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DIRAC COTANGENT BUNDLE REDUCTION

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Abstract. The authors' recent paper in *Reports in Mathematical Physics* develops Dirac reduction for cotangent bundles of Lie groups, which is called *Lie–Dirac reduction*. This procedure simultaneously includes Lagrangian, Hamiltonian, and a variational view of reduction. The goal of the present paper is to generalize Lie–Dirac reduction to the case of a general configuration manifold; we refer to this as *Dirac cotangent bundle reduction*. This reduction procedure encompasses, in particular, a reduction theory for Hamiltonian as well as implicit Lagrangian systems, including the case of degenerate Lagrangians.

First of all, we establish a reduction theory starting with the Hamilton-Pontryagin variational principle, which enables one to formulate an implicit analogue of the Lagrange-Poincaré equations. To do this, we assume that a Lie group acts freely and properly on a configuration manifold, in which case there is an associated principal bundle and we choose a principal connection. Then, we develop a reduction theory for the canonical Dirac structure on the cotangent bundle to induce a *gauged Dirac structure*. Second, it is shown that by making use of the gauged Dirac structure, one obtains a reduction procedure for standard implicit Lagrangian systems, which is called *Lagrange-Poincaré-Dirac reduction*. This procedure naturally induces the *horizontal and vertical implicit Lagrange-Poincaré equations*, which are consistent with those derived from the reduced Hamilton-Pontryagin principle. Further, we develop the case in which a Hamiltonian is given (perhaps, but not necessarily, coming from a regular Lagrangian); namely, *Hamilton-Poincaré-Dirac reduction* for the *horizontal and vertical Hamilton-Poincaré equations*. We illustrate the reduction procedures by an example of a satellite with a rotor.

The present work is done in a way that is consistent with, and may be viewed as a specialization of the larger context of Dirac reduction, which allows for *Dirac reduction by stages*. This is explored in a paper in preparation by Cendra, Marsden, Ratiu and Yoshimura.

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1. **Introduction.** This paper develops a reduction procedure for the canonical Dirac structure defined on the cotangent bundle T^*Q of a configuration manifold Q. In this paper we restrict ourselves to the case in which a Lie group G acts freely and properly on Q; that is, one has a principal bundle $Q \to Q/G$. The reduction procedure includes—in one overarching construction—the case of implicit Lagrangian systems and Hamiltonian systems as well as reduction of the Hamilton-Pontryagin principle to produce, for instance, the Hamilton-Poincaré variational principle and the associated implicit analogue of the Lagrange-Poincaré and Hamilton-Poincaré equations.

Our approach may be viewed as a specialization of the larger context of Dirac reduction of anchored vector bundles, as developed by [26]. That provides an interesting context that is closed under reduction and so allows for *Dirac reduction by stages*. Our cotangent bundle reduction procedure may be viewed as a nontrivial special case in the same way as the important case of cotangent bundle reduction may be viewed as a nontrivial special context for symplectic or Poisson reduction. Unlike some other approaches to Dirac reduction, our approach also allows for reduction of variational principles and includes, simultaneously, both the Lagrangian and Hamiltonian sides. This will be explained in further detail below.

The notion of implicit Lagrangian systems, their relation to Dirac structures and the Hamilton-Pontryagin principle were developed in our earlier papers ([87, 88]). This development made use of a Dirac structure that is induced on T^*Q from a given constraint distribution Δ_Q on Q and its variational links were clarified by employing the Hamilton-Pontryagin principle, through which one can treat the case of degenerate Lagrangian systems with holonomic constraints as well as mechanical systems with nonholonomic constraints. Applications to electric circuits were shown in [89] and the link with the generalized Legendre transformation was developed in [90].

The present paper focuses on reduction of the canonical cotangent Dirac structure without nonholonomic constraints. In accompanying papers, we shall extend this reduction procedure to the case of systems with nonholonomic constraints $\Delta_Q \subset TQ$ as well as making the link with the Dirac anchored vector bundle approach of [26].

Since this paper is not dealing with systems with nonholonomic constraints, we will be reducing integrable Dirac structures. Using the Courant bracket, one can readily show that the corresponding reduced Dirac structure is also integrable. However, we do not emphasize this point since one of our key future goals is to deal with nonholonomic constraints, in which case one does not have integrability.

We recall a few facts about the important case in which $Q = G$ that was treated in our previous paper ([91]). That paper also dealt with the case of a nonholonomic distribution Δ_G on G. When $\Delta_G = TG$, and with the canonical Dirac structure $D \subset TT^*G \oplus T^*T^*G$ on T^*G, the reduction procedure was called *Lie-Dirac reduction.* In this procedure, using the isomorphism $T^*G \cong G \times \mathfrak{g}^*$, one constructs a trivialized Dirac structure $\bar{D}$ on $G \times \mathfrak{g}^*$. By taking the quotient of $\bar{D}$ by the action of G, one obtains a reduced Dirac structure $[\bar{D}]_G := \bar{D}/G$ on the bundle $\mathfrak{g}^* \times V$ thought of as a vector bundle over $\mathfrak{g}^*$, where $V = \mathfrak{g} \oplus \mathfrak{g}^*$ and where $\mathfrak{g} = T_eG$ is the Lie algebra and $\mathfrak{g}^*$ is its dual. Thus, one ends up with a Dirac structure on V that is parameterized by a choice of $\mu \in \mathfrak{g}^*$. In this context, one gets a reduction procedure for an associated implicit Lagrangian system, resulting in the *implicit Euler-Poincaré equations* as well as its Hamiltonian counterpart, the *Lie-Poisson* *equations*. In addition, we established a general case of Dirac reduction with nonholonomic constraints in which a constraint distribution $\Delta_G \subset TG$ is given, and this case was illustrated by the *implicit Euler-Poincaré-Suslov equations*, which can be thought of as a rigid body system with nonholonomic constraints. Some related Suslov problems in nonholonomic systems were discussed in [11].

The goal of the present paper is to generalize the results just described of [91] *(without the distribution Δ_Q—that is, taking the case $\Delta_Q = TQ$) by replacing the case $Q = G$ with a general Q and assuming that we have a free and proper group action of G on Q.*

One of the works most relevant to the present paper is that of [24], which developed a context for Lagrangian reduction and in particular, the Lagrange-Poincaré equations. Moreover, that work shows that the setting of *Lagrange-Poincaré bundles*, which may be regarded as the Lagrangian analogue of a Poisson manifold in symplectic geometry, enables one to perform *Lagrangian reduction by stages*. Lagrangian reduction and the Lagrange-Poincaré equations may be regarded as the generalization of *Euler-Poincaré reduction* and its associated *Euler-Poincaré equations*. Another work that was a key to the viewpoint of the present paper was [23], where reduced variational principles for the *Lie-Poisson* and *Hamilton-Poincaré equations* were developed, namely, the *Lie-Poisson variational principle* and the *Hamilton-Poincaré variational principle*. This Hamilton-Poincaré variational principle is closely related with *Poisson cotangent bundle reduction* (see, [71, 73]). For the details of the general theory of cotangent bundle reduction, see [58].

Goal. The main goal of this paper is to develop a Dirac reduction theory starting with the canonical Dirac structure on T^*Q, under the assumption of a free and proper action of a Lie group G on Q; that is, the case of a principal bundle $Q \to Q/G$. The resulting technique will be called *Dirac cotangent bundle reduction*. Associated with this *geometric reduction*, we also develop the *reduction of dynamics*, by reducing a standard implicit Lagrangian system as well as its associated Hamilton-Pontryagin variational principle. We further show how a Hamiltonian analogue of this reduction can be carried out for a standard implicit Hamiltonian system (that is a Hamiltonian system associated with the canonical Dirac structure) and that it is consistent with the Hamilton-Poincaré variational principle.

As mentioned above, our theory of Dirac cotangent bundle reduction provides both the Lagrangian and Hamiltonian points of view; it yields *Lagrange-Poincaré-Dirac reduction* when one takes the Lagrangian view, and *Hamilton-Poincaré-Dirac reduction* when one takes the Hamiltonian view. This dual structure in reduction of standard implicit Lagrangian and Hamiltonian systems may be regarded as a generalization of *Euler-Poincaré-Dirac reduction* and *Lie-Poisson-Dirac reduction* developed in [91].

1.1. **Some history and background of reduction.** Let us briefly review some background of reduction theory relevant with Dirac cotangent bundle reduction in this paper. As to the details and history of reduction theory in mechanics, refer, for instance, to [61, 62, 24, 67, 58].

Symplectic reduction. Let (P, Ω) be a symplectic manifold and let G be a Lie group that acts freely and properly on P by symplectic maps. Suppose that the action has an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ and let $G_\mu := \{g \in G \,|\, \mathrm{Ad}^*_{g^{-1}}\, \mu = \mu\}$ be the coadjoint isotropy subgroup of $\mu \in \mathfrak{g}^*$. The *symplectic reduced space* is defined to be the quotient space $P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$, which carries a reduced symplectic form Ω_μ uniquely defined by $\pi_\mu^*\Omega_\mu = i_\mu^*\Omega$, where $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ is the projection and $i_\mu : \mathbf{J}^{-1}(\mu) \to P$ is the inclusion. When one chooses $P = T^*G$ with G acting by left translation, it was shown in [65] that the symplectic reduced space $(T^*G)_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ may be identified via left translation with the coadjoint orbit $\mathcal{O}_\mu := \{\mathrm{Ad}^*_{g^{-1}}\,\mu \mid g \in G\} = G \cdot \mu$ through $\mu \in \mathfrak{g}^*$ and also that the reduced symplectic form coincides with the *coadjoint orbit symplectic form* $\Omega_\mu(\mathrm{ad}^*_\xi\mu, \mathrm{ad}^*_\eta\mu) = -\left<\mu, [\xi, \eta]\right>$.

Lie-Poisson reduction. In its simplest form, we suppose that G acts freely and properly on a Poisson manifold P, so that P/G is a smooth manifold which is endowed with the unique Poisson structure and the canonical projection $\pi : P \to P/G$ is a Poisson map. The condition that π be Poisson is that for two functions f, $h : P/G \to \mathbb{R}$, $\{f, h\}_{P/G} \circ \pi = \{f \circ \pi, h \circ \pi\}_P$ holds, where the brackets $\{,\}_{P/G}$ and $\{,\}_P$ are on P/G and P, respectively. The functions $F = f \circ \pi$ and $H = h \circ \pi$ are the unique G–invariant functions on P that project to f and h. It is shown that $\{f, h\}_{P/G}$ is well defined by proving that $\{F, H\}_P$ is G–invariant, which follows from the fact that F and H are G–invariant and the group action of G on P consists of Poisson maps. This is a standard and perhaps the most elementary form of reduction; a detailed exposition can be found in many references, such as [61].

In the particular case when $P = T^*G$ and the action of G on T^*G is by the cotangent lift of left (or right) translation of G on itself, the quotient space $(T^*G)/G$ is naturally diffeomorphic to $\mathfrak{g}^*$, namely, the dual of the Lie algebra $\mathfrak{g}$ of G. In many references, such as [66], [39], and [68] (see [61] for an exposition), it was shown how to compute the reduced Poisson bracket and associated reduced equations of motion, a procedure now called *Lie-Poisson reduction*. In fact, the quotient Poisson bracket is given by the plus (or minus) Lie-Poisson bracket as $\{f, h\}_\pm(\mu) = \mp\left<\mu, [\delta f/\delta\mu, \delta h/\delta\mu]\right>$, where f, h are arbitrary functions on $\mathfrak{g}^*$. The minus sign goes with left reduction and the plus sign with right reduction. Let H be a left-invariant Hamiltonian on T^*G and let $\bar{\lambda} : T^*G \to G \times \mathfrak{g}^*$ be the left trivialized diffeomorphism. Then, one has the trivialized Hamiltonian $\bar{H} = H \circ \bar{\lambda}^{-1}$ and it leads to the *Lie–Poisson equations*

$$\frac{d\mu}{dt} = \mathrm{ad}^*_{\frac{\delta h}{\delta\mu}}\,\mu,$$

where $h := \bar{H}|\mathfrak{g}^*$ is the reduced Hamiltonian on $\mathfrak{g}^*$ and $\mu \in \mathfrak{g}^*$.

While it was [65] who showed how to compute the coadjoint orbit symplectic form by reduction, it was [44] and [60] who explained how the *Poisson structure* on P_μ is related with that on P/G in the generalized context of Poisson reduction. While the Lie-Poisson bracket per se was discovered by Sophus Lie in the 1890s (using different techniques), the coadjoint orbit symplectic form was discovered by Souriau, Kirillov and Kostant in the early 1960s; see [61] and [85] for additional historical comments.

Cotangent bundle reduction. We shall be reviewing some aspects of reduction in the case of tangent and cotangent bundles. We shall follow the historical route here. More details and comprehensive treatments may be found in [24] for tangent bundle reduction and in [58] for the case of cotangent bundles.

We shall first consider the case of cotangent bundle reduction, namely, $P = T^*Q$. The simplest situation concerns reduction at zero; then the symplectic reduced space at $\mu = 0$ is given by $P_\mu = T^*(Q/G)$ with the canonical symplectic form $\Omega_{Q/G}$. Another basic case is the Abelian version of cotangent bundle reduction, where

$G = G_\mu$ and one has $(T^*Q)_\mu = T^*(Q/G)$ with the symplectic form $\Omega_{Q/G} + \beta_\mu$ (here β_μ is a magnetic term that is closely related to curvature). The Abelian version of cotangent bundle reduction was first developed by [79], who showed how to construct β_μ in terms of a potential one form α_μ. The relatively simple but important case of cotangent bundle reduction at the zero value of the momentum map was studied by [77]. Motivated by the work of Smale and Satzer, the generalization of cotangent bundle reduction for nonabelian groups at arbitrary values of the momentum map was done in the 1978 version of [1] and it was [51] who interpreted the result in terms of a connection, now called the *mechanical connection*.

The geometry of this situation was used to great effect in molecule dynamics, optimal control of deformable bodies inspired from the so-called "falling cat" problem, and so on; see, for example, [38], [41, 42], and [72, 74, 75, 56].

From the symplectic point of view, the principal result is that the *symplectic reduction of a cotangent bundle at* $\mu \in \mathfrak{g}^*$ *is a bundle over* $T^*(Q/G)$ *with fiber the coadjoint orbit through* μ. This result can be traced back in a preliminary form, to the work of [80] and [84] on a "Yang–Mills construction", which is, in turn, motivated by Wong's equations, i.e., the equations of a particle in a Yang–Mills field. It was refined in [71] and [72, 73], where tangent and cotangent bundle reduction were evolved into what we now term as the "*bundle picture*" or the "*gauge theory of mechanics*". See also [1] and [56]. The main result of the bundle picture gives a structure to the quotient spaces $(T^*Q)/G$ and $(TQ)/G$ when G acts by the cotangent and tangent lifted actions. The symplectic leaves in this picture were analyzed by [92], [33], and [59]. It was also shown that the symplectically reduced cotangent bundle can be symplectically *embedded* in $T^*(Q/G_\mu)$—this is the *injective version of the cotangent bundle reduction theorem*. From the Poisson viewpoint, in which one simply takes the quotient by the group action, this reads as follows: $(T^*Q)/G$ is a $\mathfrak{g}^*$–bundle over $T^*(Q/G)$, or a *Lie-Poisson bundle over* the cotangent bundle of shape space.

The bundle point of view. Let $\pi : Q \to Q/G$ be a principal bundle with G acting freely and properly on Q and choose a principal connection $A : TQ \to \mathfrak{g}$ on this *shape space bundle*. The general theory, in principle, does not require one to choose a connection. However, there are many good reasons to do so, such as applications to stability and geometric phases. Define $\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$, the *associated bundle* to $\mathfrak{g}$, where the quotient uses the given action on Q and the adjoint action on $\mathfrak{g}$.

The connection A defines a bundle isomorphism $\Psi_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ given by $\Psi_A([v_q]_G) = T\pi(v_q) \oplus [q, A(v_q)]_G$. Here, the sum is a Whitney sum of vector bundles over Q/G (the fiberwise direct sum) and the symbol $[q, A(v_q)]_G$ means the equivalence class of $(q, A(v_q)) \in Q \times \mathfrak{g}$ under the G–action. The map Ψ_A is a well-defined vector bundle isomorphism with inverse given by $\Psi_A^{-1}(u_{[q]} \oplus [q, \xi]_G) = [(u_{[q]})^h_q + \xi q]_G$, where $(u_{[q]})^h_q$ denotes the horizontal lift of $u_{[q]}$ to the point q and $\xi q = \xi_Q(q)$.

Poisson cotangent bundle reduction. The bundle view of Poisson cotangent bundle reduction considers the inverse of the fiberwise dual of Ψ_A, which defines a bundle isomorphism $(\Psi_A^{-1})^* : T^*Q/G \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, where $\tilde{\mathfrak{g}}^* = (Q \times \mathfrak{g}^*)/G$ is the vector bundle over Q/G associated with the coadjoint action of G on $\mathfrak{g}^*$. The isomorphism makes explicit the sense in which $(T^*Q)/G$ is a bundle over $T^*(Q/G)$ with fiber $\mathfrak{g}^*$. The Poisson structure on this bundle is a synthesis of the canonical bracket, the Lie-Poisson bracket, and curvature. The inherited Poisson structure on this space was derived in [71] and the details were given in [73]. For other accounts of Poisson cotangent bundle reduction, see [76] and [58].

Symplectic cotangent bundle reduction. It was [59] who showed that each symplectic reduced space of T^*Q, which are the symplectic leaves in $(T^*Q)/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, are given by a fiber product $T^*(Q/G) \times_{Q/G} \tilde{\mathcal{O}}$, where $\tilde{\mathcal{O}}$ is the associated coadjoint orbit bundle. This makes precise the sense in which the symplectic reduced spaces are bundles over $T^*(Q/G)$ with fiber a coadjoint orbit. That paper also gives an intrinsic expression for the reduced symplectic form, which involves the canonical symplectic structure on $T^*(Q/G)$, the curvature of the connection, the coadjoint orbit symplectic form, and interaction terms that pair tangent vectors to the orbit with the vertical projections of tangent vectors to the configuration space (see also [92]). It was shown by [62] that the reduced space P_μ for $P = T^*Q$ is globally diffeomorphic to the bundle $T^*(Q/G) \times_{Q/G} Q/G_\mu$, where the bundle Q/G_μ is regarded as a bundle over Q/G and in fact, the fibers of the map $\rho_\mu : Q/G_\mu \to Q/G$ are diffeomorphic to the coadjoint orbit $\mathcal{O}_\mu$ through μ for the G action on $\mathfrak{g}^*$.

These results simplify the study of these symplectic leaves. In particular, this makes the injective version of cotangent bundle reduction transparent. Indeed, there is a natural inclusion map $T^*(Q/G) \times_{Q/G} Q/G_\mu \to T^*(Q/G_\mu)$, induced by the dual of the tangent of the projection map ρ_μ. This inclusion map then realizes the reduced space P_μ as a symplectic subbundle of $T^*(Q/G_\mu)$.

1.2. Lagrangian and Hamiltonian reduction.

Routh reduction. Routh reduction for Lagrangian systems is classically associated with systems having cyclic variables (this is almost synonymous with having Abelian symmetry group); modern accounts can be found in [3, 4] and in [61]. A key feature of Routh reduction is that when one drops the Euler-Lagrange equations to the quotient space associated with the symmetry, and when the momentum map is constrained to a specified value (i.e., when the cyclic variables and their velocities are eliminated using the given value of the momentum), then the resulting equations are in Euler-Lagrange form not with respect to the Lagrangian itself, but with respect to the *Routhian*. Routh reduction is closely related to Routh's treatment of stability theory, which is a precursor to the *energy-momentum method* (see, for instance, [56]).

Euler–Poincaré reduction. Another basic ingredient in Lagrangian reduction is *Euler–Poincaré reduction*. Let $L : TG \to \mathbb{R}$ be a left invariant Lagrangian. From the viewpoint of reduced variational principles, Euler–Poincaré reduction is given in the context of a *reduced constrained variational principle* (see [64], [61], [21] and [22]):

$$\delta \int_{t_0}^{t_1} l(\xi(t))\, dt = 0,$$

where the variation of a curve $\xi(t), t \in [t_0, t_1]$ in $\mathfrak{g}$ is restricted to be of the form $\delta\xi = \dot{\eta} + [\xi, \eta]$, and where $l := L|\mathfrak{g}$ is the reduced Lagrangian (regarding $\mathfrak{g} = T_eG \subset TG$), and $\eta(t)$ is a curve in $\mathfrak{g}$ such that $\eta(t_i) = 0, i = 0, 1$. This reduced variational principle gives the *Euler–Poincaré equations*

$$\frac{d}{dt}\frac{\partial l}{\partial \xi} = \mathrm{ad}^*_\xi \frac{\partial l}{\partial \xi}.$$

Lie-Poisson variational principle. It was shown in [23] that the Lie–Poisson equations can be also derived by the *Lie–Poisson variational principle*, which is a Hamiltonian analogue of the reduced constrained variational principle for the Euler–Poincaré equations. Let $H : T^*G \to \mathbb{R}$ be a left invariant Hamiltonian with associated reduced Hamiltonian $h : \mathfrak{g}^* \to \mathbb{R}$. The reduction of Hamilton's phase space principle yields the reduced principle

$$\delta \int_{t_0}^{t_1} \{\langle \mu(t), \xi(t)\rangle - h(\mu(t))\}\, dt = 0.$$

In the above, the variation of a curve $\xi(t), t \in [t_0, t_1]$ in $\mathfrak{g}$ is given by $\delta\xi = \dot{\eta} + [\xi, \eta]$, where $\eta(t) \in \mathfrak{g}$ satisfies the boundary conditions $\eta(t_i) = 0, i = 0, 1$. Then, it leads to

$$\xi = \frac{\partial h}{\partial \mu}, \quad \frac{d\mu}{dt} = \mathrm{ad}^*_\xi \mu,$$

which are called *Lie–Poisson equations* on $V = \mathfrak{g} \oplus \mathfrak{g}^*$.

Lagrange–Poincaré reduction. It was shown in [63, 64] how to generalize the Routh theory to the nonabelian case as well as how to get the Euler-Poincaré equations for matrix groups by the important technique of *reducing variational principles*. This approach was motivated by related works of [19] and [20] . The Euler-Poincaré variational structure was extended to general Lie groups in [10], and [24] carried out a *Lagrangian reduction theory* that extends the Euler-Poincaré case to arbitrary configuration manifolds. This work was in the context of the Lagrangian analogue of Poisson reduction in the sense that no momentum map constraint is imposed.

One of the things that makes the Lagrangian side of the reduction story interesting is the lack of an obvious general category that is the Lagrangian analogue of Poisson manifolds. Such a category, that of *Lagrange-Poincaré bundles*, is developed in [24], with the tangent bundle of a configuration manifold and a Lie algebra as its most basic examples. That work also develops the Lagrangian analogue of reduction for central extensions and, as in the case of symplectic reduction by stages (see [57] and [58]), cocycles and curvatures enter in this context in a natural way.

The Lagrangian analogue of the bundle picture is the bundle $(TQ)/G$, which is a vector bundle over Q/G; this bundle was studied in [24]. In particular, the equations and variational principles are developed on this space. For $Q = G$, this reduces to the Euler-Poincaré reduction and for G Abelian, it reduces to the classical Routh procedure. A G–invariant Lagrangian L on TQ naturally induces a Lagrangian l on $(TQ)/G$. The resulting equations inherited on this space are the *Lagrange–Poincaré equations* (or the reduced Euler-Lagrange equations).

Recall that one can define a vector bundle isomorphism $\Psi_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ by choosing a principal connection A. Using the geometry of the bundle $TQ/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}}$, along with an affine connection on Q itself, one can write the Lagrange-Poincaré equations intrinsically in terms of covariant derivatives D/Dt (see [24]) as

$$\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(x, \dot{x}, \bar{\xi}) - \frac{\partial l}{\partial x}(x, \dot{x}, \bar{\xi}) = -\left\langle \frac{\partial l}{\partial \bar{\xi}}(x, \dot{x}, \bar{\xi}), \widetilde{B}(\dot{x}, \cdot)\right\rangle,$$
$$\frac{D}{Dt}\frac{\partial l}{\partial \bar{\xi}}(x, \dot{x}, \bar{\xi}) = \mathrm{ad}^*_{\bar{\xi}}\frac{\partial l}{\partial \bar{\xi}}(x, \dot{x}, \bar{\xi}),$$

where we call the first set of these equations the *horizontal Lagrange-Poincaré equations*, while the second the *vertical Lagrange-Poincaré equations*. The notation here is as follows. Points in $T(Q/G) \oplus \tilde{\mathfrak{g}}$ are denoted $(x, \dot{x}, \bar{\xi})$, where $\dot{x} = dx/dt$ and $\bar{\xi} = [q, \xi]_G = A(q, \dot{q})$, and $l(x, \dot{x}, \bar{\xi})$ denotes the reduced Lagrangian induced on the quotient space from L. The bundle $T(Q/G) \oplus \tilde{\mathfrak{g}}$ naturally inherits vector bundle connections and D/Dt indicates the associated covariant derivatives. Also, $\widetilde{B}(\dot{x}, \cdot)$ denotes the curvature of A thought of as an adjoint bundle valued two-form on Q/G.

Hamilton–Poincaré variational principle. It was shown in [23] that a Hamiltonian analogue of Lagrange-Poincaré reduction can be carried out variationally, leading to a reduced principle called the *Hamilton–Poincaré variational principle*. It is clear that the dual of the quotient bundle TQ/G, that is, $(TQ/G)^*$ is canonically identified with the quotient bundle T^*Q/G, where the action of a Lie group G on T^*Q is the cotangent lift of the action of G on Q. Recall that, choosing a principal connection A as before, the vector bundle isomorphism Ψ_A defines by duality a bundle isomorphism $(\Psi_A^{-1})^* : T^*Q/G \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, where $\tilde{\mathfrak{g}}^* = (Q \times \mathfrak{g}^*)/G$ is the associated bundle to $\mathfrak{g}^*$. For a given G–invariant Hamiltonian $H : T^*Q \to \mathbb{R}$, Hamilton's phase space principle states that

$$\delta \int_{t_0}^{t_1} \{ \langle p(t), \dot{q}(t) \rangle - H(q(t), p(t)) \} \, dt = 0,$$

in which the endpoints $q(t_0)$ and $q(t_1)$ of $q(t), t \in [t_0, t_1]$ are held fixed. This principle induces Hamilton's equations of motion. Note that the pointwise function in the integrand, namely, $F(q, \dot{q}, p) = \langle p, \dot{q} \rangle - H(q, p)$ is defined on the *Pontryagin bundle* $TQ \oplus T^*Q$ over Q and also that G acts on $TQ \oplus T^*Q$ by simultaneously left translating on each factor by the tangent and cotangent lift. Since F is invariant under the action of G because of the G–invariance of H, the function F drops to the quotient $f : TQ/G \oplus T^*Q/G \to \mathbb{R}$, or equivalently, $f : T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V} \to \mathbb{R}$, which is given by $f(x, \dot{x}, y, \bar{\xi}, \bar{\mu}) = \langle y, \dot{x} \rangle + \langle \bar{\mu}, \bar{\xi} \rangle - h(x, y, \bar{\mu})$, where $\widetilde{V} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^*$ and h is the reduction of H from T^*Q to $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$. Then, the reduction of Hamilton's phase space principle gives the following *Hamilton–Poincaré equations*:

$$\begin{aligned} \frac{Dy}{Dt} &= -\frac{\partial h}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \\ \frac{dx}{dt} &= \frac{\partial h}{\partial y}, \\ \frac{D\bar{\mu}}{Dt} &= \mathrm{ad}^*_{\bar{\xi}} \bar{\mu}, \\ \xi &= \frac{\partial h}{\partial \bar{\mu}}. \end{aligned}$$

1.3. Dirac reduction and the Hamilton-Pontryagin principle.

Some history of Dirac reduction procedures. It goes without saying that reduction theory for Dirac structures should naturally include reduction of pre-symplectic and Poisson structures and hence it has been naturally related to reduction of Hamiltonian systems. In fact, the development of reduction of Dirac structures is a natural outgrowth of pre-symplectic and Poisson reduction.

In the context of Dirac structures, it was [31] who constructed a reduced Dirac structure on the symplectic reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ that is consistent with Poisson reduction as developed by [60]. Symmetries of Dirac structures in the context of

Hamiltonian systems were discussed in [36]. Reduction of Dirac structures and implicit Hamiltonian systems was developed by [83, 6, 7] in a way that is consistent with [31]. In these constructions, it was shown that a reduced Dirac structure fits naturally into the context of symplectic reduction as well as Poisson reduction; if a Dirac structure $D_P \subset TP \oplus T^*P$ on P is given by the graph of the bundle map $\omega^\flat : TP \to T^*P$ associated with a symplectic structure ω, then a reduced Dirac structure D_{P_μ} on $P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ may be given by the graph of the reduced bundle map associated with the reduced symplectic structure on P_μ, while if D_P is the graph of the bundle map $B^\sharp : T^*P \to TP$ associated with a Poisson structure B on P, then the reduced Dirac structure can be given by the graph of the bundle map associated with the reduced Poisson structure on P_μ. In [8], singular reduction of Dirac structures was developed in the context of implicit Hamiltonian systems.

Lie–Dirac reduction. Following [91], we start with the *canonical Dirac structure* D on T^*G defined as usual by the graph of the canonical symplectic structure Ω_{T^*G} or equivalently the canonical Poisson structure B_{T^*G}. Using the left trivialization map, $\bar{\lambda} : T^*G \to G \times \mathfrak{g}^*$, we obtain a one-form θ on $G \times \mathfrak{g}^*$ and a symplectic two-form ω on $G \times \mathfrak{g}^*$ as $\theta = \bar{\lambda}_* \Theta_{T^*G}$ and $\omega = \bar{\lambda}_* \Omega_{T^*G}$, where Θ_{T^*G} is the canonical one-form on T^*G and $\Omega_{T^*G} = -\mathbf{d}\Theta_{T^*G}$. The graph of the symplectic structure $\omega = -\mathbf{d}\theta$ on $G \times \mathfrak{g}^*$, then defines a Dirac structure $\bar{D}$ on $G \times \mathfrak{g}^*$. Again using left trivialization, we may regard $\bar{D}$ as a subset as follows:

$$\bar{D} \subset T(G \times \mathfrak{g}^*) \oplus T^*(G \times \mathfrak{g}^*) \cong (G \times \mathfrak{g}^*) \times (V \oplus V^*),$$

where $V = \mathfrak{g} \oplus \mathfrak{g}^*$ and $V^* = \mathfrak{g}^* \oplus \mathfrak{g}$ is its dual. On the right hand side, we regard $G \times \mathfrak{g}^*$ as the base and $V \oplus V^*$ as the fiber.

Using left invariance of the Dirac structure, namely, $\bar{D}(hg, \mu) = \bar{D}(g, \mu)$ for all $h, g \in G$ and $\mu \in \mathfrak{g}^*$, it induces a *μ-dependent reduced Dirac structure*

$$[\bar{D}]_G := \bar{D}/G \subset \left(T(G \times \mathfrak{g}^*) \oplus T^*(G \times \mathfrak{g}^*)\right)/G \cong \mathfrak{g}^* \times (V \oplus V^*)$$

on the bundle $T(G \times \mathfrak{g}^*)/G \cong \mathfrak{g}^* \times V$ over $(G \times \mathfrak{g}^*)/G = \mathfrak{g}^*$.[1]

Explicitly, one computes that the reduced Dirac structure $[\bar{D}]_G$ is given for each $\mu \in \mathfrak{g}^*$, by

$$[\bar{D}]_G(\mu) = \{((\xi, \rho), (\nu, \eta)) \in V \oplus V^* \mid \langle \nu, \zeta \rangle + \langle \sigma, \eta \rangle = [\omega]_G(\mu)((\xi, \rho), (\zeta, \sigma)) \text{ for all } (\zeta, \sigma) \in V\},$$

where $[\omega]_G := \omega/G$ is a μ-dependent reduced symplectic form on $\mathfrak{g}^* \times V$, given for each $\mu \in \mathfrak{g}^*$ by

$$[\omega]_G(\mu)((\xi, \rho), (\zeta, \sigma)) = \langle \sigma, \xi \rangle - \langle \rho, \zeta \rangle + \langle \mu, [\xi, \zeta] \rangle .$$

The key difference between the Lie–Dirac reduction and other Dirac reduction procedures previously mentioned is the fact that

> *Lie–Dirac reduction accommodates Lagrangian, Hamiltonian, and a variational view of reduction simultaneously* (see [91]). *The goal of the present paper is to do the same for the case of general Q.*

To carry out Lie–Dirac reduction, one works with slightly larger spaces, as is already evident in the reduction of Hamilton's phase space principle, to give a variational principle for the Lie-Poisson equations, which was developed in [23]. Specifically, many schemes in the literature take the following view (or a variant of it): one starts

[1]This description of $\bar{D}$ and its quotient $[\bar{D}]_G$ is consistent with thinking of them as Dirac structures on *Dirac anchored vector bundles*, as in [26].

with a manifold M, an almost Dirac structure on M, and a group G acting on M (consistent with the Dirac structure). From these ingredients, one then constructs a Dirac structure on M/G. For example, for $M = T^*G$ with G acting by group multiplication, one gets $\mathfrak{g}^*$ with the Dirac structure associated with the Lie-Poisson structure. However, *this construction is too limited for our purposes and does not allow one to include variational principles or the Lagrangian view.* In our case of mechanics on Lie groups, the basic difference in our approach is that instead of $\mathfrak{g}^*$, the resulting reduced space is $V = \mathfrak{g} \oplus \mathfrak{g}^*$.

Euler-Poincaré-Dirac reduction. Again using the identification $T^*G \cong G \times \mathfrak{g}^*$, the μ-dependent reduced Dirac structure $[\bar{D}]_G$ gives a reduction procedure for a standard implicit Lagrangian system, called *Euler-Poincaré-Dirac reduction*, which provides *implicit Euler-Poincaré equations* (see §5 of [91] for details).

Let $L : TG \to \mathbb{R}$ be a Lagrangian, possibly degenerate, and let D be the canonical Dirac structure on T^*G, let $X : TG \oplus T^*G \to TT^*G$ be a partial vector field—that is, it assigns a vector in T_pT^*Q to each point $(g, v, p) \in TG \oplus T^*G$—and let (L, D, X) be a standard implicit Lagrangian system, which satisfies this condition: for each $(g, v) \in TG$ and setting $(g, p) = \mathbb{F}L(g, v)$,

$$(X(g, v, p), \mathbf{d}E(g, v, p)|_{TP}) \in D(g, p),$$

where $E(g, v, p) := \langle p, v\rangle - L(g, v)$ is the generalized energy on $TG \oplus T^*G$ and $P = \mathbb{F}L(TG) \subset T^*G$ (we assume that P is a smooth submanifold).

The reduced Lagrangian $l : \mathfrak{g} \to \mathbb{R}$ is defined by $l = L|\mathfrak{g}$ where we regard $\mathfrak{g} = T_eG \subset TG$. Alternatively, using the left trivialization map $\lambda : TG \to G \times \mathfrak{g}$, one can define the reduced Lagrangian $l : \mathfrak{g} \to \mathbb{R}$ by $l = \bar{L}|\mathfrak{g}$, where $\bar{L} = L \circ \lambda^{-1}$. Further, let $\bar{E} : G \times V \to \mathbb{R}$ be the *trivialized generalized energy* associated to E in view of $TG \oplus T^*G \cong G \times V$. The *reduction of a standard implicit Lagrangian system* (L, D, X) is given by a triple $(l, [\bar{D}]_G, [\bar{X}]_G)$ that satisfies, for each $\eta = g^{-1}v \in \mathfrak{g}$ and with $\mu = \partial l/\partial\eta \in [\bar{P}]_G$, the condition

$$([\bar{X}]_G(\eta, \mu), [\mathbf{d}\bar{E}]_G(\eta, \mu)|_{[T\bar{P}]_G}) \in [\bar{D}]_G(\mu),$$

where $\bar{P} = \mathbb{F}\bar{L}(G \times \mathfrak{g}) \subset G \times \mathfrak{g}^*$, and since the map $\mathbb{F}\bar{L}$ is equivariant, the reduced Legendre transformation is given by the quotient map $\mathbb{F}l := [\mathbb{F}\bar{L}]_G$ such that

$$[\bar{P}]_G = \mathbb{F}l(\mathfrak{g}) \subset \mathfrak{g}^*,$$

and hence

$$[T\bar{P}]_G \cong \mathfrak{g} \times T[\bar{P}]_G \subset \mathfrak{g}^* \times V.$$

Reduction of the trivialized partial vector field $\bar{X} : G \times V \to G \times \mathfrak{g}^* \times V$ is given by $[\bar{X}]_G := \bar{X}/G : V \to \mathfrak{g}^* \times V$, which is denoted, for each $(\eta, \mu) \in V$, by $[\bar{X}]_G(\eta, \mu) = (\mu, \xi, \dot{\mu})$, where $\xi = g^{-1}\dot{g}$ and $\dot{\mu} = d\mu/dt$ are functions of (η, μ). The reduced differential of $\bar{E}$ is the map $[\mathbf{d}\bar{E}]_G := \mathbf{d}\bar{E}/G : V \to \mathfrak{g}^* \times (V \oplus V^*)$, which is given, in coordinates, by $[\mathbf{d}\bar{E}]_G = (\eta, \mu, 0, \mu - \partial l/\partial\eta, \eta)$, where $T^*(G \times V)/G \cong \mathfrak{g}^* \times (V \oplus V^*)$. Thus, its restriction to $[T\bar{P}]_G \subset \mathfrak{g}^* \times V$ is given by $[\mathbf{d}\bar{E}]_G|_{[T\bar{P}]_G} = (\mu, 0, \eta)$. This reduction procedure produces the *implicit Euler-Poincaré equations*:

$$\mu = \frac{\partial l}{\partial \eta}, \quad \xi = \eta, \quad \frac{d\mu}{dt} = \mathrm{ad}^*_\xi \mu.$$

Lie-Poisson-Dirac reduction. Given a left invariant Hamiltonian $H : T^*G \to \mathbb{R}$, we can form a *standard implicit Hamiltonian system*, namely the triple (H, D, X), which satisfies, for each $(g,p) \in T^*G$,

$$(X(g,p), \mathbf{d}H(g,p)) \in D(g,p),$$

where $X : T^*G \to TT^*G$ is a vector field on T^*G. Let $\bar{H} : G \times \mathfrak{g}^* \to \mathbb{R}$ be the trivialized Hamiltonian defined by $\bar{H} = H \circ \bar{\lambda}^{-1}$ and let $h = \bar{H}|\mathfrak{g}^* : \mathfrak{g}^* \to \mathbb{R}$ be the reduced Hamiltonian.

Let $\bar{X} : G \times \mathfrak{g}^* \to (G \times \mathfrak{g}^*) \times V$ be the trivialized vector field and let $[\bar{X}]_G : \mathfrak{g}^* \to \mathfrak{g}^* \times V$ be the partial vector field given by $[\bar{X}]_G = (\mu, \xi(\mu), \dot{\mu})$, where $\xi(\mu) = \partial h/\partial\mu$. Then, *reduction of a standard implicit Hamiltonian system* (H, D, X) may be given by a triple $(h, [\bar{D}]_G, [\bar{X}]_G)$ that satisfies, at fixed $\mu \in \mathfrak{g}^*$,

$$([\bar{X}]_G(\mu), [\mathbf{d}\bar{H}]_G(\mu)) \in [\bar{D}]_G(\mu),$$

where $[\mathbf{d}\bar{H}]_G := \mathbf{d}\bar{H}/G : \mathfrak{g}^* \to \mathfrak{g}^* \times V^*$ is *reduction of the differential of* $\bar{H}$, which is locally given by $[\mathbf{d}\bar{H}]_G = (\mu, 0, \partial h/\partial\mu)$. This induces *Lie-Poisson equations* on $V = \mathfrak{g} \oplus \mathfrak{g}^*$ as

$$\xi = \frac{\partial h}{\partial \mu}, \quad \frac{d\mu}{dt} = \mathrm{ad}^*_{\xi}\mu.$$

This construction of Lie-Poisson equations is consistent with the Lie-Poisson variational principle mentioned before. As to the details, refer to §3 and §6 of [91].

Nonholonomic systems and applications. There are many research works on reduction of nonholonomic systems from many point of views and we can not survey them in a comprehensive way here, but we would like to mention the following works including applications to control relevant with the present paper: [48, 5, 82, 10, 49, 50, 17, 18, 28, 25, 9, 27, 11].

Here, we only mention our previous work on Lie-Dirac reduction with nonholonomic constraints; namely, it was shown in [91] that Euler-Poincaré-Dirac reduction as well as Lie-Poisson-Dirac reduction can be extended to the case with a G-invariant distribution $\Delta_G \subset TG$ associated with the Lie-Dirac reduction with nonholonomic constraints, which may appear in nonholonomic systems in rigid body mechanics, called *Suslov problems* (see, for instance [93]). This reduction procedure is called *Euler-Poincaré-Suslov reduction* that induces *implicit Euler-Poincaré-Suslov equations* as well as *Lie-Poisson-Suslov reduction* that yields *implicit Lie-Poisson-Suslov equations* respectively.

In this paper, we primarily study the case in which there are no nonholonomic constraints, since we are interested in reduction of standard implicit Lagrangian systems and standard implicit Hamiltonian systems. Of course, the present reduction theory may be readily extended to the case in which nonholonomic constraints are given as in the above case of Lie-Dirac reduction, however, we will study the Dirac cotangent bundle reduction with nonholonomic constraints in another paper.

Hamilton-Pontryagin variational principle on Lie groups. Following §3 of [91], we review the reduced Hamilton-Pontryagin principle for the special case $Q = G$.

Letting $l : \mathfrak{g} \to \mathbb{R}$ be the reduced Lagrangian, the *reduced Hamilton-Pontryagin principle* is given by

$$\delta \int_{t_0}^{t_1} \{ l(\eta(t)) + \langle \mu(t), \xi(t) - \eta(t) \rangle \} \; dt = 0,$$

where $\xi \in \mathfrak{g}$, $\eta \in \mathfrak{g}$, and $\mu \in \mathfrak{g}^*$. The variation of $\xi(t), t \in [t_0, t_1]$ is computed exactly as in Euler-Poincaré theory to be given by $\delta\xi(t) = \dot{\zeta}(t) + [\xi(t), \zeta(t)]$, so that $\zeta(t) \in \mathfrak{g}$ satisfies $\zeta(t_i) = 0, i = 0, 1$. Then, it leads to the *implicit Euler–Poincaré equations* on $V = \mathfrak{g} \oplus \mathfrak{g}^*$,

$$\mu = \frac{\partial l}{\partial \eta}, \quad \xi = \eta, \quad \frac{d\mu}{dt} = \mathrm{ad}^*_\xi \mu.$$

We remark that in this reduced variational principle for implicit Euler-Poincaré equations, the reduced Lagrangian l on $\mathfrak{g}$ may be degenerate. Needless to say, this Hamilton-Pontryagin variational principle is consistent with the Euler-Poincaré-Dirac reduction.

Discrete mechanics, variational integrators and applications. In conjunction with variational integrators in discrete mechanics (see, for instance, [69]), a numerical scheme for time integration of Lagrangian dynamical systems was proposed by [45] in the context of discrete Hamilton-Pontryagin principle and [12] developed the structure-preserving time-integrators on Lie groups for rigid body systems based on Hamilton-Pontryagin principle. Those variational integrators can incorporate holonomic constraints via Lagrange multipliers, nonconservative forces as well as discrete optimal control. The construction of geometric variational integrator for unconstrained Lagrangian systems on Lie algebroids was also shown in [55] and a variational integrator for the case of nonholonomic systems was established in [40].

Relations with Lie algebroids and Courant algebroids. It was shown by Weinstein in [86] that Lagrangian and Hamiltonian systems can be formulated on Lie algebroids and also that the discrete analogue of Lagrangian systems can be developed on Lie groupoids, where he asked whether it is possible to develop a formalism on Lie algebroids similar to the ordinary Lagrangian formalism given in [46]. This was done by [70]. Further, a geometric description on Lie algebroids of Lagrangian systems with nonholonomic constraints was given in [29].

As mentioned previously, it was shown in [24] that the Lagrange-Poincaré equations can be formulated in the context of the bundle $TQ/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}}$ over Q/G, which falls into the category of the Lagrange-Poincaré bundles. It is known that there is a Lie algebroid structure on the quotient bundle $TQ/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}} \to Q/G$, also called the *Atiyah quotient bundle* (see, [54]). However, the category of Lagrange-Poincaré bundles does not make direct use of groupoids or algebroids. Recently, Lagrangian and Hamiltonian descriptions of mechanics on Lie algebroids were shown by [34], where a geometric formulation of the Lagrange-Poincaré and Hamilton-Poincaré equations was obtained.

Following remarks in [91], let us explain a few relationships between Lie–Dirac reduction and the reduction of *Courant algebroids*. Let P be a manifold with a free and proper action by G, so that it is also a G-principal bundle over $B = P/G$. One can consider the bundle

$$F = (TP \oplus T^*P)/G = (TP/G) \oplus (T^*P/G)$$

over $B = P/G$, where $(TP/G)^* \cong T^*P/G$. It is easily checked that the natural lift of the G-action to $TP \oplus T^*P$ preserves the Courant paring on $TP \oplus T^*P$, as well as the *Courant bracket*, which was introduced in [31]. Thus, F is naturally a Courant algebroid over B in the general sense of [53]. While this Courant algebroid is not of the form $TB \oplus T^*B$, one can still talk about Dirac structures in F. In fact, it is a general fact that if D is a G-invariant Dirac structure on P; that is, D is a subset of $TP \oplus T^*P$, then $[D]_G := D/G$ is a Dirac subbundle of F (and $[D]_G$ is integrable if D is, since G preserves the Courant bracket).

For the case $P = T^*G$, in view of $T^*G \cong G \times \mathfrak{g}^*$, one gets a Courant algebroid

$$F = \mathfrak{g}^* \times (V \oplus V^*),$$

which is a trivial bundle over $B = \mathfrak{g}^*$ and where $V = \mathfrak{g} \oplus \mathfrak{g}^*$, $V^* = \mathfrak{g}^* \oplus \mathfrak{g}$ is the dual of V, and $V \oplus V^*$ is just a fiber of F over $\mathfrak{g}^*$. Let $\bar{D}$ be the Dirac structure on $G \times \mathfrak{g}^*$. Then, the quotient $[\bar{D}]_G := D/G$, viewed as a structure in the bundle F over B, gives a μ-dependent Dirac structure on $\mathfrak{g}^* \times V$, where $\mu \in \mathfrak{g}^*$. The reduction of a Dirac structure on $P = T^*G$ induced from a nontrivial distribution on G can be also understood in the above context as remarked in [91], although the *Lie-Dirac reduction construction itself does not use groupoids or algebroids*. Of course, the reduced Dirac structure associated with the canonical Dirac structure is integrable with respect to the natural Courant bracket in F, whereas the one associated with the induced Dirac structure from a nontrivial distribution is not. Some relevant works that deal with the reduction of Courant algebroids and associated Dirac structures can be found in, for instance, [13, 14], where the authors focus mainly on so-called "exact" Courant algebroids; in the set up above, their reduction would yield $TB \oplus T^*B$, rather than $(TP/G) \oplus (T^*P/G)$.

Outline of the paper. For readers' convenience, we summarize the reduction procedures associated with Dirac cotangent bundle reduction by the chart shown in Fig. 1.

First, we provide a brief review for Dirac structures, implicit Lagrangian systems and the Hamilton-Pontryagin principle.

Second, to develop our theory of Dirac cotangent bundle reduction for the case in which a Lie group G acts freely and properly on Q, namely, in which there is a principal bundle $Q \to Q/G$, we establish the bundle picture for $TQ \oplus T^*Q$. That is, we develop the geometric structure of the quotient space $(TQ \oplus T^*Q)/G$ via a bundle isomorphism of the form $(TQ \oplus T^*Q)/G \to T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$ associated with a chosen principal connection on the principal bundle.

Third, we explore geometry of variations of curves in the reduced Pontryagin bundle $(TQ \oplus T^*Q)/G \cong T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$ by making extensive use of the ideas associated with *horizontal and vertical variations of curves in* Q developed in [24, 23]. Using this, we establish a reduction procedure for the Hamilton-Pontryagin variational principle, which enables one to formulate an implicit analogue of Lagrange-Poincaré equations, namely, *horizontal and vertical implicit Lagrange-Poincaré equations*, corresponding to the horizontal and vertical variations respectively. This may be regarded as a natural extension of the case of the variational theory developed in [91].

Further, we develop *Dirac cotangent bundle reduction*, which is a reduction procedure for the canonical Dirac structure D on T^*Q, by extending Lie-Dirac reduction theory for the case $Q = G$ to the general case of the principal bundle $Q \to Q/G$. To

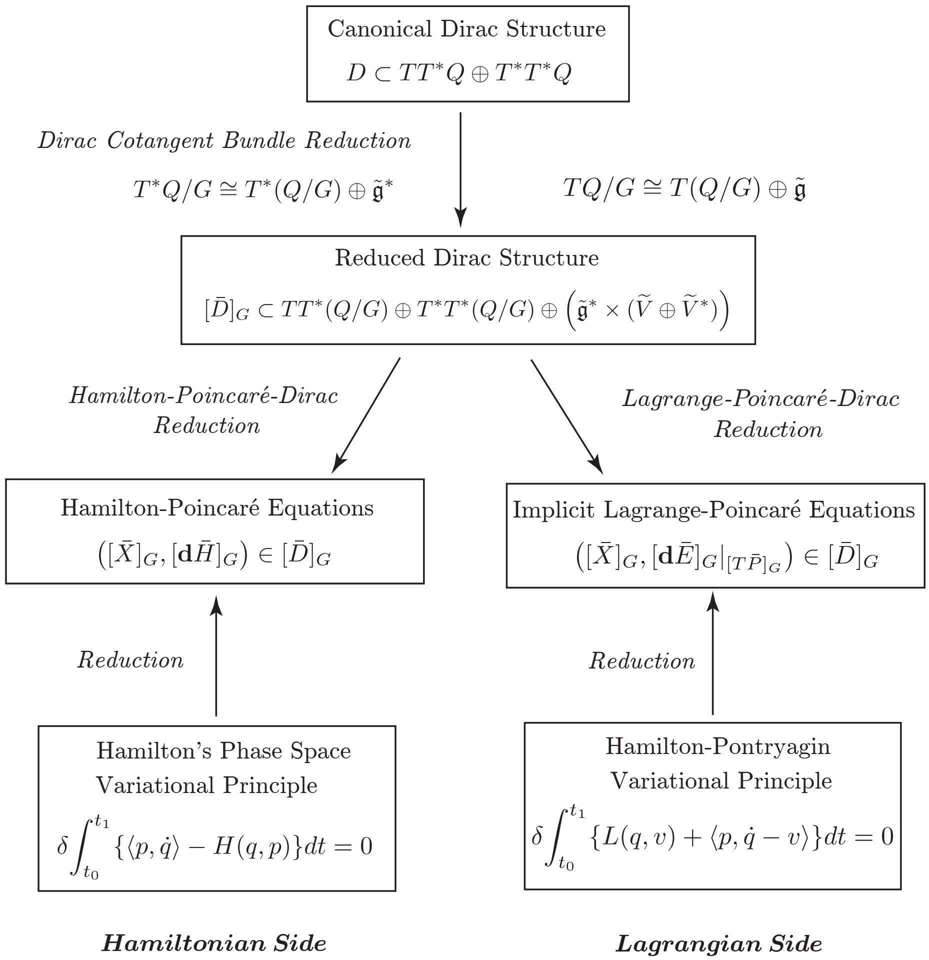


FIGURE 1. Dirac cotangent bundle reduction

do this, we introduce an isomorphism $T^*Q \cong \widetilde{Q}^* \times \mathfrak{g}^*$ (an unreduced version of Sternberg space) to develop a G-invariant Dirac structure $\bar{D} \subset T(\widetilde{Q}^* \times \mathfrak{g}^*) \oplus T^*(\widetilde{Q}^* \times \mathfrak{g}^*)$ on $\widetilde{Q}^* \times \mathfrak{g}^*$. Under the isomorphisms $T^*Q/G \cong (\widetilde{Q}^* \times \mathfrak{g}^*)/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ and $TT^*Q/G \cong T(\widetilde{Q}^* \times \mathfrak{g}^*)/G \cong TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$, it is shown that, by taking the quotient of $\bar{D}$ by the action of G, one can obtain a *gauged Dirac structure* $[\bar{D}]_G = [\bar{D}]_G^{\mathrm{Hor}} \oplus [\bar{D}]_G^{\mathrm{Ver}}$ on the bundle $TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$ over $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, where $[\bar{D}]_G^{\mathrm{Hor}}$ is a *horizontal Dirac structure* on the bundle $TT^*(Q/G)$ over $T^*(Q/G)$ and $[\bar{D}]_G^{\mathrm{Ver}}$ is a *vertical Dirac structure* on the bundle $\tilde{\mathfrak{g}}^* \times \widetilde{V}$ over $\tilde{\mathfrak{g}}^*$.

Finally, it is shown how the gauged Dirac structure $[\bar{D}]_G = [\bar{D}]_G^{\mathrm{Hor}} \oplus [\bar{D}]_G^{\mathrm{Ver}}$ accommodates a reduction procedure called *Lagrange-Poincaré-Dirac reduction*, which is the reduction of standard implicit Lagrangian systems (for which the Lagrangian is possibly degenerate) for the case of principal bundles, which naturally induces *horizontal and vertical implicit Lagrange-Poincaré equations*, consistent with the reduction of the Hamilton-Pontryagin variational principle. Making use of the gauged Dirac structure, we also develop a reduction procedure for the case in which a regular Lagrangian is given, or a Hamiltonian is given, which is called *Hamilton-Poincaré-Dirac reduction* that induces *horizontal and vertical Hamilton-Poincaré equations*, consistent with the Hamilton-Poincaré variational principle.

2. Some preliminaries.

Dirac structures. We first recall the definition of a *Dirac structure on a vector space* V, say finite dimensional for simplicity (see, [32]). Let V^* be the dual space of V, and $\langle \cdot\,, \cdot \rangle$ be the natural paring between V^* and V. Define the symmetric paring $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $V \oplus V^*$ by

$$\langle\!\langle\, (v, \alpha), (\bar{v}, \bar{\alpha})\, \rangle\!\rangle = \langle \alpha, \bar{v} \rangle + \langle \bar{\alpha}, v \rangle,$$

for $(v, \alpha), (\bar{v}, \bar{\alpha}) \in V \oplus V^*$. A *Dirac structure* on V is a subspace $D \subset V \oplus V^*$ such that $D = D^{\perp}$, where $D^{\perp}$ is the orthogonal of D relative to the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Now, let P be a given manifold and let $TP \oplus T^*P$ denote the Whitney sum bundle over P, namely, the bundle over the base P and with fiber over the point $x \in P$ equal to $T_xP \times T_x^*P$. In this paper, we shall call a subbundle $D \subset TP \oplus T^*P$ a *Dirac structure on the manifold* P, or a *Dirac structure on the bundle* $TP \to P$, when $D(x)$ is a Dirac structure on the vector space T_xP at each point $x \in P$. A given two-form Ω_P on P together with a distribution Δ_P on P determines a Dirac structure on P as follows: let $x \in P$, and define

$$\begin{aligned} D(x) = \{(v_x, \alpha_x) \in T_xP \times T_x^*P \;\mid\; & v_x \in \Delta_P(x), \text{ and} \\ & \alpha_x(w_x) = \Omega_{\Delta_P}(v_x, w_x) \;\text{ for all }\; w_x \in \Delta_P(x)\}, \end{aligned} \tag{1}$$

where Ω_{Δ_P} is the restriction of Ω_P to Δ_P.

We call a Dirac structure D *closed* or *integrable* if the condition

$$\langle \pounds_{X_1}\alpha_2, X_3 \rangle + \langle \pounds_{X_2}\alpha_3, X_1 \rangle + \langle \pounds_{X_3}\alpha_1, X_2 \rangle = 0 \tag{2}$$

is satisfied for all pairs of vector fields and one-forms (X_1, α_1), (X_2, α_2), (X_3, α_3) that take values in D, where $\pounds_X$ denotes the Lie derivative along the vector field X on P.

Remark 1. Let $\Gamma(TP \oplus T^*P)$ be a space of local sections of $TP \oplus T^*P$, which is endowed with the skew-symmetric bracket $[\;,\;] : \Gamma(TP \oplus T^*P) \times \Gamma(TP \oplus T^*P) \to \Gamma(TP \oplus T^*P)$ defined by

$$\begin{aligned} [(X_1, \alpha_1), (X_2, \alpha_2)] &:= \left([X_1, X_2]\,, \pounds_{X_1}\alpha_2 - \pounds_{X_2}\alpha_1 + \mathbf{d}\,\langle \alpha_2, X_1 \rangle\right) \\ &= \left([X_1, X_2]\,, \mathbf{i}_{X_1}\mathbf{d}\alpha_2 - \mathbf{i}_{X_2}\mathbf{d}\alpha_1 + \mathbf{d}\,\langle \alpha_2, X_1 \rangle\right). \end{aligned}$$

This bracket is the one originally developed in [31] and it does not necessarily satisfy the Jacobi identity. It was shown by [36] that the integrability condition of the Dirac structure $D \subset TP \oplus T^*P$ given in equation (2) can be expressed as

$$[\Gamma(D), \Gamma(D)] \subset \Gamma(D),$$

which is the closedness condition including the Courant bracket (see also [37] and [43]).

Induced Dirac structures. One of the most important and interesting Dirac structures in mechanics is one that is induced from kinematic constraints, whether holonomic or nonholonomic. Such constraints are generally given by a distribution on a configuration manifold.

Let Q be a configuration manifold. Let TQ be the tangent bundle and T^*Q be the cotangent bundle. Let $\Delta_Q \subset TQ$ be a regular distribution on Q and define a lifted distribution on T^*Q by

$$\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q,$$

where $\pi_Q : T^*Q \to Q$ is the canonical projection so that its tangent is a map $T\pi_Q : TT^*Q \to TQ$. Let Ω_{T^*Q} be the canonical two-form on T^*Q. The *induced Dirac structure* D_{Δ_Q} on T^*Q is the subbundle of $TT^*Q \oplus T^*T^*Q$, whose fiber is given for each $p_q \in T^*Q$ as

$$\begin{aligned} D_{\Delta_Q}(p_q) = \{(v_{p_q}, \alpha_{p_q}) \in T_{p_q}T^*Q \times T^*_{p_q}T^*Q \mid v_{p_q} \in \Delta_{T^*Q}(p_q), \text{ and} \\ \alpha_{p_q}(w_{p_q}) = \Omega_{\Delta_Q}(p_q)(v_{p_q}, w_{p_q}) \text{ for all } w_{p_q} \in \Delta_{T^*Q}(p_q)\}, \end{aligned}$$

where Ω_{Δ_Q} is defined by restricting Ω_{T^*Q} to Δ_{T^*Q}. This is, of course, a special instance of the construction in equation (1).

Implicit Lagrangian systems. Let us recall the definition of implicit Lagrangian systems (for further details, see [87]).

Let $L : TQ \to \mathbb{R}$ be a Lagrangian, possibly degenerate. Given an induced Dirac structure D_{Δ_Q} on T^*Q and a partial vector field $X : TQ \oplus T^*Q \to TT^*Q$, an *implicit Lagrangian system* is the triple (L, D_{Δ_Q}, X) that satisfies, for each $(q, v, p) \in TQ \oplus T^*Q$ and with $P = \mathbb{F}L(TQ)$, namely, $(q, p) = (q, \partial L/\partial v)$,

$$(X(q, v, p), \mathbf{d}E(q, v, p)|_{TP}) \in D_{\Delta_Q}(q, p), \tag{3}$$

where $E : TQ \oplus T^*Q \to \mathbb{R}$ is the generalized energy defined by $E(q, v, p) = \langle p, v\rangle - L(q, v)$ and the differential of E is the map $\mathbf{d}E : TQ \oplus T^*Q \to T^*(TQ \oplus T^*Q)$ is given by $\mathbf{d}E = (q, v, p, -\partial L/\partial q, p - \partial L/\partial v, v)$. Since $p = \partial L/\partial v$ holds on P, the restriction $\mathbf{d}E(q, v, p)|_{T_{(q,p)}P} = (-\partial L/\partial q, v)$ is a function on TP, which may be understood in the sense that $T_{(q,p)}P$ is naturally included in $T_{(q,v,p)}(TQ \oplus T^*Q)$.

Local representation. Recall that the partial vector field $X : TQ \oplus T^*Q \to TT^*Q$ is a map that assigns to each point $(q, v, p) \in TQ \oplus T^*Q$, a vector in TT^*Q at the point $(q, p) \in T^*Q$; we write X as

$$X(q, v, p) = (q, p, \dot{q}, \dot{p}),$$

where $\dot{q} = dq/dt$ and $\dot{p} = dp/dt$ are understood to be functions of (q, v, p).

Using the local expression of the canonical two-form Ω_{T^*Q} on T^*Q, namely

$$\Omega_{T^*Q}((q, p, u_1, \alpha_1), (q, p, u_2, \alpha_2)) = \langle \alpha_2, u_1\rangle - \langle \alpha_1, u_2\rangle,$$

one sees that the induced Dirac structure may be locally expressed by

$$D_{\Delta_Q}(q, p) = \{((q, p, \dot{q}, \dot{p}), (q, p, \alpha, w)) \mid \dot{q} \in \Delta(q), \;\; w = \dot{q}, \;\; \text{and} \;\; \alpha + \dot{p} \in \Delta^\circ(q)\},$$

where $\Delta^\circ(q) \subset T^*_qQ$ is the polar of $\Delta(q)$. Employing the local expressions for the canonical symplectic form and the Dirac differential, the condition for an implicit Lagrangian system $(X, \mathbf{d}E|_{TP}) \in D_{\Delta_Q}$ reads

$$\left\langle -\frac{\partial L}{\partial q}, u \right\rangle + \langle \alpha, v\rangle = \left\langle \alpha, \frac{dq}{dt} \right\rangle - \left\langle \frac{dp}{dt}, u \right\rangle$$

for all $u \in \Delta(q)$ and all α, where $p = \partial L/\partial v$ (equality of base points) and (u, α) are the local representatives of a point in $T_{(q,p)}T^*Q$. Then, we obtain the local representation of an implicit Lagrangian system is given by

$$p = \frac{\partial L}{\partial v}, \quad \frac{dq}{dt} = v \in \Delta(q), \quad \frac{dp}{dt} - \frac{\partial L}{\partial q} \in \Delta^\circ(q).$$

Notice that if a partial vector field $X(q, v, p) = (q, p, \dot{q}, \dot{p})$ satisfies the Dirac condition $(X, \mathbf{d}E|_{TP}) \in D_{\Delta_Q}$, then the Legendre transformation $p = \partial L/\partial v$ is consistent with the equality of base points and that the Dirac condition itself gives the second order condition $\dot{q} = v$.

In this paper, we are primarily interested in the reduction theory for the case of the canonical Dirac structure on a cotangent bundle and, given a Lagrangian, with its associated *standard implicit Lagrangian systems*; namely, the case in which no kinematic constraint is imposed, i.e., $\Delta_Q = TQ$, and the general reduction theory of implicit Lagrangian systems with nonholonomic constraints will be developed in another paper.

The standard implicit Lagrangian system can be locally expressed as

$$p = \frac{\partial L}{\partial v}, \quad \frac{dq}{dt} = v, \quad \frac{dp}{dt} = \frac{\partial L}{\partial q},$$

which we shall call the *implicit Euler–Lagrange equations*. Note that the implicit Euler–Lagrange equations include the Euler–Lagrange equations $\dot{p} = \partial L/\partial q$, the Legendre transformation $p = \partial L/\partial v$ and the second–order condition $\dot{q} = v$.

Remark 2. In [87], an implicit Lagrangian system is given by a triple (L, D_{Δ_Q}, X), which satisfies, for each $v \in \Delta_Q(q)$,

$$(X(q, v, p), \mathfrak{D}L(q, v)) \in D_{\Delta_Q}(q, p), \tag{4}$$

where $(q, p) = \mathbb{F}L(q, v)$ for $v \in \Delta_Q(q)$ and $\mathfrak{D}L = \gamma_Q \circ \mathbf{d}L : TQ \to T^*T^*Q$ is the *Dirac differential of the Lagrangian*, where $\gamma_Q : T^*TQ \to T^*T^*Q$; $(q, \delta q, \delta p, p) \mapsto (q, p, -\delta p, \delta q)$ is the natural symplectomorphism originally developed by [81]. Since the Dirac differential of a given Lagrangian L, namely, $\mathfrak{D}L$ takes its value in $P = \mathbb{F}L(\Delta_Q) \subset T^*Q$ such that, for each $(q, v) \in \Delta_Q \subset TQ$ and with the Legendre transformation $(q, p) = (q, \partial L/\partial v)$,

$$\mathfrak{D}L(q, v) = \left(-\frac{\partial L}{\partial q}\right) dq + v\, dp \in T^*P \subset T^*T^*Q,$$

the use of the Dirac differential of L implies that it contains *the generalized Legendre transformation*. It follows from [88] that the Dirac differential of a given Lagrangian L can be also understood in terms of the differential of $E(q, v, p) = \langle p, v\rangle - L(q, v)$ on the Pontryagin bundle $M = TQ \oplus T^*Q$ restricted to TP, namely, $\mathfrak{D}L(q, v) = \mathbf{d}E(q, v, p)|_{TP}$. That is, the condition given in equation (4) is equivalent with the one given in equation (3).

Pullback and pushforward of Dirac structures. Following [16] and [15], we introduce the *pushforward and pullback of Dirac structures*.

Let V and W be vector spaces and let $\mathrm{Dir}(V)$ and $\mathrm{Dir}(W)$ be sets of Dirac structures on V and W respectively. Let $\phi : V \to W$ be a linear map.

Now, the *forward map* $\mathcal{F}\phi : \mathrm{Dir}(V) \to \mathrm{Dir}(W)$ is defined by, for $D_V \in \mathrm{Dir}(V)$,

$$\mathcal{F}\phi(D_V) = \{(\phi(x), \beta) \mid x \in V,\ \beta \in W^*,\ (x, \phi^*\beta) \in D_V\}$$

and the *backward map* $\mathcal{B}\phi : \mathrm{Dir}(W) \to \mathrm{Dir}(V)$ is defined by, for $D_W \in \mathrm{Dir}(W)$,

$$\mathcal{B}\phi(D_W) = \{(x, \phi^*\beta) \mid x \in V,\ \beta \in W^*,\ (\phi(x), \beta) \in D_W\}.$$

Given $D_V \in \mathrm{Dir}(V)$ and $D_W \in \mathrm{Dir}(W)$, a linear map $\phi : V \to W$ is called a *forward Dirac map* if

$$\mathcal{F}\phi(D_V) = D_W.$$

On the other hand, $\phi : V \to W$ is called a *backward Dirac map* if

$$\mathcal{B}\phi(D_W) = D_V.$$

Note that the maps $\mathcal{F}\phi$ and $\mathcal{B}\phi$ are not, in general, inverse to each other and also that the *pushforward of the Dirac structure* D_V sometimes may be denoted as $\phi_* D_V = \mathcal{F}\phi(D_V)$ and the *pullback of the Dirac structure* D_W as $\phi^* D_W = \mathcal{B}\phi(D_W)$.

Dirac structures on the Pontryagin bundle. Let $TQ \oplus T^*Q$ be the Pontryagin bundle over Q. Let $T\varphi : T(TQ \oplus T^*Q) \to TT^*Q$ be the tangent map of the projection $\varphi : TQ \oplus T^*Q \to T^*Q$. Given the induced Dirac structure D_{Δ_Q} on T^*Q, we can define a Dirac structure $D_{TQ\oplus T^*Q}$ on $TQ \oplus T^*Q$ by using the backward Dirac map, namely, $\mathcal{B}T\varphi : \mathrm{Dir}(TT^*Q) \to \mathrm{Dir}(T(TQ \oplus T^*Q))$ as

$$D_{TQ\oplus T^*Q} = \mathcal{B}T\varphi(D_{\Delta_Q}),$$

which is given, for each point $(q, v, p) \in TQ \oplus T^*Q$, by

$$\begin{aligned} D_{TQ\oplus T^*Q}(q, v, p) = \{\, &((\dot{q}, \dot{v}, \dot{p}), T^*\varphi(\alpha, u)) \mid \\ &(\dot{q}, \dot{v}, \dot{p}) \in T_{(q,v,p)}(TQ \oplus T^*Q),\ (\alpha, u) \in T^*_{\varphi(q,v,p)}T^*Q, \\ &(T\varphi(\dot{q}, \dot{v}, \dot{p}), (\alpha, u)) \in D_{\Delta_Q}(\varphi(q, v, p)) \,\}. \end{aligned} \tag{5}$$

In the above, we note that the Dirac structure D_{Δ_Q} on T^*Q can be regarded as a linear Dirac structure on the vector space $T_{p_q}T^*Q$ at each $p_q \in T^*Q$.

Recall that an implicit Lagrangian system is given by a triple (L, D_{Δ_Q}, X), which satisfies the condition, for each $(q, v, p) \in TQ \oplus T^*Q$,

$$(X(q, v, p), \mathbf{d}E(q, v, p)|_{TP}) \in D_{\Delta_Q}(\varphi(q, v, p)),$$

where $E(q, v, p) = \langle p, v\rangle - L(q, v)$ is the generalized energy on $TQ \oplus T^*Q$. Notice that the partial vector field $X : TQ \oplus T^*Q \to TT^*Q$ has the property

$$X(q, v, p) = T\varphi(\dot{q}, \dot{v}, \dot{p}) = (q, p, \dot{q}, \dot{p}) \in TT^*Q,$$

while

$$\mathbf{d}E(q, v, p)|_{TP} = \left(-\frac{\partial L}{\partial q}\right) dq + v\, dp \in T^*P \subset T^*T^*Q,$$

where $p = \partial L/\partial v$ holds on $P = \mathbb{F}L(\Delta_Q)$. Hence, in view of equation (5), it follows that, for each point $(\dot{q}, \dot{v}, \dot{p}) \in T_{(q,v,p)}(TQ \oplus T^*Q)$ and with $p = \partial L/\partial v$,

$$((\dot{q}, \dot{v}, \dot{p}), T^*\varphi\, \mathbf{d}E(q, v, p)|_{TP}) \in D_{TQ\oplus T^*Q}(q, v, p)$$

and thus

$$((\dot{q}, \dot{v}, \dot{p}), \mathbf{d}E(q, v, p)) \in D_{TQ\oplus T^*Q}(q, v, p).$$

Remark 3. In the above formulation, the Dirac structure $D_{TQ\oplus T^*Q} \subset T(TQ \oplus T^*Q) \oplus T^*(TQ \oplus T^*Q)$ on the Pontryagin bundle $TQ \oplus T^*Q$ can be also defined by using a presymplectic form $\Omega_{TQ\oplus T^*Q}$ on $TQ \oplus T^*Q$ that is defined by pulling back the canonical symplectic two-form from T^*Q to $TQ \oplus T^*Q$ as $\Omega_{TQ\oplus T^*Q} = \varphi^*\,\Omega_{T^*Q}$. In this context, it is worth noting that our approach to implicit Lagrangian systems is closely related with the approach developed in [78] for the study of degenerate Lagrangian systems. It is also noteworthy that [30] developed the geometric approach to vakonomic and nonholonomic mechanics in the general context of [78]. It will be shown that the Dirac structure on the Pontryagin bundle and implicit Lagrangian systems can be regarded in the general context of Dirac anchored vector bundles and the associated Dirac dynamical systems in [26].

The Hamilton-Pontryagin principle. We next show how the implicit Euler–Lagrange equations can be also obtained from the Hamilton-Pontryagin principle. Define the action integral on the space of curves $(q(t), v(t), p(t))$, $t \in [t_0, t_1]$ in $TQ \oplus T^*Q$ by

$$\mathfrak{F}(q(t), v(t), p(t)) = \int_{t_0}^{t_1} \{L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t)\rangle\}\, dt,$$

where $\dot{q}(t) = dq(t)/dt$. The *Hamilton-Pontryagin* principle is the condition of stationarity of $\mathfrak{F}$: $\delta\mathfrak{F} = 0$. It follows that

$$\begin{aligned}\delta\mathfrak{F}(q, v, p) &= \int_{t_0}^{t_1} \left\{\langle \delta p, \dot{q} - v\rangle + \left\langle -\dot{p} + \frac{\partial L}{\partial q}, \delta q\right\rangle + \left\langle -p + \frac{\partial L}{\partial v}, \delta v\right\rangle\right\} dt + \langle p, \delta q\rangle\Big|_{t_0}^{t_1} \\ &= 0,\end{aligned}$$

which is to be satisfied for all $\delta q, \delta v$ and δp with appropriate boundary conditions. Keeping the endpoints $q(t_0)$ and $q(t_1)$ of $q(t)$ fixed, we obtain the *implicit Euler–Lagrange equations*.

Remark 4. Using the generalized energy $E(q, v, p) = \langle p, v\rangle - L(q, v)$, it is clear that the Hamilton-Pontryagin principle can be also represented by the condition of stationarity of the action integral

$$\mathfrak{F}(q(t), v(t), p(t)) = \int_{t_0}^{t_1} \{\langle p(t), \dot{q}(t)\rangle - E(q(t), v(t), p(t))\}\, dt.$$

3. **The bundle pictures of Pontryagin bundles.** This section develops the fundamental ingredients that will be employed in this paper to study the bundle pictures associated with the reduced Pontryagin bundle $(TQ \oplus T^*Q)/G$. Key amongst these is an isomorphism (to be defined) $\widetilde{\Psi}_A = \Psi_A \oplus (\Psi_A^{-1})^* : (TQ \oplus T^*Q)/G \to T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$, where $\widetilde{V} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^*$ is the associated bundle to $V = \mathfrak{g} \oplus \mathfrak{g}^*$. Using this and building on the ideas in the theory of *Lagrangian reduction* ([64, 24]) as well as of the *Hamiltonian reduction* ([23, 58]), we will develop "*Dirac cotangent bundle reduction*", together with reduction of implicit Lagrangian systems. Specifically, we note that the isomorphism $\Psi_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ is used in the Lagrangian reduction by stages, while the isomorphism $(\Psi_A^{-1})^* : T^*Q/G \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ is used in Hamiltonian reduction.

Principal bundle and principal connection. Let G be a Lie group acting freely and properly on a manifold Q and $\pi : Q \to Q/G$ be the corresponding principal bundle. Let $\Phi : G \times Q \to Q$ be the left action: for $g \in G$ and $q \in Q$,

$$\Phi(g, q) \equiv \Phi_g(q) \equiv L_g q \equiv g \cdot q \equiv gq.$$

In this paper, the concatenation notation gq will be most commonly used. Further, the tangent and cotangent lift of this action will be denoted $gv_q \equiv T_q\Phi_g \cdot v_q$ and $g\alpha_q \equiv T^*_{gq}\Phi_{g^{-1}} \cdot \alpha_q$, where $v_q \in T_qQ$ and $\alpha_q \in T^*_qQ$. We will also use the notations

$\pi(q) = [q]_G = [q]$ for the equivalence class of $q \in Q$. A *principal connection* A on Q is a Lie algebra valued one form $A : TQ \to \mathfrak{g}$ with the properties:

(i) $A(\xi q) = \xi$ for all $\xi \in \mathfrak{g}$; namely, A takes infinitesimal generators of a given Lie algebra element to that element, and

(ii) A is equivariant; that is, $A(T_q\Phi_g \cdot v_q) = \mathrm{Ad}_g(A(v_q))$, where Ad_g denotes the adjoint action of G on $\mathfrak{g}$.

In the above, $\xi q \equiv \xi_Q(q)$ and

$$\xi q = \left.\frac{d}{dt}\right|_{t=0} (\exp t\xi) q.$$

The restriction of A to the tangent space T_qQ is denoted by $A_q \equiv A(q) : T_qQ \to \mathfrak{g}$ and the horizontal space of the connection defined at $q \in Q$ is given by $\mathrm{Hor}_q = \mathrm{Ker}\, A_q$, namely,

$$\mathrm{Hor}_q = \{v_q \in T_qQ \mid A_q(v_q) = 0\}.$$

On the other hand, the vertical space at $q \in Q$ is

$$\mathrm{Ver}_q = \mathrm{Ker}\ T_q\pi.$$

Then, at any point $q \in Q$, the tangent space T_qQ can be decomposed as

$$T_qQ = \mathrm{Hor}_q \oplus \mathrm{Ver}_q.$$

Hence the vertical and horizontal components of a vector $v_q \in T_qQ$ may be given as

$$\mathrm{Ver}_q(v_q) = A_q(v_q)q \quad \text{and} \quad \mathrm{Hor}_q(v_q) = v_q - A_q(v_q)q,$$

which induces a decomposition $TQ = \mathrm{Hor}(TQ) \oplus \mathrm{Ver}(TQ)$, where $\mathrm{Hor}(TQ) = \cup_{q \in Q}\mathrm{Hor}_q$ and $\mathrm{Ver}(TQ) = \cup_{q \in Q}\mathrm{Ver}_q$ respectively denote the horizontal and vertical subbundle of TQ, which are invariant under the action of G. A vector v_q is called horizontal if its vertical component is zero, namely, $A(v_q) = 0$, while it is called vertical if its horizontal component is zero, i.e., $T_q\pi(v_q) = 0$.

Horizontal lifts. The projection map $T\pi : TQ \to T(Q/G)$ defines, at each point $q \in Q$, an isomorphism from the horizontal space Hor_q to the tangent space to the base $T_{\pi(q)}(Q/G)$ as

$$T_q\pi|\mathrm{Hor}_q : \mathrm{Hor}_q \to T_{\pi(q)}(Q/G)$$

and its inverse is called the *horizontal lift*. Hence, the horizontal lift of a tangent vector $v_x \in T_x(Q/G)$ at $q = \pi^{-1}(x) \in Q$ is given by

$$v_q^h = (T_q\pi|\mathrm{Hor}_q)^{-1}(v_x).$$

Given any vector field X on Q/G, there is a unique vector field X^h on Q that is horizontal and that is π-related to X such that at each point q,

$$T_q\pi \cdot X^h(q) = X(\pi(q)),$$

where the vertical part is zero as

$$A(X_q^h)q = 0.$$

The above relation of being π-related induces bracket-preserving as

$$\mathrm{Hor}[X^h, Y^h] = [X, Y]^h,$$

where X and Y are vector fields on Q/G.

For any curve $x(t)$ in Q/G, where $t \in [t_0, t_1]$, the family of horizontal lifts is denoted x^h. The definition is given as follows. For any point $q_0 \in \pi^{-1}(x_0)$, where $x_0 = x(\tau_0)$, for some $\tau_0 \in [t_0, t_1]$, the horizontal lift of $x(t)$, which at $t = \tau_0$ coincides with q_0, is uniquely determined by requiring its tangent to be a horizontal vector. We shall denote this curve by $x^h_{q_0}$, and is defined on $[t_0, t_1]$.

Let $q(t)$, $t \in [t_0, t_1]$ be a curve in Q, and choose $\tau_0 \in [t_0, t_1]$. Then, there is a unique horizontal curve $q_h(t)$ such that $q_h(\tau_0) = q(\tau_0)$ and $\pi(q_h(t)) = \pi(q(t))$ for all $t \in [t_0, t_1]$. Therefore, we can define a curve $g_q(t)$, $t \in [t_0, t_1]$ in G by the decomposition

$$q(t) = g_q(t) q_h(t) \tag{6}$$

for all $t \in [t_0, t_1]$. Note that $g_q(\tau_0)$ is the identity and also that if $x(t) = \pi(q(t))$ and $q_0 = q(\tau_0)$ then $q_h(t) = x^h_{q_0}(t)$.

The time derivative of equation (6) with respect to t induces

$$\dot{q}(t) = \dot{g}_q(t) q_h(t) + g_q(t) \dot{q}_h(t).$$

In the above, for $u_g \in T_g G$ and $q \in Q$, $u_g q$ means the derivative of the orbit map $g \mapsto gq$ in the direction of u_g to give an element of $T_{gq} Q$.

It follows from definition of a horizontal vector that $A(g_q(t)\dot{q}_h(t)) = 0$. Recall that $A(\xi q) = \xi$ for $\xi \in \mathfrak{g}$ and $q \in Q$, and we obtain, for $\xi = \dot{g}_q g_q^{-1}$ and $q(t) = g_q(t) q_h(t)$,

$$A(\dot{g}_q(t) q_h(t)) = A(\dot{g}_q(t) g_q^{-1}(t) g_q(t) q_h(t)) = \dot{g}_q g_q^{-1}.$$

Thus, one can check that for any curve $q(t), t \in [t_0, t_1]$ in Q,

$$A(q, \dot{q}) = \dot{g}_q g_q^{-1}.$$

Curvature and Cartan structure equations. The covariant exterior derivative $\mathbf{D}$ of a Lie algebra-valued one-form α is defined by applying the usual exterior derivative $\mathbf{d}$ to the horizontal parts of vectors $\mathbf{D}\alpha(X, Y) = \mathbf{d}\alpha(\operatorname{Hor} X, \operatorname{Hor} Y)$ for two vector fields X, Y on Q. The curvature B of a connection A is its covariant exterior derivative—the Lie algebra valued two-form given by $B(X, Y) = \mathbf{d}A(\operatorname{Hor} X, \operatorname{Hor} Y)$. Using the identity $(\mathbf{d}\alpha)(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y])$, where the bracket denotes the Jacobi-Lie bracket of vector fields, it follows that

$$B(X, Y) = -A([\operatorname{Hor} X, \operatorname{Hor} Y]),$$

since $A(\operatorname{Hor} X) = 0$ and $A(\operatorname{Hor} Y) = 0$. *Cartan's structure equations* state that, for vector fields X, Y (not necessarily horizontal),

$$B(X, Y) = \mathbf{d}A(X, Y) - [A(X), A(Y)],$$

where the bracket on the right hand side is the Lie bracket in $\mathfrak{g}$.

Associated bundles. Let $\Phi : G \times M \to M$ be a left action of the Lie group G on a vector space M. The *associated vector bundle* with standard fiber M is by definition,

$$Q \times_G M = (Q \times M)/G,$$

where the action of G on $Q \times M$ is given by $g(q, m) = (gq, gm)$. The class of (q, m) is denoted $[q, m]_G$ or simply $[q, m]$. The projection $\pi_M : Q \times_G M \to Q/G$ is defined by $\pi_M([q, m]_G) = \pi(q)$ and it is easy to check that it is well defined and is a surjective submersion.

Parallel transport in associated bundles. Let $[q_0, m_0]_G \in Q \times_G M$ and let $x_0 = \pi(q_0) \in Q/G$. Let $x(t), t \in [t_0, t_1]$ be a curve in Q/G and let $\tau_0 \in [t_0, t_1]$ be such that $x(\tau_0) = x_0$. The *parallel transport* of $[q_0, m_0]_G$ along the curve $x(t)$ is defined to be the curve

$$[q, m]_G(t) = [x^h_{q_0}(t), m_0]_G.$$

For $t, t+s \in [t_0, t_1]$, we adopt the notation $\tau^t_{t+s} : \pi_M^{-1}(x(t)) \to \pi_M^{-1}(x(t+s))$ for the parallel transport map along the curve $x(s)$ of any point

$$[q(t), m(t)]_G \in \pi_M^{-1}(x(t))$$

to the corresponding point

$$\tau^t_{t+s}[q(t), m(t)]_G \in \pi_M^{-1}(x(t+s)).$$

Thus,

$$\tau^t_{t+s}[q(t), m(t)]_G = [x^h_{q(t)}(t+s), m(t)]_G.$$

We shall sometimes employ the notation $\Phi'(\xi)$ for the second component of the infinitesimal generator of an element $\xi \in \mathfrak{g}$, namely, $\xi m = (m, \Phi'(\xi)m)$, where we utilize the identification $TM = M \times M$ for a vector space M. Then, the infinitesimal generator may be thought of as a map $\Phi' : \mathfrak{g} \to \mathrm{End}(M)$ (the linear vector fields on M are identified with the space of linear maps of M to itself). Thus, we have a linear representation of $\mathfrak{g}$ on M.

Let $[q(t), m(t)]_G$, $t \in [t_0, t_1]$ be a curve in $Q \times_G M$ given by

$$x(t) = \pi_M([q(t), m(t)]_G) = \pi(q(t)),$$

and let, as above, τ^t_{t+s}, where $t, t+s \in [t_0, t_1]$, denote the parallel transport along $x(t)$ from t to $t+s$.

The covariant derivative in associated bundles. The *covariant derivative* of $[q(t), m(t)]_G$ along $x(t)$ is defined by

$$\frac{D[q(t), m(t)]_G}{Dt} = \lim_{s \to 0} \frac{\tau^{t+s}_t([q(t+s), m(t+s)]_G) - [q(t), m(t)]_G}{s} \in \pi_M^{-1}(x(t)).$$

Note that if $[q(t), m(t)]_G$ is a vertical curve, then its base point is constant; that is, for each $t \in [t_0, t_1]$,

$$x(t+s) = \pi_M([q(t+s), m(t+s)]_G) = x(t),$$

so that $x^h_{q(t)}(t+s) = q(t)$ for all s. Therefore,

$$\tau^{t+s}_t([q(t+s), m(t+s)]_G) = [x^h_{q(t+s)}(t), m(t+s)]_G = [q(t), m(t+s)]_G$$

and then we obtain the well-known fact that the *covariant derivative of a vertical curve in the associated bundle is just the fiber derivative*. Namely,

$$\frac{D[q(t), m(t)]_G}{Dt} = [q(t), m'(t)]_G,$$

where $m'(t)$ is the time derivative of m.

Affine connections on vector bundles. Recall from [47] and [2] that an affine connection ∇ on a vector bundle $\tau : V \to Q$ is a map $\nabla : \mathfrak{X}^\infty(Q) \times \Gamma(V) \to \Gamma(V)$, say $(X, v) \mapsto \nabla_X v$, having the following properties:

1. $\nabla_{fX+gY} v = f\nabla_X v + g\nabla_Y v$,
2. $\nabla_X(v + w) = \nabla_X v + \nabla_X w$, and
3. $\nabla_X(fv) = f\nabla_X v + X[f]v$ for all $X \in \mathfrak{X}^\infty(Q)$ (the space of smooth vector fields on Q), $f, g \in C^\infty(Q)$ (the space of smooth real valued functions on Q), and $v, w \in \Gamma(V)$ (the space of smooth sections of the vector bundle $\tau : V \to Q$),

where $X[f] = \mathbf{d}f \cdot X$ denotes the derivative of f in the direction of the vector field X. Now, the *parallel transport* of a vector $v_q \in \tau^{-1}(q_0)$ along a curve $q(t)$, $t \in [t_0, t_1]$ in Q such that $q(\tau_0) = q_0$ for a fixed $\tau_0 \in [t_0, t_1]$ is the unique vector $v(t)$ such that $v(t) \in \tau^{-1}(q(t))$ for all t, $v(\tau_0) = v_0$, which satisfies $\nabla_{\dot{q}(t)} v(t) = 0$ for all t. The operation of parallel transport is a map given by, for $t, s \in [t_0, t_1]$,

$$T^t_{t+s} : \tau^{-1}(q(t)) \to \tau^{-1}(q(t + s))$$

associated to each curve $q(t) \in Q$. Then, we can define the *covariant derivative* of curves $v(t) \in V$ by

$$\frac{Dv(t)}{Dt} = \frac{d}{ds} T^{t+s}_t v(t + s)\bigg|_{s=0}.$$

Then, the covariant derivative of curves in a vector bundle $\tau : V \to Q$ is related to the affine connection ∇ as

$$\nabla_X v(q_0) = \frac{Dv(t)}{Dt}\bigg|_{t=t_0},$$

where, for each $q_0 \in Q$, each $X \in \mathfrak{X}^\infty(Q)$, and each $v \in \Gamma(V)$, we have, by definition, that $q(t)$ is any curve in Q such that $\dot{q}(\tau_0) = X(q_0)$ and $v(t) = v(q(t))$ for all t. This property establishes, in particular, the uniqueness of the connection associated to the covariant derivative D/Dt.

Affine connections on associated bundles. We have the following formula that gives the relation between the covariant derivative of the affine connection and the principal connection:

$$\frac{D[q(t), m(t)]_G}{Dt} = [q(t), -\Phi'(A(q(t), \dot{q}(t)))\, m(t) + \dot{m}(t))]_G\,.$$

The previous definition of the covariant derivative of a curve in the associated vector bundle $Q \times_G M$ thus leads to an affine connection on $Q \times_G M$, which we shall call $\tilde{\nabla}^A$ or simply $\tilde{\nabla}$. Let $\varphi : Q/G \to Q \times_G M$ be a section of the associated bundle and let $X(x) \in T_x(Q/G)$ be a given vector tangent to Q/G at x. Let $x(t)$ be a curve in Q/G such that $\dot{x}(\tau_0) = X(x)$; thus, $\varphi(x(t))$ is a curve in $Q \times_G M$. The covariant derivative of the section φ with respect to X at x is, by definition,

$$\tilde{\nabla}^A_{X(x)} \varphi = \frac{D\varphi(x(t))}{Dt}\bigg|_{t=\tau_0}.$$

The notion of a *horizontal curve* $[q(t), m(t)]_G$ on $Q \times_G M$ is defined by the condition that its covariant derivative vanishes. A vector tangent to $Q \times_G M$ is called horizontal if it is tangent to a horizontal curve. Correspondingly, the *horizontal space* at a point $[q, m]_G \in Q \times_G M$ is the space of all horizontal vectors at $[q, m]_G$.

The adjoint bundle. One of the most interesting cases in this paper is the one which is given as $M = \mathfrak{g}$ and $\Phi_g = \mathrm{Ad}_g$.

The associated bundle with standard fiber $\mathfrak{g}$, where the action of G on $\mathfrak{g}$ is the adjoint action, is called the *adjoint bundle* $\tilde{\mathfrak{g}}$, which is defined by $\tilde{\mathfrak{g}} := (Q \times \mathfrak{g})/G$. Let $\tilde{\pi}_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \to Q/G$ be the projection given by $\tilde{\pi}_{\tilde{\mathfrak{g}}}([q, \xi]_G) = [q]$.

Let $[q(s), \eta(s)]_G$ be any curve in $\tilde{\mathfrak{g}}$. Then, noting that $\Phi = \mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$, one can easily check

$$\frac{D[q(s), \eta(s)]_G}{Ds} = [q(s), -[\xi(s),\, \eta(s)] + \dot{\eta}(s))]_G\,, \tag{7}$$

where $\xi(s) = A(q(s), \dot{q}(s))$ and we utilized the fact $\Phi'(\xi) = \mathrm{ad}_\xi$. For the derivation of equation (7), see Lemma 2.3.2 and Lemma 2.3.4 in [24].

The adjoint bundle is a *Lie algebra bundle*, namely, each fiber $\tilde{\mathfrak{g}}_x$ of $\tilde{\mathfrak{g}}$ carries a natural Lie algebra structure defined by

$$[[q, \xi]_G, [q, \eta]_G] = [q, [\xi, \eta]]_G,$$

since

$$\begin{aligned} [[gq, \mathrm{Ad}_g\xi]_G, [gq, \mathrm{Ad}_g\eta]_G] &= [gq, [\mathrm{Ad}_g\xi, \mathrm{Ad}_g\eta]]_G = [gq, \mathrm{Ad}_g[\xi, \eta]]_G \\ &= [q, [\xi, \eta]]_G = [[q, \xi]_G, [q, \eta]_G]. \end{aligned}$$

The bundle isomorphism between TQ/G and $T(Q/G) \oplus \tilde{\mathfrak{g}}$. Let $\Phi : G \times Q \to Q$ be a free and proper action of G on Q, as before, so that there is a principal bundle $\pi : Q \to Q/G$. The tangent lift of this action of G on Q defines an action of G on TQ and so we can form the quotient $(TQ)/G =: TQ/G$. Let $\tau_Q : TQ \to Q$ be the tangent bundle projection and there is a well defined map $\tau_Q/G : TQ/G \to Q/G$ induced by the tangent of the projection map $\pi : Q \to Q/G$ and given by $[v_q]_G \to [q]_G$. The vector bundle structure of TQ is inherited by this bundle. One can express reduced variational principles in a natural way in terms of this bundle without any reference to a connection on Q. However, it is also interesting to introduce an arbitrary chosen connection on Q relative to which one can concretely realize the space TQ/G, which is one of the main tools in this paper.

Let A be a principal connection on Q, which defines a bundle isomorphism:

$$\Psi_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$$

defined by

$$\Psi_A([v_q]_G) := (T\pi(v_q), [q, A(v_q)]_G)\,, \tag{8}$$

where $v_q \in T_qQ$ and $[v_q]_G$ indicates the equivalent class of the quotient TQ/G. We can easily check that the bundle isomorphism is well defined, since $T_{gq}\pi(gv_q) = T_q\pi(v_q)$ and

$$[gq, A(gq, gv_q)]_G = [gq, \mathrm{Ad}_g A(q, v_q)] = [q, A(q, v_q)]_G.$$

Then, one has

$$\Psi_A([gq, gv_q]_G) = \Psi_A([q, v_q])_G.$$

In the above, we employ the *concatenation notation* gv_q for the tangent lift action of the group element $g \in G$ on TQ. The inverse map $\Psi_A^{-1} : T(Q/G) \oplus \tilde{\mathfrak{g}} \to TQ/G$ is given by

$$\Psi_A^{-1}(u_{[q]}, [q, \eta]_G) = [(u_{[q]})^h_q + \eta q]_G,$$

where $[q]=\pi(q)\in Q/G$ and $(u_{[q]})^h_q$ is the horizontal lift of $u_{[q]}$ at the point $q\in Q$. The map $\Psi_A^{-1}:T(Q/G)\oplus\tilde{\mathfrak{g}}\to TQ/G$ is well defined, noting that $(u_{[q]})^h_{gq}=g(u_{[q]})^h_q$ and that $(\mathrm{Ad}_g\eta)gq=g\eta q$, as

$$\Psi_A^{-1}(u_{[q]},[gq,\mathrm{Ad}_g\eta]_G)=\Psi_A^{-1}(u_{[q]},[q,\eta]_G).$$

Thus, we have the isomorphism $TQ/G\cong T(Q/G)\oplus\tilde{\mathfrak{g}}$.

Remark 5. The quotient bundle $TQ/G\to Q/G$ is known as the *Atiyah quotient bundle* (see, [54]). In [24], it was shown that the *Lagrange-Poincaré equations* can be formulated in the context of the bundle $T(Q/G)\oplus\tilde{\mathfrak{g}}$, which falls into the category of the *Lagrange-Poincaré bundles*.

Associated one-forms. The principal connection A is a Lie algebra valued one-form, which is denoted by the linear map $A(q):T_qQ\to\mathfrak{g}$ for each $q\in Q$. Then, we can define its dual map $A(q)^*:\mathfrak{g}^*\to T_q^*Q$, and evaluating $A(q)^*$ at $\mu\in\mathfrak{g}^*$ induces an ordinary one-form

$$\alpha_\mu(q)=A(q)^*(\mu). \tag{9}$$

Now, G acts on T^*Q by the cotangent lift of the action $\Phi:G\times Q\to Q$ and this lifted action is symplectic with respect to the canonical symplectic form on T^*Q. Recall that the equivariant momentum map $\mathbf{J}:T^*Q\to\mathfrak{g}^*$ is defined by $\langle\mathbf{J}(\alpha_\mu(q)),\xi\rangle=\langle\alpha_\mu(q),\xi_Q(q)\rangle$, where $\alpha_\mu(q)\in T_q^*Q$ and $\xi\in\mathfrak{g}$. Since $\mathbf{J}$ is equivariant, it follows that $\mathbf{J}(g\alpha_q)=\mathrm{Ad}^*_{g^{-1}}\mathbf{J}(\alpha_q)$.

Then, one can check that for any connection A and $\mu\in\mathfrak{g}^*$, the associated one-form α_μ defined by equation (9) has the following two properties:

1. α_μ takes values in $\mathbf{J}^{-1}(\mu)$,
2. α_μ is G-equivariant, namely, $\Phi_g^*\alpha_\mu=\alpha_{\mathrm{Ad}_g^*\mu}$.

Remark 6. It is easy to check those properties as follows (see [58]): As to the first property, one has

$$\begin{aligned}\langle\mathbf{J}(\alpha_\mu(q)),\xi\rangle&=\langle\alpha_\mu(q),\xi_Q(q)\rangle=\langle A(q)^*(\mu),\xi_Q(q)\rangle\\&=\langle\mu,A(q)(\xi_Q(q))\rangle=\langle\mu,\xi\rangle,\end{aligned}$$

which satisfies for arbitrary ξ, and hence $\mathbf{J}(\alpha_\mu(q))=\mu$. Thus, α_μ takes values in $\mathbf{J}^{-1}(\mu)$. As to the second, letting $v\in T_qQ$ and $g\in G$, by employing the definition of α_μ and the definition of the adjoint, it follows that

$$\begin{aligned}(\Phi_g^*\alpha_\mu)(v)&=\alpha_\mu(gq)(T_q\Phi_g(v))=\langle A(gq)^*(\mu),T_q\Phi_g(v)\rangle\\&=\langle\mu,A(gq)(T_q\Phi_g(v))\rangle.\end{aligned}$$

Make use of equivariance of A, namely, $A(gq)(T_q\Phi_g(v))=\mathrm{Ad}_g(A(q)(v))$ and covert the preceding expression back to one involving α_μ to get

$$\begin{aligned}(\Phi_g^*\alpha_\mu)(v)&=\langle\mu,\mathrm{Ad}_g(A(q)(v))\rangle=\left\langle\mathrm{Ad}_g^*(\mu),A(q)(v)\right\rangle\\&=\left\langle A(q)^*\mathrm{Ad}_g^*(\mu),v\right\rangle=\alpha_{\mathrm{Ad}_g^*\mu}(q)(v).\end{aligned}$$

Thus, we obtain the equivariance property of α_μ.

The bundle isomorphism between T^*Q/G and $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$. Let us consider the inverse of the fiberwise dual of Ψ_A given in equation (8), i.e., a bundle isomorphism

$$(\Psi_A^{-1})^* : T^*Q/G \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*.$$

We can compute this bundle isomorphism as follows (see also [58]):

$$\begin{aligned}\left\langle (\Psi_A^{-1})^*([\alpha_q]_G), (u_{[q]}, [q,\xi]_G) \right\rangle &= \left\langle [\alpha_q]_G, [(u_{[q]})^h_q + \xi_Q(q)]_G \right\rangle \\ &= \left\langle \alpha_q, (u_{[q]})^h_q \right\rangle + \left\langle \alpha_q, \xi_Q(q) \right\rangle \\ &= \left\langle (\alpha_q)^{h^*}_q, u_{[q]} \right\rangle + \left\langle \mathbf{J}(\alpha_q), \xi \right\rangle .\end{aligned}$$

The bundle isomorphism $(\Psi_A^{-1})^* : T^*Q/G \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ is given by

$$(\Psi_A^{-1})^*([\alpha_q]_G) = \left((\alpha_q)^{h^*}_q , [q, \mathbf{J}(\alpha_q)]_G \right). \tag{10}$$

In the above, $[\alpha_q]_G$ is the equivalence class of the quotient T^*Q/G, the map $(\cdot)^{h^*}_q : T^*_qQ \to T^*_{[q]}(Q/G)$ is the dual of the horizontal lift map $(\cdot)^h_q : T_{[q]}(Q/G) \to T_qQ$, and $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is the momentum map of the lifted action,

$$\left\langle \mathbf{J}(\alpha_q), \xi \right\rangle = \left\langle \alpha_q, \xi_Q(q) \right\rangle ,$$

for $\xi \in \mathfrak{g}$. The space $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ is called the *Weinstein space* (see, [76]).

The reduced Pontryagin bundle $(TQ \oplus T^*Q)/G$. As was shown, both TQ/G and T^*Q/G are bundles over Q/G, and the Pontryagin bundle $TQ \oplus T^*Q \cong TQ \times_Q T^*Q$ is the Whitney bundle over Q. The quotient space of the Pontryagin bundle $TQ \oplus T^*Q$ by G is a bundle over Q/G as

$$(TQ \oplus T^*Q)/G \cong (TQ \times_Q T^*Q)/G \cong (TQ)/G \times_{Q/G} (T^*Q)/G.$$

Using the isomorphisms $\Psi_A : (TQ)/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ and $(\Psi_A^{-1})^* : (T^*Q)/G \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ that are respectively given in equations (8) and (10), we can construct a bundle isomorphism in a natural way:

$$\widetilde{\Psi}_A = \Psi_A \oplus (\Psi_A^{-1})^* : (TQ \oplus T^*Q)/G \to T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V},$$

which is given by, for each $(v_q, \alpha_q) \in TQ \oplus T^*Q$,

$$\begin{aligned}\widetilde{\Psi}_A([v_q]_G, [\alpha_q]_G) &= \left(\Psi_A([v_q]_G), (\Psi_A^{-1})^*([\alpha_q]_G) \right) \\ &= \left(T_q\pi(v_q), (\alpha_q)^{h^*}_q, [q, A(v_q)]_G, [q, \mathbf{J}(\alpha_q)]_G \right).\end{aligned}$$

In the above,

$$\tilde{V} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^* = (Q \times V)/G$$

is the associated bundle to $V = \mathfrak{g} \oplus \mathfrak{g}^*$, and we identify $(\widetilde{V}^*)^* = \tilde{\mathfrak{g}}^* \oplus \tilde{\mathfrak{g}}$ with $\tilde{V}$ and $(\tilde{\mathfrak{g}}^*)^*$ with $\tilde{\mathfrak{g}}$.

4. **Reduction of the Hamilton-Pontryagin principle.** This section develops reduction of the Hamilton-Pontryagin principle for the case of a principal bundle $\pi : Q \to Q/G$ (so the Lie group G acts freely and properly on the configuration manifold Q). We begin with the geometry of variations of curves in the Pontryagin bundle $TQ \oplus T^*Q$. Following this, it will be shown that arbitrary variations of curves in Q can be decomposed into vertical and horizontal components, which eventually yield two reduced equations of motion, namely, *horizontal implicit*

Lagrange-Poincaré equations corresponding to horizontal variations and *vertical implicit Lagrange-Poincaré equations* corresponding to vertical variations.

Curves in the Pontryagin bundle. Curves in the Pontryagin bundle $TQ \oplus T^*Q$ will be written as $(q(t), v(t), p(t)),\ t \in [t_0, t_1]$. Note that, in general, $v(t) \in T_{q(t)}Q$ is not necessarily equal to the tangent vector $\dot{q}(t)$ to $q(t)$, where $\dot{q}(t)$ denotes the time derivative of $q(t)$, namely, $\dot{q}(t) \equiv dq(t)/dt$. Here, let us first consider the case in which one has the restriction $v(t) = \dot{q}$; namely, the curves $(q(t), \dot{q}(t), p(t)),\ t \in [t_0, t_1]$ in $TQ \oplus T^*Q$.

The action of an element $h \in G$ on a curve $(q(t), \dot{q}(t), p(t)) \in TQ \oplus T^*Q$ is given by

$$h \cdot (q(t), \dot{q}(t), p(t)) = (hq(t), T_{q(t)}L_h \dot{q}(t), T^*_{hq(t)}L_{h^{-1}}p(t)),$$

where $T_{q(t)}L_h : T_{q(t)}Q \to T_{hq(t)}Q$ is the tangent of the left translation map $L_h : Q \to Q;\ q(t) \mapsto hq(t)$ at the point $q(t)$ and $T^*_{hq(t)}L_{h^{-1}} : T^*_{q(t)}Q \to T^*_{hq(t)}Q$ is the dual of the map $T_{hq(t)}L_{h^{-1}} : T_{hq(t)}Q \to T_{q(t)}Q$.

Let $A : TQ \to \mathfrak{g}$ be a principal connection on $\pi : Q \to Q/G$, which induces the isomorphism

$$\widetilde{\Psi}_A = \Psi_A \oplus (\Psi_A^{-1})^* : (TQ \oplus T^*Q)/G \to T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V},$$

which is given, for each $(q, \dot{q}, p) \in TQ \oplus T^*Q$, by

$$\widetilde{\Psi}_A([q(t), \dot{q}(t), p(t)]_G) = \left(x(t), \dot{x}(t), y(t), \bar{\xi}(t), \bar{\mu}(t)\right),$$

where $x = [q] \in Q/G$, $\dot{x} = T_q\pi(\dot{q}) \in T_x(Q/G)$, $y = (p_q)^{h^*}_q \in T^*_x(Q/G)$, $\bar{\xi} = [q, \xi]_G = [q, A(q, \dot{q})]_G \in \tilde{\mathfrak{g}}$ and $\bar{\mu} = [q, \mu]_G = [q, \mathbf{J}(q, p)]_G \in \tilde{\mathfrak{g}}^*$.

Let $(x_0, \dot{x}_0, \bar{\xi}_0)$ be a given element of $T(Q/G) \oplus \tilde{\mathfrak{g}}$. For any curve $x(s)$ on Q/G, let $(x(s), u(s))$ be the horizontal lift of $x(s)$ with respect to the connection ∇ such that $(x(0), u(0)) = (x_0, \dot{x}_0)$. Notice that $(x(s), u(s))$ is *not* the tangent vector $(x(s), \dot{x}(s))$ to $x(s)$ in general. Let $(x(s), \bar{\xi}(s))$ be the horizontal lift of $x(s)$ with respect to the connection $\tilde{\nabla}^A$ on $\tilde{\mathfrak{g}}$ such that $(x(0), \bar{\xi}(0)) = (x_0, \bar{\xi}_0)$. Thus, $(x(s), u(s), \bar{\xi}(s))$ is a horizontal curve with respect to the connection $C = \nabla \oplus \tilde{\nabla}^A$ naturally defined on $T(Q/G) \oplus \tilde{\mathfrak{g}}$ in terms of the connection ∇ on $T(Q/G)$ and the connection $\tilde{\nabla}^A$ on $\tilde{\mathfrak{g}}$.

Spaces of curves and deformations of curves. Let us denote the space of all smooth curves from a fixed interval $[t_0, t_1]$ to Q by $\Omega(Q)$. For given $q_i \in Q,\ i = 0, 1$, we denote by $\Omega(Q;\, q_0, q_1)$ the space of curves $q(t)$ on Q such that $q(t_i) = q_i$. We denote by $\Omega(Q; x_0, x_1)$ the space of all curves in $\Omega(Q)$ such that $\pi(q(t_i)) = x_i$ for $q_i \in Q$.

Recall that a *deformation* of a curve $q(t),\ t \in [t_0, t_1]$ on a manifold Q is a smooth function $q(t, \epsilon)$ such that $q(t, 0) = q(t)$ for all t and that the corresponding variation is defined by

$$\delta q(t) = \left.\frac{\partial q(t, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}$$

and curves in $\Omega(Q; q_0, q_1)$ satisfy the fixed endpoint conditions, i.e., $\delta q(t_i) = 0$ for $t = 0, 1$.

Let $\tau : V \to Q$ be a vector bundle and let $v(t, \epsilon)$ be a deformation in V of a curve $v(t)$ in V. If $\tau(v(t, \epsilon)) = q(t)$ does not depend on ϵ, we call $v(t, \epsilon)$ a *V-fiber deformation* of $v(t)$, or simply, a *fiber deformation* of $v(t)$. For each t, the variation

$$\delta v(t) = \left.\frac{\partial v(t, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}$$

may be naturally identified with an element, also called $\delta v(t)$, of $\tau^{-1}(q(t))$. In this case, the curve δv in V is, by definition, a *V-fiber variation* of the curve v, or, simply, a *fiber variation* of the curve v.

Horizontal and vertical variations. Given a curve $q \in \Omega(Q; q_0, q_1)$, a *vertical variation* of δq of q satisfies the condition $\delta q(t) = \mathrm{Ver}(\delta q(t))$ for all t. Similarly, a *horizontal variation* satisfies $\delta q(t) = \mathrm{Hor}(\delta q(t))$ for all t. It is apparent that any variation δq can be uniquely decomposed into the vertical and horizontal components as

$$\delta q(t) = \mathrm{Hor}(\delta q(t)) + \mathrm{Ver}(\delta q(t))$$

for all t, where $\mathrm{Ver}(\delta q(t)) = A(q(t), \delta q(t))q(t)$ and $\mathrm{Hor}(\delta q(t)) = \delta q(t) - \mathrm{Ver}(\delta q(t))$.

The structure of vertical variations. Let $\xi = A(q, \dot{q}) = \dot{g}_q g_q^{-1} \in \mathfrak{g}$. Variations δq of $q(t)$ induce corresponding variations

$$\delta\xi(t) = \left.\frac{\partial A(q(t,\epsilon), \dot{q}(t,\epsilon))}{\partial \epsilon}\right|_{\epsilon=0} \in \mathfrak{g}.$$

As was shown in equation (6), one has the decomposition $q = g_q q_h$, where $q_h(t)$ is the horizontal curve $q_h(t)$ such that $q_h(t_0) = q(t_0)$ and $\pi(q_h(t)) = \pi(q(t))$ for all t and $g_q(t) \in G$. A *vertical deformation* $q(t,\epsilon)$ can be written as $q(t,\epsilon) = g_q(t,\epsilon)q_h(t)$. The corresponding variation $\delta q(t) = \delta g_q(t) q_h(t)$ is also vertical. Define the curve

$$\zeta(t) = \delta g_q(t) g_q(t)^{-1} \in \mathfrak{g}$$

with the fixed endpoints $\zeta(t_i) = 0$, $i = 0, 1$. Then, we can construct the following relation:

$$\delta q(t) = \delta g_q(t) q_h(t) = \zeta(t) g_q(t) q_h(t) = \zeta(t) q(t).$$

It follows from Lemma 3.1.1 in [24] that for any vertical variation $\delta q = \zeta q$ of a curve $q \in \Omega(Q; q_0, q_1)$, the corresponding variation $\delta\xi$ of $\xi = A(q, \dot{q})$ is given by

$$\delta\xi = \dot{\zeta} + [\zeta, \xi]$$

with $\zeta(t_i) = 0, i = 0, 1$.

Remark 7. Note that $\delta\xi = \dot{\zeta} + [\zeta, \xi]$ in the above is *not* computing the same object as $\delta\xi = \dot{\zeta} + [\xi, \zeta]$ in the variational formulation of the Euler-Poincaré equations.

The structure of horizontal variations. Let $\xi = A(q, \dot{q}) = \dot{g}_q g_q^{-1} \in \mathfrak{g}$ and let us calculate variations $\delta\xi$ corresponding to a horizontal variation δq of a curve $q \in \Omega(Q; q_0, q_1)$. Let $q(t,\epsilon)$ be a horizontal deformation of $q(t)$, that is, $\epsilon \mapsto q(t,\epsilon)$ is a horizontal curve for each t. In a local trivialization of the bundle, we write the connection A as

$$\xi = A(q, \dot{q}) = \langle A(q), \dot{q} \rangle .$$

Then, we can compute using the chain rule:

$$\delta\xi = \left.\frac{\partial \xi}{\partial \epsilon}\right|_{\epsilon=0} = \left.\left\langle DA(q) \cdot \frac{\partial q}{\partial \epsilon}, \frac{\partial q}{\partial t} \right\rangle\right|_{\epsilon=0} + \left.\left\langle A(q), \frac{\partial^2 q}{\partial \epsilon \partial t} \right\rangle\right|_{\epsilon=0}.$$

On the other hand, since $\epsilon \mapsto q(t,\epsilon)$ is horizontal,

$$\left\langle A(q), \frac{\partial q}{\partial \epsilon} \right\rangle = 0,$$

and so, by differentiating with respect to t,

$$\left.\left\langle DA(q) \cdot \frac{\partial q}{\partial t}, \frac{\partial q}{\partial \epsilon} \right\rangle\right|_{\epsilon=0} + \left.\left\langle A(q), \frac{\partial^2 q}{\partial \epsilon \partial t} \right\rangle\right|_{\epsilon=0} = 0.$$

Then, we obtain, by subtraction,

$$\delta\xi = \mathbf{d}A(q)\left(\frac{\partial q}{\partial \epsilon}, \frac{\partial q}{\partial t}\right)\bigg|_{\epsilon=0}.$$

Since $\partial q/\partial\lambda$ is horizontal, Cartan's structure equation implies

$$\delta\xi = B(q)\left(\frac{\partial q}{\partial \epsilon}, \frac{\partial q}{\partial t}\right)\bigg|_{\epsilon=0}, \tag{11}$$

or, in other words, $\delta\xi = B(q)(\delta q, \dot{q})$.

The covariant variation on the adjoint bundle. It follows that any curve in $q \in \Omega(Q; q_0, q_1)$ induces a curve in $\tilde{\mathfrak{g}}$ in a natural way, namely,

$$[q, \xi]_G(t) = [q(t), \xi(t)]_G.$$

Observe that, for each t, $[q,\xi]_G(t) \in \tilde{\mathfrak{g}}_{x(t)}$ (the fiber over $x(t)$), where $x(t) = \pi(q(t))$ for all t. Let us see variations $\delta[q,\xi]_G$ corresponding to vertical and also to horizontal variations δq of q.

While vertical variations δq give rise to vertical variations $\delta[q,\xi]_G$, horizontal variations δq *need not give rise to horizontal variations* $\delta[q,\xi]_G$. The deviation of any variation $\delta[q,\xi]_G$ from being horizontal is measured by the *covariant variation* $\delta^A[q,\xi]_G(t)$, which is defined for any given deformation $q(t,\epsilon)$ of $q(t)$, by

$$\delta^A[q,\xi]_G(t) = \frac{D[q(t,\epsilon), \xi(t,\epsilon)]_G}{D\epsilon}\bigg|_{\epsilon=0}.$$

Vertical variations and the adjoint bundle. Let us see the case of vertical variations. The covariant variation $\delta^A[q,\xi]_G(t)$ corresponding to a vertical variation $\delta q = \zeta q$ is given by

$$\delta^A[q,\xi]_G(t) = \frac{D[q(t), \zeta(t)]_G}{Dt} + [q, [\xi, \zeta]]_G, \tag{12}$$

where

$$[q, [\xi,\zeta]]_G = [[q,\xi]_G, [q,\zeta]_G].$$

In the above, for the special case $Q = G$, we can regard it as a principal bundle over a point and the identification of $\mathfrak{g}$ with $T(Q/G) \oplus \tilde{\mathfrak{g}}$ is given by $\xi \mapsto [e,\xi]_G$. This equivalence defines $\delta\xi \equiv \delta^A[e,\xi]_G$, which induces $\delta\xi = \dot{\zeta} + [\xi,\zeta]$ that is the same type of variations as the constrained variations for the Euler-Poincaré equations.

The reduced curvature form. The curvature 2-form $B \equiv B^A$ of A induces a $\tilde{\mathfrak{g}}$-valued 2-form $\widetilde{B} \equiv \widetilde{B}^A$ on Q/G called the *reduced curvature form* given by

$$\begin{aligned} \langle \bar{\mu}, \widetilde{B}(x)(\delta x, \dot{x})\rangle &= \langle [q,\mu]_G, [q, B(q)(\delta q, \dot{q})]_G\rangle \\ &= \langle [gq, \mathrm{Ad}^*_{g^{-1}}\mu]_G, [gq, \mathrm{Ad}_g B(q)(\delta q, \dot{q})]_G\rangle, \end{aligned} \tag{13}$$

where for each $(x,\dot{x})$ and $(x,\delta x)$ in $T_x(Q/G)$, $(q,\dot{q})$ and $(q,\delta q)$ are any elements of T_qQ such that $x = \pi(q)$, $(x,\dot{x}) = T\pi(q,\dot{q})$ and $(x,\delta x) = T\pi(q,\delta q)$. This is easily shown by checking that the right hand side does not depend on the choice of $(q,\delta q)$ and $(q,\dot{q})$ using the equivariance properties of the curvature as

$$[gq, B(gq)(g\delta q, g\dot{q})]_G = [gq, \mathrm{Ad}_g B(q)(\delta q, \dot{q})]_G = [q, B(q)(\delta q, \dot{q})]_G.$$

Horizontal variations and the adjoint bundle. It follows from equation (7) that covariant variations $\delta^A[q,\xi]_G(t)$ corresponding to horizontal variations δq

$$\delta^A[q,\xi]_G(t) = [q, -[A(q,\delta q),\xi] + \delta\xi]_G.$$

Since δq is horizontal, one has $A(q,\delta q) = 0$. Thus, by using equations (11) and (13), one can obtain covariant variations $\delta^A[q,\xi]_{G(t)}$ corresponding to horizontal variations δq that are given by

$$\delta^A[q,\xi]_G(t) = [q, B(q)\,(\delta q,\dot{q})]_G = \widetilde{B}(x)(\delta x,\dot{x})(t). \tag{14}$$

The covariant variations. Recall that arbitrary variations δq of a curve q are decomposed as

$$\delta q = \mathrm{Hor}(\delta q) + \mathrm{Ver}(\delta q),$$

where $\mathrm{Ver}(\delta q) = A(q,\delta q)q$ and $\mathrm{Hor}(\delta q) = \delta q - \mathrm{Ver}(\delta q)$. It follows from equations (12) and (14) that covariant variations $\delta^A[q,\xi]_G(t)$ associated with arbitrary variations δq are given by

$$\delta^A[q,\xi]_G(t) = \frac{D[q(t),\zeta(t)]_G}{Dt} + [q,[\xi,\zeta]]_G + [q, B(q)(\delta q,\dot{q})]_G.$$

Noting that $\bar{\xi} = [q,\xi]_G$ and $\bar{\zeta} = [q,\zeta]_G$, the *covariant variations* may be restated as

$$\delta^A\bar{\xi} = \frac{D\bar{\zeta}}{Dt} + [\bar{\xi},\bar{\zeta}] + \widetilde{B}(x)(\delta x,\dot{x}).$$

Variations of reduced curves in $T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$. Let us consider variations of the reduced curves

$$[q(t),\dot{q}(t),p(t)]_G \cong (x(t),u(t),y(t),\bar{\xi}(t),\bar{\mu}(t))$$

in $(TQ \oplus T^*Q)/G \cong T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$. Now, the variations of curves $[q(t),\dot{q}(t)]_G = (x(t),\dot{x}(t),\bar{\xi}(t))$ in $T(Q/G) \oplus \tilde{\mathfrak{g}}$ are given as

$$\delta[q(t),\dot{q}(t)]_G = \delta x(t) \oplus \delta^A\bar{\xi}(t),$$

where $\delta x(t) \in T_{x(t)}(Q/G)$, and $\delta^A\bar{\xi}(t) \in T_{\bar{\xi}(t)}\tilde{\mathfrak{g}}$ denotes the covariant variations on the adjoint bundle.

Further, arbitrary variations of the curves $[q(t),p(t)]_G = (x(t),y(t),\bar{\mu}(t))$ in $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ induce arbitrary fiber variations $(\delta x(t),\delta y(t)) \in T_{(x(t),y(t))}T^*(Q/G)$ and $\delta\bar{\mu} \in T_{\bar{\mu}}\tilde{\mathfrak{g}}^*$, and it follows

$$\delta[q(t),p(t)]_G = (\delta x(t),\delta y(t),\delta\bar{\mu}(t)).$$

Thus, variations of the curves in the reduced Pontryagin space are given by

$$\delta[q(t),\dot{q}(t),p(t)]_G \cong (\delta x(t),\delta y(t),\delta^A\bar{\xi}(t),\delta\bar{\mu}(t)).$$

Geometry of reduced variations. In general, the second slot of a curve

$$(q(t),v(t),p(t)),\ t \in [t_0,t_1]$$

in $TQ \oplus T^*Q$, namely, $v(t)$ is *not* equal to $\dot{q}(t)$. Then, the curve in the reduced Pontryagin space $(TQ \oplus T^*Q)/G \cong T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$ is given by

$$[q(t),v(t),p(t)]_G \cong (x(t),u(t),y(t),\bar{\eta}(t),\bar{\mu}(t)),$$

where $(x,u,\bar{\eta}) = [q,v]_G \in T(Q/G) \oplus \tilde{\mathfrak{g}}$, $u = T_q\pi(v_q) \in T_x(Q/G)$ and $\bar{\eta} = [q,\eta]_G = [q,A(q,v)]_G \in \tilde{\mathfrak{g}}$, and where $[q,p]_G = (x,y,\bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, $y = (p_q)_q^{h^*} \in$

$T_x^*(Q/G)$ and $\bar{\mu} = [q, \mu]_G = [q, \mathbf{J}(q,p)]_G \in \tilde{\mathfrak{g}}^*$. Hence, it follows that a general variation of the reduced curve $[q(t), v(t), p(t)]_G$ is given by

$$\delta[q(t), v(t), p(t)]_G \cong (\delta x(t), \delta u(t), \delta y(t), \delta\bar{\eta}(t), \delta\bar{\mu}(t)),$$

where $\delta x(t) = T_q\pi(\delta q(t)) \in T_x(Q/G)$, $\delta u = T_{T\pi(v_q)}T\pi(\delta v) \in T_{u_x}T(Q/G)$, $\delta y \in T_{y_x}T^*(Q/G)$ and $\delta\bar{\mu} \in T_{\bar{\mu}}\tilde{\mathfrak{g}}^*$ are arbitrary variations, and the variation $\delta\bar{\eta}(t)$ is given by

$$\delta\bar{\eta}(t) = \left.\frac{\partial\bar{\eta}(t,s)}{\partial s}\right|_{s=0}.$$

In the above, we choose a family of curves $\bar{\eta}(t,s)$ in $\tilde{\mathfrak{g}}$ such that $\bar{\eta}(t,0) = \bar{\eta}(t)$ and it follows that $\delta\bar{\eta}(t)$ is, for each t, an element of $T\tilde{\mathfrak{g}}$.

Remark 8. For the case in which a curve in $TQ \oplus T^*Q$ is given by $(q(t), \dot{q}(t), p(t))$, $t \in [t_0, t_1]$, we consider the special kind of deformations $\bar{\xi}(t,s)$ of the curve

$$\bar{\xi}(t) = [q(t), A(q(t), \dot{q}(t))]_G$$

in which the projection $\tilde{\pi}_{\tilde{\mathfrak{g}}}(\bar{\xi}(t,s)) = x(t,s)$ does not depend on s, that is, deformations that take place only in the fiber of $\tilde{\mathfrak{g}}$ over $x(t) = \tilde{\pi}_{\tilde{\mathfrak{g}}}(\bar{\xi}(t))$, where $\tilde{\pi}_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \to Q/G$. Thus, for each fixed t, the curve $s \mapsto \bar{\xi}(t,s)$ is a curve in the fiber over $x(t)$. Then, since $\tilde{\mathfrak{g}}$ is a vector bundle, the variation $\delta\bar{\xi}(t)$ induced by such a deformation $\bar{\xi}(t,s)$, is naturally identified with a curve, also called $\delta\bar{\xi}(t)$, in $\tilde{\mathfrak{g}}$, a $\tilde{\mathfrak{g}}$-*fiber variation.* For a curve $(x(t), \bar{\xi}(t))$ in $Q/G \oplus \tilde{\mathfrak{g}}$, and a given arbitrary deformation $(x(t,\epsilon), \bar{\xi}(t,\epsilon))$, with $x(t,0) \oplus \bar{\xi}(t,0) = x(t) \oplus \bar{\xi}(t)$, of it, the corresponding *covariant variation* $\delta x(t) \oplus \delta^A\bar{\xi}(t)$ is, by definition,

$$\delta x(t) \oplus \delta^A\bar{\xi}(t) = \left.\frac{\partial x(t,s)}{\partial s}\right|_{s=0} \oplus \left.\frac{D\bar{\xi}(t,s)}{Ds}\right|_{s=0},$$

where $\delta^A\bar{\xi}(t)$ is a $\tilde{\mathfrak{g}}$-fiber deformation of $\bar{\xi}(t)$.

The most important example of a covariant variation $\delta x(t) \oplus \delta^A\bar{\xi}(t)$ is the one to be described next. Let $q(t,s)$ be a deformation of a curve $q(t) = q(t,0)$ in Q. This induces a deformation $x(t,s) \oplus \bar{\xi}(t,s)$ of the curve $x(t) \oplus \bar{\xi}(t)$ by taking $x(t,s) = [q(t,s)]_G$ and $\bar{\xi}(t,s) = [q(t,s), A(q(t,s), \dot{q}(t,s))]_G$, where $\dot{q}(t,s)$ represents the derivative with respect to t. It follows that the covariant variation corresponding to this deformation of $x(t) \oplus \bar{\xi}(t)$ is given by $\delta x(t) \oplus \delta^A\bar{\xi}(t)$, where

$$\delta^A\bar{\xi}(t) = \frac{D\bar{\zeta}(t)}{Dt} + [\bar{\xi}(t), \bar{\zeta}(t)] + \widetilde{B}(\delta x(t), \dot{x}(t)),$$

where $\bar{\zeta} = [q, \zeta]_G = [q, A(q, \delta q)]_G \in \tilde{\mathfrak{g}}$.

Reduction of the Hamilton-Pontryagin principle. From the action of G on Q, we get an action of G on curves $(q(t), v(t), p(t)) \in TQ \oplus T^*Q$ in the action integral $\mathfrak{F}(q,v,p)$ in the Hamilton-Pontryagin principle by simultaneously left translating on each factor by the tangent and cotangent lift. Let $L : TQ \to \mathbb{R}$ be a left invariant Lagrangian (possibly, degenerate). Recall that the *Hamilton-Pontryagin principle* requires stationarity of the action integral on the space of curves $(q(t), v(t), p(t))$, $t \in [t_0, t_1]$ in $TQ \oplus T^*Q$ given by

$$\mathfrak{F}(q,v,p) = \int_{t_0}^{t_1} \{L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t)\rangle\}\, dt;$$

that is, $\delta\mathfrak{F} = 0$ with the endpoints of $q(t)$ fixed. This principle gives the implicit Euler–Lagrange equations on $TQ \oplus T^*Q$:

$$p = \frac{\partial L}{\partial v}, \quad \dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}.$$

By the G-invariance of $L : TQ \to \mathbb{R}$, that is,

$$L(T_qL_g(v_q)) = L(v_q),$$

where $g \in G$, $q \in Q$ and $v_q \in T_qQ$, we can define the reduced Lagrangian $l : T(Q/G) \oplus \tilde{\mathfrak{g}} \to \mathbb{R}$ by

$$l([q, v]_G) = L(q, v).$$

Recall that the curves in the reduced Pontryagin bundle $(TQ \oplus T^*Q)/G$ are isomorphic to the curves in $T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$, namely,

$$[q(t), v(t), p(t)]_G \cong (x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t)).$$

Then, the action integral in the Hamilton-Pontryagin principle may be reduced as

$$\begin{aligned}
&[\mathfrak{F}]_G(x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t)) \\
&\quad = \int_{t_0}^{t_1} \left\{ l(x(t), u(t), \bar{\eta}(t)) + \langle y(t), \dot{x}(t) - u(t)\rangle + \left\langle \bar{\mu}(t), \bar{\xi}(t) - \bar{\eta}(t)\right\rangle \right\} dt.
\end{aligned}$$

Remark 9. The action integral in the Hamilton-Pontryagin principle can be restated by using the generalized energy $E(q, v, p) = \langle p, v\rangle - L(q, v)$ on $TQ \oplus T^*Q$ as

$$\mathfrak{F}(q(t), v(t), p(t)) = \int_{t_0}^{t_1} \{\langle p(t), \dot{q}(t)\rangle - E(q(t), v(t), p(t))\}\, dt.$$

Since E is G-invariant, the reduction of the action integral in the Hamilton-Pontryagin principle is also represented by

$$\begin{aligned}
&[\mathfrak{F}]_G(x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t)) \\
&\quad = \int_{t_0}^{t_1} \left\{ \langle y(t), \dot{x}(t)\rangle + \left\langle \bar{\mu}(t), \bar{\xi}(t)\right\rangle - \mathcal{E}(x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t)) \right\} dt,
\end{aligned}$$

where $\mathcal{E} : T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V} \to \mathbb{R}$ is the reduced generalized energy given by

$$\mathcal{E}(x, u, y, \bar{\eta}, \bar{\mu}) = \langle y, u\rangle + \langle \bar{\mu}, \bar{\eta}\rangle - l(x, u, \bar{\eta}).$$

Proposition 1. *The variation of the action integral in the reduced Hamilton-Pontryagin principle is given by*

$$\begin{aligned}
&\delta[\mathfrak{F}]_G(x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t)) \\
&\quad = \delta \int_{t_0}^{t_1} \left\{ l(x(t), u(t), \bar{\eta}(t)) + \langle y(t), \dot{x}(t) - u(t)\rangle + \left\langle \bar{\mu}(t), \bar{\xi}(t) - \bar{\eta}(t)\right\rangle \right\} dt \\
&\quad = \int_{t_0}^{t_1} \left\{ \left\langle \frac{\partial l}{\partial x} - \frac{Dy}{Dt} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot)\right\rangle, \delta x \right\rangle + \left\langle \frac{\partial l}{\partial u} - y, \delta u \right\rangle + \left\langle \frac{\partial l}{\partial \bar{\eta}} - \bar{\mu}, \delta\bar{\eta} \right\rangle \right. \\
&\qquad \left. + \langle \delta y, \dot{x} - u\rangle + \left\langle \delta\bar{\mu}, \bar{\xi} - \bar{\eta}\right\rangle + \left\langle -\frac{D\bar{\mu}}{Dt} + \mathrm{ad}^*_{\bar{\xi}}\bar{\mu}, \bar{\zeta} \right\rangle \right\} dt,
\end{aligned}$$

for arbitrary variations δu, δy, $\delta\bar{\eta}$ and $\delta\bar{\mu}$ and for variations $\delta x \oplus \delta^A\bar{\xi}$, where

$$\delta^A\bar{\xi} = \frac{D\bar{\zeta}}{Dt} + [\bar{\xi}, \bar{\zeta}] + \widetilde{B}(\delta x, \dot{x})$$

and with the boundary conditions

$$\delta x(t_i) = 0 \text{ and } \bar{\zeta}(t_i) = 0 \text{ for } i = 0, 1.$$

Proof. It follows that the variation of the action integral in the reduced Hamilton-Pontryagin principle is given by direct computations as

$$\begin{aligned}
&\delta[\mathfrak{F}]_G(x, u, y, \bar{\eta}, \bar{\mu}) \\
&\quad = \delta \int_{t_0}^{t_1} \left\{ l(x, u, \bar{\eta}) + \langle y, \dot{x} - u \rangle + \left\langle \bar{\mu}, \bar{\xi} - \bar{\eta} \right\rangle \right\} dt \\
&\quad = \int_{t_0}^{t_1} \left\{ \left\langle \frac{\partial l}{\partial x}, \delta x \right\rangle + \left\langle \frac{\partial l}{\partial u}, \delta u \right\rangle + \left\langle \frac{\partial l}{\partial \bar{\eta}}, \delta \bar{\eta} \right\rangle + \langle \delta y, \dot{x} - u \rangle + \langle y, \delta \dot{x} - \delta u \rangle \right. \\
&\qquad\qquad \left. + \left\langle \delta \bar{\mu}, \bar{\xi} - \bar{\eta} \right\rangle + \left\langle \bar{\mu}, \delta^A \bar{\xi} - \delta \bar{\eta} \right\rangle \right\} dt \\
&\quad = \int_{t_0}^{t_1} \left\{ \left\langle \frac{\partial l}{\partial x} - \frac{Dy}{Dt} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \delta x \right\rangle + \left\langle \frac{\partial l}{\partial u} - y, \delta u \right\rangle + \left\langle \frac{\partial l}{\partial \bar{\eta}} - \bar{\mu}, \delta \bar{\eta} \right\rangle \right. \\
&\qquad\qquad \left. + \langle \delta y, \dot{x} - u \rangle + \left\langle \delta \bar{\mu}, \bar{\xi} - \bar{\eta} \right\rangle + \left\langle -\frac{D\bar{\mu}}{Dt} + \mathrm{ad}^*_{\bar{\xi}} \bar{\mu}, \bar{\zeta} \right\rangle \right\} dt.
\end{aligned}$$

□

Remark 10. Since $T(Q/G)$ and $\tilde{\mathfrak{g}}$ are vector bundles, we can interpret the derivatives $\partial l/\partial \dot{x}$ and $\partial l/\partial \bar{\xi}$ in a standard way of *fiber derivatives* as being elements of the dual bundles $T^*(Q/G)$ and $\tilde{\mathfrak{g}}^*$, for each choice of $(x, \dot{x}, \bar{\xi})$ in $T(Q/G) \oplus \tilde{\mathfrak{g}}$. In other words, for given $(x_0, \dot{x}_0, \bar{\xi}_0)$ and $(x_0, \dot{x}', \bar{\xi}')$, we define

$$\frac{\partial l}{\partial \dot{x}}(x_0, \dot{x}_0, \bar{\xi}_0) \cdot x' = \left. \frac{d}{ds} \right|_{s=0} l(x_0, \dot{x}_0 + s x', \bar{\xi}_0)$$

and

$$\frac{\partial l}{\partial \bar{\xi}}(x_0, \dot{x}_0, \bar{\xi}_0) \cdot \bar{\xi}' = \left. \frac{d}{ds} \right|_{s=0} l(x_0, \dot{x}_0, \bar{\xi}_0 + s \bar{\xi}').$$

To define the derivative $\partial l/\partial x$, one uses the connection ∇ on the manifold Q/G. The *covariant derivative* of l with respect to x at $(x_0, \dot{x}_0, \bar{\xi}_0)$ in the direction of $(x(0), \dot{x}(0))$ is defined by

$$\frac{\partial^C l}{\partial x}(x_0, \dot{x}_0, \bar{\xi}_0) \cdot (x(0), \dot{x}(0)) = \left. \frac{d}{ds} \right|_{s=0} l(x(s), \dot{x}(s), \bar{\xi}(s)),$$

where we often write

$$\frac{\partial^C l}{\partial x} \equiv \frac{\partial l}{\partial x},$$

whenever there is no danger of confusion.

The covariant derivative on $\tilde{\mathfrak{g}}$ induces a corresponding covariant derivative on the dual bundle $\tilde{\mathfrak{g}}^*$. Namely, let $\bar{\mu}(t)$ be a curve in $\tilde{\mathfrak{g}}^*$. We can define the covariant derivative of $\bar{\mu}(t)$ in such a way that for any curve $\bar{\xi}(t)$ in $\tilde{\mathfrak{g}}$, such that both $\bar{\mu}(t)$ and $\bar{\xi}(t)$ project on the same curve $x(t)$ in Q/G, we have

$$\frac{d}{dt} \left\langle \bar{\mu}(t), \bar{\xi}(t) \right\rangle = \left\langle \frac{D\bar{\mu}(t)}{Dt}, \bar{\xi}(t) \right\rangle + \left\langle \bar{\mu}(t), \frac{D\bar{\xi}}{Dt} \right\rangle.$$

Likewise, we can define the covariant derivative on the vector bundle $T^*(Q/G)$. Then, we obtain a covariant derivative on the vector bundle $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$.

In the sense of this definition, the term Dy/Dt (as will be shown in the horizontal implicit Lagrange-Poincaré equations) means the covariant derivative on $T^*(Q/G)$

and the term $D\bar{\mu}/Dt$ (as will be shown in the vertical implicit Lagrange-Poincaré equations) denotes the covariant derivative on the bundle $\tilde{\mathfrak{g}}^*$.

Remark 11. Note that variations $\delta x \oplus \delta^A \bar{\xi}$ such that

$$\delta^A \bar{\xi} = \widetilde{B}(\delta x, \dot{x})$$

with $\delta x(t_i) = 0$ for $i = 0, 1$ exactly correspond to *horizontal variations* δq of the curve $q(t)$ such that $\delta q(t_i) = 0$ for $t_i = 0, 1$, while variations $\delta x \oplus \delta^A \bar{\xi}$ such that

$$\delta^A \bar{\xi} = \frac{D\bar{\zeta}}{Dt} + [\bar{\xi}, \bar{\zeta}] \equiv \frac{D[q, \zeta]_G}{Dt} + [q, [\xi, \zeta]]_G$$

with $\bar{\zeta}(t_i) = 0$ (or, equivalently $\zeta(t_i) = 0$) for $i = 0, 1$ exactly correspond to *vertical variations* δq of the curve $q(t)$ such that $\delta q(t_i) = 0$ for $i = 0, 1$.

Proposition 2. *The stationary condition for the reduced Hamilton-Pontryagin principle is given by*

$$\begin{aligned} &\delta[\mathfrak{F}]_G(x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t)) \\ &\quad = \delta \int_{t_0}^{t_1} \left\{ l(x(t), u(t), \bar{\eta}(t)) + \langle y(t), \dot{x}(t) - u(t) \rangle + \left\langle \bar{\mu}(t), \bar{\xi}(t) - \bar{\eta}(t) \right\rangle \right\} dt \\ &\quad = 0, \end{aligned}$$

for arbitrary variations δu, δy, $\delta \bar{\eta}$ and $\delta \bar{\mu}$ and for variations $\delta x \oplus \delta^A \bar{\xi}$, where

$$\delta^A \bar{\xi} = \frac{D\bar{\zeta}}{Dt} + [\bar{\xi}, \bar{\zeta}] + \widetilde{B}(\delta x, \dot{x})$$

and with the boundary conditions

$$\delta x(t_i) = 0 \;\; \textit{and} \;\; \bar{\zeta}(t_i) = 0 \;\; \textit{for} \;\; i = 0, 1.$$

This induces **horizontal implicit Lagrange-Poincaré equations**

$$\frac{Dy}{Dt} = \frac{\partial l}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \quad \dot{x} = u, \quad y = \frac{\partial l}{\partial u}, \tag{15}$$

and **vertical implicit Lagrange-Poincaré equations**

$$\frac{D\bar{\mu}}{Dt} = \mathrm{ad}^*_{\bar{\xi}} \bar{\mu}, \quad \bar{\xi} = \bar{\eta}, \quad \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}. \tag{16}$$

Proof. It follows from Proposition 1 that the stationarity condition

$$\begin{aligned} &\delta[\mathfrak{F}]_G(x, u, y, \bar{\eta}, \bar{\mu}) \\ &\quad = \int_{t_0}^{t_1} \left\{ \left\langle \frac{\partial l}{\partial x} - \frac{Dy}{Dt} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \delta x \right\rangle + \left\langle \frac{\partial l}{\partial u} - y, \delta u \right\rangle + \left\langle \frac{\partial l}{\partial \bar{\eta}} - \bar{\mu}, \delta \bar{\eta} \right\rangle \right. \\ &\qquad \left. + \langle \delta y, \dot{x} - u \rangle + \langle \delta \bar{\mu}, \bar{\xi} - \bar{\eta} \rangle + \left\langle -\frac{D\bar{\mu}}{Dt} + \mathrm{ad}^*_{\bar{\xi}} \bar{\mu}, \bar{\zeta} \right\rangle \right\} dt \\ &\quad = 0 \end{aligned}$$

is satisfied for arbitrary variations δu, δy, $\delta \bar{\eta}$ and $\delta \bar{\mu}$ and for variations $\delta x \oplus \delta^A \bar{\xi}$, where

$$\delta^A \bar{\xi} = \frac{D\bar{\zeta}}{Dt} + [\bar{\xi}, \bar{\zeta}] + \widetilde{B}(\delta x, \dot{x})$$

and with the boundary conditions

$$\delta x(t_i) = 0 \;\; \text{and} \;\; \bar{\zeta}(t_i) = 0 \;\; \text{for} \;\; i = 0, 1.$$

Thus, we obtain the horizontal and vertical implicit Lagrange-Poincaré equations as in equations (15) and (16). □

Now we summarize what we have obtained in the following theorem.

Theorem 4.1. *The following statements are equivalent:*

(i) **Hamilton-Pontryagin principle.** *The curve* $(q(t), v(t), p(t))$ *in* $TQ \oplus T^*Q$ *is a critical point of the action integral*

$$\mathfrak{F}(q(t), v(t), p(t)) = \delta \int_{t_0}^{t_1} \{L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t)\rangle\}\, dt$$

for all variations $\delta q, \delta v$ *and* δp *under the endpoint conditions* $\delta q(t_0) = 0$ *and* $\delta q(t_1) = 0$.

(ii) **The Reduced Hamilton-Pontryagin principle.** *The reduced curve*

$$[q(t), v(t), p(t)]_G \cong (x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t)),$$

in the reduced Pontryagin bundle $(TQ \oplus T^*Q)/G \cong T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$ *is a critical point of the reduced action integral*

$$\begin{aligned}&\delta[\mathfrak{F}]_G(x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t))\\ &\quad = \delta \int_{t_0}^{t_1} \left\{l(x(t), u(t), \bar{\eta}(t)) + \langle y(t), \dot{x}(t) - u(t)\rangle + \left\langle \bar{\mu}(t), \bar{\xi}(t) - \bar{\eta}(t)\right\rangle\right\} dt\\ &\quad = 0,\end{aligned}$$

for arbitrary variations δu, δy, $\delta\bar{\eta}$ *and* $\delta\bar{\mu}$ *and for variations* $\delta x \oplus \delta^A \bar{\xi}$, *where*

$$\delta^A \bar{\xi} = \frac{D\bar{\zeta}}{Dt} + [\bar{\xi}, \bar{\zeta}] + \widetilde{B}(\delta x, \dot{x})$$

and with the boundary conditions

$$\delta x(t_i) = 0 \ \text{ and } \ \bar{\zeta}(t_i) = 0 \ \text{ for } \ i = 0, 1.$$

(iii) **The implicit Euler-Lagrange equations** *hold:*

$$p = \frac{\partial L}{\partial v}, \quad \dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}.$$

(iv) *The* **horizontal implicit Lagrange-Poincaré equations**, *corresponding to horizontal variations, hold:*

$$\frac{Dy}{Dt} = \frac{\partial l}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot)\right\rangle, \quad \dot{x} = u, \quad y = \frac{\partial l}{\partial u},$$

and the **vertical implicit Lagrange-Poincaré equations**, *corresponding to vertical variations, hold:*

$$\frac{D\bar{\mu}}{Dt} = \mathrm{ad}^*_{\bar{\xi}}\bar{\mu}, \quad \bar{\xi} = \bar{\eta}, \quad \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}.$$

Hamilton's phase space principle. For the case in which a given G-invariant Lagrangian $L : TQ \to \mathbb{R}$ is regular, one can define a G-invariant Hamiltonian H on T^*Q by $H = E \circ (\mathbb{F}L)^{-1}$, where $E(v_q) = \langle \mathbb{F}L(v_q), v_q\rangle - L(v_q)$ and $\mathbb{F}L : TQ \to T^*Q; v_q \mapsto p_q = \partial L/\partial v_q$ is the Legendre transformation. Then the Hamilton-Pontryagin principle may be replaced by *Hamilton's phase space principle*.

Recall that Hamilton's phase space principle states that the stationary condition of the action integral on the space of curves $(q(t), p(t))$, $t \in [t_0, t_1]$ in T^*Q given by

$$\int_{t_0}^{t_1} \{\langle p(t), \dot{q}(t)\rangle - H(q(t), p(t))\}\, dt$$

with the endpoints $q(t_0)$ and $q(t_1)$ of $q(t)$ fixed, gives Hamilton's equations on T^*Q as

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

The Hamilton-Poincaré variational principle. Let us see how we can develop a Hamiltonian analogue of Lagrange-Poincaré reduction variationally, namely, a reduced principle called the *Hamilton–Poincaré variational principle*. For details, see [23].

It is clear that the dual of the quotient bundle TQ/G, that is, $(TQ/G)^*$ is canonically identified with the quotient bundle T^*Q/G. Recall that the vector bundle isomorphism Ψ_A defines by duality a bundle isomorphism $(\Psi_A^{-1})^* : T^*Q/G \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$. In Hamilton's phase space principle, the pointwise function in the integrand, namely,

$$F(q, \dot{q}, p) = \langle p, \dot{q}\rangle - H(q, p)$$

is defined on $TQ \oplus T^*Q$. The group G acts on $TQ \oplus T^*Q$ by simultaneously left translating on each factor by the tangent and cotangent lift and it induces $[q, \dot{q}, p]_G = (x, \dot{x}, \bar{\xi}, \bar{\mu})$, where $x = [q]$, $\bar{\xi} = [q, \xi]_G = [q, A(q, \dot{q})]_G$ and $\bar{\mu} = [q, \mu]_G = [q, \mathbf{J}(q, p)]_G$. Since the function F is invariant under the action of G, assuming invariance of H, the function F drops to the quotient, namely, to the function $f : TQ/G \oplus T^*Q/G \to \mathbb{R}$, or equivalently, $f : T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V} \to \mathbb{R}$, which is given by

$$f(x, \dot{x}, y, \bar{\xi}, \bar{\mu}) = \langle y, \dot{x}\rangle + \left\langle \bar{\mu}, \bar{\xi}\right\rangle - h(x, y, \bar{\mu}),$$

where $\widetilde{V} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^*$ and h is the reduction of H from T^*Q to $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$.

When the function F is used in the phase space variational principle, one is varying curves $(q(t), p(t))$ and one of course insists that the slot $\dot{q}$ actually is the time derivative of $q(t)$. This restriction induces in a natural way a restriction on the variation of the curve $[q(t), \dot{q}(t)]_G = (x(t), \dot{x}(t), \bar{\xi}(t))$ and the variation of $x(t) \oplus \bar{\xi}(t)$ is given by $\delta x(t) \oplus \delta^A \bar{\xi}(t)$, where

$$\delta^A \bar{\xi}(t) = \frac{D\bar{\zeta}(t)}{Dt} + [\bar{\xi}(t), \bar{\zeta}(t)] + \widetilde{B}(\delta x(t), \dot{x}(t)),$$

with the conditions $\delta x(t_i) = 0$ and $\bar{\zeta}(t_i) = 0$, $i = 0, 1$, where $\bar{\zeta} = [q, \zeta]_G = [q, A(q, \delta q)]_G \in \tilde{\mathfrak{g}}$. On the other hand, arbitrary variations δp induce arbitrary fiber variations δy and $\delta \bar{\mu}$.

Using the same kind of argument, based on reducing the action and the variations that we have used to derive the reduced Hamilton-Pontryagin variational principle and implicit Lagrange–Poincaré equations, we can easily show that Hamilton's phase space variational principle can also be reduced. In fact, we can easily obtain reduction of Hamilton's phase space principle by applying the usual integration by parts argument to the action

$$\int_{t_0}^{t_1} \left\{ \langle y, \dot{x} \rangle + \left\langle \bar{\mu}, \bar{\xi} \right\rangle - h(x, y, \bar{\mu}) \right\} dt,$$

with variations

$$(\delta x, \delta^A \bar{\xi}) \oplus (\delta y, \delta \bar{\mu}) = \left(\delta x, \frac{D\bar{\zeta}}{Dt} + [\bar{\xi}, \bar{\zeta}] + \widetilde{B}(\delta x, \dot{x}) \right) \oplus (\delta y, \delta \bar{\mu}),$$

with restrictions explained above.

In this way, we obtain the **horizontal Hamilton–Poincaré equations**

$$\frac{Dy}{Dt} = -\frac{\partial h}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \quad \frac{dx}{dt} = \frac{\partial h}{\partial y}$$

as well as the **vertical Hamilton–Poincaré equations**

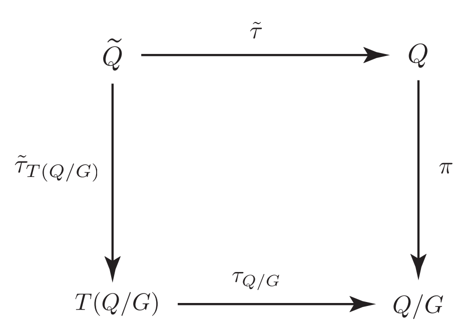
$$\frac{D\bar{\mu}}{Dt} = \mathrm{ad}^*_{\bar{\xi}} \bar{\mu}, \quad \bar{\xi} = \frac{\partial h}{\partial \bar{\mu}}.$$

5. **Dirac cotangent bundle reduction.** In this section, we develop a reduction procedure for the canonical Dirac structure $D \subset TT^*Q \oplus T^*T^*Q$ on the cotangent bundle T^*Q. Choosing a principal connection $A : TQ \to \mathfrak{g}$, we introduce a G-principal bundle $\widetilde{Q}^*$ with the base $T^*(Q/G)$ by pulling back the principal bundle $\pi : Q \to Q/G$ by the cotangent bundle projection $\pi_{Q/G} : T^*(Q/G) \to Q/G$, and then we introduce an isomorphism $\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*$. Further, we develop a G-invariant Dirac structure $\bar{D}$ on $\widetilde{Q}^* \times \mathfrak{g}^*$ using $\bar{\lambda}$ as a forward Dirac map. Under the isomorphism $(\widetilde{Q}^* \times \mathfrak{g}^*)/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, it is shown that taking the quotient of $\bar{D} \subset T(\widetilde{Q}^* \times \mathfrak{g}^*) \oplus T^*(\widetilde{Q}^* \times \mathfrak{g}^*)$ by the action of G leads to a *gauged Dirac structure* $[\bar{D}]_G = [\bar{D}]_G^{\mathrm{Hor}} \oplus [\bar{D}]_G^{\mathrm{Ver}}$ on the bundle $TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$ over $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, where $[\bar{D}]_G^{\mathrm{Hor}}$ is a *horizontal Dirac structure* on the bundle $TT^*(Q/G)$ over $T^*(Q/G)$ and $[\bar{D}]_G^{\mathrm{Ver}}$ is a *vertical Dirac structure* on $\tilde{\mathfrak{g}}^* \times \widetilde{V}$ over $\tilde{\mathfrak{g}}^*$.

A trivialized isomorphism $\lambda : TQ \to \widetilde{Q} \times \mathfrak{g}$. Let us introduce an isomorphism $\lambda : TQ \to \widetilde{Q} \times \mathfrak{g}$. First, let us pull back the G-principal bundle $\pi : Q \to Q/G$ by the tangent bundle projection $\tau_{Q/G} : T(Q/G) \to Q/G$ to obtain the G-principal bundle

$$\widetilde{Q} = \left\{ (q, u_{[q]}) \mid \tau_{Q/G}(u_{[q]}) = \pi(q) = [q],\ q \in Q,\ u_{[q]} \in T_{[q]}(Q/G) \right\}$$

with the base $T(Q/G)$ whose fiber over $u_{[q]}$ is diffeomorphic to $\pi^{-1}([q])$.



It follows that the free and proper G-action on Q induces a free and proper G-action on $\widetilde{Q}$: for each $(q, u_{[q]}) \in \widetilde{Q}$ and for $g \in G$,

$$g \cdot (q, u_{[q]}) = (gq, u_{[q]}),$$

and we have the two projections

$$\tilde{\tau}_{T(Q/G)} : \widetilde{Q} \to T(Q/G);\ (q, u_{[q]}) \mapsto u_{[q]}$$

and

$$\tilde{\tau} : \widetilde{Q} \to Q;\ (q, u_{[q]}) \mapsto q.$$

Now, define the isomorphism λ by

$$\lambda : TQ \to \widetilde{Q} \times \mathfrak{g};\ \ v_q \mapsto (q, u_{[q]} = T\pi(v_q), \eta = A(v_q)).$$

We think of $\lambda : TQ \to \widetilde{Q} \times \mathfrak{g}$ as a *right trivialization*. The action of G on the product space $\widetilde{Q} \times \mathfrak{g}$ is given by, for $g \in G$ and $(q, u_{[q]}, \eta) \in \widetilde{Q} \times \mathfrak{g}$,

$$g \cdot (q, u_{[q]}, \eta) = (gq, u_{[q]}, g\eta) = (gq, u_{[q]}, \mathrm{Ad}_g \eta).$$

In the above, we employ the concatenation notation $g\eta \equiv \mathrm{Ad}_g \eta$ for the adjoint action, which is given by $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$. The inverse map of λ is given by, for each $(q, u_{[q]}, \eta) \in \widetilde{Q} \times \mathfrak{g}$,

$$\lambda^{-1} : \widetilde{Q} \times \mathfrak{g} \to TQ;\ \ (q, u_{[q]}, \eta) \mapsto (q, (u_{[q]})^h_q + \eta q).$$

Let $\tau_Q : TQ \to Q$ be the tangent bundle projection and we can naturally define the canonical projection $\tilde{\tau}_Q = \tau_Q \circ \lambda^{-1} : \widetilde{Q} \times \mathfrak{g} \to Q$ given by

$$\tilde{\tau}_Q : \widetilde{Q} \times \mathfrak{g} \to Q;\ (q, u_{[q]}, \eta) \mapsto q$$

and the differential map of $\tilde{\tau}_Q$ may be given by

$$T\tilde{\tau}_Q : T(\widetilde{Q} \times \mathfrak{g}) \to TQ;\ (q, u_{[q]}, \eta, \delta q, \delta u_{[q]}, \delta\eta) \to (q, \delta q).$$

The quotient space $(\widetilde{Q} \times \mathfrak{g})/G$. Recall that the principal connection $A : TQ \to \mathfrak{g}$ satisfies the equivariance condition

$$A(g \cdot v_q) = \mathrm{Ad}_g(A(v_q))$$

and the quotient of the right trivialized isomorphism $\lambda : TQ \to \widetilde{Q} \times \mathfrak{g}$ by G yields the isomorphism $[\lambda]_G : (TQ)/G \to (\widetilde{Q} \times \mathfrak{g})/G$ as a *left trivialization*, which is given by, for $v_q \in Q$,

$$[v_q]_G \mapsto (T\pi(v_q), [q, A(v_q)]_G) = \left(u_{[q]}, [q, \eta]_G\right).$$

This map is well defined since the equivariance condition gives

$$\left(u_{[q]}, [q, \eta]_G\right) = \left(T\pi(v_q), [gq, \mathrm{Ad}_g A(v_q)]_G\right).$$

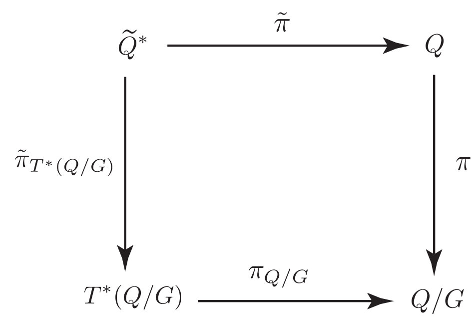
It follows that the quotient space $(\widetilde{Q} \times \mathfrak{g})/G$ is isomorphic to the space $T(Q/G) \oplus \tilde{\mathfrak{g}}$. Since one has the isomorphism $\Psi_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ (see equation (8)), one eventually has the following isomorphisms:

$$TQ/G \cong (\widetilde{Q} \times \mathfrak{g})/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}}.$$

A trivialized isomorphism $\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*$. Let us consider a trivialized isomorphism $\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*$ by following [76]. First, let us pull back the G-principal bundle $\pi : Q \to Q/G$ by the cotangent bundle projection $\pi_{Q/G} : T^*(Q/G) \to Q/G$ to obtain the G-principal bundle

$$\widetilde{Q}^* = \left\{ (q, y_{[q]}) \mid \pi_{Q/G}(y_{[q]}) = \pi(q) = [q],\ q \in Q,\ y_{[q]} \in T^*_{[q]}(Q/G) \right\}$$

with the base $T^*(Q/G)$ whose fiber over $y_{[q]}$ is diffeomorphic to $\pi^{-1}([q])$.



It follows that the free and proper G-action on Q induces a free and proper G-action on $\widetilde{Q}^*$ given, for each $(q, y_{[q]}) \in \widetilde{Q}^*$ and for $g \in G$, by $g \cdot (q, y_{[q]}) = (gq, y_{[q]})$, and we have the two projections

$$\tilde{\pi}_{T^*(Q/G)} : \widetilde{Q}^* \to T^*(Q/G);\ \ (q, y_{[q]}) \mapsto y_{[q]}$$

and

$$\tilde{\pi} : \widetilde{Q}^* \to Q;\ \ (q, y_{[q]}) \mapsto q.$$

Then, it is easily shown that $\widetilde{Q}^*$ is also a vector bundle over Q, which is isomorphic to the annihilator $V(Q)^\circ \subset T^*Q$ of the vertical bundle $V(Q) = \operatorname{Ker} T\pi \subset TQ$, where the fibers of these vector subbundles are given by, for each $q \in Q$, $V(Q)_q := \operatorname{Ker} T_q\pi = \{\xi_Q(q) \mid \xi \subset \mathfrak{g}\} \subset T_qQ$ and

$$V(Q)^\circ_q := \left\{ p_q \in T^*_qQ \mid \langle p_q, \xi_Q(q) \rangle = 0 \right\} \subset T^*_qQ.$$

Let us consider the product space $\widetilde{Q}^* \times \mathfrak{g}^*$, which is isomorphic to T^*Q by

$$\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*;\ \ p_q \mapsto (q, y_{[q]} = (p_q)^{h^*}_q, \mu = \mathbf{J}(p_q)),$$

where the map $(\cdot)^h_q : T_{[q]}(Q/G) \to T_qQ$ is the horizontal lift map, $(\cdot)^{h^*}_q : T^*Q \to T^*_{[q]}(Q/G)$ denotes its dual, and $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is the equivariant momentum map associated with the lifted action $\langle \mathbf{J}(p_q), \xi \rangle = \langle p_q, \xi_Q(q) \rangle$ for $p_q \in T^*Q$ and $\xi \in \mathfrak{g}$.

The action of G on the product space $\widetilde{Q}^* \times \mathfrak{g}^*$ is given by, for $g \in G$ and $(q, y_{[q]}, \mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$,

$$g \cdot (q, y_{[q]}, \mu) = (gq, y_{[q]}, g\mu) = (gq, y_{[q]}, \operatorname{Ad}^*_{g^{-1}}\mu).$$

In the above, we employ the concatenation notation $g\mu \equiv \operatorname{Ad}^*_{g^{-1}}\mu$ for the coadjoint action. The inverse map of $\bar{\lambda}$ is given by, for each $(q, y_{[q]}, \mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$,

$$\bar{\lambda}^{-1} : \widetilde{Q}^* \times \mathfrak{g}^* \to T^*Q;\ \ (q, y_{[q]}, \mu) \mapsto (q, T^*_q\pi(y_{[q]}) + A^*(q)\mu).$$

We can naturally define the canonical projection $\tilde{\pi}_Q = \pi_Q \circ \bar{\lambda}^{-1} : \widetilde{Q}^* \times \mathfrak{g}^* \to Q$ given by

$$\tilde{\pi}_Q : \widetilde{Q}^* \times \mathfrak{g}^* \to Q;\ (q, y_{[q]}, \mu) \mapsto q,$$

where $\pi_Q : T^*Q \to Q$ is the cotangent bundle projection. Then, the differential map of $\tilde{\pi}_Q$ may be given by

$$T\tilde{\pi}_Q : T(\widetilde{Q}^* \times \mathfrak{g}^*) \to TQ;\ (q, y_{[q]}, \mu, \delta q, \delta y_{[q]}, \delta\mu) \to (q, \delta q).$$

The quotient space $(\widetilde{Q}^* \times \mathfrak{g}^*)/G$. Recall that the momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ for the cotangent lift of *left translation* of G on Q is given by

$$\langle \mathbf{J}(p_q), \xi \rangle = \langle p_q, \xi_Q(q) \rangle ,$$

where $\xi \in \mathfrak{g}$. Note that the isomorphism $\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*;\ p_q \mapsto (q, y_{[q]} = (p_q)_q^{h^*}, \mu = \mathbf{J}(p_q))$ is a *right trivialization*. Taking the quotient of $\bar{\lambda}$ by the action of G, one obtains the isomorphism $[\bar{\lambda}]_G : (T^*Q)/G \to (\widetilde{Q}^* \times \mathfrak{g}^*)/G$ given by

$$[p_q]_G \mapsto \left((p_q)_q^{h^*}, [q, \mathbf{J}(p_q)]_G \right) = \left(y_{[q]}, [q, \mu]_G \right).$$

This map is well defined because of equivariance of the momentum map; namely,

$$\left(y_{[q]}, [q, \mu]_G \right) = \left((p_q)_q^{h^*}, \left[gq, \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(p_q) \right]_G \right).$$

This observation shows that taking the quotient of the right trivialization $\widetilde{Q}^* \times \mathfrak{g}^*$ by the action of G leads to the isomorphism $T^*Q/G \cong (\widetilde{Q}^* \times \mathfrak{g}^*)/G$ as a *left trivialization*, from which one has the following isomorphisms:

$$T^*Q/G \cong (\widetilde{Q}^* \times \mathfrak{g}^*)/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*.$$

The quotient space $(\widetilde{Q}^* \times \mathfrak{g}^*)/G$ is known as the *Sternberg space*.

The canonical Dirac structure on T^*Q. Recall that the canonical one-form Θ_{T^*Q} on T^*Q is defined by, for $p_q \in T_q^*Q$ and $W_{p_q} \in T_{p_q}T^*Q$,

$$\Theta_{T^*Q}(p_q)(W_{p_q}) = \left\langle p_q, T_{p_q}\pi(W_{p_q}) \right\rangle$$

and also that the canonical two-form on T^*Q is given by $\Omega_{T^*Q} = -\mathbf{d}\Theta_{T^*Q}$. Then, the canonical Dirac structure on T^*Q, namely,

$$D \subset TT^*Q \oplus T^*T^*Q$$

is given by, for each $p_q \in T_q^*Q$,

$$\begin{aligned} D(p_q) = \{ &(V_{p_q}, \beta_{p_q}) \in T_{p_q}T^*Q \times T^*_{p_q}T^*Q \mid \\ &\beta_{p_q}(W_{p_q}) = \ \Omega_{T^*Q}(p_q)(V_{p_q}, W_{p_q}) \ \text{ for all } \ W_{p_q} \in T_{p_q}T^*Q \}. \end{aligned} \tag{17}$$

Symplectic forms on $\widetilde{Q}^* \times \mathfrak{g}^*$. Using the isomorphism $\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*$, let us define a one-form Θ on $\widetilde{Q}^* \times \mathfrak{g}^*$ by $\Theta = \bar{\lambda}_* \Theta_{T^*Q}$ and a symplectic two-form Ω on $\widetilde{Q}^* \times \mathfrak{g}^*$ by $\Omega = \bar{\lambda}_* \Omega_{T^*Q}$, where, needless to say, $\Omega = -\mathbf{d}\Theta$ holds since $\mathbf{d}$ and $\bar{\lambda}_*$ commute.

Proposition 3. *A one-form Θ on $\widetilde{Q}^* \times \mathfrak{g}^*$ can be defined by $\Theta = \bar{\lambda}_* \Theta_{T^*Q}$, which is given by*

$$\Theta = \gamma^* \Theta_{T^*(Q/G)} + \tilde{\pi}_Q^* \alpha_\mu,$$

where $\tilde{\pi}_Q = \pi_Q \circ \bar{\lambda}^{-1} : \widetilde{Q}^* \times \mathfrak{g}^* \to Q;\ (q, y, \mu) \mapsto q$ is the natural projection, $\gamma : \widetilde{Q}^* \times \mathfrak{g}^* \to T^*(Q/G)$ is the projection given by $(q, y, \mu) \mapsto ([q], y)$, $\Theta_{T^*(Q/G)}$ is the canonical one-form on $T^*(Q/G)$, α_μ is the one-form on Q associated to $\mu \in \mathfrak{g}^*$, which is defined by $\alpha_\mu(q) = A^*(q)\mu$, namely, $\langle \alpha_\mu(q), \delta q \rangle = \langle \mu, A(q)(\delta q) \rangle$ for all $\delta q \in T_qQ$.

A two-form Ω on $\widetilde{Q}^* \times \mathfrak{g}^*$ can be defined by $\Omega = -\mathbf{d}\Theta = \bar{\lambda}_* \Omega_{T^*Q}$, which is given by

$$\Omega = \gamma^* \Omega_{T^*(Q/G)} - \tilde{\pi}_Q^* B_\mu + \omega,$$

where $\Omega_{T^*(Q/G)}$ is the canonical symplectic structure on $T^*(Q/G)$, ω is a symplectic two-form on $\widetilde{Q}^* \times \mathfrak{g}^*$, and B_μ is a two-form defined by, for $q \in Q$ and $\mu \in \mathfrak{g}^*$, $B_\mu(q)(v_q, w_q) = \langle \mu, B(q)(v_q, w_q) \rangle$, where B is the curvature two-form of A.

In the above, the symplectic two-form ω is given, in a local trivialization $(x, g, y, \mu) \in U \times G \times U^* \times \mathfrak{g}^*$ for $(q, y, \mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$, by

$$\begin{aligned} &\omega(x, g, y, \mu)\left((\dot{x}, \dot{g}, \dot{y}, \dot{\mu}), (\delta x, \delta g, \delta y, \delta\mu)\right) \\ &\qquad = \left\langle \delta\mu, \dot{g}g^{-1} \right\rangle - \left\langle \dot{\mu}, \delta g g^{-1} \right\rangle + \left\langle \mu, [\dot{g}g^{-1}, \delta g g^{-1}] \right\rangle, \end{aligned}$$

where U is an open subset of $\mathbb{R}^r$ with $r = \dim(Q/G)$.

Proof. By direct computations using $(x, g, y, \mu) \cong (q, y, \mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$, it follows that the one-form Θ on $\widetilde{Q}^* \times \mathfrak{g}^*$ may be given by

$$\begin{aligned} \Theta(x, g, y, \mu)(\delta x, \delta g, \delta y, \delta\mu) &= \bar{\lambda}_* \Theta_{T^*Q}(x, g, y, \mu)(\delta x, \delta g, \delta y, \delta\mu) \\ &= \Theta_{T^*Q}(\bar{\lambda}^{-1}(x, g, y, \mu)) \left(T_{(x,g,y,\mu)} \bar{\lambda}^{-1}(\delta x, \delta g, \delta y, \delta\mu)\right) \\ &= \bar{\lambda}^{-1}(x, g, y, \mu) \left(T_{\bar{\lambda}^{-1}(x,g,y,\mu)} \pi_Q \circ T_{(x,g,y,\mu)} \bar{\lambda}^{-1}(\delta x, \delta g, \delta y, \delta\mu)\right) \\ &= \left(T^*_{(x,g)} \pi(y) + A^*(x, g)\mu\right) \left(T_{(x,g,y,\mu)}(\pi_Q \circ \bar{\lambda}^{-1})(\delta x, \delta g, \delta y, \delta\mu)\right) \\ &= \left(T^*_{(x,g)} \pi(y) + \alpha_\mu(x, g)\right) \left(T_{(x,g,y,\mu)} \tilde{\pi}_Q(\delta x, \delta g, \delta y, \delta\mu)\right) \\ &= T^*_{(x,g)} \pi(y) \left(T_{(x,g,y,\mu)} \tilde{\pi}(\delta x, \delta g, \delta y, \delta\mu)\right) + \tilde{\pi}_Q^* \alpha_\mu(x, g)(\delta x, \delta g, \delta y, \delta\mu) \\ &= \left\langle y, T_{(x,g)} \pi(\delta x, \delta g) \right\rangle + \tilde{\pi}_Q^* \alpha_\mu(x, g)(\delta x, \delta g, \delta y, \delta\mu) \\ &= \gamma^* \Theta_{T^*(Q/G)}(x, g, y, \mu)(\delta x, \delta g, \delta y, \delta\mu) + \tilde{\pi}_Q^* \alpha_\mu(x, g)(\delta x, \delta g, \delta y, \delta\mu). \end{aligned}$$

In the above, $\Theta_{T^*(Q/G)}$ is the canonical two-form on $T^*(Q/G)$, which is given by, for $(x, y) \in T^*(Q/G)$ and $(\delta x, \delta y) \in T_{(x,y)} T^*(Q/G)$,

$$\Theta_{T^*(Q/G)}(x, y)(\delta x, \delta y) = \left\langle y, T_{(x,y)} \pi_{Q/G}(\delta x, \delta y) \right\rangle,$$

where $x = [q]$ and the commutative relation $\pi \circ \tilde{\pi}_Q = \pi_{Q/G} \circ \gamma$ and $T_{(x,g)} \pi(\delta x, \delta g) = T_{(x,y)} \pi_{Q/G}(\delta x, \delta y)$ holds. Thus, we can write the one-form Θ on $\widetilde{Q}^* \times \mathfrak{g}^*$ as

$$\Theta = \gamma^* \Theta_{T^*(Q/G)} + \tilde{\pi}_Q^* \alpha_\mu.$$

Next, let us compute the two-form Ω on $\widetilde{Q}^* \times \mathfrak{g}^*$ and recall that, for vector fields $X, Y \in \mathfrak{X}(\widetilde{Q}^* \times \mathfrak{g}^*)$, one has

$$\mathbf{d}\Theta(X, Y) = X[\Theta(Y)] - Y[\Theta(X)] - \Theta([X, Y]).$$

The vector fields X, Y on $\widetilde{Q}^* \times \mathfrak{g}^*$ may be locally decomposed as

$$X = (X^1, X^2),\ Y = (Y^1, Y^2) \in \mathfrak{X}(U \times G) \times \mathfrak{X}(U^* \times \mathfrak{g}^*),$$

which are given by, for $(x, g, y, \mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$,

$$\begin{aligned} X(x, g, y, \mu) &= \left(X^1_{(x,g,y,\mu)}(x, g), X^2_{(x,g,y,\mu)}(y, \mu)\right) = ((\dot{x}, \dot{g}), (\dot{y}, \dot{\mu})), \\ Y(x, g, y, \mu) &= \left(Y^1_{(x,g,y,\mu)}(x, g), Y^2_{(x,g,y,\mu)}(y, \mu)\right) = ((\delta x, \delta g), (\delta y, \delta\mu)). \end{aligned}$$

So, the flow of X is given by

$$(t,x,g,y,\mu)\mapsto(\phi_t^1(x,g),\phi_t^2(y,\mu))=((x+t\,\dot{x},\exp(\dot{g}g^{-1})t\cdot g),(y+t\,\dot{y},\ \mu+t\,\dot{\mu})).$$

Hence, it follows that

$$\begin{aligned}X[\Theta(Y)](x,g,y,\mu)&=\pounds_X(\Theta(Y))(x,g,y,\mu)\\&=\left.\frac{d}{dt}\right|_{t=0}\Theta(\phi_t^1(x,g),\phi_t^2(y,\mu))(Y^1_{(x,g,y,\mu)}(x,g),Y^2_{(x,g,y,\mu)}(y,\mu))\\&=\left.\frac{d}{dt}\right|_{t=0}\left\langle\phi_t^2(y,\mu),\,Y^1_{(x,g,y,\mu)}(x,g)g^{-1}\right\rangle\\&=\langle\dot{y},\delta x\rangle+\left\langle\dot{\mu},\delta g g^{-1}\right\rangle.\end{aligned}$$

Similarly, one can easily obtain

$$\begin{aligned}Y[\Theta(X)](x,g,y,\mu)&=\pounds_Y(\Theta(X))(x,g,y,\mu)\\&=\langle\delta y,\dot{x}\rangle+\left\langle\delta\mu,\dot{g}g^{-1}\right\rangle.\end{aligned}$$

The horizontal and vertical parts of $X^1(x,g)=(\dot{x},\dot{g})$ and $Y^1(x,g)=(\delta x,\delta g)$ are given by

$$\begin{aligned}&\mathrm{Hor}(X^1(x,g))=X^1(x,g)-\mathrm{Ver}(X^1(x,g))=\dot{x},\quad \mathrm{Ver}(X^1(x,g))=\dot{g},\\&\mathrm{Hor}(Y^1(x,g))=Y^1(x,g)-\mathrm{Ver}(Y^1(x,g))=\delta x,\quad \mathrm{Ver}(Y^1(x,g))=\delta g.\end{aligned}$$

For $(x,g,\dot{x},\dot{g})$ and $(x,g,\delta x,\delta g)\in T_{(x,g)}(U\times G)$,

$$\begin{aligned}&A(x,g)\cdot(\dot{x},\dot{g})=\mathrm{Ad}_g(A(x,e)\cdot\dot{x}+g^{-1}\dot{g})=A(x,g)\cdot\dot{x}+\dot{g}g^{-1},\\&A(x,g)\cdot(\delta x,\delta g)=\mathrm{Ad}_g(A(x,e)\cdot\delta x+g^{-1}\delta g)=A(x,g)\cdot\delta x+\delta g g^{-1},\end{aligned}$$

where $A(x,e)\cdot\dot{x}=A(x,e,\dot{x},0)$, $A(x,e)\cdot\delta x=A(x,e,\delta x,0)$, $\mathrm{Ad}_g g^{-1}\dot{g}=\dot{g}g^{-1}$ and $\mathrm{Ad}_g g^{-1}\delta g=\delta g g^{-1}$.

Hence, we can compute the term $\Theta([X,Y])$ as

$$\begin{aligned}&\Theta([X,Y])(x,g,y,\mu)\\&\quad=\Theta(x,g,y,\mu)([X^1,Y^1]\left(x,g),[X^2,Y^2](y,\mu)\right)\\&\quad=\left\langle y_x,T_{(x,g)}\pi([X^1,Y^1](x,g))\right\rangle+\left\langle\mu,A(x,g)([X^1,Y^1](x,g))\right\rangle\\&\quad=\left\langle y_x,[T_{(x,g)}\pi(X^1(x,g)),T_{(x,g)}\pi(Y^1(x,g))]\right\rangle\\&\qquad\quad+\left\langle\mu,A(x,g)([\mathrm{Hor}(X^1(x,g)),\mathrm{Hor}(Y^1(x,g))])\right\rangle\\&\qquad\qquad\qquad+\left\langle\mu,A(x,g)([\mathrm{Ver}(X^1(x,g)),\mathrm{Ver}(Y^1(x,g))])\right\rangle\\&\quad=-\left\langle\mu,B(x,g)((\dot{x},\dot{g}),(\delta x,\delta g))\right\rangle+\left\langle\mu,[\dot{g}g^{-1},\delta g g^{-1}]\right\rangle.\end{aligned}$$

In the above, we note

$$\begin{aligned}B(x,g)((\dot{x},\dot{g}),(\delta x,\delta g))&=-A(x,g)([\mathrm{Hor}(X^1(x,g)),\mathrm{Hor}(Y^1(x,g))])\\&=-A(x,g)([(\dot{x},0),(\delta x,0)])\end{aligned}$$

and

$$[\dot{g}g^{-1},\delta g g^{-1}]=A(x,g)([\mathrm{Ver}(X^1(x,g)),\mathrm{Ver}(Y^1(x,g))]).$$

Since one has

$$\Omega=-\mathbf{d}\Theta(X,Y)=-X[\Theta(Y)]+Y[\Theta(X)]+\Theta([X,Y]),$$

it follows that, for $(x,g,y,\mu)\in\widetilde{Q}^*\times\mathfrak{g}^*$,

$$\begin{aligned}&\Omega(x,g,y,\mu)\left((\dot x,\dot g,\dot y,\dot\mu),(\delta x,\delta g,\delta y,\delta\mu)\right)\\&\quad=\Omega_{T^*(Q/G)}(x,y)\left((\dot x,\dot y),(\delta x,\delta y)\right)-B_\mu(x,g)((\dot x,\dot g),(\delta x,\delta g))\\&\qquad\qquad+\omega(x,g,y,\mu)\left((\dot x,\dot g,\dot y,\dot\mu),(\delta x,\delta g,\delta y,\delta\mu)\right),\end{aligned}$$

where

$$\begin{aligned}\Omega_{T^*(Q/G)}(x,y)\left((\dot x,\dot y),(\delta x,\delta y)\right)&=\langle\delta y,\dot x\rangle-\langle\dot y,\delta x\rangle\,,\\B_\mu(x,g)((\dot x,\dot g),(\delta x,\delta g))&=\langle\mu,B(x,g)((\dot x,\dot g),(\delta x,\delta g))\rangle\,,\end{aligned}$$

and

$$\begin{aligned}&\omega(x,g,y,\mu)\left((\dot x,\dot g,\dot y,\dot\mu),(\delta x,\delta g,\delta y,\delta\mu)\right)\\&\qquad=\left\langle\delta\mu,\dot g g^{-1}\right\rangle-\left\langle\dot\mu,\delta g g^{-1}\right\rangle+\left\langle\mu,[\dot g g^{-1},\delta g g^{-1}]\right\rangle.\end{aligned}$$

Thus, we obtain

$$\Omega=\gamma^*\Omega_{T^*(Q/G)}-\tilde\pi_Q^*B_\mu+\omega.$$

□

A Dirac structure on $\widetilde{Q}^*\times\mathfrak{g}^*$. A Dirac structure $\bar D$ on $\widetilde{Q}^*\times\mathfrak{g}^*$ can be defined from the canonical Dirac structure D on T^*Q in equation (17) by using the *forward Dirac map* $\mathcal{F}T\bar\lambda:\mathrm{Dir}(TT^*Q)\to\mathrm{Dir}(T(\widetilde{Q}^*\times\mathfrak{g}^*))$ associated with the tangent map $T\bar\lambda:TT^*Q\to T(\widetilde{Q}^*\times\mathfrak{g}^*)$ of the right trivialization $\bar\lambda:T^*Q\to\widetilde{Q}^*\times\mathfrak{g}^*$ as

$$\bar D=\mathcal{F}T\bar\lambda(D),$$

which is given by, for each $p_q\in T^*Q$,

$$\begin{aligned}\bar D(\bar\lambda(p_q))=\{(T\bar\lambda(w_{p_q}),\alpha_{\bar\lambda(p_q)})\mid w_{p_q}\in T_{p_q}T^*Q,\ \alpha_{\bar\lambda(p_q)}\in T^*_{\bar\lambda(p_q)}(\widetilde{Q}^*\times\mathfrak{g}^*),\\(w_{p_q},T^*\bar\lambda(\alpha_{\bar\lambda(p_q)}))\in D(p_q)\},\end{aligned}$$

where $\bar\lambda(p_q)=(q,y,\mathbf{J}(p_q))=(q,y,\mu)\in\widetilde{Q}^*\times\mathfrak{g}^*$.

The Dirac structure $\bar D$ on $\widetilde{Q}^*\times\mathfrak{g}^*$ is given, for $(x,g,y,\mu)\cong(q,y,\mu)\in\widetilde{Q}^*\times\mathfrak{g}^*$, by

$$\begin{aligned}&\bar D(x,g,y,\mu)\\&=\{((\dot x,\dot g,\dot y,\dot\mu),(\kappa,\nu,v,\eta))\in T_{(x,g,y,\mu)}(\widetilde{Q}^*\times\mathfrak{g}^*)\times T^*_{(x,g,y,\mu)}(\widetilde{Q}^*\times\mathfrak{g}^*)\mid\\&\langle\kappa,\delta x\rangle+\langle\nu,\delta g\rangle+\langle\delta y,v\rangle+\langle\delta\mu,\eta\rangle=\Omega(x,g,y,\mu)((\dot x,\dot g,\dot y,\dot\mu),(\delta x,\delta g,\delta y,\delta\mu))\\&\qquad\text{for all}\quad(\delta x,\delta g,\delta y,\delta\mu)\in T_{(x,g,y,\mu)}(\widetilde{Q}^*\times\mathfrak{g}^*)\}.\end{aligned}\tag{18}$$

Invariance of Dirac structures. We recall the natural definition of invariant Dirac structures (see, for instance, [35, 36, 53] and [7]). Let P be a manifold and $D_P\subset TP\oplus T^*P$ be a Dirac structure on P with a Lie group G acting freely and properly on P. We denote this action by $\Phi:G\times P\to P$ and the action of a group element $h\in G$ on a point $x\in P$ by $h\cdot x=\Phi(h,x)=\Phi_h(x)$, so that $\Phi_h:P\to P$. Then, a *Dirac structure D_P is G–invariant* if

$$(\Phi_{h^*}X,(\Phi_h^*)^{-1}\alpha)\in D_P$$

for all $h\in G$ and $(X,\alpha)\in D_P$.

Next, we show that the Dirac structure $\bar{D}$ on $\widetilde{Q}^* \times \mathfrak{g}^*$ is G–invariant. To do this, let $\Phi : G \times (\widetilde{Q}^* \times \mathfrak{g}^*) \to \widetilde{Q}^* \times \mathfrak{g}^*$ denote the G–action on $\widetilde{Q}^* \times \mathfrak{g}^*$, so that holding $h \in G$ fixed, $\Phi_h : \widetilde{Q}^* \times \mathfrak{g}^* \to \widetilde{Q}^* \times \mathfrak{g}^*$ is given by, for each $(x, g, y, \mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$,

$$\Phi_h(x, g, y, \mu) = (x, hg, y, h\mu),$$

where G acts on the components of $\widetilde{Q}^*$ by left multiplication and it acts on the component of $\mathfrak{g}^*$ by the coadjoint group action.

Since the action on T^*Q is canonical, the corresponding symplectic structure Ω on $\widetilde{Q}^* \times \mathfrak{g}^*$ is also G–invariant; namely,

$$\Phi_h^* \Omega = \Omega$$

for all $h \in G$. The G-invariance of Dirac structures is given by, for all $(X, \alpha) \in \bar{D}$,

$$(\Phi_{h^*} X, (\Phi_h^*)^{-1}\alpha) \in \bar{D},$$

which is restated by

$$\mathcal{F}\tilde{\Phi}_h(\bar{D}) = \bar{D},$$

where $\tilde{\Phi}_h : T(\widetilde{Q}^* \times \mathfrak{g}^*) \to T(\widetilde{Q}^* \times \mathfrak{g}^*)$ is the tangent lift of $\Phi_h : \widetilde{Q}^* \times \mathfrak{g}^* \to \widetilde{Q}^* \times \mathfrak{g}^*$, which is given by, for each $(x, g, y, \mu, \dot{x}, \dot{g}, \dot{y}, \dot{\mu}) \in T(\widetilde{Q}^* \times \mathfrak{g}^*)$,

$$h \cdot (x, g, y, \mu, \dot{x}, \dot{g}, \dot{y}, \dot{\mu}) \mapsto (x, hg, y, h\mu, \dot{x}, h\dot{g}, \dot{y}, h\dot{\mu}),$$

where $h\mu = \mathrm{Ad}^*_{h^{-1}}\mu$, $h\dot{g} = T_g L_h \dot{g}$ and $h\dot{\mu} = \mathrm{Ad}^*_{h^{-1}}\dot{\mu}$.

In fact, the left G-invariance of the Dirac structure can be represented by

$$\bar{D}(x, hg, y, h\mu) = \bar{D}(x, g, y, \mu),$$

for all $h \in G$ and $(x, g, y, \mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$, which is given, in view of equation (18), by

$$\begin{aligned} &\bar{D}(x, hg, y, h\mu) \\ &\quad = \{((\dot{x}, h\dot{g}, \dot{y}, h\dot{\mu}), (\kappa, h\nu, v, h\eta)) \in T_{(x,hg,y,h\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*) \times T^*_{(x,hg,y,h\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*) \mid \\ &\qquad \langle \kappa, \delta x\rangle + \langle h\nu, h\delta g\rangle + \langle \delta y, v\rangle + \langle h\delta\mu, h\eta\rangle \\ &\qquad\quad = \Omega(x, hg, y, h\mu)((\dot{x}, h\dot{g}, \dot{y}, h\dot{\mu}), (\delta x, h\delta g, \delta y, h\delta\mu)) \\ &\qquad\qquad \text{for all } \ (\delta x, h\delta g, \delta y, h\delta\mu) \in T_{(x,hg,y,h\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*)\}. \end{aligned}$$

In the above, we can easily verify that the symplectic structure Ω is G–invariant as follows; for each $(x, g, y, \mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$,

$$\begin{aligned} &\Omega(x, hg, y, h\mu)((\dot{x}, h\dot{g}, \dot{y}, h\dot{\mu}), (\delta x, h\delta g, \delta y, h\delta\mu)) \\ &\quad = \langle \delta y, \dot{x}\rangle - \langle \dot{y}, \delta x\rangle - \langle h\mu, B(x, hg)((\dot{x}, h\dot{g}), (\delta x, h\delta g))\rangle \\ &\qquad + \omega(x, hg, y, h\mu)\left((\dot{x}, h\dot{g}, \dot{y}, h\dot{\mu}), (\delta x, h\delta g, \delta y, h\delta\mu)\right), \end{aligned}$$

where

$$\begin{aligned} \langle h\mu, B(x, hg)((\dot{x}, h\dot{g}), (\delta x, h\delta g))\rangle &= \langle \mathrm{Ad}^*_{h^{-1}}\mu, \mathrm{Ad}_h B(x, g)((\dot{x}, \dot{g}), (\delta x, \delta g))\rangle \\ &= \langle \mu, B(x, g)((\dot{x}, \dot{g}), (\delta x, \delta g))\rangle \end{aligned}$$

and

$$\begin{aligned} &\omega(x, hg, y, h\mu)\left((\dot{x}, h\dot{g}, \dot{y}, h\dot{\mu}), (\delta x, h\delta g, \delta y, h\delta\mu)\right) \\ &\quad = \left\langle h\delta\mu, h(\dot{g}g^{-1})\right\rangle - \left\langle h\dot{\mu}, h(\delta g g^{-1})\right\rangle + \left\langle h\mu, [h(\dot{g}g^{-1}), h(\delta g g^{-1})]\right\rangle \\ &\quad = \left\langle \mathrm{Ad}^*_{h^{-1}}\delta\mu, \mathrm{Ad}_h(\dot{g}g^{-1})\right\rangle - \left\langle \mathrm{Ad}^*_{h^{-1}}\dot{\mu}, \mathrm{Ad}_h(\delta g g^{-1})\right\rangle \\ &\qquad\qquad + \left\langle \mathrm{Ad}^*_{h^{-1}}\mu, \mathrm{Ad}_h[\dot{g}g^{-1}, \delta g g^{-1}]\right\rangle \\ &\quad = \left\langle \delta\mu, \dot{g}g^{-1}\right\rangle - \left\langle \dot{\mu}, \delta g g^{-1}\right\rangle + \left\langle \mu, [\dot{g}g^{-1}, \delta g g^{-1}]\right\rangle. \end{aligned}$$

The quotient space of the Pontryagin bundle $TT^*Q \oplus T^*T^*Q$. Recall that one has the isomorphism

$$\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*; \quad p_q \mapsto (q, (p_q)_q^{h^*}, \mathbf{J}(p_q)) = (q, y_{[q]}, \mu),$$

where $y_{[q]} = (p_q)_q^{h^*}$ and $\mu = \mathbf{J}(p_q)$.

Using the map $\bar{\lambda}$, one has an isomorphism regarding with $TT^*Q \oplus T^*T^*Q$ as

$$TT^*Q \oplus T^*T^*Q \cong T(\widetilde{Q}^* \times \mathfrak{g}^*) \oplus T^*(\widetilde{Q}^* \times \mathfrak{g}^*).$$

The action of an element $h \in G$ on an element $(x, g, y, \mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$ is given by

$$\begin{aligned} h \cdot (x, g, y, \mu) &= (x, hg, y, h\mu) \\ &= (x, hg, y, \mathrm{Ad}^*_{h^{-1}}\mu) \in \widetilde{Q}^* \times \mathfrak{g}^*, \end{aligned}$$

the action on an element $(\dot{x}, \dot{g}, \dot{y}, \dot{\mu}) \in T_{(x,g,y,\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*)$ is denoted by

$$\begin{aligned} h \cdot (\dot{x}, \dot{g}, \dot{y}, \dot{\mu}) &= (\dot{x}, h\dot{g}, \dot{y}, h\dot{\mu}) \\ &= (\dot{x}, h\dot{g}, \dot{y}, \mathrm{Ad}^*_{h^{-1}}\dot{\mu}) \in T_{(x,hg,y,h\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*), \end{aligned}$$

and the action on $(\kappa, \nu, v, \eta) \in T^*_{(x,g,y,\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*)$ is given by

$$\begin{aligned} h \cdot (\kappa, \nu, v, \eta) &= (\kappa, h\nu, v, h\eta) \\ &= (\kappa, h\nu, v, \mathrm{Ad}_h \eta) \in T^*_{(x,hg,y,h\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*). \end{aligned}$$

Recall that taking the quotient of $\widetilde{Q}^* \times \mathfrak{g}^*$ by the action of G leads to the identification

$$T^*Q/G \cong (\widetilde{Q}^* \times \mathfrak{g}^*)/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*,$$

and one has also the following isomorphisms:

$$\begin{aligned} (TQ \oplus T^*Q)/G &\cong \left((\widetilde{Q} \times \mathfrak{g}) \oplus (\widetilde{Q}^* \times \mathfrak{g}^*) \right) /G \\ &\cong T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}, \\ (TT^*Q)/G &\cong T(\widetilde{Q}^* \times \mathfrak{g}^*)/G \\ &\cong TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}), \\ (T^*T^*Q)/G &\cong T^*(\widetilde{Q}^* \times \mathfrak{g}^*)/G \\ &\cong T^*T^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}^*). \end{aligned}$$

Further, the quotient of $TT^*Q \oplus T^*T^*Q$ by the action of G induces the isomorphisms:

$$\begin{aligned} (TT^*Q \oplus T^*T^*Q)/G &\cong T(\widetilde{Q}^* \times \mathfrak{g}^*)/G \oplus T^*(\widetilde{Q}^* \times \mathfrak{g}^*)/G \\ &\cong TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}) \oplus T^*T^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}^*) \\ &\cong TT^*(Q/G) \oplus T^*T^*(Q/G) \oplus \left(\tilde{\mathfrak{g}}^* \times (\widetilde{V} \oplus \widetilde{V}^*) \right), \end{aligned}$$

where we employed $(\tilde{\mathfrak{g}}^* \times \widetilde{V}) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}^*) \cong \tilde{\mathfrak{g}}^* \times (\widetilde{V} \oplus \widetilde{V}^*)$.

Dirac cotangent bundle reduction. In view of the isomorphism $T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*$, it goes without saying that the canonical Dirac structure D on T^*Q can be identified with $\bar{D}$ on $\widetilde{Q}^* \times \mathfrak{g}^*$.

By the left G-invariance of $\bar{D}$, it follows that, for $h \in G$ and $(x, g, y, \mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$,

$$\bar{D}(x, hg, y, h\mu) = \bar{D}(x, g, y, \mu).$$

Then, one can uniquely determine $\bar{D}$ by its value at the point $(x, e, y, g^{-1}\mu)$ as

$$\begin{aligned}\bar{D}(x, e, y, g^{-1}\mu) = \{ &\big((\dot{x}, g^{-1}\dot{g}, \dot{y}, g^{-1}\dot{\mu}), (\kappa, g^{-1}\nu, v, g^{-1}\eta)\big) \mid \\ &\langle \kappa, \delta x\rangle + \langle g^{-1}\nu, g^{-1}\delta g\rangle + \langle \delta y, v\rangle + \langle g^{-1}\delta\mu, g^{-1}\eta\rangle \\ &\quad = \Omega(x, e, y, g^{-1}\mu)((\dot{x}, g^{-1}\dot{g}, \dot{y}, g^{-1}\dot{\mu}), (\delta x, g^{-1}\delta g, \delta y, g^{-1}\delta\mu)) \\ &\qquad \text{for all } (\delta x, g^{-1}\delta g, \delta y, g^{-1}\delta\mu)\}.\end{aligned} \tag{19}$$

In the above, the symplectic two-form Ω on $\widetilde{Q}^* \times \mathfrak{g}^*$ takes its value at $(x, e, y, g^{-1}\mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$ as

$$\begin{aligned}&\Omega(x, e, y, g^{-1}\mu)((\dot{x}, g^{-1}\dot{g}, \dot{y}, g^{-1}\dot{\mu}), (\delta x, g^{-1}\delta g, \delta y, g^{-1}\delta\mu)) \\ &\quad = \langle \delta y, \dot{x}\rangle - \langle \dot{y}, \delta x\rangle - \left\langle g^{-1}\mu, B(x, e)((\dot{x}, g^{-1}\dot{g}), (\delta x, g^{-1}\delta g))\right\rangle \\ &\qquad + \omega(x, e, y, g^{-1}\mu)\left((\dot{x}, g^{-1}\dot{g}, \dot{y}, g^{-1}\dot{\mu}), (\delta x, g^{-1}\delta g, \delta y, g^{-1}\delta\mu)\right),\end{aligned}$$

where

$$\begin{aligned}&\omega(x, e, y, g^{-1}\mu)\left((\dot{x}, g^{-1}\dot{g}, \dot{y}, g^{-1}\dot{\mu}), (\delta x, g^{-1}\delta g, \delta y, g^{-1}\delta\mu)\right) \\ &\quad = \left\langle g^{-1}\delta\mu, g^{-1}\dot{g}\right\rangle - \left\langle g^{-1}\dot{\mu}, g^{-1}\delta g\right\rangle + \left\langle g^{-1}\mu, [g^{-1}\dot{g}, g^{-1}\delta g]\right\rangle.\end{aligned}$$

It follows from equation (19) that, by taking the quotient of $\bar{D}$ by the action of G, one can develop a *structure* $[\bar{D}]_G := \bar{D}/G$ on the bundle $TT^*Q/G \cong TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$ over $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, which is given by, for each $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$,

$$\begin{aligned}&[\bar{D}]_G(x, y, \bar{\mu}) \\ &= \Big\{ \big((\dot{x}, \dot{y}, \bar{\xi}, \dot{\bar{\mu}}), (\kappa, v, \bar{\nu}, \bar{\eta})\big) \in (T_{(x,y)}T^*(Q/G) \times \widetilde{V}) \times (T^*_{(x,y)}T^*(Q/G) \times \widetilde{V}^*) \mid \\ &\langle \kappa, \delta x\rangle + \langle \delta y, v\rangle + \langle \bar{\nu}, \bar{\zeta}\rangle + \langle \delta\bar{\mu}, \bar{\eta}\rangle = [\Omega]_G(x, y, \bar{\mu})((\dot{x}, \dot{y}, \bar{\xi}, \dot{\bar{\mu}}), (\delta x, \delta y, \bar{\zeta}, \delta\bar{\mu})) \\ &\qquad \text{for all } (\delta x, \delta y, \bar{\zeta}, \delta\bar{\mu}) \in T_{(x,y)}T^*(Q/G) \times \widetilde{V} \Big\},\end{aligned} \tag{20}$$

where $\bar{\xi} = [q, \xi]_G = [q, \dot{g}g^{-1}]_G \in \tilde{\mathfrak{g}}$, $\bar{\zeta} = [q, \zeta]_G = [q, \delta g g^{-1}]_G \in \tilde{\mathfrak{g}}$, $\bar{\eta} = [q, \eta]_G \in \tilde{\mathfrak{g}}$, $\bar{\nu} = [q, \nu]_G \in \tilde{\mathfrak{g}}^*$, and $\bar{\mu} = [q, \mu]_G \in \tilde{\mathfrak{g}}^*$. Further, we note that $\widetilde{V} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^*$ and its dual $\widetilde{V}^* = \tilde{\mathfrak{g}}^* \oplus \tilde{\mathfrak{g}}$ are the bundles over $\tilde{\mathfrak{g}}^*$, where we recall $\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$ and $\tilde{\mathfrak{g}}^* = (Q \times \mathfrak{g}^*)/G$ are the associated bundles over Q/G.

Furthermore, $[\Omega]_G := \Omega/G$ is given by, for each $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$,

$$\begin{aligned}&[\Omega]_G(x, y, \bar{\mu})((\dot{x}, \dot{y}, \bar{\xi}, \dot{\bar{\mu}}), (\delta x, \delta y, \bar{\zeta}, \delta\bar{\mu})) \\ &\quad = \langle \delta y, \dot{x}\rangle - \langle \dot{y}, \delta x\rangle - \left\langle \bar{\mu}, \widetilde{B}(x)(\dot{x}, \delta x)\right\rangle + [\omega]_G(\bar{\mu})\big((\bar{\xi}, \dot{\bar{\mu}}), (\bar{\zeta}, \delta\bar{\mu})\big) \\ &\quad = \omega_{T^*(Q/G)}(x, y)((\dot{x}, \dot{y}), (\delta x, \delta y)) \oplus [\omega]_G(\bar{\mu})\big((\bar{\xi}, \dot{\bar{\mu}}), (\bar{\zeta}, \delta\bar{\mu})\big).\end{aligned} \tag{21}$$

In the above, $\omega_{T^*(Q/G)}(x, y) = \Omega_{T^*(Q/G)}(x, y) - \mathcal{B}_{\bar{\mu}}(x, y)$ can be regarded as a *reduced symplectic structure* that is fiberwisely defined on $T_{(x,y)}T^*(Q/G)$, where $\Omega_{T^*(Q/G)}$ is the canonical symplectic structure on $T^*(Q/G)$ and $\mathcal{B}_{\bar{\mu}} = \pi^*_{Q/G}\widetilde{B}_{\bar{\mu}}$ is a $\tilde{\mathfrak{g}}$*-valued reduced curvature form on* $T^*(Q/G)$, where $\widetilde{B}_{\bar{\mu}}(\dot{x}, \cdot) := \langle \bar{\mu}, \widetilde{B}(x)(\dot{x}, \cdot)\rangle$ and we note that the curvature two-form $B \equiv B^A$ of the connection A is reduced to a $\tilde{\mathfrak{g}}$–valued two-form $\widetilde{B} \equiv \widetilde{B}^A$ on Q/G given by $\widetilde{B}(x)(\delta x, \dot{x}) = [q, B(q)(\delta q, \dot{q})]_G$.

Similarly, $[\omega]_G := \omega/G$ is a *reduced symplectic structure* on $\tilde{\mathfrak{g}}^* \times \widetilde{V}$ that is fiber-wisely defined by, for each $\tilde{\mu} \in \tilde{\mathfrak{g}}^*$,

$$[\omega]_G(\bar{\mu})((\bar{\xi}, \dot{\bar{\mu}}), (\bar{\zeta}, \delta\bar{\mu})) = \left\langle \delta\bar{\mu}, \bar{\xi} \right\rangle - \left\langle \dot{\bar{\mu}}, \bar{\zeta} \right\rangle + \left\langle \bar{\mu}, [\bar{\xi}, \bar{\zeta}] \right\rangle, \tag{22}$$

where $(\bar{\xi}, \dot{\bar{\mu}}), (\bar{\zeta}, \delta\bar{\mu}) \in \widetilde{V}$. It is worth to note that the reduced symplectic structure $[\omega]_G$ on $\tilde{\mathfrak{g}}^* \times \widetilde{V}$ is the extended structure of the one that we derived for the case $Q = G$ (see [91]).

Notice that $[\Omega]_G = \omega_{T^*(Q/G)} \oplus [\omega]_G$ is a **reduced symplectic structure on the bundle** $TT^*Q/G \cong TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$, which is defined at each point $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, since it is skew symmetric and non-degenerate, as it can be easily checked.

Proposition 4. *The structure $[\bar{D}]_G$ given in equation (4) is restated by, for each $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$,*

$$\begin{aligned}
&[\bar{D}]_G(x, y, \bar{\mu}) \\
&= \Big\{ \big((\dot{x}, \dot{y}, \bar{\xi}, \dot{\bar{\mu}}), (\kappa, v, \bar{\nu}, \bar{\eta})\big) \in (T_{(x,y)}T^*(Q/G) \times \widetilde{V}) \times (T^*_{(x,y)}T^*(Q/G) \times \widetilde{V}^*) \mid \\
&\qquad \dot{x} = v, \quad \dot{y} + \kappa = -\widetilde{B}_{\bar{\mu}}(\dot{x}, \cdot), \quad \bar{\xi} = \bar{\eta}, \quad \dot{\bar{\mu}} + \bar{\nu} = \mathrm{ad}^*_{\bar{\xi}}\bar{\mu} \Big\}.
\end{aligned} \tag{23}$$

Proof. Assume that the condition (4) holds and it follows that, for each $(x, y, \bar{\mu})$,

$$\langle \kappa, \delta x \rangle + \langle \delta y, v \rangle + \langle \bar{\nu}, \bar{\zeta} \rangle + \langle \delta\bar{\mu}, \bar{\eta} \rangle = [\Omega]_G(x, y, \bar{\mu})((\dot{x}, \dot{y}, \bar{\xi}, \dot{\bar{\mu}}), (\delta x, \delta y, \bar{\zeta}, \delta\bar{\mu}))$$

for all $(\delta x, \delta y, \bar{\zeta}, \delta\bar{\mu}) \in T_{(x,y)}T^*(Q/G) \times \widetilde{V}$. First, by utilizing (21) and (22) and setting $\delta x = 0$ and $\delta\bar{\mu} = 0$, one has

$$\langle \delta y,\, v - \dot{x} \rangle + \left\langle \bar{\nu} + \dot{\bar{\mu}} - \mathrm{ad}^*_{\bar{\xi}}\bar{\mu},\, \bar{\zeta} \right\rangle = 0, \quad \text{for all } \ \delta y \text{ and } \bar{\zeta},$$

and it follows that $v = \dot{x}$ and $\dot{\bar{\mu}} + \bar{\nu} = \mathrm{ad}^*_{\bar{\xi}}\bar{\mu}$. Next, setting $\delta y = 0$ and $\bar{\zeta} = 0$, one has

$$\left\langle \dot{y} + \kappa + \widetilde{B}_{\bar{\mu}}(\dot{x}, \cdot), \delta x \right\rangle + \left\langle \delta\bar{\mu}, \bar{\eta} - \bar{\xi} \right\rangle = 0, \quad \text{for all } \ \delta x \text{ and } \delta\bar{\mu},$$

which leads to $\dot{y} + \kappa = -\widetilde{B}_{\bar{\mu}}(\dot{x}, \cdot)$ and $\bar{\eta} = \bar{\xi}$. The converse is shown in the same way. □

The gauged Dirac structure on $TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$. We have the proposition that the structure $[\bar{D}]_G$ on $TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$ over $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ can be understood as the direct sum of $[D]_G^{\mathrm{Hor}}$ on $TT^*(Q/G)$ over $T^*(Q/G)$ and $[D]_G^{\mathrm{Ver}}$ on $\tilde{\mathfrak{g}}^* \times \widetilde{V}$ over $\tilde{\mathfrak{g}}^*$ as shown below.

Proposition 5. *For each $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, one can express the structure*

$$[\bar{D}]_G(x, y, \bar{\mu}) = [\bar{D}]_G^{\mathrm{Hor}}(x, y) \oplus [\bar{D}]_G^{\mathrm{Ver}}(\bar{\mu}).$$

In the above, $[\bar{D}]_G^{\mathrm{Hor}}$ is given by, for each $(x, y) \in T^*(Q/G)$,

$$\begin{aligned}
[\bar{D}]_G^{\mathrm{Hor}}(x, y) = \Big\{ &((\dot{x}, \dot{y}), (\kappa, v)) \in T_{(x,y)}T^*(Q/G) \times T^*_{(x,y)}T^*(Q/G) \mid \\
&\langle \kappa, \delta x \rangle + \langle \delta y, v \rangle = \omega_{T^*(Q/G)}(x, y)((\dot{x}, \dot{y}), (\delta x, \delta y)) \\
&\qquad\qquad \textit{for all } \ (\delta x, \delta y) \in T_{(x,y)}T^*(Q/G) \Big\},
\end{aligned}$$

or equivalently,

$$[\bar{D}]_G^{\rm Hor}(x,y) = \Big\{ ((\dot{x},\dot{y}),(\kappa,v)) \in T_{(x,y)}T^*(Q/G) \times T^*_{(x,y)}T^*(Q/G) \mid \dot{x} = v, \quad \dot{y} + \kappa = -\widetilde{B}_{\bar{\mu}}(\dot{x},\cdot) \Big\}. \tag{24}$$

Further, $[\bar{D}]_G^{\rm Ver}$ *is given by, for each* $\bar{\mu} \in \tilde{\mathfrak{g}}^*$,

$$[\bar{D}]_G^{\rm Ver}(\bar{\mu}) = \Big\{ \big((\bar{\xi},\dot{\bar{\mu}}),(\bar{\nu},\bar{\eta})\big) \in \widetilde{V} \oplus \widetilde{V}^* \mid \langle \bar{\nu},\bar{\zeta}\rangle + \langle \delta\bar{\mu},\bar{\eta}\rangle = [\omega]_G(\bar{\mu})((\bar{\xi},\dot{\bar{\mu}}),(\bar{\zeta},\delta\bar{\mu})) \ \textit{for all} \ (\bar{\zeta},\delta\bar{\mu}) \in \widetilde{V} \Big\},$$

or equivalently,

$$[\bar{D}]_G^{\rm Ver}(\bar{\mu}) = \Big\{ \big((\bar{\xi},\dot{\bar{\mu}}),(\bar{\nu},\bar{\eta})\big) \in \widetilde{V} \oplus \widetilde{V}^* \mid \bar{\xi} = \bar{\eta}, \quad \dot{\bar{\mu}} + \bar{\nu} = \mathrm{ad}^*_{\bar{\xi}}\bar{\mu} \Big\}. \tag{25}$$

Proof. It is clear from Proposition 4 and also checked by direct computations. □

Thus, we have the following theorem, which is associated with Dirac cotangent bundle reduction.

Theorem 5.1. *For* $(y_x,\bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$*, the structure* $[\bar{D}]_G := \bar{D}/G$ *given by equation (4), or equivalently, equation (23) is a Dirac structure on the bundle* $TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$ *over* $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$*, which is the natural reduction of the canonical Dirac structure* D *on the cotangent bundle* T^*Q*. Further, the reduced Dirac structure* $[\bar{D}]_G$ *can be expressed by a direct sum of a Dirac structure* $[\bar{D}]_G^{\rm Hor}$ *on the bundle* $TT^*(Q/G)$ *over* $T^*(Q/G)$ *given in equation (24) and a Dirac structure* $[\bar{D}]_G^{\rm Ver}$ *on the bundle* $\tilde{\mathfrak{g}}^* \times \widetilde{V}$ *over* $\tilde{\mathfrak{g}}^*$ *given in equation (25).*

Proof. One can simply prove that for each $(y_x,\bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, equation (4) is a special case of the construction of a Dirac structure given by equation (1) on the reduced symplectic manifold $TT^*Q/G = TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$ with a symplectic two form $[\Omega]_G$. Similarly, one can show that $[\bar{D}]_G^{\rm Hor}$ is a Dirac structure on $TT^*(Q/G)$ and $[\bar{D}]_G^{\rm Ver}$ is a Dirac structure on $\tilde{\mathfrak{g}}^* \times \widetilde{V}$. □

Notice that $[\bar{D}]_G$ for the case $Q = G$, as in [91], is the reduced Dirac structure on the bundle $TT^*G \cong \mathfrak{g}^* \times V$, which is the special case of $[\bar{D}]_G^{\rm Ver}$ for the general Q.

Let us call the above reduced Dirac structure $[\bar{D}]_G = [\bar{D}]_G^{\rm Hor} \oplus [\bar{D}]_G^{\rm Ver}$ a **gauged Dirac structures** on the bundle $TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$ over $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, associated with the reduction of the canonical Dirac structure D on T^*Q. Here, $[\bar{D}]_G^{\rm Hor}$ is a **horizontal Dirac structure** on the bundle $TT^*(Q/G)$ over $T^*(Q/G)$ and $[\bar{D}]_G^{\rm Ver}$ is a **vertical Dirac structure** on the bundle $\tilde{\mathfrak{g}}^* \times \widetilde{V}$ over $\tilde{\mathfrak{g}}^*$.

Needless to say, we note that the *"gauged Dirac structure"* depends on a principal connection $A : TQ \to \mathfrak{g}$ and also that Q/G is a *shape space* and $\tilde{\mathfrak{g}}^* = (G \times \mathfrak{g}^*)/G$ is the associated bundle to $\mathfrak{g}^*$, regarded as a bundle over the shape space Q/G.

This construction of the reduced Dirac structure is quite consistent with the *reduced variational structures* shown in §4.

Relationship with reduction of Courant algebroids. This paragraph considers the relationship between Dirac reduction and the reduction of Courant algebroids for the case of $P = T^*Q$. By choosing a principal connection and using $\lambda : TQ \to \widetilde{Q} \times \mathfrak{g}$ and $\bar{\lambda} : P \to \widetilde{Q}^* \times \mathfrak{g}^*$, one can observe that

$$F = (TP \oplus T^*P)/G \cong TT^*(Q/G) \oplus T^*T^*(Q/G) \oplus \left(\tilde{\mathfrak{g}}^* \times (\widetilde{V} \oplus \widetilde{V}^*)\right)$$

is a Courant algebroid over $B = P/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$. Given the canonical Dirac structure D on $P = T^*Q$ as in equation (17), the quotient $[D]_G = D/G$ viewed as a subbundle of F gives the *gauged Dirac structure* on the bundle $TP/G \cong TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$. This is consistent with the case $P = T^*G$.

Another interesting thing relevant with Courant algebroids is that because the gauged Dirac structure $[\bar{D}]_G$ on the bundle $TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$ can be represented by a direct sum of the *horizontal Dirac structure* $[\bar{D}]_G^{\mathrm{Hor}}$ and the *vertical Dirac structure* $[\bar{D}]_G^{\mathrm{Ver}}$, namely,

$$[\bar{D}]_G = [\bar{D}]_G^{\mathrm{Hor}} \oplus [\bar{D}]_G^{\mathrm{Ver}},$$

one can regard F as

$$F = F^{\,\mathrm{Hor}} \oplus F^{\,\mathrm{Ver}}$$

over the bundle $B = B^{\,\mathrm{Hor}} \oplus B^{\,\mathrm{Ver}}$, where

$$F^{\,\mathrm{Hor}} = TT^*(Q/G) \oplus T^*T^*(Q/G)$$

is a **horizontal Courant algebroid** over the shape space $B^{\,\mathrm{Hor}} = T^*(Q/G)$ and

$$F^{\,\mathrm{Ver}} = \tilde{\mathfrak{g}}^* \times \left(\widetilde{V} \oplus \widetilde{V}^*\right)$$

is a **vertical Courant algebroid** over the associated bundle $B^{\,\mathrm{Ver}} = \tilde{\mathfrak{g}}^*$.

6. **Lagrange-Poincaré-Dirac reduction.** This section shows how reduction of standard implicit Lagrangian systems can be incorporated into the Dirac cotangent bundle reduction and then how an implicit analogue of Lagrange-Poincaré equations can be established in the context of the reduction procedure called *Lagrange-Poincaré-Dirac reduction*. We also illustrate the reduction procedure by an example of a satellite with a rotor.

Standard implicit Lagrangian systems. Here, we consider the case in which there is no constraint, namely, the case of a *standard implicit Lagrangian system*.

Let $L : TQ \to \mathbb{R}$ be a Lagrangian, possibly degenerate. Given the canonical Dirac structure D on T^*Q and a partial vector field $X : TQ \oplus T^*Q \to TT^*Q$, a standard implicit Lagrangian system is the triple (L, D, X) that satisfies, for each $(q, v, p) \in TQ \oplus T^*Q$ and with $P = \mathbb{F}L(TQ)$, namely, $(q, p) = (q, \partial L/\partial v)$,

$$(X(q, v, p), \mathbf{d}E(q, v, p)|_{TP}) \in D(q, p),$$

where $E : TQ \oplus T^*Q \to \mathbb{R}$ is the generalized energy defined by $E(q, v, p) = \langle p, v\rangle - L(q, v)$ and the differential of E is the map $\mathbf{d}E : TQ \oplus T^*Q \to T^*(TQ \oplus T^*Q)$ is given by $\mathbf{d}E = (q, v, p, -\partial L/\partial q, p - \partial L/\partial v, v)$. Noting that $p = \partial L/\partial v$ holds on P, the restriction of $\mathbf{d}E$ to TP is given, in coordinates, by $\mathbf{d}E(q, v, p)|_{T_{(q,p)}P} = (q, p, -\partial L/\partial q, v)$, which may be understood in the sense that $T_{(q,p)}P$ is naturally included in $T_{(q,v,p)}(TQ \oplus T^*Q)$.

Reduced Lagrangians. Suppose that L be left G-invariant, namely, for $v_q \in TQ$,

$$L(T_qL_h \cdot v_q) = L(v_q),$$

where $h \in G$ and $v_q \in T_qQ$. Let $A : TQ \to \mathfrak{g}$ be a chosen principal connection on $\pi : Q \to Q/G$ and we recall that the isomorphism $\lambda : TQ \to \widetilde{Q} \times \mathfrak{g}$ is given by

$$v_q \mapsto (q, T\pi(v_q), A(v_q)) = (x, g, u, \eta),$$

where $q \in Q$, $g \in G$, $x = [q] \in Q/G$, $u = T\pi(v_q) \in T_{[q]}(Q/G)$ and $\eta = A(v_q) \in \mathfrak{g}$. Then, one can define a Lagrangian on $\widetilde{Q} \times \mathfrak{g}$ by

$$\bar{L} = L \circ \lambda^{-1}.$$

Because L is G–invariant, so is $\bar{L}$ as

$$\bar{L}(x, hg, u, h\eta) = \bar{L}(x, hg, u, \mathrm{Ad}_h\eta).$$

One can write this as

$$\bar{L}(x, e, u, g^{-1}\eta) = \bar{L}(x, e, u, \mathrm{Ad}_{g^{-1}}\eta) = l(x, u, \bar{\eta}),$$

where $\bar{\eta} = [q, \eta]_G \in \tilde{\mathfrak{g}}$ and l is the reduced Lagrangian on $TQ/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}}$.

The reduced differential of generalized energies. Let us define the right trivialized isomorphism

$$\Lambda = \lambda \oplus \bar{\lambda} : TQ \oplus T^*Q \to (\widetilde{Q} \times \mathfrak{g}) \oplus (\widetilde{Q}^* \times \mathfrak{g}^*) = (\widetilde{Q} \oplus \widetilde{Q}^*) \times V$$

by, for each $(q, v, p) \in TQ \oplus T^*Q$,

$$(q, v, p) \mapsto (q, T\pi(v_q), (p_q)_q^{h^*}, A(v_q), \mathbf{J}(p_q)) = (x, g, u, y, \eta, \mu),$$

where $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is an equivariant momentum map, $(q, T\pi(v_q), (p_q)_q^{h^*}) = (x, g, u, y) \in \widetilde{Q} \oplus \widetilde{Q}^*$ and $(A(v_q), \mathbf{J}(p_q)) = (\eta, \mu) \in V = \mathfrak{g} \oplus \mathfrak{g}^*$. The quotient of Λ by the action of G is the map

$$[\Lambda]_G := \Lambda/G : (TQ \oplus T^*Q)/G \to T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V},$$

which is given by

$$[q, v, p]_G \mapsto ([q], T\pi(v_q), (p_q)_q^{h^*}, [q, A(v_q)]_G, [q, \mathbf{J}(p_q)]_G) = (x, u, y, \bar{\eta}, \bar{\mu}).$$

Recall that we can define a generalized energy E on $TQ \oplus T^*Q$ by

$$E(q, v, p) = \langle p, v \rangle - L(q, v),$$

which is G-invariant because of the G-invariance of L. The trivialized expression of the generalized energy E can be defined by

$$\bar{E} = E \circ \Lambda^{-1} : (\widetilde{Q} \oplus \widetilde{Q}^*) \times V \to \mathbb{R},$$

which is given by

$$\bar{E}(x, g, u, y, \eta, \mu) = \langle y, u \rangle + \langle \mu, \eta \rangle - \bar{L}(x, g, u, \eta).$$

So, the differential of $\bar{E}$ is the map $\mathbf{d}\bar{E} : (\widetilde{Q} \oplus \widetilde{Q}^*) \times V \to T^*\left((\widetilde{Q} \oplus \widetilde{Q}^*) \times V\right)$, which is given by

$$\begin{aligned}\mathbf{d}\bar{E}(x, g, u, y, \eta, \mu) &= \left(\frac{\partial \bar{E}}{\partial x}, \frac{\partial \bar{E}}{\partial g}, \frac{\partial \bar{E}}{\partial u}, \frac{\partial \bar{E}}{\partial y}, \frac{\partial \bar{E}}{\partial \eta}, \frac{\partial \bar{E}}{\partial \mu}\right)\\ &= \left(-\frac{\partial \bar{L}}{\partial x}, -\frac{\partial \bar{L}}{\partial g}, y - \frac{\partial \bar{L}}{\partial u}, u, \mu - \frac{\partial \bar{L}}{\partial \eta}, \eta\right).\end{aligned}$$

The map $\mathbf{d}\bar{E}$ is G-invariant as

$$\begin{aligned} h\cdot \mathbf{d}\bar{E}(x,g,u,y,\eta,\mu) &= \mathbf{d}\bar{E}(x,hg,u,y,h\eta,h\mu) \\ &= \left(\frac{\partial \bar{E}}{\partial x}, h\frac{\partial \bar{E}}{\partial g}, \frac{\partial \bar{E}}{\partial u}, \frac{\partial \bar{E}}{\partial y}, h\frac{\partial \bar{E}}{\partial \eta}, h\frac{\partial \bar{E}}{\partial \mu}\right) \\ &= \left(\frac{\partial \bar{E}}{\partial x}, h\frac{\partial \bar{E}}{\partial g}, \frac{\partial \bar{E}}{\partial u}, \frac{\partial \bar{E}}{\partial y}, \mathrm{Ad}^*_{h^{-1}}\frac{\partial \bar{E}}{\partial \eta}, \mathrm{Ad}_h\frac{\partial \bar{E}}{\partial \mu}\right), \end{aligned}$$

where $h\eta = \mathrm{Ad}_h \eta$ and $h\mu = \mathrm{Ad}^*_{h^{-1}}\mu$.

Then, the *quotient of the generalized energy* $\mathbf{d}\bar{E}$ by the action of G is given by the map

$$\begin{aligned} [\mathbf{d}\bar{E}]_G := & \mathbf{d}\bar{E}/G : T(Q/G)\oplus T^*(Q/G)\oplus \widetilde{V} \to \\ & T^*T(Q/G)\oplus T^*T^*(Q/G)\oplus \left(\tilde{\mathfrak{g}}^*\times(\widetilde{V}\oplus\widetilde{V}^*)\right), \end{aligned}$$

which is represented by, for each $(x,u,y,\bar{\eta},\bar{\mu})\in T(Q/G)\oplus T^*(Q/G)\oplus\widetilde{V}$,

$$\begin{aligned} [\mathbf{d}\bar{E}]_G(x,u,y,\bar{\eta},\bar{\mu}) &= \left(x,u,y,\frac{\partial \mathcal{E}}{\partial x},\frac{\partial \mathcal{E}}{\partial u},\frac{\partial \mathcal{E}}{\partial y},\bar{\mu},\bar{\eta},\frac{\partial \mathcal{E}}{\partial \bar{\eta}},g^{-1}\frac{\partial \mathcal{E}}{\partial g},\frac{\partial \mathcal{E}}{\partial \bar{\mu}}\right) \\ &= \left(x,u,y,-\frac{\partial l}{\partial x}, y-\frac{\partial l}{\partial u},u,\bar{\mu},\bar{\eta},\bar{\mu}-\frac{\partial l}{\partial \bar{\eta}},0,\bar{\eta}\right), \end{aligned}$$

where $\mathcal{E}$ is the reduced generalized energy on $T(Q/G)\oplus T^*(Q/G)\oplus\widetilde{V}$ defined by $\mathcal{E}(x,u,y,\bar{\eta},\bar{\mu}) = \bar{E}(x,e,u,y,g^{-1}\eta,g^{-1}\mu)$.

The reduced Legendre transformation. In view of the isomorphisms $\lambda : TQ \to \widetilde{Q}\times\mathfrak{g}$ and $\bar{\lambda} : T^*Q \to \widetilde{Q}^*\times\mathfrak{g}^*$, let us define the *trivialized Legendre transformation* by

$$\mathbb{F}\bar{L} : \widetilde{Q}\times\mathfrak{g}\to\bar{P};\quad (x,g,u,\eta)\to\left(x,g,\,y=\frac{\partial \bar{L}}{\partial u},\,\mu=\frac{\partial \bar{L}}{\partial \eta}\right).$$

In the above, $\bar{P}=\mathbb{F}\bar{L}(\widetilde{Q}\times\mathfrak{g})\subset\widetilde{Q}^*\times\mathfrak{g}^*$ corresponds to the trivialized expression of the subspace $P=\mathbb{F}L(TQ)\subset T^*Q$. By the equivariance of the map $\mathbb{F}\bar{L}$, we can define the *reduced Legendre transformation* by the quotient of $\mathbb{F}\bar{L}$ as

$$\mathbb{F}l := [\mathbb{F}\bar{L}]_G : T(Q/G)\oplus\tilde{\mathfrak{g}}\to[\bar{P}]_G\subset T^*(Q/G)\oplus\tilde{\mathfrak{g}}^*,$$

which is given by

$$(x,u,\bar{\eta})\to\left(x,y=\frac{\partial l}{\partial u},\,\bar{\mu}=\frac{\partial l}{\partial \bar{\eta}}\right).$$

On the other hand, the reduced Legendre transformation induces the condition

$$\frac{\partial \mathcal{E}}{\partial u}=y-\frac{\partial l}{\partial u}=0,\quad \frac{\partial \mathcal{E}}{\partial \bar{\eta}}=\bar{\mu}-\frac{\partial l}{\partial \bar{\eta}}=0.$$

Hence, the *restriction of the quotient of the generalized energy* $[\mathbf{d}\bar{E}]_G$ to the subbundle $[T\bar{P}]_G\subset TT^*(Q/G)\oplus(\tilde{\mathfrak{g}}^*\times\widetilde{V})$ is given by

$$\begin{aligned} [\mathbf{d}\bar{E}]_G(x,u,y,\bar{\eta},\bar{\mu})|_{[T\bar{P}]_G} &= \left(x,y,\frac{\partial \mathcal{E}}{\partial x},\frac{\partial \mathcal{E}}{\partial y},\bar{\mu},g^{-1}\frac{\partial \mathcal{E}}{\partial g},\frac{\partial \mathcal{E}}{\partial \bar{\mu}}\right) \\ &= \left(x,y,-\frac{\partial l}{\partial x},u,\bar{\mu},0,\bar{\eta}\right). \end{aligned}$$

Notice that the subbundle $[T\bar{P}]_G$ is the bundle over $[\bar{P}]_G\subset T^*(Q/G)\oplus\tilde{\mathfrak{g}}^*$.

Remark 12. In view of the isomorphisms $\lambda : TQ \to \widetilde{Q} \times \mathfrak{g}$ and $\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*$, the quotient of $T^*(TQ \oplus T^*Q) \cong T^*TQ \oplus T^*T^*Q$ by the action of G induces the isomorphisms:

$$
\begin{aligned}
(T^*TQ \oplus T^*T^*Q)/G &\cong (T^*TQ/G) \oplus (T^*T^*Q/G) \\
&\cong T^*(\widetilde{Q} \times \mathfrak{g})/G \oplus T^*(\widetilde{Q}^* \times \mathfrak{g}^*)/G \\
&\cong T^*T(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}) \oplus T^*T^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}^*) \\
&\cong T^*T(Q/G) \oplus T^*T^*(Q/G) \oplus \left(\tilde{\mathfrak{g}}^* \times (\widetilde{V} \oplus \widetilde{V}^*)\right),
\end{aligned}
$$

where $T^*TQ/G \cong T^*(\widetilde{Q} \times \mathfrak{g})/G \cong T^*T(Q/G) \oplus (\tilde{\mathfrak{g}} \times (\tilde{\mathfrak{g}}^* \oplus \tilde{\mathfrak{g}}^*)) \cong T^*T(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$.

Reduction of partial vector fields. Let $X : TQ \oplus T^*Q \to TT^*Q$ be a *partial vector field* on T^*Q; namely, a map that assigns to a point $(q, v, p) \in TQ \oplus T^*Q$, a vector in $T_{p_q}T^*Q$ as

$$X(q, v, p) = (q, p, \dot{q}, \dot{p}),$$

where $\dot{q}$ and $\dot{p}$ are functions of (q, v, p). Since X is left invariant, one has

$$h \cdot X(q, v, p) = X(hq, T_qL_h v, T^*_{hq}L_{h^{-1}} p).$$

Using the trivialized isomorphism $\Lambda = \lambda \oplus \bar{\lambda} : TQ \oplus T^*Q \to (\widetilde{Q} \oplus \widetilde{Q}^*) \times V$, the trivialized expression of the partial vector field $X : TQ \oplus T^*Q \to TT^*Q$ may be given by

$$\bar{X} : (\widetilde{Q} \oplus \widetilde{Q}^*) \times V \to T(\widetilde{Q}^* \times \mathfrak{g}^*),$$

which is expressed as

$$\bar{X}(x, g, u, y, \eta, \mu) = (x, g, y, \mu, \dot{x}, \dot{g}, \dot{y}, \dot{\mu}) \in T_{(x,g,y,\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*).$$

Since X is G-invariant and $\Lambda = \lambda \oplus \bar{\lambda} : TQ \oplus T^*Q \to (\widetilde{Q} \oplus \widetilde{Q}^*) \times V$ is equivariant, $\bar{X}$ is also G-invariant:

$$h \cdot \bar{X}(x, g, u, y, \eta, \mu) = \bar{X}(x, hg, u, y, h\eta, h\mu) \in T_{(x,hg,y,h\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*).$$

In the above, $\bar{X}(x, hg, u, y, h\eta, h\mu) = \bar{X}(x, hg, u, y, \mathrm{Ad}_h \eta, \mathrm{Ad}^*_{h^{-1}}\mu)$ and it follows that

$$
\begin{aligned}
\bar{X}(x, hg, u, y, h\eta, h\mu) &= (x, hg, y, h\mu, \dot{x}, h\dot{g}, \dot{y}, h\dot{\mu}) \\
&= \left(x, hg, y, \mathrm{Ad}^*_{h^{-1}}\mu, \dot{x}, T_gL_h\dot{g}, \dot{y}, \mathrm{Ad}^*_{h^{-1}}\dot{\mu}\right).
\end{aligned}
$$

Hence, $\bar{X}$ takes its value at the identity as

$$\bar{X}(x, e, u, y, g^{-1}\eta, g^{-1}\mu) = (x, e, y, g^{-1}\mu, \dot{x}, g^{-1}\dot{g}, \dot{y}, g^{-1}\dot{\mu}).$$

Since $\bar{X} : (\widetilde{Q} \oplus \widetilde{Q}^*) \times V \to T(\widetilde{Q}^* \times \mathfrak{g}^*)$ is equivariant, it drops to the quotient by G, and we can define a *reduced partial vector field*, denoted

$$[\bar{X}]_G := \bar{X}/G : T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V} \to TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}),$$

which is given by, for each $(x, u, y, \bar{\eta}, \bar{\mu}) \in T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$,

$$[\bar{X}]_G(x, u, y, \bar{\eta}, \bar{\mu}) = \left(x, y, \dot{x}, \dot{y}, \bar{\mu}, \bar{\xi}, \dot{\bar{\mu}}\right),$$

where $\bar{\eta} = [q, \eta]_G = [q, A(q, v)]_G, \bar{\xi} = [q, \xi]_G = [q, A(q, \dot{q})]_G \in \tilde{\mathfrak{g}}$ and $\bar{\mu} = [q, \mu]_G \in \tilde{\mathfrak{g}}^*$ such that $\bar{\mu} = \partial l/\partial \bar{\eta}$.

When we have an integral curve of $[\bar{X}]_G$, then $\dot{x} \equiv dx/dt$ will be the time derivative of x, $\dot{y} \equiv Dy/Dt$ will be the covariant derivative of y in the bundle $T^*(Q/G)$ and $\dot{\bar{\mu}} \equiv D\bar{\mu}/Dt$ will be the covariant derivative of $\bar{\mu}$ in $\tilde{\mathfrak{g}}^*$; note that $(\dot{x}, \dot{y}, \bar{\xi}, \dot{\bar{\mu}})$ are functions of $(x, u, y, \bar{\eta}, \bar{\mu})$.

Lagrange-Poincaré-Dirac reduction. Let us show reduction of a standard implicit Lagrangian system, called *Lagrange-Poincaré-Dirac reduction.*

Definition 6.1. The **reduction of the standard implicit Lagrangian system** (L, D, X) is given by a triple

$$(l, [\bar{D}]_G, [\bar{X}]_G),$$

which satisfies, for each $(x, u, y, \bar{\eta}, \bar{\mu}) \in T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$, the condition

$$\left([\bar{X}]_G\,(x, u, y, \bar{\eta}, \bar{\mu})\,, [\mathbf{d}\bar{E}]_G(x, u, y, \bar{\eta}, \bar{\mu})|_{[T\bar{P}]_G}\right) \in [\bar{D}]_G(x, y, \bar{\mu}), \tag{26}$$

where $l : T(Q/G) \oplus \tilde{\mathfrak{g}} \to \mathbb{R}$ is the reduced Lagrangian and the reduced Legendre transformation holds as $(x, y = \partial l/\partial u, \bar{\mu} = \partial l/\partial\bar{\eta}) \in [\bar{P}]_G = \mathbb{F}l(T(Q/G) \oplus \tilde{\mathfrak{g}}) \subset T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$.

Definition 6.2. A solution curve of the reduced standard implicit Lagrangian system $(l, [\bar{D}]_G, [\bar{X}]_G)$ is a curve $(x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t)), t \in [t_0, t_1]$ in $T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$, which is an integral curve of the reduced partial vector field $[\bar{X}]_G$ that takes its value for each point of the curve as

$$[\bar{X}]_G\,(x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t)) = \left(x(t), y(t), \frac{dx(t)}{dt}, \frac{Dy(t)}{Dt}, \bar{\mu}(t), \bar{\xi}(t), \frac{D\bar{\mu}(t)}{Dt}\right),$$

where $y(t) = (\partial l/\partial u)(t)$ and $\bar{\mu}(t) = (\partial l/\partial\bar{\eta})(t)$.

Theorem 6.3. *Let $(l, [\bar{D}]_G, [\bar{X}]_G)$ be the reduction of the standard implicit Lagrangian system that satisfies equation (26). Let $(x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t)), t \in [t_0, t_1]$ be a solution curve in $T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$ of $(l, [\bar{D}]_G, [\bar{X}]_G)$. Then, the solution curve satisfies the* **implicit Lagrange-Poincaré equations***:*

$$\begin{aligned}\frac{Dy}{Dt} &= \frac{\partial l}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot)\right\rangle, \\ \frac{dx}{dt} &= u, \\ y &= \frac{\partial l}{\partial u}, \\ \frac{D\bar{\mu}}{Dt} &= \mathrm{ad}^*_{\bar{\xi}}\bar{\mu}, \\ \bar{\xi} &= \bar{\eta}, \\ \bar{\mu} &= \frac{\partial l}{\partial\bar{\eta}}.\end{aligned} \tag{27}$$

Proof. The reduction of the standard implicit Lagrangian system $(l, [\bar{D}]_G, [\bar{X}]_G)$ satisfies, for each $(x, u, y, \bar{\eta}, \bar{\mu}) \in T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$ such that $y = \partial l/\partial u$ and $\bar{\mu} = \partial l/\partial\bar{\eta}$,

$$\left([\bar{X}]_G(x, u, y, \bar{\eta}, \bar{\mu}), [\mathbf{d}\bar{E}]_G(x, u, y, \bar{\eta}, \bar{\mu})|_{[T\bar{P}]_G}\right) \in [\bar{D}]_G(x, y, \bar{\mu}),$$

which is represented by

$$\left(\left(\frac{dx}{dt}, \frac{Dy}{Dt}, \bar{\xi}, \frac{D\bar{\mu}}{Dt}\right), \left(-\frac{\partial l}{\partial x}, u, 0, \bar{\eta}\right)\right) \in [\bar{D}]_G(x, y, \bar{\mu}).$$

Then, it follows

$$\left\langle -\frac{\partial l}{\partial x}, \delta x \right\rangle + \langle \delta y, u \rangle + \langle 0, \bar{\zeta} \rangle + \langle \delta \bar{\mu}, \bar{\eta} \rangle$$
$$= \left\langle \delta y, \frac{dx}{dt} \right\rangle - \left\langle \frac{Dy}{Dt}, \delta x \right\rangle - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \delta x) \right\rangle + \langle \delta \bar{\mu}, \bar{\xi} \rangle - \left\langle \frac{D\bar{\mu}}{Dt}, \bar{\zeta} \right\rangle + \langle \bar{\mu}, [\bar{\xi}, \bar{\zeta}] \rangle .$$

Hence, one has

$$\left\langle -\frac{Dy}{Dt} + \frac{\partial l}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \delta x \right\rangle + \left\langle \delta y, \frac{dx}{dt} - u \right\rangle$$
$$+ \left\langle -\frac{D\bar{\mu}}{Dt} + \mathrm{ad}^*_{\bar{\xi}} \bar{\mu}, \bar{\zeta} \right\rangle + \langle \delta \bar{\mu}, \bar{\xi} - \bar{\eta} \rangle = 0$$

for all $\delta x, \delta y, \bar{\zeta}$ and $\delta \bar{\mu}$.

Thus, we can obtain the *implicit Lagrange-Poincaré equations* in equation (27) as the reduction of the standard implicit Lagrangian system. □

Coordinate expressions. Suppose that Q has dimension n, so that Q/G has dimension $r = n - \dim G$. We choose a local trivialization of the principal bundle $Q \to Q/G$ to be $S \times G$, where S is an open set of $\mathbb{R}^r$. Thus, we consider the trivial principal bundle $\pi : S \times G \to S$ with the structure group G acting only on the second factor by left multiplication. Let A be a given principal connection on the bundle $Q \to Q/G$, or, in local representation, on the bundle $S \times G \to S$. Write local coordinates for a point $q = (x, g) \in Q \cong S \times G$ by (x^α, g^a), where $\alpha = 1, ..., r$ and $a = 1, ..., \dim G$. The principal connection one-form is locally written by $\mathrm{Ad}_g(A_e(x)dx + g^{-1}dg)$. Then, at any tangent vector $(x, g, \dot{x}, \dot{g}) \in T_{(x,g)}(S \times G)$, one has

$$A(x, g, \dot{x}, \dot{g}) = \mathrm{Ad}_g(A_e(x) \cdot \dot{x} + \xi) = A(x, g) \cdot \dot{x} + \dot{g} g^{-1},$$

where A_e is the $\mathfrak{g}$-valued one-form on S defined by $A_e(x) \cdot \dot{x} = A(x, e, \dot{x}, 0)$ and $\xi = g^{-1}\dot{g}$. By the bundle isomorphism $\Psi_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$,

$$\Psi_A([x, g, \dot{x}, \dot{g}]_G) = (x, \dot{x}, \bar{\xi}),$$

where $\bar{\xi} = (x, A_e(x) \cdot \dot{x} + \xi)$. Let $A^a_\alpha(x)$ be the local coordinate expression of A_e on the bundle $S \times G \to S$. Then, we simply write $\bar{\xi}^a = \xi^a + A^a_\alpha \dot{x}^\alpha$. Let C^b_{cd} be the structure constants of the Lie algebra $\mathfrak{g}$ and the components of the curvature of A are given, in coordinates, by

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\beta}{\partial x^\alpha} - \frac{\partial A^b_\alpha}{\partial x^\beta} - C^b_{c\,d} A^c_\alpha A^d_\beta \right).$$

Thus, the **coordinate expression of the implicit Lagrange–Poincaré equations** in (27) may be represented by

$$\begin{aligned}
\frac{dy_\alpha}{dt} &= \frac{\partial l}{\partial x^\alpha} + \bar{\mu}_a \left(B^a_{\alpha\beta} \dot{x}^\beta - C^a_{db} A^b_\alpha \bar{\xi}^d \right), \\
\frac{dx^\alpha}{dt} &= u^\alpha, \\
y_\alpha &= \frac{\partial l}{\partial u^\alpha}, \\
\frac{d\bar{\mu}_b}{dt} &= \bar{\mu}_a \left(C^a_{db} \bar{\xi}^d - C^a_{db} A^d_\alpha \dot{x}^\alpha \right), \\
\bar{\xi}^b &= \bar{\eta}^b, \\
\bar{\mu}_b &= \frac{\partial l}{\partial \bar{\eta}^b}.
\end{aligned}$$

Gauged Dirac structures in Lagrange-Poincaré-Dirac reduction. Let us denote by $(l, [\bar{D}]_G, [\bar{X}]_G)$ the reduced implicit Lagrangian system that satisfies equation (26). Recall the reduced partial vector field

$$[\bar{X}]_G : T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V} \to TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$$

is locally given by, for each $(x, u, y, \bar{\eta}, \bar{\mu}) \in T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$,

$$[\bar{X}]_G(x, u, y, \bar{\eta}, \bar{\mu}) = \left(x, y, \frac{dx}{dt}, \frac{Dy}{Dt}, \bar{\mu}, \bar{\xi}, \frac{D\bar{\mu}}{Dt} \right).$$

Notice that $[\bar{X}]_G$ can be represented by the direct sum of the horizontal and vertical parts such that

$$[\bar{X}]_G(x, u, y, \bar{\eta}, \bar{\mu}) = [\bar{X}]^{\mathrm{Hor}}_G(x, u, y) \oplus [\bar{X}]^{\mathrm{Ver}}_G(\bar{\eta}, \bar{\mu}),$$

where $[\bar{X}]^{\mathrm{Hor}}_G : T(Q/G) \oplus T^*(Q/G) \to TT^*(Q/G)$ denotes the *horizontal partial vector field*, which is given by

$$[\bar{X}]^{\mathrm{Hor}}_G(x, u, y) = \left(x, y, \frac{dx}{dt}, \frac{Dy}{Dt} \right) \in TT^*(Q/G)$$

and $[\bar{X}]^{\mathrm{Ver}}_G : \widetilde{V} \to \tilde{\mathfrak{g}}^* \times \widetilde{V}$ indicates the *vertical partial vector field*, which is given by

$$[\bar{X}]^{\mathrm{Ver}}_G(\bar{\eta}, \bar{\mu}) = \left(\bar{\mu}, \bar{\xi}, \frac{D\bar{\mu}}{Dt} \right) \in \tilde{\mathfrak{g}}^* \times \widetilde{V}.$$

On the other hand, the quotient of the differential of the generalized energy, namely,

$$[\mathbf{d}\bar{E}]_G : T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V} \to T^*T(Q/G) \oplus T^*T^*(Q/G) \oplus \left(\tilde{\mathfrak{g}}^* \times (\widetilde{V} \oplus \widetilde{V}^*) \right)$$

can be decomposed into the horizontal and vertical parts as

$$[\mathbf{d}\bar{E}]_G(x, u, y, \bar{\eta}, \bar{\mu}) = [\mathbf{d}\bar{E}]^{\mathrm{Hor}}_G(x, u, y) \oplus [\mathbf{d}\bar{E}]^{\mathrm{Ver}}_G(\bar{\eta}, \bar{\mu}),$$

where $[\mathbf{d}\bar{E}]^{\mathrm{Hor}}_G : T(Q/G) \oplus T^*(Q/G) \to T^*T(Q/G) \oplus T^*T^*(Q/G)$ is the *horizontal differential of the generalized energy* given by, for each $(x, u, y) \in T(Q/G) \oplus T^*(Q/G)$,

$$[\mathbf{d}\bar{E}]^{\mathrm{Hor}}_G(x, u, y) = \left(x, u, y, \frac{\partial \mathcal{E}}{\partial x}, \frac{\partial \mathcal{E}}{\partial u}, \frac{\partial \mathcal{E}}{\partial y} \right) = \left(x, u, y, -\frac{\partial l}{\partial x}, y - \frac{\partial l}{\partial u}, u \right),$$

and $[\mathbf{d}\bar{E}]_G^{\mathrm{Ver}} : \widetilde{V} \to \tilde{\mathfrak{g}}^* \times (\widetilde{V} \oplus \widetilde{V}^*)$ is the *vertical differential of the generalized energy* given by, for each $(\bar{\eta}, \bar{\mu}) \in \widetilde{V}$,

$$[\mathbf{d}\bar{E}]_G^{\mathrm{Ver}}(\bar{\eta}, \bar{\mu}) = \left(\bar{\mu}, \bar{\eta}, \frac{\partial \mathcal{E}}{\partial \bar{\eta}}, g^{-1}\frac{\partial \mathcal{E}}{\partial g}, \frac{\partial \mathcal{E}}{\partial \bar{\mu}}\right) = \left(\bar{\mu}, \bar{\eta}, \bar{\mu} - \frac{\partial l}{\partial \bar{\eta}}, 0, \bar{\eta}\right).$$

Recall the reduced Legendre transformation $[\bar{P}]_G = \mathbb{F}l(T(Q/G) \oplus \tilde{\mathfrak{g}}) \subset T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ that is given by

$$(x, u, \bar{\eta}) \mapsto \left(x, y = \frac{\partial l}{\partial u}, \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}\right)$$

induces the conditions $\partial \mathcal{E}/\partial u = y - \partial l/\partial u = 0$ and $\partial \mathcal{E}/\partial \bar{\eta} = \bar{\mu} - \partial l/\partial \bar{\eta} = 0$.

This reduced Legendre transformation may be naturally decomposed into the horizontal and vertical parts such that

$$\mathbb{F}l = \mathbb{F}l^{\mathrm{Hor}} \oplus \mathbb{F}l^{\mathrm{Ver}},$$

where the *horizontal Legendre transformation*

$$\mathbb{F}l^{\mathrm{Hor}} : T(Q/G) \to T^*(Q/G); \quad (x, u) \mapsto \left(x, y = \frac{\partial l}{\partial u}\right)$$

induces the condition $\partial \mathcal{E}/\partial u = y - \partial l/\partial u = 0$, while the *vertical Legendre transformation*

$$\mathbb{F}l^{\mathrm{Ver}} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*; \quad \bar{\eta} \mapsto \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}$$

satisfies the condition $\partial \mathcal{E}/\partial \bar{\eta} = \bar{\mu} - \partial l/\partial \bar{\eta} = 0$. One has $[\bar{P}]_G^{\mathrm{Hor}} = \mathbb{F}l^{\mathrm{Hor}}(T(Q/G)) \subset T^*(Q/G)$ and $[\bar{P}]_G^{\mathrm{Ver}} = \mathbb{F}l^{\mathrm{Ver}}(\tilde{\mathfrak{g}}) \subset \tilde{\mathfrak{g}}^*$ such that

$$[\bar{P}]_G = [\bar{P}]_G^{\mathrm{Hor}} \oplus [\bar{P}]_G^{\mathrm{Ver}}.$$

Further, setting

$$[T\bar{P}]_G^{\mathrm{Hor}} = T[\bar{P}]_G^{\mathrm{Hor}} \subset TT^*(Q/G) \quad \text{and} \quad [T\bar{P}]_G^{\mathrm{Ver}} = \tilde{\mathfrak{g}} \times T[\bar{P}]_G^{\mathrm{Ver}} \subset \tilde{\mathfrak{g}}^* \times \widetilde{V},$$

one has

$$[T\bar{P}]_G = [T\bar{P}]_G^{\mathrm{Hor}} \oplus [T\bar{P}]_G^{\mathrm{Ver}} \subset TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}).$$

Thus, the restriction of $[\mathbf{d}\bar{E}]_G$ to $[T\bar{P}]_G$ can be also decomposed as

$$[\mathbf{d}\bar{E}]_G(x, u, y, \bar{\eta}, \bar{\mu})|_{[T\bar{P}]_G} = [\mathbf{d}\bar{E}]_G^{\mathrm{Hor}}(x, u, y)|_{[T\bar{P}]_G^{\mathrm{Hor}}} \oplus [\mathbf{d}\bar{E}]_G^{\mathrm{Ver}}(\bar{\eta}, \bar{\mu})|_{[T\bar{P}]_G^{\mathrm{Ver}}},$$

where

$$[\mathbf{d}\bar{E}]_G^{\mathrm{Hor}}(x, u, y)|_{[T\bar{P}]_G^{\mathrm{Hor}}} = \left(x, y, \frac{\partial \mathcal{E}}{\partial x}, \frac{\partial \mathcal{E}}{\partial y}\right) = \left(x, y, -\frac{\partial l}{\partial x}, u\right),$$

and

$$[\mathbf{d}\bar{E}]_G^{\mathrm{Ver}}(\bar{\eta}, \bar{\mu})|_{[T\bar{P}]_G^{\mathrm{Ver}}} = \left(\bar{\mu}, g^{-1}\frac{\partial \mathcal{E}}{\partial g}, \frac{\partial \mathcal{E}}{\partial \bar{\mu}}\right) = (\bar{\mu}, 0, \bar{\eta}).$$

Recall that the reduced Dirac structure $[\bar{D}]_G$ is a *gauged Dirac structure*, which may be decomposed into the horizontal and vertical Dirac structures as, for each $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}$,

$$[\bar{D}]_G(x, y, \bar{\mu}) = [\bar{D}]_G^{\mathrm{Hor}}(x, y) \oplus [\bar{D}]_G^{\mathrm{Ver}}(\bar{\mu}).$$

Note that the *horizontal Dirac structure* $[\bar{D}]_G^{\mathrm{Hor}}$ is shown in equation (24), while the *vertical Dirac structure* $[\bar{D}]_G^{\mathrm{Ver}}$ in equation (25).

Thus, we obtain the following theorem associated with the gauged Dirac structure.

Theorem 6.4. *Let (L, D, X) be a standard implicit Lagrangian system, which satisfies the condition, for each $(q, v, p) \in TQ \oplus T^*Q$,*

$$(X(q, v, p), \mathbf{d}E(q, v, p) \mid_{TP}) \in D(q, p),$$

where $P = \mathbb{F}L(TQ)$, namely, $(q, p) = (q, \partial L/\partial v)$. Let $(l, [\bar{D}]_G, [\bar{X}]_G)$ be the reduction of (L, D, X), which satisfies the condition, for each $(x, u, y, \bar{\eta}, \bar{\mu}) \in T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$,

$$([\bar{X}]_G\,(x, u, y, \bar{\eta}, \bar{\mu})\,, [\mathbf{d}\bar{E}]_G(x, u, y, \bar{\eta}, \bar{\mu})|_{[T\bar{P}]_G}) \in [\bar{D}]_G(x, y, \bar{\mu}),$$

together with the reduced Legendre transformation

$$[\bar{P}]_G = \mathbb{F}l(T(Q/G) \oplus \tilde{\mathfrak{g}}) \subset T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*,$$

that is,

$$(x, u, \bar{\eta}) \mapsto \left(x, y = \frac{\partial l}{\partial u}, \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}\right).$$

Then, $(l, [\bar{D}]_G, [\bar{X}]_G)$ can be decomposed into the horizontal and vertical parts as

$$(l, [\bar{D}]_G, [\bar{X}]_G) = (l, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}}) \oplus (l, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}}).$$

In the above, $(l, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}})$ is the **horizontal implicit Lagrangian system***, which satisfies, for each $(x, u, y) \in T(Q/G) \oplus T^*(Q/G)$,*

$$([\bar{X}]_G^{\mathrm{Hor}}\,(x, u, y)\,, [\mathbf{d}\bar{E}]_G^{\mathrm{Hor}}(x, u, y)|_{[T\bar{P}]_G^{\mathrm{Hor}}}) \in [\bar{D}]_G^{\mathrm{Hor}}(x, y),$$

together with the horizontal Legendre transformation $[\bar{P}]_G^{\mathrm{Hor}} = \mathbb{F}l^{\mathrm{Hor}}(T(Q/G)) \subset T^*(Q/G)$; namely,

$$(x, u) \mapsto \left(x, y = \frac{\partial l}{\partial u}\right).$$

This induces the horizontal implicit Lagrange-Poincaré equations:

$$\frac{Dy}{Dt} = \frac{\partial l}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \quad \frac{dx}{dt} = u, \quad y = \frac{\partial l}{\partial u}.$$

Moreover, $(l, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}})$ is the **vertical implicit Lagrangian system***, which satisfies, for each $(\bar{\eta}, \bar{\mu}) \in \widetilde{V}$,*

$$([\bar{X}]_G^{\mathrm{Ver}}\,(\bar{\eta}, \bar{\mu})\,, [\mathbf{d}\bar{E}]_G^{\mathrm{Ver}}(\bar{\eta}, \bar{\mu})|_{[T\bar{P}]_G^{\mathrm{Ver}}}) \in [\bar{D}]_G^{\mathrm{Ver}}(\bar{\mu}),$$

together with the vertical Legendre transformation $[\bar{P}]_G^{\mathrm{Ver}} = \mathbb{F}l^{\mathrm{Ver}}(\tilde{\mathfrak{g}}) \subset \tilde{\mathfrak{g}}^*$;

$$\bar{\eta} \mapsto \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}.$$

This induces the vertical implicit Lagrange-Poincaré equations:

$$\frac{D\bar{\mu}}{Dt} = \mathrm{ad}^*_{\bar{\xi}}\,\bar{\mu}, \quad \bar{\xi} = \bar{\eta}, \quad \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}.$$

Example: Satellite with a rotor. Let us consider an illustrative example of a satellite with a rotor aligned with the third principal axis of the body (see, [64, 9, 27]).

The satellite with a rotor is modeled by a rigid body of a carrier and a rigid rotor, whose configuration manifold is given by $Q = S^1 \times SO(3)$, with the first factor being the rotor relative angle and the second factor the rigid body attitude. Consider the case in which there exists no torque at the rotor. Then, it immediately follows that the Lie group $G = SO(3)$ only acts on the second factor of Q and hence that $Q/G = S^1$.

Now, let $q = (\theta, R)$ be local coordinates for $Q = S^1 \times SO(3)$ and $(q, v) = (\theta, R, u, U)$ for TQ. Let us take a *trivialized connection* on $Q \to Q/G$ such that $TQ/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}} = TS^1 \times \mathfrak{so}(3)$.

Let $L : TQ \to \mathbb{R}$ be a left invariant Lagrangian that is given by the total kinetic energy of the system (rigid carrier plus rotor) and let $E : TQ \oplus T^*Q \to \mathbb{R}$ be a left invariant generalized energy that is defined, for each $(q, v, p) \in TQ \oplus T^*Q$, by $E(q, v, p) = \langle p, v\rangle - L(q, v)$. In view of $TQ/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}}$ and the G-invariance associated with the Lagrangian, one has

$$L(q, v) = l([q, v]_G),$$

where $l : T(Q/G) \oplus \tilde{\mathfrak{g}} = TS^1 \times \mathfrak{so}(3) \to \mathbb{R}$ is the reduced Lagrangian, which is given by, for each $[q, v]_G = (\theta, u, \Sigma) \in TS^1 \times \mathbb{R}^3 \cong TS^1 \times \mathfrak{so}(3)$,

$$l(\theta, u, \Sigma) = \frac{1}{2}\left(\lambda_1\Sigma_1^2 + \lambda_2\Sigma_2^2 + I_3\Sigma_3^2 + J_3(\Sigma_3 + u)^2\right).$$

In the above, $I_1 > I_2 > I_3$ are the rigid body moments of inertia, $J_1 = J_2$ and J_3 are the rotor moments of inertia, $\lambda_i = I_i + J_i$, θ is the relative angle of the rotor, $(\theta, u) \in TS^1$, $\Sigma = (\Sigma^1, \Sigma^2, \Sigma^3) \in \mathbb{R}^3$ is the body angular velocity that is defined by $\hat{\Sigma} = R^{-1}U \in \mathfrak{so}(3)$ such that $\hat{\Sigma}\, w \cong \Sigma \times w$ for all $w \in \mathbb{R}^3$, where $(R, U) \in TSO(3)$.

In view of $(TQ \oplus T^*Q)/G \cong T(Q/G) \oplus T^*(Q/G) \oplus \widetilde{V}$ and the G-invariance associated with the generalized energy E, one has

$$E(q, v, p) = \mathcal{E}([q, v, p]_G),$$

where $\widetilde{V} = \mathfrak{so}(3) \oplus \mathfrak{so}(3)^*$ and $\mathcal{E} : TS^1 \oplus T^*S^1 \oplus \widetilde{V} \to \mathbb{R}$ is the reduced generalized energy, which is given by, for each $[q, v, p]_G = (\theta, u, y, \Sigma, \Pi) \in TS^1 \oplus T^*S^1 \oplus \widetilde{V}$,

$$\begin{aligned}\mathcal{E}(\theta, u, y, \Sigma, \Pi) &= \langle y, u\rangle + \langle \Pi, \Sigma\rangle - l(\theta, u, \Sigma)\\ &= \Pi_1\Sigma_1 + \Pi_2\Sigma_2 + \Pi_3\Sigma_3 - \frac{1}{2}\left(\lambda_1\Sigma_1^2 + \lambda_2\Sigma_2^2 + I_3\Sigma_3^2 + J_3(\Sigma_3 + u)^2\right),\end{aligned}$$

where $y \in T^*_\theta S^1$ indicates the momentum and $\Pi = (\Pi_1, \Pi_2, \Pi_3) \in (\mathbb{R}^3)^* \cong \mathfrak{so}(3)^*$ denotes the body angular momentum and where $\widetilde{V} = \mathbb{R}^3 \oplus (\mathbb{R}^3)^* \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)^*$.

The quotient of the differential of the generalized energy is the map

$$[\mathbf{d}E]_G : TS^1 \oplus T^*S^1 \oplus \widetilde{V} \to T^*TS^1 \oplus T^*T^*S^1 \oplus \left(\mathfrak{so}(3)^* \times (\widetilde{V} \oplus \widetilde{V}^*)\right).$$

Since the Lagrangian in this example is regular, the Legendre transform holds as

$$\left(\theta,\ y = \frac{\partial l}{\partial u},\ \Pi = \frac{\partial l}{\partial \Sigma}\right) \in T^*S^1 \times (\mathbb{R}^3)^* \cong T^*S^1 \times \mathfrak{so}(3)^*$$

and the restriction of $[\mathbf{d}E]_G$ to $TT^*S^1 \oplus (\mathfrak{so}(3)^* \times \widetilde{V})$ is given by

$$[\mathbf{d}E]_G(\theta, u, y, \Sigma, \Pi) \mid_{TT^*S^1 \oplus (\mathfrak{so}(3)^* \times \widetilde{V})} = \left(\theta, y, -\frac{\partial l}{\partial \theta}, u, \Pi, 0, \Sigma\right). \tag{28}$$

It follows that the conjugate momentum y of θ and the body angular momentum Π may be represented by

$$y = J_3(\Sigma_3 + u), \;\; \Pi_1 = \lambda_1 \Sigma_1, \;\; \Pi_2 = \lambda_2 \Sigma_2, \;\; \Pi_3 = \lambda_3 \Sigma_3 + J_3 u.$$

The reduction of the partial vector field $X : TQ \oplus T^*Q \to TT^*Q$, namely, the map $[\bar{X}]_G : TS^1 \oplus T^*S^1 \oplus \widetilde{V} \to TT^*S^1 \oplus (\mathfrak{so}(3)^* \times \widetilde{V})$ is given by, for each $(\theta, u, y, \Sigma, \Pi) \in TS^1 \oplus T^*S^1 \oplus \widetilde{V}$,

$$[\bar{X}]_G\,(\theta, u, y, \Sigma, \Pi) = \left(\theta, y, \dot{\theta}, \dot{y}, \Pi, \Omega, \dot{\Pi}\right), \tag{29}$$

where $\Omega \in \mathbb{R}^3$ is defined by $\hat{\Omega} = R^{-1}\dot{R} \in \mathfrak{so}(3)$ such that $\hat{\Omega}\, w \cong \Omega \times w$ for $w \in \mathbb{R}^3$.

The reduction of the standard implicit Lagrangian system (L, D, X) is a triple

$$(l, [\bar{D}]_G, [\bar{X}]_G),$$

which satisfies, for each $(\theta, u, y, \Sigma, \Pi) \in TS^1 \oplus T^*S^1 \oplus \widetilde{V}$, the condition

$$\left([\bar{X}]_G\,(\theta, u, y, \Sigma, \Pi)\,, [\mathbf{d}E]_G(\theta, u, y, \Sigma, \Pi) \mid_{TT^*S^1 \oplus (\mathfrak{so}(3) \times \widetilde{V})}\right) \in [\bar{D}]_G(\theta, y, \Pi), \tag{30}$$

together with $(\theta,\; y = \partial l/\partial u,\; \Pi = \partial l/\partial \Sigma) \in T^*S^1 \oplus \mathfrak{so}(3)^*$.

Using equations (28), (29) and (30), one can easily derive the implicit Lagrange-Poincaré equations for the satellite with a rotor, which consist of the horizontal implicit Lagrange-Poincaré equations

$$\frac{dy}{dt} = 0, \quad \frac{d\theta}{dt} = u, \quad y = \frac{\partial l}{\partial u}$$

and the vertical implicit Lagrange-Poincaré equations

$$\frac{d\Pi}{dt} = \Pi \times \Omega, \quad \Omega = \Sigma, \quad \Pi = \frac{\partial l}{\partial \Sigma}.$$

In the above, it follows that the momentum $y = J_3(\Sigma_3 + u) = \text{constant}$.

7. **Hamilton-Poincaré-Dirac reduction.** We can define a standard implicit Hamiltonian system from a given regular Lagrangian via the Legendre transformation. In this section, we show the *reduction procedure of standard implicit Hamiltonian systems*, called *Hamilton-Poincaré-Dirac reduction*, where it is shown how *Hamilton-Poincaré equations* can be constructed in the context of the Dirac cotangent bundle reduction. We also demonstrate this reduction procedure by an example of a satellite with a rotor.

Standard implicit Hamiltonian systems. As in Lagrange-Poincaré-Dirac reduction, we consider the case in which there is no constraint, namely the case of a *standard implicit Hamiltonian system*.

Let $L : TQ \to \mathbb{R}$ be a given left G–invariant Lagrangian and let D be the canonical Dirac structure on the cotangent bundle T^*Q of a configuration manifold Q. Denote a vector field on T^*Q by X. For the case in which L is regular, we can define a left G–invariant Hamiltonian by

$$H = E \circ (\mathbb{F}L)^{-1},$$

where $E : TQ \to \mathbb{R}$ is the energy defined, for each $(q, v) \in TQ$, by $E(q, v) = \langle \mathbb{F}L(q, v), v_q \rangle - L(q, v)$. Then, a standard implicit Hamiltonian system can be defined as a triple (H, D, X) which satisfies, for each $(q, p) \in T^*Q$,

$$(X(q, p), H(q, p)) \in D(q, p).$$

Local representation. The differential of H is a map $\mathbf{d}H : T^*Q \to T^*T^*Q$, which is locally expressed by, for each (q, p) for T^*Q,

$$\mathbf{d}H = \left(q, p, \frac{\partial H}{\partial q}, \frac{\partial H}{\partial p} \right).$$

The vector field X is a map from T^*Q to TT^*Q, which assigns to each point $(q, p) \in T^*Q$, a vector in $T_{(q,p)}T^*Q$ as

$$X(q, p) = \left(q, p, \frac{dq}{dt}, \frac{dp}{dt} \right).$$

Recall that the canonical two-form Ω_{T^*Q} is given by

$$\Omega_{T^*Q} \left((q, p, u_1, \alpha_1), (q, p, u_2, \alpha_2) \right) = \langle \alpha_2, u_1 \rangle - \langle \alpha_1, u_2 \rangle ,$$

and hence the canonical Dirac structure may be expressed by

$$D(q, p) = \left\{ ((q, p, \dot{q}, \dot{p}), (q, p, \alpha, w)) \mid w = \dot{q}, \ \text{ and } \ \alpha = -\dot{p} \right\}.$$

Then, the condition for the standard implicit Hamiltonian system $(X, \mathbf{d}H) \in D$ reads

$$\left\langle \frac{\partial H}{\partial q}, u \right\rangle + \left\langle \alpha, \frac{\partial H}{\partial p} \right\rangle = \left\langle \alpha, \frac{dq}{dt} \right\rangle - \left\langle \frac{dp}{dt}, u \right\rangle$$

for all u and α, where (u, α) are the local representatives of a point in $T_{(q,p)}T^*Q$. Then, we obtain the local representation

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

Reduction of the differential of Hamiltonians. Next, we consider reduction of the standard Hamiltonian system. Let $A : TQ \to \mathfrak{g}$ be a principal connection on the principal bundle $\pi : Q \to Q/G$ as before. Recall that an equivariant momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is defined by, for $p_q \in T_q^*Q$,

$$\langle \mathbf{J}(p_q), \xi \rangle = \langle p_q, \xi_Q(q) \rangle,$$

where $\xi \in \mathfrak{g}$. Since $H : T^*Q \to \mathbb{R}$ is left invariant, namely, for $p_q \in T^*Q$,

$$H(T_{hq}^* L_{h^{-1}} p_q) = H(p_q)$$

for all $h \in G$. Recall that the isomorphism $\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*$ is denoted by

$$p_q \mapsto ([q], (p_q)_q^{h^*}, \mathbf{J}(p_q)) = (x, g, y, \mu),$$

where $y = (p_q)_q^{h^*} \in T_{[q]}^*(Q/G)$ and $\mu = \mathbf{J}(p_q) \in \mathfrak{g}^*$. Then, one can define a G–invariant Hamiltonian on $\widetilde{Q}^* \times \mathfrak{g}^*$ by

$$\bar{H} = H \circ \bar{\lambda}^{-1}$$

and its differential may be represented by the map

$$\mathbf{d}\bar{H} : \widetilde{Q}^* \times \mathfrak{g}^* \to T^*(\widetilde{Q}^* \times \mathfrak{g}^*),$$

which is expressed in coordinates by

$$\mathbf{d}\bar{H} = \left(x, g, y, \mu, \frac{\partial \bar{H}}{\partial x}, \frac{\partial \bar{H}}{\partial g}, \frac{\partial \bar{H}}{\partial y}, \frac{\partial \bar{H}}{\partial \mu}\right).$$

The map $\mathbf{d}\bar{H} : \widetilde{Q}^* \times \mathfrak{g}^* \to T^*(\widetilde{Q}^* \times \mathfrak{g}^*)$ is G-invariant as

$$h \cdot \mathbf{d}\bar{H}(x, g, y, \mu) = \mathbf{d}\bar{H}(x, hg, y, h\mu) = \mathbf{d}\bar{H}(x, hg, y, \mathrm{Ad}^*_{h^{-1}}\mu),$$

and it follows that

$$\begin{aligned}\mathbf{d}\bar{H} &= \left(x, hg, y, h\mu, \frac{\partial \bar{H}}{\partial x}, h\frac{\partial \bar{H}}{\partial g}, \frac{\partial \bar{H}}{\partial y}, h\frac{\partial \bar{H}}{\partial \mu}\right)\\ &= \left(x, hg, u, \mathrm{Ad}^*_{h^{-1}}\mu, \frac{\partial \bar{H}}{\partial x}, T^*_{hg}L_{h^{-1}}\frac{\partial \bar{H}}{\partial g}, \frac{\partial \bar{H}}{\partial y}, \mathrm{Ad}_h\frac{\partial \bar{H}}{\partial \mu}\right).\end{aligned}$$

Then, one can compute it at the identity as

$$\begin{aligned}\mathbf{d}\bar{H} &= \left(x, e, y, g^{-1}\mu, \frac{\partial \bar{H}}{\partial x}, g^{-1}\frac{\partial \bar{H}}{\partial g}, \frac{\partial \bar{H}}{\partial y}, g^{-1}\frac{\partial \bar{H}}{\partial \mu}\right)\\ &= \left(x, e, y, \mathrm{Ad}^*_g\mu, \frac{\partial \bar{H}}{\partial x}, T^*_eL_g\frac{\partial \bar{H}}{\partial g}, \frac{\partial \bar{H}}{\partial y}, \mathrm{Ad}_{g^{-1}}\frac{\partial \bar{H}}{\partial \mu}\right)\\ &= \left(x, e, y, \mathrm{Ad}^*_g\mu, \frac{\partial \bar{H}}{\partial x}, 0, \frac{\partial \bar{H}}{\partial y}, \mathrm{Ad}_{g^{-1}}\frac{\partial \bar{H}}{\partial \mu}\right).\end{aligned}$$

Since $\bar{H}$ is G–invariant as

$$\bar{H}(x, hg, y, h\mu) = \bar{H}(x, hg, y, \mathrm{Ad}^*_{h^{-1}}\mu),$$

one can write

$$\bar{H}(x, e, y, g^{-1}\mu) = \bar{H}(x, e, y, \mathrm{Ad}^*_g\mu) = h(x, y, \bar{\mu}),$$

where $\bar{\mu} = [q, \mu]_G \in \tilde{\mathfrak{g}}^*$ and h is the *reduced Hamiltonian* on $T^*Q/G \cong (\widetilde{Q}^* \times \mathfrak{g}^*)/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$.

Noting the isomorphisms $T^*T^*Q/G \cong T^*(\widetilde{Q}^* \times \mathfrak{g}^*)/G \cong T^*T^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}^*)$, the quotient of the map $\mathbf{d}\bar{H} : \widetilde{Q}^* \times \mathfrak{g}^* \to T^*(\widetilde{Q}^* \times \mathfrak{g}^*)$ by the action of G is given by

$$[\mathbf{d}\bar{H}]_G := \mathbf{d}\bar{H}/G : T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* \to T^*T^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}^*),$$

which is denoted by, for each $[q, p]_G = (x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$,

$$[\mathbf{d}\bar{H}]_G(x, y, \bar{\mu}) = \left(x, y, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \bar{\mu}, 0, \frac{\partial h}{\partial \bar{\mu}}\right).$$

The reduced vector field. Let $X : T^*Q \to TT^*Q$ be a vector field on T^*Q; namely, a map that assigns to a point $(q, p) \in T^*Q$, a vector in $T_{p_q}T^*Q$ and X is locally represented by

$$X(q, p) = (q, p, \dot{q}, \dot{p}).$$

Since X is left invariant, it follows

$$h \cdot X(q, p) = X(hq, T^*_{hq}L_{h^{-1}}p).$$

By using $\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*$, the trivialized expression of X is a map

$$\bar{X} : \widetilde{Q}^* \times \mathfrak{g}^* \to T(\widetilde{Q}^* \times \mathfrak{g}^*),$$

which is expressed, for each $(x,g,y,\mu) \in \widetilde{Q}^* \times \mathfrak{g}^*$, by

$$\bar{X}(x,g,y,\mu) = (\dot{x},\dot{g},\dot{y},\dot{\mu}) \in T_{(x,g,y,\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*).$$

Because X is left invariant and $\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*$ is equivariant, $\bar{X}$ is also left invariant, and hence

$$h \cdot \bar{X}(x,g,y,\mu) = \bar{X}(x,hg,y,h\mu) \in T_{(x,hg,y,h\mu)}(\widetilde{Q}^* \times \mathfrak{g}^*),$$

where $\bar{X}(x,hg,y,h\mu) = \bar{X}(x,hg,y,\mathrm{Ad}^*_{h^{-1}}\mu)$ and so

$$\begin{aligned}\bar{X}(x,hg,y,h\mu) &= (x,hg,y,h\mu,\dot{x},h\dot{g},\dot{y},h\dot{\mu})\\ &= \Big(x,hg,y,\mathrm{Ad}^*_{h^{-1}}\mu,\dot{x},T_{hg}L_h\dot{g},\dot{y},\mathrm{Ad}^*_{h^{-1}}\dot{\mu}\Big).\end{aligned}$$

Hence, the value of $\bar{X}$ at the identity is

$$\bar{X}(x,e,y,g^{-1}\mu) = (x,e,y,g^{-1}\mu,\dot{x},g^{-1}\dot{g},\dot{y},g^{-1}\dot{\mu}).$$

The quotient of the vector field

$$\bar{X} : \widetilde{Q}^* \times \mathfrak{g}^* \to T(\widetilde{Q}^* \times \mathfrak{g}^*)$$

by the action of G defines the *reduced vector field* as

$$[\bar{X}]_G := \bar{X}/G : T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* \to TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V}),$$

which is given by, for each $(x,y,\bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$,

$$[\bar{X}]_G(x,y,\bar{\mu}) = \left(x,y,\dot{x},\dot{y},\bar{\mu},\bar{\xi},\dot{\bar{\mu}}\right).$$

In the above, $\bar{\mu} = [q,\mu]_G$ and $\bar{\xi} = [q,\xi]_G$, where $\xi = A(q,\dot{q}) = \dot{g}g^{-1}$.

When one has an integral curve of $[\bar{X}]_G$, then $\dot{x} \equiv dx/dt$ will be the time derivative of x and $\dot{y}$ will be Dy/Dt, the covariant derivative of y in the bundle $T^*(Q/G)$ and $\dot{\bar{\mu}} \equiv D\bar{\mu}/Dt$ will be the covariant derivative of $\bar{\mu}$ in $\tilde{\mathfrak{g}}^*$. Note that $\bar{\xi}$ is a function of $\bar{\mu}$, namely, $\bar{\xi}(\bar{\mu})$.

Hamilton-Poincaré-Dirac reduction. Let us show reduction of the standard implicit Hamiltonian system, called *Hamilton-Poincaré-Dirac reduction*.

Definition 7.1. The **reduction of the standard implicit Hamiltonian system** (H,D,X) is given by a triple

$$(h,[\bar{D}]_G,[\bar{X}]_G)$$

that satisfies, for each $(x,y,\bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, the condition

$$([\bar{X}]_G(x,y,\bar{\mu}),[\mathbf{d}\bar{H}]_G(x,y,\bar{\mu})) \in [\bar{D}]_G(x,y,\bar{\mu}). \tag{31}$$

Definition 7.2. A solution curve of $(h,[\bar{D}]_G,[\bar{X}]_G)$ is a curve $(x(t),y(t),\bar{\mu}(t)), t \in [t_0,t_1]$ in $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* \cong T^*Q/G$, which is an integral curve of the reduced vector field $[\bar{X}]_G : T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* \to TT^*(Q/G) \oplus (\tilde{\mathfrak{g}}^* \times \widetilde{V})$ that takes its value for each point of the curve as

$$[\bar{X}]_G(x(t),y(t),\bar{\mu}(t)) = \left(x(t),y(t),\frac{dx(t)}{dt},\frac{Dy(t)}{Dt},\bar{\mu}(t),\bar{\xi}(t),\frac{D\bar{\mu}(t)}{Dt}\right).$$

Theorem 7.3. *Let* $(h, [\bar{D}]_G, [\bar{X}]_G)$ *be the reduction of the standard implicit Hamiltonian system that satisfies equation* (31). *Let* $(x(t), y(t), \bar{\mu}(t)), t \in [t_0, t_1]$ *in* $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ *be a solution curve of* $(h, [\bar{D}]_G, [\bar{X}]_G)$. *Then, the solution curve satisfies the* **Hamilton-Poincaré equations***:*

$$\begin{aligned} \frac{Dy}{Dt} &= -\frac{\partial h}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \\ \frac{dx}{dt} &= \frac{\partial h}{\partial y}, \\ \frac{D\bar{\mu}}{Dt} &= \mathrm{ad}^*_{\bar{\xi}} \bar{\mu}, \\ \bar{\xi} &= \frac{\partial h}{\partial \bar{\mu}}. \end{aligned} \tag{32}$$

Proof. The condition for $(h, [\bar{D}]_G, [\bar{X}]_G)$ is given by, for each $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$,

$$([\bar{X}]_G(x, y, \bar{\mu}), [\mathbf{d}\bar{H}]_G(x, y, \bar{\mu})) \in [\bar{D}]_G(x, y, \bar{\mu}),$$

which is represented by

$$\left(\left(\frac{dx}{dt}, \frac{Dy}{Dt}, \bar{\xi}, \frac{D\bar{\mu}}{Dt} \right), \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, 0, \frac{\partial h}{\partial \bar{\mu}} \right) \right) \in [\bar{D}]_G \left(x, y, \bar{\mu} \right),$$

together with $\bar{\xi} = \partial h / \partial \bar{\mu}$. Then, it follows

$$\begin{aligned} &\left\langle \frac{\partial h}{\partial x}, \delta x \right\rangle + \left\langle \delta y, \frac{\partial h}{\partial y} \right\rangle + \langle 0, \bar{\zeta} \rangle + \left\langle \delta \bar{\mu}, \frac{\partial h}{\partial \bar{\mu}} \right\rangle \\ &= \left\langle \delta y, \frac{dx}{dt} \right\rangle - \left\langle \frac{Dy}{Dt}, \delta x \right\rangle - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \delta x) \right\rangle + \langle \delta \bar{\mu}, \bar{\xi} \rangle - \left\langle \frac{D\bar{\mu}}{Dt}, \bar{\zeta} \right\rangle + \langle \bar{\mu}, [\bar{\xi}, \bar{\zeta}] \rangle . \end{aligned}$$

Hence, one has

$$\begin{aligned} \left\langle -\frac{Dy}{Dt} - \frac{\partial h}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \delta x \right\rangle + \left\langle \delta y, \frac{dx}{dt} - \frac{\partial h}{\partial y} \right\rangle \\ + \left\langle -\frac{D\bar{\mu}}{Dt} + \mathrm{ad}^*_{\bar{\xi}} \bar{\mu}, \bar{\zeta} \right\rangle + \left\langle \delta \bar{\mu}, \bar{\xi} - \frac{\partial h}{\partial \bar{\mu}} \right\rangle = 0 \end{aligned}$$

for all $\delta x, \delta y, \bar{\zeta}$ and $\delta \bar{\mu}$.

Thus, we can obtain the *Hamilton-Poincaré equations* in equation (32) as the reduction of the standard implicit Hamiltonian system. □

Remark 13. The derivative $\partial h / \partial x$ can be interpreted in a covariant way, in a similar way that the partial derivative $\partial l / \partial x$ is defined as was previously shown, by making use of the covariant derivatives in the bundles $T^*(Q/G)$ and $\tilde{\mathfrak{g}}^*$ induced by duality in correspondence with the derivatives in the bundle $T(Q/G)$ and $\tilde{\mathfrak{g}}$.

Coordinate expressions. As previously mentioned, suppose that Q has dimension n, so that Q/G has dimension $r = n - \dim G$. Choose a local trivialization of the principal bundle $Q \to Q/G$ to be $S \times G \to S$, where S is an open set of $\mathbb{R}^r$. Coordinates for a point $q = (x, g) \in Q \cong S \times G$ are written (x^α, g^a). The principal connection one-form on $Q \cong S \times G$ is locally written by $\mathrm{Ad}_g(A_e(x)dx + g^{-1}dg)$. Then, at any tangent vector $(x, g, \dot{x}, \dot{g}) \in T_{(x,g)}(S \times G)$, one has

$$A(x, g, \dot{x}, \dot{g}) = \mathrm{Ad}_g(A_e(x) \cdot \dot{x} + \xi) = A(x, g) \cdot \dot{x} + \dot{g}g^{-1},$$

where A_e is the $\mathfrak{g}$-valued one-form on S defined by $A_e(x)\cdot\dot{x} = A(x,e,\dot{x},0)$ and $\xi = g^{-1}\dot{g}$. By the bundle isomorphism $\Psi_A : TQ/G \to T(Q/G)\oplus\tilde{\mathfrak{g}}$,

$$\Psi_A([x,g,\dot{x},\dot{g}]_G) = (x,\dot{x},\bar{\xi}),$$

where $\bar{\xi} = (x, A_e(x)\cdot\dot{x}+\xi)$. Let $A^a_\alpha(x)$ be the local coordinate expression of A_e on the bundle $S\times G\to S$. Then, we simply write $\bar{\xi}^a = \xi^a + A^a_\alpha\dot{x}^\alpha$. Let C^a_{bd} be the structure constants of $\mathfrak{g}$. Recall that the components of the curvature of A are given, in coordinates, by

$$B^b_{\alpha\,\beta} = \left(\frac{\partial A^b_\beta}{\partial x^\alpha} - \frac{\partial A^b_\alpha}{\partial x^\beta} - C^b_{c\,d}A^c_\alpha A^d_\beta\right).$$

Thus, the **coordinate expression of the Hamilton–Poincaré equations** in (32) may be represented by

$$\begin{aligned}
\frac{dy_\alpha}{dt} &= -\frac{\partial h}{\partial x^\alpha} + \bar{\mu}_a\left(B^a_{\alpha\beta}\dot{x}^\beta - C^a_{db}A^b_\alpha\bar{\xi}^d\right),\\
\frac{dx^\alpha}{dt} &= \frac{\partial h}{\partial y_\alpha},\\
\frac{d\bar{\mu}_b}{dt} &= \bar{\mu}_a\left(C^a_{db}\bar{\xi}^d - C^a_{db}A^d_\alpha\dot{x}^\alpha\right),\\
\bar{\xi}^b &= \frac{\partial h}{\partial\bar{\mu}^b}.
\end{aligned}$$

Gauged Dirac structures in Hamilton-Poincaré-Dirac reduction. Let us denote by $(h,[\bar{D}]_G,[\bar{X}]_G)$ the reduced standard implicit Hamiltonian system that satisfies equation (31). Recall the reduced vector field $[\bar{X}]_G : T^*(Q/G)\oplus\tilde{\mathfrak{g}}^* \to TT^*(Q/G)\oplus(\tilde{\mathfrak{g}}^*\times\widetilde{V})$ is locally given by, for each $(x,y,\bar{\mu})\in T^*(Q/G)\oplus\tilde{\mathfrak{g}}^*$,

$$[\bar{X}]_G(x,y,\bar{\mu}) = \left(x,y,\frac{dx}{dt},\frac{Dy}{Dt},\bar{\mu},\bar{\xi},\frac{D\bar{\mu}}{Dt}\right),$$

and $[\bar{X}]_G$ can be decomposed into the horizontal and vertical parts such that

$$[\bar{X}]_G(x,y,\bar{\mu}) = [\bar{X}]^{\mathrm{Hor}}_G(x,y)\oplus[\bar{X}]^{\mathrm{Ver}}_G(\bar{\mu}).$$

In the above, $[\bar{X}]^{\mathrm{Hor}}_G : T^*(Q/G)\to TT^*(Q/G)$ denotes the *horizontal partial vector field*, which is given by

$$[\bar{X}]^{\mathrm{Hor}}_G(x,y) = \left(x,y,\frac{dx}{dt},\frac{Dy}{Dt}\right)\in TT^*(Q/G),$$

while $[\bar{X}]^{\mathrm{Ver}}_G : \tilde{\mathfrak{g}}^*\to\tilde{\mathfrak{g}}^*\times\widetilde{V}$ indicates the *vertical partial vector field*, which is given by

$$[\bar{X}]^{\mathrm{Ver}}_G(\bar{\mu}) = \left(\bar{\mu},\bar{\xi},\frac{D\bar{\mu}}{Dt}\right)\in\tilde{\mathfrak{g}}^*\times\widetilde{V}.$$

On the other hand, the quotient of $[\mathbf{d}\bar{H}]_G : T^*(Q/G)\oplus\tilde{\mathfrak{g}}^*\to T^*T^*(Q/G)\oplus(\tilde{\mathfrak{g}}^*\times\widetilde{V}^*)$ is given by, for each $(x,y,\bar{\mu})\in T^*(Q/G)\oplus\tilde{\mathfrak{g}}^*$,

$$[\mathbf{d}\bar{H}]_G(x,y,\bar{\mu}) = \left(x,y,\frac{\partial h}{\partial x},\frac{\partial h}{\partial y},\bar{\mu},0,\frac{\partial h}{\partial\bar{\mu}}\right),$$

which can be decomposed into the horizontal and vertical parts as

$$[\mathbf{d}\bar{H}]_G(x,y,\bar{\mu}) = [\mathbf{d}\bar{H}]^{\mathrm{Hor}}_G(x,y)\oplus[\mathbf{d}\bar{H}]^{\mathrm{Ver}}_G(\bar{\mu}).$$

In the above, $[\mathbf{d}\bar{H}]_G^{\mathrm{Hor}}$ is the *horizontal differential of the Hamiltonian* given by, for each $(x,y) \in T^*(Q/G)$,

$$[\mathbf{d}\bar{H}]_G^{\mathrm{Hor}}(x,y) = \left(x, y, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right),$$

and $[\mathbf{d}\bar{H}]_G^{\mathrm{Ver}}$ is the *vertical differential of the Hamiltonian* given by, for each $\bar{\mu} \in \tilde{\mathfrak{g}}^*$,

$$[\mathbf{d}\bar{H}]_G^{\mathrm{Ver}}(\bar{\mu}) = \left(\bar{\mu}, 0, \frac{\partial h}{\partial \bar{\mu}}\right).$$

Recall that the reduced Dirac structure $[\bar{D}]_G$ is a *gauged Dirac structure* that can be decomposed into the horizontal and vertical Dirac structures, which may be represented by, for each $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$,

$$[\bar{D}]_G(x,y,\bar{\mu}) = [\bar{D}]_G^{\mathrm{Hor}}(x,y) \oplus [\bar{D}]_G^{\mathrm{Ver}}(\bar{\mu}),$$

where the *horizontal Dirac structure* $[\bar{D}]_G^{\mathrm{Hor}}$ is shown in equation (24) and the *vertical Dirac structure* $[\bar{D}]_G^{\mathrm{Ver}}$ in equation (25).

Thus, we obtain the following theorem associated with the *gauged Dirac structures in Hamilton-Poincaré-Dirac reduction*.

Theorem 7.4. *Let (H, D, X) be a standard implicit Hamiltonian system and $h : T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* \to \mathbb{R}$ be the reduced Hamiltonian. Let $(h, [\bar{D}]_G, [\bar{X}]_G)$ be the reduction of (H, D, X), which satisfies, for each $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, the condition*

$$\left([\bar{X}]_G\,(x,y,\bar{\mu})\,, [\mathbf{d}\bar{H}]_G(x,y,\bar{\mu})\right) \in [\bar{D}]_G(x,y,\bar{\mu}).$$

Then, $(h, [\bar{D}]_G, [\bar{X}]_G)$ can be decomposed into the horizontal and vertical parts as

$$(h, [\bar{D}]_G, [\bar{X}]_G) = (h, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}}) \oplus (h, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}}).$$

In the above, $(h, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}})$ is the **horizontal implicit Hamiltonian system** *that satisfies, for each $(x, y) \in T^*(Q/G)$,*

$$\left([\bar{X}]_G^{\mathrm{Hor}}\,(x,y)\,, [\mathbf{d}\bar{H}]_G^{\mathrm{Hor}}(x,y)\right) \in [\bar{D}]_G^{\mathrm{Hor}}(x,y),$$

which induces the horizontal Hamilton-Poincaré equations:

$$\frac{Dy}{Dt} = -\frac{\partial h}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot)\right\rangle, \quad \frac{dx}{dt} = \frac{\partial h}{\partial y}.$$

On the other hand, $(h, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}})$ is the **vertical implicit Hamiltonian system** *that satisfies, for each $\bar{\mu}$,*

$$\left([\bar{X}]_G^{\mathrm{Ver}}\,(\bar{\mu})\,, [\mathbf{d}\bar{H}]_G^{\mathrm{Ver}}(\bar{\mu})\right) \in [\bar{D}]_G^{\mathrm{Ver}}(\bar{\mu}),$$

which induces the vertical Hamilton-Poincaré equations:

$$\frac{D\bar{\mu}}{Dt} = \mathrm{ad}^*_{\bar{\xi}}\,\bar{\mu}, \quad \bar{\xi} = \frac{\partial h}{\partial \bar{\mu}}.$$

Equivalence with the implicit Lagrange-Poincaré equations. We have established the Hamilton-Poincaré equations in the context of Dirac cotangent bundle reduction, where a Hamiltonian H is defined on the cotangent bundle T^*Q via the Legendre transformation $\mathbb{F}L : TQ \to T^*Q$ from a given *regular* Lagrangian L on the tangent bundle TQ. On the other hand, one can define the *reduced Hamiltonian* h on $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* \cong T^*Q/G$ from the *reduced Lagrangian* l on $T(Q/G) \oplus \tilde{\mathfrak{g}} \cong TQ/G$, which is given by, for each $(x, u, \bar{\eta}) \in T(Q/G) \oplus \tilde{\mathfrak{g}}$,

$$h(x, y, \bar{\mu}) = \langle y, u \rangle + \langle \bar{\mu}, \bar{\eta} \rangle - l(x, u, \bar{\eta}),$$

where the *reduced Legendre transformation*

$$\mathbb{F}l : T(Q/G) \oplus \tilde{\mathfrak{g}} \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

holds as $(x, u, \bar{\eta}) \mapsto (x, y = \partial l/\partial u, \bar{\mu} = \partial l/\partial \bar{\eta})$. Recall that the reduced Legendre transformation can be expressed by a direct sum of the *horizontal* and *vertical* parts as

$$\mathbb{F}l = \mathbb{F}l^{\mathrm{Hor}} \oplus \mathbb{F}l^{\mathrm{Ver}},$$

where the *horizontal Legendre transformation* $\mathbb{F}l^{\mathrm{Hor}} : T(Q/G) \to T^*(Q/G)$ is given by

$$(x, u) \mapsto \left(x, y = \frac{\partial l}{\partial u} \right),$$

while the *vertical Legendre transformation* $\mathbb{F}l^{\mathrm{Ver}} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*$ is given by

$$\bar{\eta} \mapsto \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}.$$

It follows from Theorems 6.4 and 7.4 that for the case in which a given Lagrangian is regular, the horizontal implicit Lagrange-Poincaré equations are transformed into the horizontal Hamilton-Poincaré equations via the horizontal Legendre transformation, while the vertical implicit Lagrange-Poincaré equations are transformed into the vertical Hamilton-Poincaré equations via the vertical Legendre transformation.

Thus, we can show the *equivalence between the implicit Lagrange-Poincaré equations and the Hamilton-Poincaré equations via the reduced Legendre transformation.*

Example: Satellite with a rotor. Again, let us consider the same example as in the Lagrange-Poincaré-Dirac reduction, namely, a satellite with a rotor aligned with the third principal axis of the body. Recall the satellite with a rotor is modeled by a rigid body of a carrier and a rigid rotor, whose configuration manifold is given by $Q = S^1 \times SO(3)$, with the first factor being the rotor relative angle and the second factor the rigid body attitude. Consider the case in which there exists no torque at the rotor. Then, the Lie group $G = SO(3)$ only acts on the second factor of Q and hence $Q/G = S^1$ as before.

As was shown, let $q = (\theta, R)$ be local coordinates for $Q = S^1 \times SO(3)$ and $(q, v) = (\theta, R, u, U)$ for TQ. Let us take a *trivialized connection* on $Q \to Q/G$ such that $TQ/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}} = TS^1 \times \mathfrak{so}(3)$, while $T^*Q/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* = T^*S^1 \times \mathfrak{so}(3)^*$.

Recall that a given Lagrangian $L : TQ \to \mathbb{R}$ is regular in this example and a left invariant Hamiltonian $H : T^*Q \to \mathbb{R}$ can be naturally defined from L via the Legendre transformation. Hence, H is to be given by the total kinetic energy of the system (rigid carrier plus rotor). By the G-invariance of H, it follows that, for each $(q, p) \in T^*Q$,

$$H(q, p) = h([q, p]_G).$$

In the above, $h : T^*S^1 \times \mathfrak{so}(3)^* \to \mathbb{R}$ is the reduced Hamiltonian, which is given by, for $[q,p]_G = (\theta, y, \Pi) \in T^*S^1 \times (\mathbb{R}^3)^* \cong T^*S^1 \times \mathfrak{so}(3)^*$,

$$h(\theta, y, \Pi) = \frac{1}{2}\left(\frac{\Pi_1^2}{\lambda_1} + \frac{\Pi_2^2}{\lambda_2} + \frac{\Pi_3^2}{\lambda_3} - \frac{2\Pi_3 y}{I_3} - \frac{y^2}{I_3} + \frac{2\lambda_3 y^2}{I_3 J_3} - \frac{y^2}{J_3}\right),$$

where $I_1 > I_2 > I_3$ are the rigid body moments of inertia, $J_1 = J_2$ and J_3 are the rotor moments of inertia, $\lambda_i = I_i + J_i$, θ is the relative angle of the rotor, $(\theta, y) \in T^*S^1$, and $\Pi = (\Pi^1, \Pi^2, \Pi^3) \in (\mathbb{R}^3)^* \cong \mathfrak{so}(3)^*$ is the body angular momentum.

The quotient of the differential of the Hamiltonian, namely, $[\mathbf{d}H]_G : T^*S^1 \times \mathfrak{so}(3)^* \to T^*T^*S^1 \oplus (\mathfrak{so}(3)^* \times \widetilde{V}^*)$ is given by, for each $(\theta, y, \Pi) \in T^*S^1 \times \mathfrak{so}(3)^*$,

$$[\mathbf{d}H]_G(\theta, y, \Pi) = \left(\theta, y, \frac{\partial h}{\partial \theta}, \frac{\partial h}{\partial y}, \Pi, 0, \frac{\partial h}{\partial \Pi}\right), \tag{33}$$

where $\widetilde{V}^* = \mathfrak{so}(3)^* \oplus \mathfrak{so}(3)$, while the reduction of the partial vector field $X : T^*Q \to TT^*Q$ is the quotient map $[\bar{X}]_G : T^*S^1 \oplus \mathfrak{so}(3)^* \to TT^*S^1 \oplus (\mathfrak{so}(3)^* \times \widetilde{V})$, which is given by, for each $(\theta, y, \Pi) \in T^*S^1 \times \mathfrak{so}(3)^*$,

$$[\bar{X}]_G\,(\theta, y, \Pi) = \left(\theta, y, \dot{\theta}, \dot{y}, \Pi, \Omega, \dot{\Pi}\right), \tag{34}$$

where $\hat{\Omega} = R^{-1}\dot{R} \in \mathfrak{so}(3)$.

Then, reduction of the standard implicit Hamiltonian system (H, D, X) is a triple

$$(h, [\bar{D}]_G, [\bar{X}]_G),$$

which satisfies, for each $(\theta, y, \Pi) \in T^*S^1 \oplus \mathfrak{so}(3)^*$, the condition

$$([\bar{X}]_G(\theta, y, \Pi), [\mathbf{d}\bar{H}]_G(\theta, y, \Pi)) \in [\bar{D}]_G(\theta, y, \Pi). \tag{35}$$

It follows from equation (23), (33), (34) and (35) that one can easily derive the Hamilton-Poincaré equations for the satellite with a rotor, which consist of the horizontal Hamilton-Poincaré equations

$$\frac{dy}{dt} = 0, \quad \frac{d\theta}{dt} = \frac{\partial h}{\partial y},$$

as well as the vertical Hamilton-Poincaré equations

$$\frac{d\Pi}{dt} = \Pi \times \Omega, \quad \Omega = \frac{\partial h}{\partial \Pi}.$$

In the above, notice that the momentum $y = J_3(\Sigma_3 + u) =$ constant.

8. **Conclusions and future directions.** This paper has developed a Dirac reduction theory, called *Dirac cotangent bundle reduction*, which is applicable to the reduction of a cotangent bundle T^*Q with its canonical Dirac structure when one has a Lie group G acting freely and properly on Q.

We have shown that *Dirac cotangent bundle reduction accommodates Lagrangian, Hamiltonian and a variational view simultaneously.* Further, we have shown that the resulting reduced structure is given by a *gauged Dirac structure* that consists of the direct sum of a *horizontal* and a *vertical Dirac structure.* Associated with this reduction procedure, we have developed a reduced Hamilton-Pontryagin variational principle, which yields an implicit analogue of the Lagrange-Poincaré equations. Correspondingly, we have established a reduction procedure for standard implicit Lagrangian systems called *Lagrange-Poincaré-Dirac reduction*, including the case of degenerate Lagrangians. This theory also allows one to develop *horizontal and* *vertical implicit Lagrange-Poincaré equations* in the context of gauged Dirac structures, which is consistent with the reduction of the Hamilton-Pontryagin variational principle. We have also explored the Hamiltonian side (for the case in which the given Lagrangian is regular); namely, we have developed a reduction procedure for standard implicit Hamiltonian systems called *Hamilton-Poincaré-Dirac reduction*. This can be also incorporated into the context of the Dirac cotangent bundle reduction, and, as with the Lagrangian side, one gets *horizontal and vertical Hamilton-Poincaré equations*, consistent with the *Hamilton-Poincaré variational principle*. We have finally demonstrated how the Dirac cotangent bundle reduction theory accommodates Lagrange-Poincaré-Dirac reduction as well as Hamilton-Poincaré-Dirac reduction. We have illustrated the theory using the example of a satellite with a rotor.

In the future, we expect to be able to develop a similar theory for systems with nonholonomic constraints as well as including the present theory into the broader context of reduction theory for Dirac anchored vector bundles, as in [26]. We will investigate some concrete examples of degenerate Lagrangian systems by the present theory. We are also exploring Dirac structures and the Hamilton-Pontryagin variational principle for field theories. Another interesting future direction would be to develop these reduction methods for discrete Dirac structures as well as the Hamilton-Pontryagin variational integrator, as shown in [12] and [52].

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