explicit expression is listed below in (47).

2.2.4. *Dynamics of the Fluid-Solid System.* As stated in the introduction, SMBK [43] were the first to derive the equations of motion for a rigid cylinder interacting with N point vortices. See also [42]. These equations generalize both the Kirchhoff equations for a rigid body in a potential flow and the equations for N point vortices in a bounded flow. Rather remarkably, SMBK established by direct inspection that these equations are Hamiltonian with respect to the canonical Poisson structure on $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$ (*i.e.* the sum of the Poisson structures on both factors), and a Hamiltonian given below involving the kinetic energy and interaction terms.

The SMBK equations are given by

$$\frac{d\mathbf{L}}{dt} = 0, \quad \frac{d\mathbf{A}}{dt} + \mathbf{V} \times \mathbf{L} = 0, \quad \text{and} \quad \Gamma_k \frac{d\mathbf{X}_k}{dt} = -J\frac{\partial H}{\partial \mathbf{X}_k}, \tag{29}$$

where $\mathbf{L}$ and $\mathbf{A} = A\mathbf{e}_3$ are the translational and angular momenta of the system, defined by

$$\begin{aligned} \mathbf{L} &= c\mathbf{V} + \sum_{k=1}^{N} \Gamma_k \mathbf{X}_k \times \mathbf{e}_3 + \sum_{k=1}^{N} \Gamma_k \mathbf{e}_3 \times \frac{\mathbf{X}_k}{\|\mathbf{X}_k\|^2} \\ A &= \mathbb{I}\Omega - \frac{1}{2}\sum_{i=1}^{N} \Gamma_i \|\mathbf{X}_i\|^2, \end{aligned} \tag{30}$$

and H is the Hamiltonian:

$$\begin{aligned} H(\mathbf{L}, \mathbf{X}_k) = -W(\mathbf{L}, \mathbf{X}_k) + \frac{1}{2c}\|\mathbf{L}\|^2 - \frac{1}{c}\Big(&\sum \Gamma_k (\mathbf{L} \times \mathbf{X}_k)\cdot \mathbf{e}_3 \\ &- \frac{1}{2}\sum \Gamma_k^2 \|\mathbf{X}_k\|^2 - \sum_{j>k} \Gamma_k \Gamma_j \mathbf{X}_k \cdot \mathbf{X}_j \\ &+ \frac{1}{2}\left\langle \sum \Gamma_k \frac{\mathbf{X}_k}{\|\mathbf{X}_k\|^2}, \sum \Gamma_k \frac{\mathbf{X}_k}{\|\mathbf{X}_k\|^2} \right\rangle \Big). \end{aligned} \tag{31}$$

Here, c is the total mass of the cylinder, consisting of the intrinsic mass m and the added mass πR^2 (due to the presence of the fluid): $c = m + \pi R^2 = 2\pi R^2$.

Theorem 2.1. *The SMBK equations are Hamiltonian on the space $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$ equipped with the Poisson bracket*

$$\{F, G\}_{\mathfrak{se}(2)^* \times \mathbb{R}^{2N}} = \{F_{|\mathfrak{se}(2)^*}, G_{|\mathfrak{se}(2)^*}\}_{\text{l.p}} + \{F_{|\mathbb{R}^{2N}}, G_{|\mathbb{R}^{2N}}\}_{\text{vortex}}. \tag{32}$$

Here, F and G are functions on $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$.

In the theorem above, $\{\cdot,\cdot\}_{\text{l.p}}$ is the Lie-Poisson bracket on $\mathfrak{se}(2)^*$:

$$\{f_1, f_2\}_{\mathfrak{se}(2)^*} = (\nabla f_1)^T \Lambda \nabla f_2, \quad \text{where} \quad \Lambda = \begin{pmatrix} 0 & -P_y & P_x \\ P_y & 0 & 0 \\ -P_x & 0 & 0 \end{pmatrix}, \tag{33}$$

for arbitrary functions f_1, f_2 on $\mathfrak{se}(2)^*$. Similarly, $\{\cdot,\cdot\}_{\text{vortex}}$ is the vortex bracket, given by:

$$\{g_1, g_2\}_{\text{vortex}} = \sum_{i=1}^{N} \frac{1}{\Gamma_i}\left(\frac{\partial g_1}{\partial X_i}\frac{\partial g_2}{\partial Y_i} - \frac{\partial g_2}{\partial X_i}\frac{\partial g_1}{\partial Y_i} \right), \tag{34}$$

where g_1, g_2 are arbitrary functions on $\mathbb{R}^{2N}$.

2.2.5. *The BMR Equations.* A completely different perspective on the rigid body interacting with N point vortices is offered by BMR [7]. From their point of view, the equations of motion are again written in Hamiltonian form on $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$, but now with a *noncanonical* Poisson bracket:

$$\dot{F} = \{F, H\}_{\text{BMR}}$$

for all functions F on $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$. The Hamiltonian is the sum of the kinetic energies of the subsystems, without interaction terms:

$$H = \frac{c}{2}\langle \mathbf{V}, \mathbf{V}\rangle - W_G(\mathbf{V}, \mathbf{X}_k), \tag{35}$$

whereas the Poisson bracket is determined by its value on the coordinate functions:

$$\begin{aligned}
\{V_1, V_2\}_{\mathrm{BMR}} &= \frac{\Gamma}{c^2} - \sum \frac{\Gamma}{c^2}\frac{R_i^4 - R^4}{R_i^4}, & \{V_1, X_i\}_{\mathrm{BMR}} &= \frac{1}{c}\frac{R_i^4 - R^2(X_i^2 - Y_i^2)}{R_i^4}, \\
\{V_1, Y_i\}_{\mathrm{BMR}} &= -\frac{2R^2}{c}\frac{X_iY_i}{R_i^4}, & \{V_2, X_i\}_{\mathrm{BMR}} &= -\frac{2R^2}{c}\frac{X_iY_i}{R_i^4}, \\
\{V_2, Y_i\}_{\mathrm{BMR}} &= \frac{1}{c}\frac{R_i^4 + R^2(X_i^2 - Y_i^2)}{R_i^4}, & \{X_i, Y_i\}_{\mathrm{BMR}} &= -\frac{1}{\Gamma_i},
\end{aligned} \tag{36}$$

where $R_i^2 = \|\mathbf{X}_i\|^2$, $\mathbf{V} = (V_1, V_2)$, $\mathbf{X}_i = (X_i, Y_i)$, and Γ is the total vortex strength:

$$\Gamma = \sum_{i=1}^{N} \Gamma_i.$$

We summarize the situation so far in the following theorem.

Theorem 2.2. *The BMR equations, specified by the Hamiltonian (35) and the Poisson brackets (36), form a non-canonical Hamiltonian system describing the interaction between a circular rigid body and N point vortices.*

Note that the version of the BMR Poisson bracket listed here differs from the one in [7] by an overall factor of 2π. Shashikanth [41] showed that this discrepancy can be attributed to the way in which BMR define the fluid density.

2.2.6. *The Link between the SMBK and the BMR Equations.* By explicit calculation, Shashikanth [41] constructed a Poisson map taking the SMBK equations into the BMR equations. His result is listed below.

Theorem 2.3. *The map $\mathcal{S} : (\mathbf{L}, A; \mathbf{X}_1, \ldots, \mathbf{X}_N) \mapsto (\mathbf{V}, \Omega; \mathbf{X}_1, \ldots, \mathbf{X}_N)$, where $\mathbf{L}$ and A are given by (30), is a Poisson map from $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$ equipped with the bracket $\{\cdot,\cdot\}_{\mathfrak{se}(2)^* \times \mathbb{R}^{2N}}$ to $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$ with the bracket $\{\cdot,\cdot\}_{\mathrm{BMR}}$.*

Even though this result asserts that both sets of equations are equivalent, it leaves open the question as to *why* this is so. By re-deriving the equations of motion using symplectic reduction, not only do we obtain both sets of equations, but the map $\mathcal{S}$ also follows naturally.

3. **The Neumann connection.** The bundle $\pi : Q \to \mathrm{SE}(2)$ is equipped with a principal fibre bundle connection, called the ***Neumann connection*** by Montgomery [34]. There are many ways of describing this connection, but from a physical point of view, the definition using the horizontal lift operator is perhaps most appealing. From this point of view, the Neumann connection is a map

$$\mathbf{h} : Q \times T\,\mathrm{SE}(2) \to TQ, \quad \mathbf{h}(\varphi, g, \dot{g}) = \nabla\Phi \circ \varphi, \tag{37}$$

where Φ is the solution of the Neumann problem (20) associated to $(g, \dot{g})$. In other words, the Neumann connection associates to each motion $(g, \dot{g})$ the corresponding induced velocity field of the fluid, and hence encodes the effect of the body on the fluid. It is important to note that the Neumann connection does not depend on the point vortex model and is valid for any vorticity field.

Similar connections as this one have been described before (see for example [27, 23, 19]) but a complete overview of its definition and properties seems to be lacking. In this section, we give an outline of the properties of the Neumann connection which are relevant for the developments in this paper, leaving detailed proofs for the appendix.

3.1. **Invariance of the Kirchhoff decomposition.** Before introducing the Neumann connection, we prove that the velocity potential $\Phi(g, \dot{g}; \mathbf{x})$ is left SE(2)-invariant, expressing the fact that the dynamics is invariant under translations and rotations of the combined solid-fluid system.

Proposition 1. *The velocity potential Φ is left* SE(2)*-invariant in the sense that*

$$\Phi(hg, TL_h(\dot{g}); h\mathbf{x}) = \Phi(g, \dot{g}; \mathbf{x}) \tag{38}$$

for all $h \in \mathrm{SE}(2)$ and $(g, \dot{g}; \mathbf{x}) \in T\,\mathrm{SE}(2) \times \mathbb{R}^2$.

Proof. This assertion can be proved in a number of different ways. The easiest is to use the assumption that the body is circular and solve the equation for the elementary potentials explicitly. Recall that these elementary potentials are given by (23). It is then straightforward to check that (38) holds, using the transformation properties of the velocity in the inertial frame. □

3.2. **The connection one-form.** For our purposes, it is convenient to define the Neumann connection through its connection one-form $\mathcal{A} : TQ \to \mathfrak{X}_{\mathrm{vol}}$ given by

$$\mathcal{A}_{(\varphi,g)}(\dot{g}, \dot{\varphi}) = \varphi^* \mathbf{u}_{\mathrm{v}}, \tag{39}$$

where $\mathbf{u}_{\mathrm{v}}$ is the divergence-free part in the Helmholtz-Hodge decomposition (19) of the Eulerian velocity $\mathbf{u} = \dot{\varphi} \circ \varphi^{-1}$. A proof that this prescription determines a well-defined connection form can be found in the appendix, proposition 9, where it is also shown that this prescription agrees with the horizontal lift operator (37).

3.3. **The curvature of the Neumann connection.** It will be convenient in what follows to have an expression for the μ-component of the curvature of the Neumann connection, where for now μ is an arbitrary element of $\mathfrak{X}^*_{\mathrm{vol}}$. Later on, μ will be the vorticity (25) associated to N point vortices.

The curvature of a principal fiber bundle connection is a two-form $\mathcal{B}$ whose definition is listed in the appendix. For the Neumann connection we have

$$\mathcal{B}_{(g,\varphi)}((\dot{g}_1, \dot{\varphi}_1), (\dot{g}_2, \dot{\varphi}_2)) = -\mathcal{A}_{(g,\varphi)}([X^H, Y^H]), \tag{40}$$

where X^H and Y^H are horizontal vector fields on Q such that $X^H(g, \varphi) = (\dot{g}_1, \dot{\varphi}_1)$ and $Y^H(g, \varphi) = (\dot{g}_2, \dot{\varphi}_2)$. Here $\mathcal{A}$ is the connection one-form of the Neumann connection, given by (39).

Proposition 2. *Let $(\dot{g}_1, \dot{\varphi}_1)$ and $(\dot{g}_2, \dot{\varphi}_2)$ be elements of $T_{(g,\varphi)}Q$ and denote the solutions of the Neumann problem (20) associated to $(g, \dot{g}_1)$ resp. $(g, \dot{g}_2)$ by Φ_1 and Φ_2. Then the μ-component of the curvature $\mathcal{B}$ is given by*

$$\left\langle \mu, \mathcal{B}_{(g,\varphi)}((\dot{g}_1, \dot{\varphi}_1), (\dot{g}_2, \dot{\varphi}_2)) \right\rangle = \langle\!\langle \mu, \mathbf{d}\Phi_1 \wedge \mathbf{d}\Phi_2 \rangle\!\rangle_{\mathcal{F}} - \int_{\partial \mathcal{F}} \alpha \wedge *(\mathbf{d}\Phi_1 \wedge \mathbf{d}\Phi_2), \tag{41}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathcal{F}}$ is the metric on the space of forms on $\mathcal{F}$, induced by the Euclidian metric on $\mathcal{F}$.

Proof. Let μ be equal to $\varphi^* \mathbf{d}\alpha$, and pick α such that $\alpha = \mathbf{u}_{\mathrm{v}}^\flat$, where $\mathbf{u}_{\mathrm{v}}$ is a divergence-free vector field on $\mathcal{F}$ tangent to $\partial\mathcal{F}$. For example, in the case of the vorticity due to N point vortices, $\mathbf{u}_{\mathrm{v}}$ is the velocity field due to the vortices and their images (see [40]). The calculation of the curvature involves computing the Jacobi-Lie bracket of two horizontal vector fields and taking the divergence-free part of the result. Because of the special form of α we can dispense with the latter step, since α is chosen to be L_2-orthogonal to gradient vector fields. Therefore, the μ-component of the curvature is given by

$$\begin{aligned}\left\langle \mu, \mathcal{B}_{(g,\varphi)}((\dot{g}_1, \dot{\varphi}_1), (\dot{g}_2, \dot{\varphi}_2)) \right\rangle &= -\left\langle \mu, \mathcal{A}_{(g,\varphi)}([X^H, Y^H]) \right\rangle \\ &= -\int_{\mathcal{F}_0} (\varphi^*\alpha)([X^H, Y^H])\, d\mathbf{x}_0,\end{aligned}$$

where the bracket on the left-hand side is the Jacobi-Lie bracket, which is the negative of the usual commutator of vector fields.

The bracket can be made more explicit by noting that (as vector fields on Q)

$$[X^H, Y^H](g, \varphi) = ([(\mathbf{d}\Phi_1)^\flat, (\mathbf{d}\Phi_2)^\flat] \circ \varphi, \cdots),$$

where the dots denote a term in $T_g\,\mathrm{SE}(2)$ whose explicit form doesn't matter. The curvature then becomes

$$\left\langle \mu, \mathcal{B}_{(g,\varphi)}((\dot{g}_1, \dot{\varphi}_1), (\dot{g}_2, \dot{\varphi}_2)) \right\rangle = -\int_{\mathcal{F}} \alpha([(\mathbf{d}\Phi_1)^\flat, (\mathbf{d}\Phi_2)^\flat])\, d\mathbf{x}.$$

The remainder of the proof relies on the following formula for the codifferential of a wedge product [46, formula 1.34]:

$$\delta(\alpha \wedge \beta) = \delta\alpha \wedge \beta + (-1)^p \alpha \wedge \delta\beta - [\alpha^\flat, \beta^\flat]^\sharp, \tag{42}$$

where $p = \deg \alpha$. Applying (42) with $\alpha = \mathbf{d}\Phi_1$ and $\beta = \mathbf{d}\Phi_2$ gives

$$[(\mathbf{d}\Phi_1)^\flat, (\mathbf{d}\Phi_2)^\flat]^\sharp = \Delta\Phi_1 \mathbf{d}\Phi_2 - \Delta\Phi_2 \mathbf{d}\Phi_1 - \delta(\mathbf{d}\Phi_1 \wedge \mathbf{d}\Phi_2),$$

where Δ is the Laplace-Beltrami operator on differential forms. Since both Φ_1 and Φ_2 are harmonic, the first two terms of the right-hand side vanish. Consequently, the μ-component of the curvature becomes

$$\begin{aligned}\left\langle \mu, \mathcal{B}_{(g,\varphi)}((\dot{g}_1, \dot{\varphi}_1), (\dot{g}_2, \dot{\varphi}_2)) \right\rangle &= \langle\!\langle \alpha, \delta(\mathbf{d}\Phi_1 \wedge \mathbf{d}\Phi_2) \rangle\!\rangle_{\mathcal{F}} \\ &= \langle\!\langle \mathbf{d}\alpha, \mathbf{d}\Phi_1 \wedge \mathbf{d}\Phi_2 \rangle\!\rangle_{\mathcal{F}} - \int_{\partial\mathcal{F}} \alpha \wedge *(\mathbf{d}\Phi_1 \wedge \mathbf{d}\Phi_2),\end{aligned}$$

using the adjointness of $\mathbf{d}$ and δ for manifolds with boundary (this is a simple consequence of Stokes' theorem). □

In traditional fluid mechanics notation, formula (41) becomes

$$\langle \mu, \mathcal{B} \rangle = \int_{\mathcal{F}} (\nabla \times \mathbf{u}_V) \cdot (\nabla\Phi_1 \times \nabla\Phi_2) dx - \int_{\partial\mathcal{F}} \mathbf{u}_V \cdot (\mathbf{n} \times (\nabla\Phi_1 \times \nabla\Phi_2)) dA,$$

a formula first derived in [34].

4. **Reduction with respect to the diffeomorphism group.** As mentioned in the introduction, the fluid-solid system on the space $Q \subset \mathrm{SE}(2) \times \mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2)$ is invariant under the particle relabeling group $\mathrm{Diff}_{\mathrm{vol}}$. We now perform symplectic reduction to eliminate that symmetry. In order to do so, we need to fix a value of the momentum map associated to the $\mathrm{Diff}_{\mathrm{vol}}$-symmetry. From a physical point of view, this boils down to fixing the vorticity of the system; it is at this point that the assumption is used that the vorticity is concentrated in N point vortices.

Before tackling symplectic reduction in the context of the fluid-solid system, we first give a general overview of cotangent bundle reduction following [30]. Roughly speaking, applying symplectic reduction to a cotangent bundle yields a space which is diffeomorphic to the product of a reduced cotangent bundle and a co-adjoint orbit of the group. The dynamics on the reduced space is governed by a reduced Hamiltonian and a magnetic symplectic form: the reduced symplectic form is the sum of the canonical symplectic forms on the individual factors and an additional magnetic term.

As we shall see below, in the case of the fluid-solid system, the reduced phase space is equal to $T^* \mathrm{SE}(2) \times \mathbb{R}^{2N}$, where the first factor describes the rigid body, while the second factor determines the configuration of the vortices. At first sight, it may therefore appear that the intermediate fluid is completely gone. Yet, the vortices act on the rigid body, and vice versa, through the surrounding fluid. The answer to this apparent contradiction is that the effect of the fluid is concentrated in the magnetic symplectic form, for which we derive a convenient expression below.

4.1. **Cotangent bundle reduction: Review.** In this section, we collect some relevant results from [30]. We consider a manifold Q on which a Lie group G, with Lie algebra $\mathfrak{g}$, acts from the right and we denote the action by $\sigma : Q \times G \to Q$. In addition, we assume that we are given a connection one-form $\mathcal{A} : TQ \to \mathfrak{g}$ with curvature two-form $\mathcal{B}$. In the rest of this paper, Q will be the configuration space of the solid-fluid system, while the structure group G will be $\mathrm{Diff}_{\mathrm{vol}}$, and the connection $\mathcal{A}$ the Neumann connection.

4.1.1. *The Curvature as a Two-form on the Reduced Space.* Let μ be an element of $\mathfrak{g}^*$, and denote its isotropy subgroup under the co-adjoint action by G_μ:

$$G_\mu = \{g \in G : \mathrm{CoAd}_g \mu = \mu\}\,.$$

Consider the contraction $\langle \mu, \mathcal{B}\rangle$ of μ with the curvature $\mathcal{B}$: due to the G-equivariance and the fact that $\mathcal{B}$ vanishes on vertical vectors (see (70)), we may show that $\langle \mu, \mathcal{B}\rangle$ is a G_μ-invariant form.

Proposition 3. *The μ-component of the curvature $\mathcal{B}$ is a G_μ-invariant two-form:*

1. $\sigma_g^* \langle \mu, \mathcal{B}\rangle = \langle \mu, \mathcal{B}\rangle$*, for all* $g \in G_\mu$*;*
2. $i_{\xi_Q} \langle \mu, \mathcal{B}\rangle = 0$ *for all* $\xi \in \mathfrak{g}_\mu$*.*

Hence, $\langle \mu, \mathcal{B}\rangle$ *induces a two-form* $\bar{\mathcal{B}}_\mu$ *on* Q/G_μ *such that*

$$\pi^*_{Q,G_\mu} \bar{\mathcal{B}}_\mu = \langle \mu, \mathcal{B}\rangle\,,$$

where $\pi_{Q,G_\mu} : Q \to Q/G_\mu$ *is the quotient map.*

Proof. For the first property, we have

$$\sigma_g^* \langle \mu, \mathcal{B}\rangle = \left\langle \mu, \sigma_g^* \mathcal{B}\right\rangle = \left\langle \mu, \mathrm{Ad}_{g^{-1}} \mathcal{B}\right\rangle = \langle \mathrm{CoAd}_g(\mu), \mathcal{B}\rangle = \langle \mu, \mathcal{B}\rangle$$

if $g \in G_\mu$. The second item follows from the corresponding property for $\mathcal{B}$. Actually, more is true: $i_{\xi_Q} \langle \mu, \mathcal{B}\rangle = 0$ for all $\xi \in \mathfrak{g}$, not just in $\mathfrak{g}_\mu$. □

4.1.2. *Cotangent Bundle Reduction.* The first stage in reducing the phase space consists of dividing out the particle relabeling symmetry. To this end, we use the framework of ***cotangent bundle reduction*** to construct the reduced phase space. The framework for cotangent bundle reduction outlined in [30] allows us to write down the reduced phase space and the modified symplectic form. We quote:

Theorem 4.1. *(Marsden* et al. *[theorem 2.2.1])*

1. *There exists a symplectic imbedding φ_μ of the reduced phase space $(T^*Q)_\mu$ into the cotangent bundle $T^*(Q/G_\mu)$ with the shifted symplectic structure $\Omega_{\mathcal{B}} := \Omega_{\text{can}} - B_\mu$.*
2. *The image of φ_μ is the vector subbundle $[T\pi_{Q,G_\mu}(V)]^\circ$ of $T^*(Q/G_\mu)$, where $V \subset TQ$ is the vector subbundle consisting of vectors tangent to the G-orbits in Q, and $^\circ$ denotes the annihilator relative to the natural duality pairing between $T(Q/G_\mu)$ and $T^*(Q/G_\mu)$.*

Here $(T^*Q)_\mu$ is the reduced phase space $J^{-1}(\mu)/G_\mu$, where G_μ is the isotropy subgroup of μ. The two-form B_μ is called the ***magnetic two-form***, and is defined as $B_\mu = \pi^*_{Q/G_\mu}\beta_\mu$, where $\pi_{Q/G_\mu} : T^*(Q/G_\mu) \to Q/G_\mu$ is the cotangent bundle projection and β_μ is a two-form on Q/G_μ.

To define β_μ, consider the one-form $\alpha_\mu := \langle \mu, \mathcal{A}\rangle$ on Q. One can show that $\mathbf{d}\alpha_\mu$ (but not α_μ itself) is G_μ-invariant, and the induced two-form on Q/G_μ is precisely β_μ, or in other words

$$\pi^*_{Q,G_\mu}\beta_\mu = \mathbf{d}\alpha_\mu,$$

where $\pi_{Q,G_\mu} : Q \to Q/G_\mu$ is the quotient map. The calculation of $\mathbf{d}\alpha_\mu$ is greatly facilitated by using the Cartan structure equation [21], which allows us to rewrite $\mathbf{d}\alpha_\mu$ as

$$\mathbf{d}\alpha_\mu = \langle \mu, \mathcal{B}\rangle - \langle \mu, [\mathcal{A}, \mathcal{A}]\rangle . \tag{43}$$

Observe the sign difference on the right-hand side with [30], which is due to the fact that we take G to be acting on Q from the *right*.

4.2. **The reduced phase space.** Theorem 4.1 provides us with an explicit prescription for the reduced phase space. Recall that the diffeomorphism group $\mathrm{Diff}_{\mathrm{vol}}$ acts on the dual Lie algebra $\mathbf{d}\Omega^1(\mathcal{F}_0)$ through push-forward. In particular, if μ represents the vorticity due to N point vortices as in (25), then

$$\phi_*\mu = \sum_{i=1}^N \Gamma_i \delta(\bar{\mathbf{x}} - \phi(\bar{\mathbf{x}}_i))\, \mathbf{d}\bar{x} \wedge \mathbf{d}\bar{y}.$$

In other words, the diffeomorphism group acts on the space of point vortices by simply moving the vortices. It also follows that the isotropy subgroup of μ consists of all diffeomorphisms ϕ for which the vortex reference locations are fixed: $\phi(\bar{\mathbf{x}}_i) = \bar{\mathbf{x}}_i$, for $i = 1, \ldots, N$. We denote the group of all such diffeomorphisms as $\mathrm{Diff}_{\mathrm{vol},\mu}$.

The group $\mathrm{Diff}_{\mathrm{vol},\mu}$ acts on Q and moreover, the quotient of Q by this action is diffeomorphic to $\mathrm{SE}(2) \times \mathbb{R}^{2N}$. To see this, note that $\mathrm{Diff}_{\mathrm{vol},\mu}$ acts on the $\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2)$ factor only, and that there exists a diffeomorphism of the quotient space $\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2)/\mathrm{Diff}_{\mathrm{vol},\mu}$ with $\mathbb{R}^{2N}$, given by

$$[\varphi] \mapsto (\varphi(\bar{\mathbf{x}}_1), \ldots, \varphi(\bar{\mathbf{x}}_N)).$$

Similarly, the projection of Q onto the quotient space $Q/\mathrm{Diff}_{\mathrm{vol},\mu} = \mathrm{SE}(2) \times \mathbb{R}^{2N}$ is given by

$$\pi_{Q,G_\mu} : (g, \varphi) \mapsto (g; \varphi(\bar{\mathbf{x}}_1), \ldots, \varphi(\bar{\mathbf{x}}_N)).$$

Proposition 4. *After reducing by the group of volume preserving diffeomorphisms, the reduced phase is given by $T^*\,\mathrm{SE}(2) \times \mathbb{R}^{2N}$.*

Proof. The vertical bundle V on Q consists of vectors $(g, 0; \varphi, \dot\varphi)$. Projecting this bundle down under $T\pi_{Q,G_\mu}$ shows that $T\pi_{Q,G_\mu}(V)$ is spanned by elements of the form

$$T\pi_{Q,G_\mu}(g, 0; \varphi, \dot\varphi) = (g, 0; \varphi(\bar{\mathbf{x}}_1), \dot\varphi(\bar{\mathbf{x}}_1); \dots; \varphi(\bar{\mathbf{x}}_N), \dot\varphi(\bar{\mathbf{x}}_N))$$

and as each of the N vectors on the right hand side can range over the whole of $T\mathbb{R}^2$, $T\pi_{Q,G_\mu}(V)$ is equal to $\mathrm{SE}(2) \times T\mathbb{R}^{2N}$. Its annihilator is therefore $T^*\,\mathrm{SE}(2) \times \mathbb{R}^{2N}$. □

The reduced symplectic structure on $T^*\,\mathrm{SE}(2) \times \mathbb{R}^{2N}$ is described in theorem 4.1. Explicitly, it is given by

$$\Omega_{\mathcal{B}} = \Omega_{\mathrm{can}} - B_\mu, \tag{44}$$

where Ω_{can} is the canonical symplectic structure on $T^*\,\mathrm{SE}(2)$ and B_μ is the pullback to $T^*\,\mathrm{SE}(2) \times \mathbb{R}^{2N}$ of the form β_μ on $\mathrm{SE}(2) \times \mathbb{R}^{2N}$. From now on, we will no longer make any notational distinction between β_μ and its pull-back B_μ and denote both by β_μ.

Apart from the reduced symplectic form, which will be determined explicitly later on, the dynamics on the reduced space is also governed by a reduced version of the Hamiltonian (17). The calculation of this Hamiltonian is the subject of the next section.

4.3. **The reduced Hamiltonian.** The kinetic energy (10) is invariant under the action of $\mathrm{Diff}_{\mathrm{vol},\mu}$ on Q and hence determines a reduced kinetic energy function on the quotient space $\mathrm{SE}(2) \times \mathbb{R}^{2N}$. Parts of the computation of the explicit form of the reduced kinetic energy can be found throughout the literature, but since no single reference has a complete picture, we briefly recall these results here.

Using the Helmholtz-Hodge decomposition (19), the kinetic energy (10) of the combined solid-fluid system can be written as

$$\begin{aligned} T(\dot g, \dot\varphi) &= \frac{1}{2}\langle\!\langle \dot g, \dot g\rangle\!\rangle_{\mathrm{SE}(2)} + \frac{1}{2}\int_{\mathcal{F}} \mathbf{u}\cdot\mathbf{u}\, d\mathbf{x} \\ &= \frac{1}{2}\langle\!\langle \dot g, \dot g\rangle\!\rangle_{\mathrm{SE}(2)} + \frac{1}{2}\int_{\mathcal{F}} \mathbf{u}_V\cdot\mathbf{u}_V\, d\mathbf{x} + \frac{1}{2}\int_{\mathcal{F}} \nabla\Phi\cdot\nabla\Phi\, d\mathbf{x}, \end{aligned} \tag{45}$$

where we have used the fact that $\mathbf{u}_V$ and $\nabla\Phi$ are L_2-orthogonal.

Following a reasoning similar to [29], it can be shown that the kinetic energy of the vortex system is nothing but the negative of the Kirchhoff-Routh function W_G (see (27)):

$$\frac{1}{2}\int_{\mathcal{F}} \mathbf{u}_V\cdot\mathbf{u}_V\, d\mathbf{x} = -W_G(\mathbf{X}_1, \dots, \mathbf{X}_N),$$

The gradient term in (45) can be rewritten using a standard procedure, going back to Kirchhoff and Lamb, and yields the familiar added masses and moments of inertia for a rigid body in potential flow. By using Green's theorem to rewrite the gradient term as an integral over the boundary, and substituting the Kirchhoff expansion (21), one can show that (see [25, 16])

$$\frac{1}{2}\int_{\mathcal{F}} \nabla\Phi\cdot\nabla\Phi\, d\mathbf{x} = \frac{1}{2}\begin{pmatrix}\Omega & \mathbf{V}\end{pmatrix}\mathbb{M}_a\begin{pmatrix}\Omega \\ \mathbf{V}\end{pmatrix}, \tag{46}$$

where $\mathbb{M}_a$ is a 3-by-3 matrix of added masses and inertia whose entries depend only on the geometry of the rigid body. For the circular cylinder, $\mathbb{M}_a$ can be evaluated explicitly:

$$\mathbb{M}_a = \begin{pmatrix} 0 & 0 \\ 0 & \pi R^2 \mathbf{I} \end{pmatrix}.$$

The important point is to note that the gradient term (46) has the same form as the kinetic energy of the rigid body. By introducing the matrix $\mathbb{M} = \mathbb{M}_m + \mathbb{M}_a$, where $\mathbb{M}_m$ is the mass matrix (11), the kinetic energy of the rigid body, together with the gradient term, can be written as

$$T = \frac{1}{2} \begin{pmatrix} \Omega & \mathbf{V} \end{pmatrix} \mathbb{M} \begin{pmatrix} \Omega \\ \mathbf{V} \end{pmatrix},$$

which determines by left extension a function, also denoted by T, on $T\,\mathrm{SE}(2)$. Putting everything together, we conclude that the total reduced kinetic energy on $T\,\mathrm{SE}(2) \times \mathbb{R}^{2N}$ is given by

$$T_{\mathrm{red}}(\Omega, \mathbf{V}; \mathbf{X}_1, \ldots, \mathbf{X}_N) = \frac{1}{2} \begin{pmatrix} \Omega & \mathbf{V} \end{pmatrix} \mathbb{M} \begin{pmatrix} \Omega \\ \mathbf{V} \end{pmatrix} - W_G(\mathbf{X}_1, \ldots, \mathbf{X}_N), \tag{47}$$

where W_G is given by (27) and (28).

Two issues are noteworthy here. First of all, we follow Marsden and Weinstein [29] and SMBK [43] in "regularizing" W_G by subtracting infinite contributions that arise when putting $\mathbf{X}_0 = \mathbf{X}_1$ in the expression for the Green's function G in (28). Secondly, we have chosen to express the kinetic energy in the body frame, which will be more convenient later on.

4.4. **The magnetic two-form β_μ.** Recall that β_μ is a two-form on $\mathbb{R}^{2N} \times \mathrm{SE}(2)$ defined as

$$\beta_\mu(\mathbf{x}_1, \ldots, \mathbf{x}_N; g)((\mathbf{v}_1, \ldots, \mathbf{v}_N, \dot{g}_1), (\mathbf{w}_1, \ldots, \mathbf{w}_N, \dot{g}_2)) = \mathbf{d}\alpha_\mu(\varphi, g)((\dot{\varphi}_1, \dot{g}_1), (\dot{\varphi}_2, \dot{g}_2)),$$

where $\mathbf{v}_i, \mathbf{w}_i \in T_{\mathbf{x}_i}\mathbb{R}^2$ $(i = 1, \ldots, N)$, and $\dot{g}_1$ and $\dot{g}_2$ are elements of $T_g\,\mathrm{SE}(2)$. The embedding φ has to satisfy $\varphi(\bar{\mathbf{x}}_i) = \mathbf{x}_i$ for $i = 1, \ldots, N$, while $\dot{\varphi}_1$ and $\dot{\varphi}_2$ should satisfy

$$\dot{\varphi}_1(\bar{\mathbf{x}}_i) = \mathbf{v}_i \quad \text{and} \quad \dot{\varphi}_2(\bar{\mathbf{x}}_i) = \mathbf{w}_i \quad (\text{for } i = 1, \ldots, N).$$

Using Cartan's structure equation (43), $\mathbf{d}\alpha_\mu$ can be rewritten as the difference of a curvature term and a "Lie-Poisson" term: if we introduce two-forms β_{curv} and $\beta_{\mathrm{l.p}}$ on $\mathbb{R}^{2N} \times \mathrm{SE}(2)$, determined by

$$\pi^*_{Q,G_\mu}\beta_{\mathrm{curv}} = \langle \mu, \mathcal{B} \rangle, \quad \text{and} \quad \pi^*_{Q,G_\mu}\beta_{\mathrm{l.p}} = \langle \mu, [\mathcal{A}, \mathcal{A}] \rangle,$$

then $\beta_\mu = \beta_{\mathrm{curv}} - \beta_{\mathrm{l.p}}$.

In the computation of these terms, we will frequently encounter expressions involving the vertical part of $(\dot{\varphi}_1, \dot{g}_1)$ and $(\dot{\varphi}_2, \dot{g}_2)$, evaluated at the vortex locations. We now derive a convenient expression for these terms.

Let Φ_1 be the solution of the Neumann problem (20) with boundary data $\dot{g}_1$ and denote by $\mathbf{u}$ the Eulerian velocity field associated to $\dot{\varphi}_1$. From the Helmholtz-Hodge decomposition $\mathbf{u} = \mathbf{u}_{\mathrm{v}} + \nabla\Phi_1$, it follows the divergence-free part $\mathbf{u}_{\mathrm{v}}$ satisfies (for $i = 1, \ldots, N$)

$$\begin{aligned} \mathbf{u}_{\mathrm{v}}(\mathbf{x}_i) &= \mathbf{u}(\mathbf{x}_i) - \nabla\Phi_1(\mathbf{x}_i) \\ &= \mathbf{v}_i - \nabla\Phi_1(\mathbf{x}_i), \end{aligned}$$

since $\mathbf{u}(\mathbf{x}_i) = \dot{\varphi}_1(\bar{\mathbf{x}}_i) = \mathbf{v}_i$. From the definition of $\mathcal{A}$ it then follows that we have proved the following fact about the vertical part of $\dot{\varphi}_1$:

$$\begin{aligned}(\mathcal{A}_{(\varphi,g)}(\dot{\varphi}_1, \dot{g}_1))(\bar{\mathbf{x}}_i) &= (\varphi^* \mathbf{u}_\mathrm{v})(\bar{\mathbf{x}}_i) \\ &= T\varphi^{-1}(\mathbf{v}_i - \nabla\Phi_1(\mathbf{x}_i)) \end{aligned} \tag{48}$$

for $i = 1, \ldots, N$. A similar property holds for $\dot{\varphi}_2$, with $\mathbf{v}_i$ replaced by $\mathbf{w}_i$ and involving Φ_2, the solution of (20) with boundary data $\dot{g}_2$.

As an aside, we note that the Neumann connection induces a connection on the trivial bundle $\mathbb{R}^2 \times \mathrm{SE}(2) \to \mathrm{SE}(2)$, and the expression between brackets on the right-hand side of (48) is the vertical component of the vector $\mathbf{v}_i$. This connection is important in the theory of *Routh reduction* (see [32]).

4.4.1. *The Curvature Term.* The curvature term β_curv is given by

$$\begin{aligned}&\beta_\mathrm{curv}(\mathbf{x}_1, \ldots, \mathbf{x}_N; g)((\mathbf{v}_1, \ldots, \mathbf{v}_N, \dot{g}_1), (\mathbf{w}_1, \ldots, \mathbf{w}_N, \dot{g}_2)) \\ &\qquad = \langle \mu, \mathcal{B}(\varphi, g)((\dot{\varphi}_1, \dot{g}_1), (\dot{\varphi}_2, \dot{g}_2))\rangle\,,\end{aligned}$$

where $\mathcal{B}$ is the curvature of the Neumann connection. It follows from (41) that β_curv does not depend on the value of $\mathbf{v}_i$ and $\mathbf{w}_i$, but only on $\dot{g}_1$ and $\dot{g}_2$. From now on, we will therefore suppress these arguments.

Proposition 5. *For the rigid body interacting with N point vortices, the curvature term is given by*

$$\beta_\mathrm{curv}(\mathbf{x}_1, \ldots, \mathbf{x}_N; g)(\dot{g}_1, \dot{g}_2) = \sum_{i=1}^N \Gamma_i \, d\boldsymbol{x}(\nabla\Phi_1(\mathbf{x}_i), \nabla\Phi_2(\mathbf{x}_i)), \tag{49}$$

where Φ_1 and Φ_2 are the solutions of (20) associated to $\dot{g}_1$ and $\dot{g}_2$, respectively, and $d\boldsymbol{x}$ is the volume element on $\mathcal{F}$.

Proof. We use the explicit form for the elementary velocity potentials derived in proposition 1. Note that β_curv depends on $\dot{g}_1$ and $\dot{g}_2$ through $g^{-1}\dot{g}_1$ and $g^{-1}\dot{g}_2$, so we can calculate its expression on the basis $\{\mathbf{e}_\Omega, \mathbf{e}_x, \mathbf{e}_y\}$ of $\mathfrak{se}(2)$ given in (6) by assuming that $g = e$ and $\dot{g}_1, \dot{g}_2 \in T_e\,\mathrm{SE}(2) = \mathfrak{se}(2)$.

First, we prove that the boundary term in (41) is zero. This is obvious if either $\dot{g}_1$ or $\dot{g}_2$ is proportional to $\mathbf{e}_\Omega$, since $\Phi_\omega = 0$. We may therefore assume that $\dot{g}_1 = \mathbf{e}_x$ and $\dot{g}_2 = \mathbf{e}_y$. In this case, the two-form $\mathbf{d}\Phi_x \wedge \mathbf{d}\Phi_y$ restricted to the boundary of $\mathcal{F}$ (the curve $(x - x_0)^2 + (y - y_0)^2 = 1$) is given by $\mathbf{d}\Phi_x \wedge \mathbf{d}\Phi_y = d\mathbf{x}$. Since $*d\mathbf{x} = 1$ in two dimensions, the boundary term reduces to

$$\int_{\partial\mathcal{F}} \alpha = \Gamma, \tag{50}$$

which is zero, since by assumption there is no circulation around the rigid body.

This leaves us with the integral over the fluid domain in (41). Plugging in the explicit form (25) of the vorticity, we have

$$\begin{aligned}\beta_\mathrm{curv}(\mathbf{x}_1, \ldots, \mathbf{x}_N; g)(\dot{g}_1, \dot{g}_2) &= \langle\!\langle \mu, \mathbf{d}\Phi_1 \wedge \mathbf{d}\Phi_2 \rangle\!\rangle_\mathcal{F} \\ &= \sum_{i=1}^N \Gamma_i \, d\mathbf{x}\,(\nabla\Phi_1(\mathbf{x}_i), \nabla\Phi_2(\mathbf{x}_i)).\end{aligned}$$

We do not work out this expression any further, since it will be cancelled out by a similar term in the computation of $\beta_\mathrm{l.p}$. ∎

It is remarkable that the contribution from the curvature to the magnetic term will be cancelled in its entirety by a similar term in the expression for Lie-Poisson term, which is computed below. However, in the case of non-zero circulation, the curvature generates an additional term (50) proportional to the circulation. The effect of this term is studied in [47]. We currently do not know whether this cancellation reflects an underlying fundamental property of the system, or if this is merely a coincidence. Shedding more light on the nature of the terms that arise in the magnetic term is an interesting project: we hope to return to this in the future when we study the interaction of solids with more intricate vortical structures.

4.4.2. *The Lie-Poisson Term.* The Lie-Poisson term $\beta_{\rm l.p}$ is given by

$$\begin{aligned}\beta_{\rm l.p}(\mathbf{x}_1, \dots, \mathbf{x}_N; g)((\mathbf{v}_1, \dots, \mathbf{v}_N, \dot{g}_1), (\mathbf{w}_1, \dots, \mathbf{w}_N, \dot{g}_2))& \\ = \left\langle \mu, [\mathcal{A}_{(\varphi,g)}(\dot{\varphi}_1, \dot{g}_1), \mathcal{A}_{(\varphi,g)}(\dot{\varphi}_2, \dot{g}_2)] \right\rangle,&\end{aligned}$$

where $(\dot{\varphi}_1, \dot{g}_1)$ and $(\dot{\varphi}_2, \dot{g}_2)$ have similar interpretations as before. The right-hand side can be made more explicit by noting that, for divergence-free vector fields $\mathbf{u}_1$ and $\mathbf{u}_2$ tangent to $\partial\mathcal{F}$ and arbitrary one-forms α, the following holds:

$$\int_{\mathcal{F}} \alpha([\mathbf{u}_1, \mathbf{u}_2])\, d\mathbf{x} = \int_{\mathcal{F}} \mathbf{d}\alpha(\mathbf{u}_1, \mathbf{u}_2)\, d\mathbf{x}. \tag{51}$$

The proof of this assertion is a straightforward application of Cartan's magic formula and parallels the proof of theorem 4.2 in [29].

Using this formula, we have

$$\begin{aligned}\left\langle \mu, [\mathcal{A}_{(\varphi,g)}(\dot{\varphi}_1, \dot{g}_1), \mathcal{A}_{(\varphi,g)}(\dot{\varphi}_2, \dot{g}_2)] \right\rangle &= \int_{\mathcal{F}} (\varphi^*\alpha)([\mathcal{A}_{(\varphi,g)}(\dot{\varphi}_1, \dot{g}_1), \mathcal{A}_{(\varphi,g)}(\dot{\varphi}_2, \dot{g}_2)])\, d\mathbf{x} \\ &= \int_{\mathcal{F}} \mu(\mathcal{A}_{(\varphi,g)}(\dot{\varphi}_1, \dot{g}_1), \mathcal{A}_{(\varphi,g)}(\dot{\varphi}_2, \dot{g}_2))\, d\mathbf{x} \\ &= \sum_{i=1}^N \Gamma_i\, (\varphi^* d\mathbf{x})\big((\mathcal{A}_{(\varphi,g)}(\dot{\varphi}_1, \dot{g}_1))(\bar{\mathbf{x}}_i), (\mathcal{A}_{(\varphi,g)}(\dot{\varphi}_2, \dot{g}_2))(\bar{\mathbf{x}}_i)\big).\end{aligned}$$

By substituting (48) in this expression, we conclude that the Lie-Poisson term is given by

$$\begin{aligned}\beta_{\rm l.p}(\mathbf{x}_1, \dots, \mathbf{x}_N; g)((\mathbf{v}_1, \dots, \mathbf{v}_N, \dot{g}_1), (\mathbf{w}_1, \dots, \mathbf{w}_N, \dot{g}_2))& \\ = \sum_{i=1}^N \Gamma_i\, d\mathbf{x}(\mathbf{v}_i - \nabla\Phi_1(\mathbf{x}_i), \mathbf{w}_i - \nabla\Phi_2(\mathbf{x}_i)).&\end{aligned}$$

4.4.3. *The Magnetic Two-form: Putting Everything Together.* Using the previously derived results, we may conclude that β_μ is given by (suppressing its arguments for the sake of clarity)

$$\begin{aligned}\beta_\mu &= \beta_{\rm curv} - \beta_{\rm l.p} \\ &= \sum_{i=1}^N \Gamma_i \big(d\mathbf{x}(\nabla\Phi_1(\mathbf{x}_i), \nabla\Phi_2(\mathbf{x}_i)) - d\mathbf{x}(\mathbf{v}_i - \nabla\Phi_1(\mathbf{x}_i), \mathbf{w}_i - \nabla\Phi_2(\mathbf{x}_i)\big) \\ &= \sum_{i=1}^N \Gamma_i \big(- d\mathbf{x}(\mathbf{v}_i, \mathbf{w}_i) + d\mathbf{x}(\mathbf{v}_i, \nabla\Phi_2(\mathbf{x}_i)) + d\mathbf{x}(\nabla\Phi_1(\mathbf{x}_i), \mathbf{w}_i)\big).\end{aligned}$$

Introducing stream functions $\Psi'_i(g, \dot{g}; \mathbf{x})$ $(i = 1, 2)$ as harmonic conjugates to $\Phi_i(g, \dot{g}; \mathbf{x})$, *i.e.* such that

$$\nabla \Phi_i \lrcorner \, d\mathbf{x} = \mathbf{d}\Psi'_i \quad i = 1, 2 \tag{52}$$

we may rewrite the expression for β_μ as

$$\beta_\mu = \sum_{i=1}^{N} \Gamma_i \big(- d\mathbf{x}(\mathbf{v}_i, \mathbf{w}_i) - \mathbf{d}\Psi'_2(\mathbf{v}_i) + \mathbf{d}\Psi'_1(\mathbf{w}_i) \big).$$

The stream function can be written in Lamb form by introducing the elementary stream functions Ψ_x, Ψ_y, and Ψ_ω as harmonic conjugates to Φ_x, Φ_y, and Φ_ω, respectively. For $i = 1, 2$, we then have

$$\begin{aligned} \Psi'_i(g, \dot{g}; \mathbf{x}) &= v_{x,i}\Psi_x(g, \mathbf{x}) + v_{y,i}\Psi_y(g, \mathbf{x}) + v_{\omega,i}\Psi_\omega(g, \mathbf{x}) \\ &= \mathbf{d}x_0(\dot{g}_i)\Psi_x(g, \mathbf{x}) + \mathbf{d}y_0(\dot{g}_i)\Psi_y(g, \mathbf{x}) + \mathbf{d}\beta(\dot{g}_i)\Psi_\omega(g, \mathbf{x}), \end{aligned}$$

and so the magnetic two-form becomes

$$\beta_\mu = \sum_{i=1}^{N} \Gamma_i \big(- d\mathbf{x}(\mathbf{v}_i, \mathbf{w}_i) - \mathbf{d}\xi_A(\dot{g}_2)\mathbf{d}\Psi'_A(\mathbf{v}_i) + \mathbf{d}\xi_A(\dot{g}_1)\mathbf{d}\Psi'_1(\mathbf{w}_i) \big),$$

where the coordinates on SE(2) are denoted by $\xi_A = \{x_0, y_0, \beta\}$, and $\Psi'_A = \{\Psi'_x, \Psi'_y, \Psi'_\omega\}$.

Theorem 4.2. *The magnetic two-form β_μ is a two-form on $\mathbb{R}^{2N} \times \mathrm{SE}(2)$ and can be written as*

$$\begin{aligned} &\beta_\mu(\mathbf{x}_1, \ldots, \mathbf{x}_N; g)((\mathbf{v}_1, \ldots, \mathbf{v}_N, \dot{g}_1), (\mathbf{w}_1, \ldots, \mathbf{w}_N, \dot{g}_2)) \\ &\qquad = \sum_{i=1}^{N} \Gamma_i \left(- d\boldsymbol{x}(\mathbf{v}_i, \mathbf{w}_i) + \mathbf{d}\Theta(g, \mathbf{x}_i)(\dot{g}_1, \mathbf{v}_i; \dot{g}_2, \mathbf{w}_i) \right). \end{aligned} \tag{53}$$

where Θ is the one-form on $\mathbb{R}^2 \times \mathrm{SE}(2)$ given by

$$\Theta(g, \mathbf{x})(\dot{g}, \mathbf{v}) = \Psi'_A(g, \mathbf{x})\mathbf{d}\xi_A(\dot{g}). \tag{54}$$

Proof. Note that the exterior derivative on the right-hand side of (53) is the exterior derivative on $\mathbb{R}^{2N} \times \mathrm{SE}(2)$. Expanding the second term, we have

$$\begin{aligned} \mathbf{d}(\Psi'_A(g, \mathbf{x})\mathbf{d}\xi_A) &= \mathbf{d}\Psi'_A \wedge \mathbf{d}\xi_A \\ &= \mathbf{d}\Psi'_x \wedge \mathbf{d}x_0 + \mathbf{d}\Psi'_y \wedge \mathbf{d}y_0 \\ &= \mathbf{d}_{\mathbb{R}^2}\Psi'_x \wedge \mathbf{d}x_0 + \mathbf{d}_{\mathbb{R}^2}\Psi'_y \wedge \mathbf{d}y_0 + \left(\frac{\partial \Psi'_y}{\partial x_0} - \frac{\partial \Psi'_x}{\partial y_0} \right) \mathbf{d}x_0 \wedge \mathbf{d}y_0, \end{aligned}$$

where the subscript '$\mathbb{R}^2$' indicates that the derivative should be taken with respect to the spatial variables only, *i.e.* that the SE(2)-variable should be kept fixed. Also, we rely on the fact that $\Psi'_\omega = 0$ (since $\Phi_\omega = 0$), and that $\Psi'_{x,y}$ does not depend on β.

The theorem is proved once we show that the term in brackets vanishes. To show this, we need the explicit form for the elementary stream functions:

$$\Psi'_x = R^2 \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} \quad \text{and} \quad \Psi'_y = -R^2 \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}, \tag{55}$$

where R is the radius of the cylinder, and it is then easily shown that this is indeed the case. $\square$

Putting everything together, we conclude that the symplectic structure $\Omega_{\mathcal{B}}$ on $\mathbb{R}^{2N} \times T^* \mathrm{SE}(2)$ (see (44)) is given by

$$\Omega_{\mathcal{B}} = \Omega_{\text{can}} - \beta_\mu, \tag{56}$$

where β_μ is given in the theorem above. The first term is the canonical symplectic form on $T^* \mathrm{SE}(2)$ while the magnetic consists of two parts, one (proportional to $d\mathbf{x}$) being the symplectic form on $\mathbb{R}^{2N}$, while the other one encodes the effects of the ambient fluid through the stream functions Ψ'_x and Ψ'_y. Note that $\mathbb{R}^{2N}$ is a co-adjoint orbit of the diffeomorphism group, and that its symplectic structure is nothing but the Konstant-Kirillov-Souriau form.

Koiller [23] considers the symplectic structure for the point vortex-solid system in the limit where the radius R of the circle goes to zero. These results can be obtained as a special case of theorem 4.2: in this limit, the term Θ vanishes and the total symplectic form is given by

$$\Omega_{\mathcal{B}} = \Omega_{\text{can}} + \sum_{i=1}^{N} \Gamma_i d\mathbf{x}_i.$$

This is precisely the symplectic form obtained directly in [23] (see Eq. 13).

5. Reduction with respect to $SE(2)$.

After reducing the fluid-solid system with respect to the $\mathrm{Diff}_{\text{vol}}$-symmetry, we are left with a system on $T^* \mathrm{SE}(2) \times \mathbb{R}^{2N}$ with a magnetic symplectic form. The group $\mathrm{SE}(2)$ acts diagonally on the reduced phase space. What remains is to carry out the reduction with respect to the group $\mathrm{SE}(2)$ and then to make the connection with the equations of motion in [43] and [6].

Two factors complicate this strategy, however: first of all, there is the presence of the magnetic term in the symplectic structure, which precludes using standard Euler-Poincaré theory, for instance. Secondly, the phase space is a Cartesian product on which the symmetry group acts diagonally. Even in the absence of magnetic terms, this would lead to the appearance of certain ***interaction terms*** in the Poisson structure (see [24]).

In order to do Poisson reduction for this kind of manifold, we begin this section by considering the product of a cotangent bundle T^*H, where H is a Lie group, and a symplectic manifold P, and we assume that the symplectic structure on $T^*H \times P$ is the sum of the canonical symplectic structures on both factors plus an additional magnetic term. This case extends both magnetic Lie-Poisson reduction in [30], for which the P-factor is absent, as well as the "coupling to a Lie group" scenario of [24], for which there is no magnetic term. Since the proofs in this section are rather lengthy, we have relegated them to Appendix B and we simply quote the relevant expressions here.

The bulk of this section is then devoted to making these results explicit for the case where $H = \mathrm{SE}(2)$, $P = \mathbb{R}^{2N}$ and the symplectic form is the magnetic symplectic form derived in the previous section; see equations (53) and (56). The induced Poisson structure with interaction terms will turn out to be nothing but the BMR Poisson structure (36). Secondly, the Poisson map induced by the momentum map J is just the Shashikanth map described in Theorem 2.3, so that the transformed Poisson structure is the SMBK Poisson structure (32).

5.1. Coupling to a Lie group.

As mentioned above, the purpose of this section is to generalize the reduction of phase spaces of the form $T^*H \times P$, where H is a Lie group acting on the symplectic manifold P. This is more general than results in the literature, since the symplectic structure on $T^*H \times P$ may contain magnetic terms. We shall need this generalization in our application. The proofs of the theorems in this section can be found in appendix B.

Specifically, if ω_P denotes the symplectic form on P, we assume that the symplectic structure Ω on $T^*H \times P$ is given by

$$\Omega = \omega_{\text{can}} - B, \tag{57}$$

where ω_{can} is the canonical symplectic structure on T^*H and B is a closed two-form on $H \times P$ such that $B_{|\{e\}\times P} = -\omega_P$, *i.e.* the restriction of B to $\{e\} \times P$ is just $-\omega_P$. The latter ensures that Ω is a well-defined symplectic form on $T^*H \times P$.

We assume furthermore that H acts on P and we denote the action of $h \in H$ on $p \in P$ by $h \cdot p$. Furthermore, we assume that this action leaves invariant ω_P. The group H then acts diagonally on $T^*H \times P$: if $h \in H$ and $(\alpha_g, p) \in T^*H \times P$, then

$$h \cdot (\alpha_g, p) = (T^*L_{h^{-1}}(\alpha_g), h \cdot p). \tag{58}$$

5.1.1. *The Momentum Map.* Consider an element ξ of $\mathfrak{h}$. We denote the fundamental vector field associated to ξ on $T^*H \times P$ by

$$\xi_{T^*H\times P}(\alpha_g, p) := \frac{d}{dt}\exp(t\xi)\cdot(\alpha_g, p)\Big|_{t=0},$$

and similar for the fundamental vector field $\xi_{H\times P}$ on $H \times P$. Under certain topological assumptions (which are satisfied for the solid-fluid system), there exists a momentum map $J : T^*H \times P \to \mathfrak{h}^*$ for the action of H, defined by

$$\mathbf{i}_{\xi_{T^*H\times P}}\Omega = \mathbf{d}J_\xi,$$

where $J_\xi = \langle J, \xi\rangle$. It follows from the explicit form for the symplectic form (57) that J can be written as $J = J_{\text{can}} - \phi$, where $J_{\text{can}} : T^*H \to \mathfrak{h}^*$ is the momentum map associated to the canonical symplectic form on T^*H:

$$\mathbf{i}_{\xi_{T^*H}}\omega_{\text{can}} = \mathbf{d}\left\langle J_{\text{can}}, \xi\right\rangle \tag{59}$$

while $\phi : H \times P \to \mathfrak{h}^*$ is the so-called $\mathcal{B}\mathfrak{g}$*-potential* (borrowing the terminology of [30]):

$$\mathbf{i}_{\xi_{H\times P}}B = \mathbf{d}\left\langle \phi, \xi\right\rangle. \tag{60}$$

The theory of $\mathcal{B}\mathfrak{g}$-potentials and Poisson reduction for cotangent bundles T^*H endowed with a magnetic symplectic form is further developed in [30]. Here, we are dealing with a product $T^*H \times P$, for which no such theory can be found in the current literature.

5.1.2. *Infinitesimal Equivariance of the Momentum Map.* Whereas the canonical momentum map J_{can} is equivariant, the same does not necessarily hold for the $\mathcal{B}\mathfrak{g}$-potential. To measure non-equivariance, we introduce a one-cocycle $\sigma : H \to L(\mathfrak{h}, \mathcal{C}(H \times P))$ by defining first a family of functions $\Gamma_{\eta,g}$ (where $\eta \in \mathfrak{h}$ and $g \in H$):

$$\Gamma_{\eta,g}(h,p) = -\left\langle \phi(gh, gp), \eta\right\rangle + \left\langle \mathrm{Ad}_g^*\phi(h,p), \eta\right\rangle,$$

for all $(h,p) \in H \times P$, and then putting $\sigma(g)\cdot\eta = \Gamma_{\eta,g}$. This definition follows the usual introduction of cocycles for momentum maps; see for instance [15, 31].

The one-cocycle σ induces a two-cocycle $\Sigma : \mathfrak{h} \times \mathfrak{h} \to C^\infty(H \times P)$ given by

$$\Sigma(\xi, \eta) = T_e\sigma_\eta(\xi),$$

where $\sigma_\eta : H \to C^\infty(H \times P)$ is defined by $\sigma_\eta(g) = \sigma(g) \cdot \eta$. It is not hard to verify that Σ is explicitly given by

$$\Sigma(\xi, \eta) = -\langle \phi, [\xi, \eta] \rangle + B(\xi_{H\times P}, \eta_{H\times P}). \tag{61}$$

5.1.3. *The Poisson Structure on $\mathfrak{h}^* \times P$.* The canonical Poisson structure on $T^*H \times P$ associated to the symplectic structure (57) gives rise to a Poisson structure on the quotient space $\mathfrak{h}^* \times P$, which we denote by $\{\cdot, \cdot\}_{\text{int}}$.

The explicit form of this Poisson structure is derived in Appendix B. Before quoting this result, we first introduce an operation $\star : \mathfrak{h} \times \mathfrak{h} \to C^\infty(H \times P)$ defined by

$$\xi \star \eta := \left\{ \phi_{\xi|P}, \phi_{\eta|P} \right\}_P,$$

where $\{\cdot, \cdot\}_P$ is the Poisson structure associated to ω_P, and an operation $\lhd : \mathfrak{h} \times C^\infty(P) \to C^\infty(H \times P)$ by putting

$$\xi \lhd F := \{\phi_{\xi|P}, F\}_P.$$

Using these two operations, the reduced Poisson structure is given in the following theorem. Note the interaction term due to curvature and the last two terms which are due to the coupling of the Lie group H with the symplectic manifold P.

Theorem 5.1. *The reduced Poisson structure on $\mathfrak{h}^* \times P$ is given by*

$$\begin{aligned} \{f, k\}_{\text{int}} = \frac{\delta f}{\delta \mu} \star \frac{\delta k}{\delta \mu} - \{f_{|P}, k_{|P}\}_P - B\left(\left(\frac{\delta f}{\delta \mu}\right)_{H\times P}, \left(\frac{\delta k}{\delta \mu}\right)_{H\times P} \right) \\ - \frac{\delta f}{\delta \mu} \lhd k + \frac{\delta k}{\delta \mu} \lhd f, \end{aligned} \tag{62}$$

for functions $f = f(\mu, x)$ and $k = k(\mu, x)$ on $\mathfrak{h}^* \times P$.

5.1.4. *Shifting Away the Interaction Terms.* The reduced Poisson structure $\{\cdot, \cdot\}_{\text{int}}$ described in theorem 5.1 is not canonical, *i.e.* it is not the sum of the Lie-Poisson structure on $\mathfrak{h}^*$ and the canonical Poisson structure on $\mathbb{R}^{2N}$, but rather contains a number of ***interaction terms***. However, using the $\mathcal{B}\mathfrak{g}$-potential ϕ, we can define a ***shift map*** $\hat{\mathcal{S}}$ from $\mathfrak{h}^* \times \mathbb{R}^{2N}$ to itself, taking $\{\cdot, \cdot\}_{\text{int}}$ into the canonical Poisson structure. The price we have to pay for getting rid of interaction terms is the introduction of the non-equivariance cocycle Σ of ϕ, as in the following definition.

Definition 5.2. The natural Poisson structure on $\mathfrak{h}^* \times P$ is given by

$$\{f, k\}_\Sigma = \{f_{|\mathfrak{h}^*}, k_{|\mathfrak{h}^*}\}_{\text{l.p}} - \{f_{|P}, k_{|P}\}_P - \Sigma\left(\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu} \right) \tag{63}$$

for functions $f = f(\mu, x)$ and $k = k(\mu, x)$ on $\mathfrak{h}^* \times P$.

The main result of this section is then given in the following theorem. It relates the Poisson structure with interaction terms with the natural one (possibly with a cocycle). The former will turn out the BMR Poisson structure, while the latter is nothing but the SMBK structure. The cocycle will encode the effects of nonzero circulation on the rigid body.

Theorem 5.3. *The map $\hat{\mathcal{S}} : \mathfrak{h}^* \times P \to \mathfrak{h}^* \times P$ given by*

$$\hat{\mathcal{S}}(\mu, x) = (\mu - \phi(e, x), x). \tag{64}$$

is a Poisson isomorphism taking the Poisson structure $\{\cdot,\cdot\}_{\text{int}}$ *with interaction terms into the Poisson structure* $\{\cdot,\cdot\}_\Sigma$*:*

$$\{f \circ \hat{\mathcal{S}}, k \circ \hat{\mathcal{S}}\}_{\text{int}} = \{f, k\}_\Sigma \circ \hat{\mathcal{S}}. \tag{65}$$

5.1.5. *The Symplectic Leaves.* The shifted Poisson structure $\{\cdot,\cdot\}_\Sigma$ is the sum of the Lie-Poisson structure on $\mathfrak{h}^*$, the canonical Poisson structure on P, and a cocycle term. If $\Sigma = 0$, this allows us to write down a convenient expression for the symplectic leaves in $\mathfrak{h}^* \times P$:

Proposition 6. *For* $\Sigma = 0$*, the symplectic leaves in* $\mathfrak{h}^* \times P$ *of the Poisson structure* $\{\cdot,\cdot\}_{\Sigma=0}$ *are of the form* $\mathcal{O}_\mu \times P$*, where* $\mathcal{O}_\mu$ *is the co-adjoint orbit of an element* $\mu \in \mathfrak{h}^*$*.*

The proof follows that of proposition 10.3.3 in [30] and relies on the fact that the symplectic leaves are precisely the symplectic reduced spaces. But since the Poisson structure on the reduced space is simply the sum of the Lie-Poisson and the canonical Poisson structure, the reduced space at μ is $\mathcal{O}_\mu \times P$, as above. When $\Sigma \neq 0$, we expect the co-adjoint orbit $\mathcal{O}_\mu$ to be replaced by an orbit $\mathcal{O}_\mu^\Sigma$ of a suitable ***affine*** action of H on $\mathfrak{h}^*$.

5.2. **The fluid-solid system.** Now we are ready to specialize the theory in the previous sections to the case of a solid interacting dynamically with point vortices. Recall that our goal is to find the Hamiltonian structure for this problem using reduction techniques and to use the shifting map developed in the preceding section to relate the two Hamiltonian structures in the literature.

Recall that the reduced phase space for the solid-fluid system is $T^*\operatorname{SE}(2) \times \mathbb{R}^{2N}$. Now, the group $\operatorname{SE}(2)$ acts on $\operatorname{SE}(2) \times \mathbb{R}^{2N}$ by the diagonal left action, denoted by Φ and given in inertial coordinates by

$$\Phi_h(g; \mathbf{x}_1, \ldots, \mathbf{x}_N) = (hg; h\mathbf{x}_1, \ldots, h\mathbf{x}_N).$$

Hence, $\operatorname{SE}(2)$ acts from the left on $T^*\operatorname{SE}(2) \times \mathbb{R}^{2N}$ using the cotangent lift in the first factor, and thus, this action leaves the Hamiltonian (47) invariant.

This system is of the form $T^*H \times P$, as discussed in the previous section, where $H = \operatorname{SE}(2)$ and $P = \mathbb{R}^{2N}$. We now apply the results of that section to divide out the $\operatorname{SE}(2)$-symmetry and obtain a system on $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$.

This reduction is similar to the passage from inertial to body coordinates for the rigid body (see for example [31]). To see this, notice that the space $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$ is obtained from $T^*\operatorname{SE}(2) \times \mathbb{R}^{2N}$ by dividing out by the diagonal $\operatorname{SE}(2)$-action: the quotient mapping is given by

$$\rho : T^*\operatorname{SE}(2) \times \mathbb{R}^{2N} \to \mathfrak{se}(2)^* \times \mathbb{R}^{2N},$$

defined as

$$\rho(g, \alpha_g; \mathbf{x}_1, \ldots, \mathbf{x}_N) = (T^*L_g(\alpha_g); \mathbf{X}_1, \ldots, \mathbf{X}_N),$$

where, if $g = (R, \mathbf{x}_0)$, then $\mathbf{x}_i$ and $\mathbf{X}_i$ are related by

$$\mathbf{x}_i = R\mathbf{X}_i + \mathbf{x}_0. \tag{66}$$

In other words, if $\mathbf{x}_i$ describes the location of the ith vortex in inertial coordinates, then $\mathbf{X}_i$ is its location in a frame fixed to the body.

Proposition 7. *The magnetic symplectic structure* (56) *is invariant under the action of* $\operatorname{SE}(2)$ *on* $T^*\operatorname{SE}(2) \times \mathbb{R}^{2N}$ *described above.*

Proof. Recall the expression (56) for $\Omega_{\mathcal{B}}$, where β_μ is given by (53). It is a standard result (see for instance [11]) that the canonical symplectic form Ω_{can} on $T^*\,\mathrm{SE}(2)$ is invariant under the left action of SE(2), and a similar result holds for the form $\sum_{i=1}^N \Gamma_i \mu(\mathbf{v}_i, \mathbf{w}_i)$. The only thing that remains to be shown is that Θ is SE(2)-invariant, but this follows from the SE(2)-invariance of Ψ, which is itself a consequence of proposition 1 and the fact that Φ and Ψ are harmonic conjugates. □

5.2.1. *The Momentum Map.* Recall from section 5.1 that the momentum map J for the action of SE(2) on $T^*\,\mathrm{SE}(2) \times \mathbb{R}^{2N}$ is the difference of two separate parts: $J = J_{\text{can}} - \phi$, where J_{can} is the momentum map (59) defined by the canonical symplectic form on $T^*\,\mathrm{SE}(2)$, while ϕ is the so-called $\mathcal{B}\mathfrak{g}$-potential (60).

The momentum map J is a map from $T^*\,\mathrm{SE}(2) \times \mathbb{R}^{2N}$ to $\mathfrak{se}(2)^*$. It will be convenient to identify $T^*\,\mathrm{SE}(2)$ with $\mathrm{SE}(2) \times \mathfrak{se}(2)^*$ through left translations, and to use the fact that $\mathfrak{se}(2)^*$ is isomorphic to $\mathbb{R}^3$, so that J is a collection of three functions (J^x, J^y, J^Ω) on $\mathbb{R}^{2N} \times \mathrm{SE}(2) \times \mathfrak{se}(2)^*$. A typical element of that space will be denoted as $(\mathbf{x}_1, \ldots, \mathbf{x}_N; g, \mathbf{\Pi})$, where $g = (R, \mathbf{x}_0)$ and $\mathbf{\Pi} = (\Pi_x, \Pi_y, \Pi_\omega)$, but for the sake of clarity we will usually suppress the argument of J.

Proposition 8. *The momentum map J associated to the* SE(2)*-symmmetry represents the spatial momentum of the solid-fluid system, and is given by $J = (J_x, J_y, J_\Omega)$, where*

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = R \begin{pmatrix} \Pi_x \\ \Pi_y \end{pmatrix} + \sum_{i=1}^N \Gamma_i \mathbf{x}_i \times \mathbf{e}_3 - \sum_{i=1}^N \Gamma_i \begin{pmatrix} \Psi_x(\mathbf{x}_i) \\ \Psi_y(\mathbf{x}_i) \end{pmatrix} + \Gamma \mathbf{x}_0 \times \mathbf{e}_3 \tag{67}$$

$$J_\Omega = \Pi_\Omega - \sum_{i=1}^N \frac{\Gamma_i}{2}(x_i^2 + y_i^2) - \sum_{i=1}^N \Gamma_i \Psi_\omega(\mathbf{x}_i),$$

where $\Gamma = \sum_{i=1}^N \Gamma_i$ is the total vortex strength.

Proof. The canonical part J_{can} can be obtained through a standard calculation for cotangent lifted actions (see [31]). The result is

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = R \begin{pmatrix} \Pi_x \\ \Pi_y \end{pmatrix} \quad \text{and} \quad J_\Omega = \Pi_\Omega,$$

for $g = (R, \mathbf{x}_0)$ and $\mathbf{\Pi} = (\Pi_x, \Pi_y, \Pi_\Omega)$.

For the $\mathcal{B}\mathfrak{g}$-potential, note that the form β_μ is the sum of a "pure vortex" part (*i.e.* not involving the fluid) and a part involving the stream functions (*i.e.* the form Θ).

The $\mathcal{B}\mathfrak{g}$-potential corresponding to the pure-vortex part is given by

$$\begin{pmatrix} \phi^x_{\text{vortex}} \\ \phi^y_{\text{vortex}} \end{pmatrix} = -\sum_{i=1}^N \mathbf{x}_i \times \mathbf{e}_3 \quad \text{and} \quad \phi^\Omega_{\text{vortex}} = \sum_{i=1}^N \frac{\Gamma_i}{2}(x_i^2 + y_i^2).$$

These expressions coincide up to sign with those in [2], which is a consequence of the fact that our symplectic structure is the negative of theirs.

Finally, for the stream function term $\mathbf{d}\Theta$ in β_μ, observe that Θ is an SE(2)-invariant one-form on $\mathrm{SE}(2) \times \mathbb{R}^{2N}$. The $\mathcal{B}\mathfrak{g}$-potential associated to $\mathbf{d}\Theta$ is hence given by (see the remark following Theorem 7.1.1 in [30])

$$\langle \phi_{\text{stream}}, \xi \rangle = \mathbf{i}_{\xi_{\mathrm{SE}(2)\times\mathbb{R}^{2N}}} \Theta \quad \text{for all } \xi \in \mathfrak{se}(2).$$

Explicitly,

$$\begin{pmatrix} \phi^x_{\text{stream}} \\ \phi^y_{\text{stream}} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^N \Gamma_i \Psi_x(g, \mathbf{x}_i) \\ \sum_{i=1}^N \Gamma_i \Psi_y(g, \mathbf{x}_i) \end{pmatrix} \quad \text{and} \quad \phi^\Omega_{\text{stream}} = \sum_{i=1}^N \Gamma_i \Psi_\omega(g, \mathbf{x}_i),$$

where the elementary stream functions are given by (55).

The full momentum map J is then the sum of these three contributions:

$$J = J_{\text{can}} - \phi_{\text{vortex}} - \phi_{\text{stream}},$$

and this is precisely (67). □

The expression for the momentum map can be rewritten by introducing the momentum map of the solid-fluid system in ***body coordinates*** [41, 17]:

$$\begin{pmatrix} J_X \\ J_Y \end{pmatrix} = \begin{pmatrix} \Pi_x \\ \Pi_y \end{pmatrix} + \sum_{i=1}^N \Gamma_i \mathbf{X}_i \times \mathbf{e}_3 - \sum_{i=1}^N \Gamma_i \begin{pmatrix} \Psi_X(\mathbf{X}_i) \\ \Psi_Y(\mathbf{X}_i) \end{pmatrix}$$

$$J_\Omega = \Pi_\Omega - \sum_{i=1}^N \frac{\Gamma_i}{2} \|\mathbf{X}_i\|^2 - \sum_{i=1}^N \Gamma_i \Psi_\Omega(\mathbf{X}_i),$$

where $\Psi_X, \Psi_Y, \Psi_\Omega$ are the expression for the elementary stream functions in body coordinates. By using the relation (66) between inertial and body coordinates, one can show that the spatial and body momentum are related by

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = R \begin{pmatrix} J_X \\ J_Y \end{pmatrix} + \Gamma \mathbf{x}_0 \times \mathbf{e}_3 \quad \text{and} \quad J_\omega = J_\Omega + (\mathbf{x}_0 \times \mathbf{J_x}) \cdot \mathbf{e}_3, \tag{68}$$

where $\mathbf{J_x} = (J_x, J_y)^T$. Note that the spatial momentum is a function on $\mathrm{SE}(2) \times \mathbb{R}^{2N}$, whereas the body momentum is a function on $\mathbb{R}^{2N}$, and that the SE(2)-dependence of J is determined by the relation above. In particular, if we evaluate J at the identity, putting $R = \mathbf{1}$ and $\mathbf{x}_0 = 0$ in (68), then we obtain just the body momentum:

$$J(e, \mathbf{x}) = J(\mathbf{X}).$$

5.2.2. *Non-equivariance of the Momentum Map.* Using the definition (61), the non-equivariance two-cocycle of the momentum map is a map $\Sigma : \mathfrak{se}(2) \times \mathfrak{se}(2) \to C^\infty(\mathrm{SE}(2) \times \mathbb{R}^{2N})$. To compute Σ, we recall the basis $\{\mathbf{e}_\Omega, \mathbf{e}_x, \mathbf{e}_y\}$ of $\mathfrak{se}(2)$ given in (6) and let $\{\mathbf{e}^*_\Omega, \mathbf{e}^*_x, \mathbf{e}^*_y\}$ be the corresponding dual basis of $\mathfrak{se}(2)^*$.

Theorem 5.4. *The non-equivariance two-cocycle Σ of the momentum map J is given by*

$$\Sigma = -\Gamma \mathbf{e}^*_x \wedge \mathbf{e}^*_y,$$

where $\Gamma = \sum_{i=1}^N \Gamma_i$ is the total vortex strength.

Proof. In order to use (61) to compute Σ, we need the infinitesimal generators corresponding to the basis elements $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_\Omega$. The infinitesimal generator of any element $\xi \in \mathfrak{se}(2)$, evaluated at $(e; \mathbf{x}_1, \ldots, \mathbf{x}_N)$, will be denoted by $\tilde{\xi}$ and is given by

$$\tilde{\xi} := \xi_{\mathrm{SE}(2) \times \mathbb{R}^{2N}}(e; \mathbf{x}_1, \ldots, \mathbf{x}_N) = (\xi; \xi_{\mathbb{R}^{2N}}(\mathbf{x}_1), \ldots, \xi_{\mathbb{R}^{2N}}(\mathbf{x}_N)),$$

where $\xi_{\mathbb{R}^{2N}}$ is the infinitesimal generator of ξ for the fundamental action of SE(2) on $\mathbb{R}^2$. Explicitly,

$$(\mathbf{e}_\Omega)_{\mathbb{R}^2} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \quad (\mathbf{e}_x)_{\mathbb{R}^2} = \frac{\partial}{\partial x}, \quad \text{and} \quad (\mathbf{e}_y)_{\mathbb{R}^2} = \frac{\partial}{\partial y}.$$

Evaluating β_μ on these vectors, we obtain

$$\beta_\mu(\tilde{\mathbf{e}}_x, \tilde{\mathbf{e}}_y) = -\Gamma,$$
$$\beta_\mu(\tilde{\mathbf{e}}_x, \tilde{\mathbf{e}}_\Omega) = \sum_{i=1}^N \Gamma_i \left(-\mathbf{X}_i + \frac{1}{\|\mathbf{X}_i\|^2} \right),$$
$$\beta_\mu(\tilde{\mathbf{e}}_y, \tilde{\mathbf{e}}_\Omega) = \sum_{i=1}^N \Gamma_i \left(-\mathbf{Y}_i + \frac{1}{\|\mathbf{Y}_i\|^2} \right).$$

Remarkably, when calculating Σ, the last two expressions are cancelled entirely by opposite contributions from the remaining terms:

$$\begin{aligned}\Sigma(\mathbf{e}_x, \mathbf{e}_\Omega) &= -\left\langle \phi, [\mathbf{e}_x, \mathbf{e}_\Omega] \right\rangle + \beta_\mu(\tilde{\mathbf{e}}_x, \tilde{\mathbf{e}}_\Omega) \\ &= \phi_y + \beta_\mu(\tilde{\mathbf{e}}_x, \tilde{\mathbf{e}}_\Omega) \\ &= 0,\end{aligned}$$

and similar for $\Sigma(\mathbf{e}_y, \mathbf{e}_\Omega)$. The only non-zero term is given by $\Sigma(\mathbf{e}_x, \mathbf{e}_y) = -\Gamma$. $\square$

In its current form, Σ has the same form as the non-equivariance two-cocycle for the N-vortex problem in an unbounded fluid (see [2]).

5.3. **Poisson structures.** Now we come to the conclusion of this paper. Using the theory developed in the previous sections, we derive an explicit form for the reduced Poisson bracket on $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$ associated to the magnetic symplectic form (56): this will turn out to be the BMR bracket. In the terminology of section 5.1, this is the bracket $\{\cdot,\cdot\}_{\rm int}$ with interaction terms. Secondly, we then use the shift map $\hat{\mathcal{S}}$ (see (64)) associated to the momentum map (67) to obtain the Poisson structure $\{\cdot,\cdot\}_\Sigma$ where the interaction terms are absent, at the expense of a non-equivariance cocycle. When made more explicit, the latter bracket turns out to be the SMBK bracket.

5.3.1. *The BMR Poisson Structure.* The bracket obtained by Poisson reduction of the magnetic symplectic structure on $T^*\,\mathrm{SE}(2) \times \mathbb{R}^{2N}$ was described in theorem 5.1. The explicit computation of this Poisson bracket boils down to substituting the explicit expresssion (67) for the momentum map into the Poisson bracket $\{\cdot,\cdot\}$ in theorem 5.1. After a long, but straightforward calculation, one obtains the BMR bracket (36). As an illustration, we compute one bracket element, leaving the others to the reader.

We consider the functions $F = \Pi_x$ and $G = \Pi_y$ on $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$ and compute $\{\Pi_x, \Pi_y\}_{\rm int}$. Note that, considered as elements of $\mathfrak{se}(2)$,

$$\frac{\delta F}{\delta \mu} = \mathbf{e}_x \quad \text{and} \quad \frac{\delta G}{\delta \mu} = \mathbf{e}_y.$$

Computing (36) term by term, we have first of all that

$$\Pi_x \star \Pi_y = \{\phi_x, \phi_y\}_{\rm vortex} = -\sum_{i=1}^N \Gamma_i \frac{\|\mathbf{X}_i\|^4 - R^4}{\|\mathbf{X}_i\|^4},$$

where $\{\cdot,\cdot\}_{\rm vortex}$ is the vortex bracket defined in (34). Secondly,

$$\beta_\mu(\tilde{\mathbf{e}}_x, \tilde{\mathbf{e}}_y) = -\Gamma,$$

while the other terms are zero. Finally, it follows that

$$\{\Pi_x, \Pi_y\}_{\text{int}} = \Gamma - \sum_{i=1}^{N} \Gamma_i \frac{\|\mathbf{X}_i\|^4 - R^4}{\|\mathbf{X}_i\|^4},$$

which is precisely the first bracket element of (36). The computation of the other elements is similar.

5.3.2. *The SMBK Poisson Structure.* By subjecting the BMR Poisson structure to the shift map $\hat{\mathcal{S}}$, we can eliminate the interaction terms from the Poisson structure, at the expense of introducing a non-equivariance cocycle. As shown in Theorem B.3, the result is a Poisson structure consisting of the sum of the Lie-Poisson and the vortex Poisson structure on the individual factors, together with a cocycle term:

$$\{F, G\}_\Sigma = \{F_{|\mathfrak{se}(2)^*}, G_{|\mathfrak{se}(2)^*}\}_{\text{l.p}} + \{F_{|\mathbb{R}^{2N}}, G_{|\mathbb{R}^{2N}}\}_{\text{vortex}} - \Sigma\left(\frac{\delta F}{\delta \mu}, \frac{\delta F}{\delta \mu}\right).$$

The last term is explicitly given by

$$\Sigma\left(\frac{\delta F}{\delta \mu}, \frac{\delta F}{\delta \mu}\right) = -\Gamma\left(\frac{\partial F}{\partial \Pi_x}\frac{\partial G}{\partial \Pi_y} - \frac{\partial G}{\partial \Pi_x}\frac{\partial F}{\partial \Pi_y}\right).$$

In the case where the total vorticity Γ is zero, this term vanishes and the bracket reduces to the SMBK bracket (B.3).

5.3.3. *The Symplectic Leaves.* In the case where the two-cocycle Σ vanishes, or equivalently, when the total circulation Γ is zero, a convenient expression for the symplectic leaves in $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$ can be read off from proposition 6.

Recall that the symplectic leaves for the Lie-Poisson structure in $\mathfrak{se}(2)^* \cong \mathbb{R}^3$ come in two varieties. One class consists of cylinders whose axis is the Ω-axis:

$$\mathcal{O} = \{(\Pi_x, \Pi_y, \Omega) : \Pi_x^2 + \Pi_y^2 = \text{constant}\},$$

while the other class consists of the individual points $(0, 0, \Omega)$ of the Ω-axis. The symplectic leaves in $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$ are then the product of the symplectic leaves of the Lie-Poisson structure with $\mathbb{R}^{2N}$.

6. **Conclusions and future work.** In this paper, we have used reduction theory to give a systematic derivation of the equations of motion for a circular rigid body in a perfect fluid with point vortices. Among other things, we have derived the Poisson structures that govern this problem, and related them to the Poisson structures in the literature. However, the usefulness of our geometric method is not limited to merely reproducing the correct Poisson structure for one specific problem: by uncovering fundamental geometric structures such as the Neumann connection, we have shed new light on the precise nature of the solid-fluid interaction. Moreover, even though a number of simplifying assumptions were made in the beginning, it is expected that the present method can be extended without too much trouble to cover more general cases. Below, we have listed a number of open questions for which our method seems especially suited.

6.1. **Non-zero circulation.** As we have seen in this paper, if the total strength of the point vortices is non-zero, a cocycle term appears in the equations of motion. In a previous paper (see [47]), we studied the case of a rigid body moving in a potential flow with circulation and found that the circulation manifests itself through the curvature of the Neumann connection.

From a physical point of view, both phenomena are very similar. Indeed, having a non-zero circulation around the rigid body amounts to placing a vortex at the center of the rigid body. Hence, mathematically speaking, there appears to be a duality between the description in terms of cocycles and in terms of curvatures. Having a better mathematical understanding of this duality would prove to be useful for other physical theories as well, for example the dynamics of spin glasses in [9]. In addition, it would be interesting to investigate the link between this theory with the work of Gay-Balmaz and Ratiu [14].

6.2. **Arbitrary body shapes.** The pioneering work of SMBK was extended to cover the case of rigid bodies with arbitrary shapes by Shashikanth [41], and studied further by Kanso and Oskouei [17] with a view towards stability and bio-locomotion.

By using methods of complex analysis, the methods of the present paper can easily be extended to cover the case of rigid bodies of arbitrary shape as well: any closed curve enclosing a simply-connected area of the plane can be mapped to the unit circle by a suitable conformal transformation and by pulling back the geometric objects in this paper along that map, arbitrary body shapes can be treated. For the case of *multiple* rigid bodies, one can use the conformal techniques in [10].

Moreover, the objects defined in this paper transform naturally under conformal transformations. The magnetic form β_μ, for instance, will be mapped into a form β'_μ which has the same appearance as before, but with now Ψ_A the elementary stream functions for a body of that shape. As is known from the classical fluid dynamics literature, these stream functions also transform naturally under conformal mappings.

6.3. **Three-dimensional bodies.** In contrast to calculations using only standard, ad-hoc methods, our setup naturally generalizes to the case of three-dimensional flows interacting with rigid bodies. Shashikanth *et al.* [44] consider a rigid body interacting with ***vortex rings***. The space of point vortices $\mathbb{R}^{2N}$ then is replaced by the space $\mathcal{M}$ of N vortex rings, and much of the analysis in the 2D case carries through, at the expense of significant computational work.

From a geometric point of view, however, $\mathcal{M}$ plays a similar role as $\mathbb{R}^{2N}$: both are coadjoint orbits of the group $\mathrm{Diff}_{\mathrm{vol}}$ of volume-preserving diffeomorphisms. Hence, $\mathcal{M}$ is equipped with a natural symplectic structure (see [5]), which is of importance for the calculation of β_μ in this case. Also, the definition of the Neumann connection remains valid in three dimensions. We therefore expect that the geometric approach can be extended without too much difficulty to this case, and will lead to a conceptually much clearer picture.

6.4. **Elastic bodies.** From a physical perspective, our attention goes out foremost to the case of *elastic, three-dimensional bodies* interacting with a perfect fluid. However, in contrast with the issues outlined before, this will most likely involve a number of significant new developments. One problem is the definition of the Neumann connection, which explicitly relies on the decomposition (21) of the velocity potential into translational and rotational parts. Secondly, the clarity of the reduced equations of motion in this paper ultimately hinges on the concise form for the curvature of the Neumann connection. It remains to be seen whether similar expressions exist for deformable bodies. A special case that deserves close attention is the motion of an elastic body in potential flow rather than arbitrary vortical flow. This would entail doing reduction at zero momentum, which would obviate the need to compute the curvature of the Neumann connection.

In general, we expect the literature on swimmers in *Stokes flow* (see [22] and the references therein) to be useful. Furthermore, we mention here the work of Galper and Miloh [13], who study the motion of a deformable body in an ideal fluid. It would be interesting, as a starting point for further investigations, to identify the geometric structures in their work.

6.5. **Routhian reduction by stages and Dirac reduction.** It has been known for some time that the Lagrangian analogue of symplectic reduction is ***Routhian reduction*** [32]. By starting from the Lagrangian of an incompressible fluid together with the rigid body Lagrangian, one could perform reduction on the Lagrangian side using this theory. The advantage would be that Routhian reduction preserves the variational nature of the system. However, as the reduction procedure for the fluid-solid system consists of two succesive reductions, one would need a theory of ***Routhian reduction by stages***. Developing such a theory would be of considerable interest.

A related approach consists of using Dirac structures. Through the Hamilton-Pontryagin principle, Dirac structures offer a way of incorporating both the Lagrangian and Hamiltonian formalisms making them ideally suited for systems with degenerate Lagrangians; see [49]. Since the point vortex Lagrangian is degenerate, Dirac structures are likely to be useful here.

Acknowledgments. It is a pleasure to thank Jair Koiller, Richard Montgomery, Paul Newton, Tudor Ratiu and Banavara Shashikanth, as well as Frans Cantrijn, Scott Kelly, Bavo Langerock, Jim Radford, and Clancy Rowley, for useful suggestions and interesting discussions.

J. Vankerschaver is a Postdoctoral Fellow from the Research Foundation – Flanders (FWO-Vlaanderen) and a Fulbright Research Scholar, and wishes to thank both agencies for their support. Additional financial support from the Fonds Professor Wuytack is gratefully acknowledged. E. Kanso's work is partially supported by NSF through the award CMMI 06-44925. J. E. Marsden is partially supported by NSF Grant DMS-0505711.

Appendix A. Further properties of the Neumann connection. Even though the Neumann connection is used in many problems from geometric fluid dynamics or differential geometry, a systematic presentation of its properties seems to be lacking. In this appendix we prove some basic theorems related to the Neumann connection.

A.1. **Fiber bundles and connections.** Recall that if $\pi : Q \to Q/G$ is a principal fibre bundle with structure group G, then a connection one-form on Q is a $\mathfrak{g}$-valued one-form $\mathcal{A}$ on Q satisfying the following two properties:

1. $\mathcal{A}$ is ***G-equivariant***: $\sigma_g^* \mathcal{A} = \mathrm{Ad}_{g^{-1}} \circ \mathcal{A}$, where $\sigma_g : Q \to Q$ denotes the G-action, and Ad is the adjoint action on $\mathfrak{g}$.

2. Let ξ_Q be the ***infinitesimal generator*** associated to an element $\xi \in \mathfrak{g}$, *i.e.*
$$\xi_Q(q) = \frac{d}{dt}\sigma(\exp t\xi, q)\Big|_{t=0}, \quad \text{where } \sigma(g,q) = \sigma_g(q).$$
Then $\mathcal{A}(\xi_Q) = \xi$.

At each point $q \in Q$, the connection $\mathcal{A}$ induces complementary projection operators $P_V, P_H : TQ \to TQ$, given by
$$P_V(v_q) = [\mathcal{A}(v_q)]_Q(q), \quad \text{and} \quad P_H = 1 - P_V,$$
and referred to as the ***vertical*** and ***horizontal*** projection operators, respectively. More information on principal fiber bundles and connections can be found in [21].

A.1.1. *The Curvature.* The ***curvature*** of a principal fiber bundle connection $\mathcal{A}$ is the $\mathfrak{g}$-valued two-form $\mathcal{B}$ defined as follows. Let $u_q, v_q \in T_qQ$ and consider vector fields X_u and X_v extending u_q and v_q, *i.e.* such that $X_u(q) = u_q$ and $X_v(q) = v_q$. Denote the horizontal part of X_u as $X_u^H := P_H \circ X_u$, and similar for X_v^H. Then, $\mathcal{B}$ can be conveniently expressed as
$$\mathcal{B}_q(u_q, v_q) = -\mathcal{A}_q([X_u^H, Y_v^H]). \tag{69}$$
With this definition, $\mathcal{B}$ is G-equivariant and vanishes on vertical vector fields:
$$\sigma_g^*\mathcal{B} = \mathrm{Ad}_{g^{-1}}\mathcal{B} \quad \text{and} \quad i_{\xi_Q}\mathcal{B} = 0, \tag{70}$$
for all $g \in G$ and $\xi \in \mathfrak{g}$.

A.2. **The Neumann connection.**

A.2.1. *The Connection One-form.* We recall from section 3 that the connection one-form for the Neumann connection is given by
$$\mathcal{A}_{(\varphi,g)}(\dot{g}, \dot{\varphi}) = \varphi^*\mathbf{u}_\mathrm{v},$$
where $\mathbf{u}_\mathrm{v}$ is the divergence-free part in the Helmholtz-Hodge decomposition (19) of the Eulerian velocity $\mathbf{u}$; see (39).

Before showing that this prescription indeed yields a well-defined connection form, we note that each divergence-free vector field $\mathbf{u} \in \mathfrak{X}_\mathrm{vol}$ defines a vertical vector field on Q, denoted by $\mathbf{u}_Q$ and given by
$$\mathbf{u}_Q(g, \varphi) = (0, T\varphi \circ \mathbf{u}).$$
The vector field $\mathbf{u}_Q$ is the infinitesimal generator associated to $\mathbf{u}$ under the action of $\mathrm{Diff}_\mathrm{vol}$ on Q.

Proposition 9. *The one-form $\mathcal{A}$ defined in (39) is a connection one-form. In other words,*

1. *$\mathcal{A}$ is $\mathrm{Diff}_\mathrm{vol}$-equivariant: for all $\phi \in \mathrm{Diff}_\mathrm{vol}$ and $(g, \dot{g}; \varphi, \dot{\varphi}) \in TQ$,*
$$\mathcal{A}_{(g,\varphi\circ\phi)}(\dot{g}, \dot{\varphi}\circ\phi) = \phi^*\mathcal{A}_{(g,\varphi)}(\dot{g}, \dot{\varphi});$$
2. *If $\mathbf{u}$ is an element of $\mathfrak{X}_\mathrm{vol}$ with associated fundamental vector field $\mathbf{u}_Q$, then $\mathcal{A}(\mathbf{u}_Q) = \mathbf{u}$.*

Proof. To prove equivariance, note that the Eulerian velocity associated to $\dot{\varphi}\circ\phi$ is equal to that associated to $\dot{\varphi}$:
$$\mathbf{u}' = (\dot{\varphi}\circ\phi)\circ(\varphi\circ\phi)^{-1} = \dot{\varphi}\circ\varphi^{-1} = \mathbf{u}.$$
Hence,
$$\mathcal{A}_{(g,\varphi\circ\phi)}(\dot{g}, \dot{\varphi}\circ\phi) = (\varphi\circ\phi)^*\mathbf{u}_\mathrm{v} = \phi^*(\varphi^*\mathbf{u}_\mathrm{v}) = \phi^*\mathcal{A}_{(g,\varphi)}(\dot{g}, \dot{\varphi}).$$

To prove the second property, note that the push-forward $X = \varphi_* \mathbf{u}$ is divergence-free and tangent to the boundary of $\mathcal{F}$ since $\mathbf{u}$ is divergence-free and tangent to $\partial \mathcal{F}_0$ and φ is volume-preserving.

Evaluating the connection one-form on $\mathbf{u}$ therefore gives

$$\mathcal{A}_{(g,\varphi)}(\mathbf{u}_Q(g,\varphi)) = \varphi^* X = \mathbf{u},$$

which concludes the proof that $\mathcal{A}$ is a well-defined connection one-form. □

A.2.2. *The Horizontal Lift.* The horizontal lift operator associates to each (g, φ) a linear map $\mathbf{h}_{(g,\varphi)} : T_g \,\mathrm{SE}(2) \to T_{(g,\varphi)} Q$ with the following properties:

1. The composition $T_{(g,\varphi)}\pi \circ \mathbf{h}_{(g,\varphi)}$ is the identity in $T_g \,\mathrm{SE}(2)$;
2. The image of $\mathbf{h}_{(g,\varphi)}$ consists of horizontal vectors: $\mathcal{A}_{(g,\varphi)} \circ \mathbf{h}_{(g,\varphi)} = 0$.

For the Neumann connection, the horizontal lift is given by

$$\mathbf{h}_{(g,\varphi)}(\dot{g}) = (\dot{g}, \nabla\Phi \circ \varphi),$$

where Φ is the solution of the Neumann problem (20) associated to $(g, \dot{g})$; *i.e.* $(\omega, \mathbf{v}) = \dot{g} g^{-1}$. It is easy to show that $\mathbf{h}$ satisfies both properties of the horizontal lift, and hence that $\mathbf{h}$ is indeed the horizontal lift of the Neumann connection as we claimed in section 3.

A.2.3. *The Neumann Connection as a Mechanical Connection.* When given a group-invariant metric on the total space of a principal fiber bundle, the ***mechanical connection*** is defined by declaring its horizontal subspaces to be orthogonal to the vertical bundle. The terminology stems from the fact that the underlying metric is usually the kinetic-energy metric of a mechanical system.

Since there exists a natural metric (18) on Q, it should therefore come as no surprise that the Neumann connection can also be viewed as a mechanical connection.

The horizontal subspace of the Neumann connection at an element $(g, \varphi) \in Q$ is given by the kernel of $\mathcal{A}_{(g,\varphi)}$. Explicitly, a tangent vector $(g, \dot{g}; \varphi, \dot{\varphi})$ is in the kernel of $\mathcal{A}_{(g,\varphi)}$ if and only if $\dot{\varphi} = \nabla\Phi \circ \varphi$, where Φ is the solution of the Neumann problem (20) associated to $(g, \dot{g})$.

On the other hand, the vertical subspace at (g, φ) is generated by elements of the form $\mathbf{u}_Q(g, \varphi)$. Verifying that the Neumann connection is indeed the mechanical connection now boils down to checking that the vertical and horizontal subspaces are orthogonal with respect to the metric (18).

Let $(g, \dot{g}; \varphi, \nabla\Phi \circ \varphi)$ and $\mathbf{u}_Q(g, \varphi)$ be horizontal and vertical tangent vectors, respectively. Then we have

$$\begin{aligned}\langle\!\langle (g, \dot{g}; \varphi, \nabla\Phi \circ \varphi), \mathbf{u}_Q(g,\varphi) \rangle\!\rangle &= \langle \nabla\Phi \circ \varphi, T\varphi \circ \mathbf{u} \rangle_{\mathrm{Emb}} \\ &= \int_{\mathcal{F}_0} (\nabla\Phi \circ \varphi) \cdot (T\varphi \circ \mathbf{u})\, d\mathbf{x}_0 = \int_{\mathcal{F}} \nabla\Phi \cdot (\varphi_* \mathbf{u})\, \eta = 0,\end{aligned}$$

where the last equality follows from the L_2-orthogonality between gradient vector fields and divergence-free vector fields tangent to the boundary of $\mathcal{F}$.

Appendix B. **Proofs of the Theorems in section 5.1.** In this technical appendix, we provide proofs for the theorems in section 5.1 regarding the reduced Poisson structures with or without interaction terms. We do so by using the diffeomorphism of $T^*H \times P$ with $H \times \mathfrak{h}^* \times P$ given by left translation. More specifically, we introduce a diffeomorphism $\Lambda : T^*H \times P \to H \times \mathfrak{h}^* \times P$ by

$$\Lambda(g, \alpha_g; p) = (g, T^* L_g(\alpha_g), g^{-1} \cdot p),$$

and a diffeomorphism $\lambda : H \times P \to H \times P$ by $\lambda(g,p) = (g, g^{-1} \cdot p)$ such that the following diagram commutes:

$$\begin{tikzcd} T^*H \times P \arrow[r, "\Lambda"] \arrow[d, "\pi_H"'] & H \times \mathfrak{h}^* \times P \arrow[d, "\mathrm{pr}"] \\ H \times P \arrow[r, "\lambda"'] & H \times P \end{tikzcd}$$

Here, $\pi_H : T^*H \times P \to H \times P$ is given by $\pi_H(g, \alpha_g; p) = (g,p)$, while pr simply forgets the second factor: $\mathrm{pr}(g, \mu, x) = (g, x)$.

The advantage of this construction is that under Λ the action (58) becomes left translation on the first factor. That is, if $\Lambda(\alpha_g; p) = (g, \mu, x)$, then $\Lambda(h \cdot (\alpha_g; p)) = (hg, \mu, x)$. The fact that the diagonal action (58) reduces in this way greatly simplifies Poisson reduction.

B.1. **Symplectic forms in body coordinates.** We start by deriving in lemma B.1 an explicit expression for the push-forward $\Lambda_* \Omega$ of the symplectic form. We then define a natural ***shift map*** $\mathcal{S}$ from $H \times \mathfrak{h}^* \times P$ to itself, given by

$$\mathcal{S}(g, \mu, x) = (g, \mu - \phi(e, x), x),$$

where $\phi : H \times P \to \mathfrak{h}^*$ is the $\mathcal{B}\mathfrak{g}$-potential associated to the action of H on $H \times P$, *i.e.* the solution of (60). This map takes the symplectic structure (72) with interaction terms of lemma B.1 into one where the interaction terms are absent. The explicit form of the resulting symplectic form is derived in lemma B.2.

For future reference, we will refer to the push-forwards $\Lambda_* \omega_{\mathrm{can}}$ and $\Lambda_* \Omega$ as ω_{body} and Ω_{body}, respectively:

$$\omega_{\mathrm{body}} = \Lambda_* \omega_{\mathrm{can}} \quad \text{and} \quad \Omega_{\mathrm{body}} = \Lambda_* \Omega. \tag{71}$$

Lemma B.1. *The push-forward $\Lambda_* \Omega$ of the symplectic form Ω to $H \times \mathfrak{h}^* \times P$ is given by*

$$\begin{aligned} (\Lambda_*\Omega)(g, \mu, x)((g \cdot \xi, \rho, v_x), (g \cdot \eta, \sigma, w_x)) = \\ - \langle \rho, \eta \rangle + \langle \sigma, \xi \rangle + \langle \mu, [\xi, \eta] \rangle \\ - B(e,x)(\xi_{H \times P}(e,x), \eta_{H \times P}(e,x)) - B(e,x)((0, v_x), (0, w_x)) \\ - \langle \mathbf{d}\phi_\xi(e,x), (0, w_x) \rangle + \langle \mathbf{d}\phi_\eta(e,x), (0, v_x) \rangle , \end{aligned} \tag{72}$$

where (g, μ, x) is an element of $H \times \mathfrak{h}^* \times P$ and $(g \cdot \xi, \rho, v_x)$, $(g \cdot \eta, \sigma, w_x)$ are elements of $T_{(g,\mu,x)}(H \times \mathfrak{h}^* \times P) \cong T_g H \times \mathfrak{h}^* \times T_x P$.

Proof. Notice first that $\Lambda_* \Omega = \Lambda_* \omega_{\mathrm{can}} - \mathrm{pr}^*(\lambda_* B)$. The first term is the expression for the canonical symplectic form in body coordinates calculated by R. Cushman (quoted in [1], proposition 4.4.1) and is given by

$$(\Lambda_* \omega_{\mathrm{can}})(g, \mu, x)((g \cdot \xi, \rho, v_x), (g \cdot \eta, \sigma, w_x)) = - \langle \rho, \eta \rangle + \langle \sigma, \xi \rangle + \langle \mu, [\xi, \eta] \rangle .$$

For the second term, we have

$$\begin{aligned} &(\lambda_* B)(g, x)((g \cdot \xi, v_x), (g \cdot \eta, w_x)) \\ &\quad = B(\lambda^{-1}(g,x))(T\lambda^{-1}(g \cdot \xi, v_x), T\lambda^{-1}(g \cdot \eta, w_x)) \\ &\quad = B(g, g \cdot x)\big((g \cdot \xi, T\Psi_g(v_x) + T\hat{\Psi}_x(g \cdot \xi)), (g \cdot \eta, T\Psi_g(w_x) + T\hat{\Psi}_x(g \cdot \eta))\big), \end{aligned}$$

where $\Psi_g : P \to P$ denotes the action of H on P: $\Psi_g(x) = g \cdot x$, and $\hat{\Psi}_x : G \to P$ is the map defined by $\hat{\Psi}_x(g) = g \cdot x$. Note that $T\hat{\Psi}_x(g \cdot \xi) = g \cdot \xi_P(p)$. Taking this into account, as well as the H-invariance of B, allows us to rewrite the above expression as

$$\begin{aligned} &B(e,x)\big((\xi, v_x + \xi_P(x)), (\eta, w_x + \eta_P(x))\big) \\ &\qquad = B(e,x)((0,v_x),(0,w_x)) + B(e,x)((\xi,\xi_P(x)),(\eta,\eta_P(x))) \\ &\qquad + B(e,x)((0,v_x),(\eta,\eta_P(x))) + B(e,x)((\xi,\xi_P(x)),(0,w_x)). \end{aligned}$$

The third term can be rewritten as

$$B(e,x)((0,v_x),(\eta,\eta_P(x))) = -\mathbf{d}\phi_\eta(e,x)(0,v_x),$$

and similar for the last term. Putting everything together, we obtain (72). $\square$

Lemma B.2. *The push-forward $\mathcal{S}_*\Omega_{\text{body}}$ on $H \times \mathfrak{h}^* \times P$ is again a symplectic form, given by*

$$\begin{aligned} &(\mathcal{S}_*\Omega_{\text{body}})(g,\mu,x)((g\cdot\xi,\rho,v_x),(g\cdot\eta,\sigma,w_x)) = \\ &\qquad\qquad \omega_{\text{body}}(g,\mu)((g\cdot\xi,\rho),(g\cdot\eta,\sigma)) - \omega_P(v_x,w_x) - \Sigma(\xi,\eta). \quad (73) \end{aligned}$$

Here, the arguments of $\mathcal{S}_*\Omega_{\text{body}}$ have the same meaning as in lemma B.1, and Σ is the non-equivariance two-cocycle (61) of ϕ.

Proof. The pushforward of ω_{body} under $\mathcal{S}$ is given by

$$\begin{aligned} &(\mathcal{S}_*\omega_{\text{body}})(g,\mu,x)((g\cdot\xi,\rho,v_x),(g\cdot\eta,\sigma,w_x)) = \\ &\qquad \omega_{\text{body}}(g,\mu',x)((g\cdot\xi,\rho',v_x),(g\cdot\eta,\sigma',w_x)) = \\ &\qquad\qquad -\langle\rho',\eta\rangle + \langle\sigma',\xi\rangle + \langle\mu',[\xi,\eta]\rangle, \quad (74) \end{aligned}$$

where

$$\mu' = \mu + \phi(e,p), \quad \rho' = \rho + T\phi(0,v_x) \quad \text{and} \quad \sigma' = \sigma + T\phi(0,w_x).$$

Substituting this into (74), we get

$$\begin{aligned} \langle\rho',\eta\rangle &= \langle\rho,\eta\rangle + \langle T\phi(0,v_x),\eta\rangle \\ &= \langle\rho,\eta\rangle + \mathbf{d}\phi_\eta(0,v_x), \end{aligned}$$

and similar for $\langle\sigma',\xi\rangle$. For the term involving μ', we obtain

$$\begin{aligned} \langle\mu',[\xi,\eta]\rangle &= \langle\mu,[\xi,\eta]\rangle + \langle\phi(e,x),[\xi,\eta]\rangle \\ &= \langle\mu,[\xi,\eta]\rangle - \Sigma(\xi,\eta) + B(e,x)(\xi_{H\times P}(e,x),\eta_{H\times P}(e,x)), \end{aligned}$$

where we have used the definition of Σ. Hence,

$$\begin{aligned} &(\mathcal{S}_*\omega_{\text{body}})(g,\mu,x)((g\cdot\xi,\rho,v_x),(g\cdot\eta,\sigma,w_x)) = \\ &\qquad \omega_{\text{body}}(g,\mu,x)((g\cdot\xi,\rho,v_x),(g\cdot\eta,\sigma,w_x)) - \Sigma(\xi,\eta) \\ &\qquad\quad - \mathbf{d}\phi_\eta(0,v_x) + \mathbf{d}\phi_\xi(0,w_x) + B(e,x)(\xi_{H\times P}(e,x),\eta_{H\times P}(e,x)). \end{aligned}$$

The last three terms are just the interaction terms. Hence, by substituting this expression into the expression for $\mathcal{S}_*\Omega_{\text{body}}$, these terms cancel out, leaving (73). $\square$

In other words, by applying $\mathcal{S}$ we effectively get rid of the interaction terms in the symplectic form.

B.2. **The reduced Poisson structures.**

B.2.1. *The Poisson Structure on $H \times \mathfrak{h}^* \times P$.* Now that we have established the different symplectic structures on $H \times \mathfrak{h}^* \times P$, we can also find an explicit form for the associated Poisson structures.

Proposition 10. *The Poisson structure on $H \times \mathfrak{h}^* \times P$ associated to the symplectic structure (72) is given by*

$$\begin{aligned}\{F, K\}^0_{\text{int}} = &-\frac{\delta F}{\delta \mu} \star \frac{\delta K}{\delta \mu} - \{F_{|P}, K_{|P}\}_P + B\left(\left(\frac{\delta F}{\delta \mu}\right)_{H\times P}, \left(\frac{\delta K}{\delta \mu}\right)_{H\times P}\right) \\ &-\frac{\delta F}{\delta \mu} \triangleleft K + \frac{\delta K}{\delta \mu} \triangleleft F \\ &+\left\langle \mathbf{d}_g F, g \cdot \frac{\delta K}{\delta \mu}\right\rangle - \left\langle \mathbf{d}_g K, g \cdot \frac{\delta F}{\delta \mu}\right\rangle\end{aligned} \tag{75}$$

for functions $F = F(g, \mu, x)$ and $K = K(g, \mu, x)$ on $H \times \mathfrak{h}^* \times P$.

Proof. The Poisson bracket of two functions F, K on $H \times \mathfrak{h}^* \times P$ is defined as $\{F, K\}^0_{\text{int}} = X_K(F)$, where X_K is the Hamiltonian vector field associated to K, defined by $\mathbf{i}_{X_K}\Omega_{\text{body}} = \mathbf{d}K$.

Let (g, μ, x) be an element of $H \times \mathfrak{h}^* \times P$ and write $X_K(g, \mu, x) \in T_gG \times \mathfrak{h}^* \times T_xP$ as $X_K(g, \mu, x) = (g \cdot \xi, \rho, v_x)$, where $\xi \in \mathfrak{h}$, $\rho \in \mathfrak{h}^*$, and $v_x \in T_xP$. Note also that $\mathbf{d}K$ is of the form

$$\mathbf{d}K = \left(\mathbf{d}_g K, \frac{\delta K}{\delta \mu}, \mathbf{d}_x K\right),$$

where $\mathbf{d}_x K$ denotes the differential of $K(g, \mu, x)$ keeping g and μ fixed, and similar for $\mathbf{d}_g K$.

A long but straightforward calculation shows that

$$\xi = \frac{\delta K}{\delta \mu} \quad \text{and} \quad v_x = -(\mathbf{d}_x K + \mathbf{d}_x \phi_\xi)^\sharp, \tag{76}$$

where the map $\sharp : T^*P \to TP$ is given by $\sharp(\alpha_x) = w_x$ iff $\mathbf{i}_{w_x}\omega_P = \alpha_x$, while

$$\rho = -\mathbf{i}_w B \circ T_e\hat{\sigma}_{(e,x)} - T^*L_g(\mathbf{d}_g K), \quad \text{where} \quad w = \xi_{H\times P}(e, x) + v_x,$$

and v_x and ξ are given by (76). Here, $\hat{\sigma}_{(g,x)} : H \to H \times P$ is the map defined by $\hat{\sigma}_{(g,x)}(h) = (hg, h \cdot x)$, for all $g, h \in H$ and $x \in P$. Note that

$$T_e\hat{\sigma}_{(g,x)}(\xi) = \xi_{H\times P}(g, x).$$

The Poisson bracket is then given by

$$\{F, K\}^0_{\text{int}} = X_K(F) = \langle g \cdot \xi, \mathbf{d}_g F\rangle + \langle v_x, \mathbf{d}_x F\rangle + \left\langle \rho, \frac{\delta F}{\delta \mu}\right\rangle.$$

The first term is equal to

$$\langle g \cdot \xi, \mathbf{d}_g F\rangle = \left\langle \frac{\delta K}{\delta \mu}, T^*L_g(\mathbf{d}_g F)\right\rangle,$$

while for the second term, we have

$$\begin{aligned}\langle v_x, \mathbf{d}_x F\rangle &= -\left\langle (\mathbf{d}_x K + \mathbf{d}_x \phi_\xi)^\sharp, \mathbf{d}_x F\right\rangle \\ &= -\{F_{|P}, K_{|P}\}_P + \frac{\delta K}{\delta \mu} \triangleleft F,\end{aligned}$$

using the definition of the induced Poisson structure in terms of the original symplectic structure: $\{F_{|P}, K_{|P}\}_P = (\mathbf{d}_x K)^\sharp \cdot \mathbf{d}_x F$. Writing out the last term requires a little more work. Denote

$$\eta := \frac{\delta F}{\delta \mu};$$

the last term then becomes

$$\left\langle \rho, \frac{\delta F}{\delta \mu} \right\rangle = -\mathbf{i}_w B\left(\eta_{H\times P}\right) - \left\langle T^* L_g(\mathbf{d}_g K), \eta \right\rangle.$$

The first term on the right-hand side can be rewritten as

$$\begin{aligned} -\mathbf{i}_w B\left(\eta_{H\times P}\right) &= \mathbf{d}\phi_\eta(w) \\ &= \mathbf{d}\phi_\eta(\xi_{H\times P}) + \mathbf{d}\phi_\eta(v_x) \\ &= B(\eta_{H\times P}, \xi_{H\times P}) - \mathbf{d}_x\phi_\eta \cdot (\mathbf{d}_x K)^\sharp - \mathbf{d}_x \phi_\eta \cdot (\mathbf{d}_x \phi_\xi)^\sharp \\ &= B(\eta_{H\times P}, \xi_{H\times P}) - \eta \lhd K - \eta \star \xi. \end{aligned}$$

Putting everything together, we obtain (75). □

B.2.2. *The Reduced Poisson Structure with Interaction Terms.* Having established the explicit form for the interaction Poisson structure $\{\cdot,\cdot\}^0_{\text{int}}$ on $H \times \mathfrak{h}^* \times P$, we can at last show that the reduced Poisson structure on $\mathfrak{h}^* \times P$ is given by expression (62) in theorem 5.1.

Proof of theorem 5.1. According to the Poisson reduction theorem (see [37]), the reduced Poisson structure $\{\cdot,\cdot\}_{\text{int}}$ on $\mathfrak{h}^* \times P$ is defined by the prescription

$$\{f, h\}_{\text{int}} \circ \pi = \{f \circ \pi, h \circ \pi\}^0_{\text{int}},$$

where $\{\cdot,\cdot\}^0_{\text{int}}$ is the Poisson structure on $H \times \mathfrak{h}^* \times P$ described in proposition 10 and $\pi : H \times \mathfrak{h}^* \times P \to \mathfrak{h}^* \times P$ forgets the first factor: $\pi(g, \mu, x) = (\mu, x)$.

In other words, the reduced Poisson structure is the original Poisson structure (75) evaluated on functions $f = f(\mu, x)$ and $k = k(\mu, x)$ which do not depend on $g \in H$. This yields the expression (62) for $\{\cdot,\cdot\}_{\text{int}}$. □

B.2.3. *Shifting Away the Interaction Terms.* The symplectic form Ω_{body} and the Poisson structure $\{\cdot,\cdot\}_{\text{int}}$ are characterized by the presence of ***interaction terms***. By this, we mean that the symplectic structure on $H \times \mathfrak{h}^* \times P$ is not simply the sum of the symplectic form ω_{body} on $H \times \mathfrak{h}^*$ and the symplectic form ω_P on P, but contains terms involving the curvature B and the $\mathcal{B}\mathfrak{g}$-potentials ϕ_ξ, and similar for $\{\cdot,\cdot\}_{\text{int}}$. As shown in lemma B.2, one can remove the interaction terms from Ω_{body} at the expense of introducing a cocycle Σ by using the shift map $\hat{\mathcal{S}}$. The resulting symplectic structure then induces a reduced Poisson structure on $\mathfrak{h}^* \times P$, which is just the natural Poisson structure of definition 5.2.

Theorem B.3. *The symplectic form $\mathcal{S}_*\Omega_{\text{body}}$ on $H \times \mathfrak{h}^* \times P$ is H-invariant. The Poisson structure on $\mathfrak{h}^* \times P$ induced by $\mathcal{S}_*\Omega_{\text{body}}$ is the natural Poisson structure $\{\cdot,\cdot\}_\Sigma$ of definition 5.2.*

Proof. This is an immediate consequence of the fact that the Lie-Poisson structure $\{\cdot,\cdot\}_{\text{l.p}}$ is induced by ω_{body}, while $\{\cdot,\cdot\}_P$ is induced by ω_P. □

The map $\mathcal{S}$ shifts away the interacting terms in the symplectic form, and induces a map on the Lie algebra level. It is easy to see that this map is just the map $\hat{\mathcal{S}}$ defined in (64). Using this observation, the proof of theorem 5.3 now follows immediately.

Proof of theorem 5.3. The map $\hat{\mathcal{S}}$ is the quotient space map induced by $\mathcal{S}$, which is a symplectic map taking the symplectic form (72) with interaction terms into the symplectic form (73) without interaction terms. Hence, the induced map is a Poisson map with respect to the induced Poisson structures, *i.e.* (65) holds. $\square$

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Received November 2008; revised March 2009.

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