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THE MOTION OF SOLID BODIES IN POTENTIAL FLOW WITH CIRCULATION: A GEOMETRIC OUTLOOK

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ABSTRACT

The motion of a circular body in 2*D potential flow is studied using symplectic reduction. The equations of motion are obtained starting from a kinetic-energy type system on a space of embeddings and reducing by the particle relabelling symmetry group and the special Euclidian group. In the process, we give a geometric interpretation for the Kutta-Joukowski lift force in terms of the curvature of a connection on the original phase space.*

INTRODUCTION

It has been known since the pioneering work of Kirchhoff, Stokes, and Lamb that the motion of a rigid body in a potential flow has a very succinct description with the ambient fluid manifesting itself only through the appearance of added masses and added moments of inertia.

If the circulation around the body is non-zero, or if isolated point vortices are present in the fluid, additional effects have to be taken into account. In the former case, the body experiences an additional lift force, proportional to its velocity and the circulation. The resulting dynamics was first studied by Chaplygin and Lamb (see [1] and the references therein). In the case of point vortices, these effects are sufficiently subtle for the equations of motion of the system to have been derived only recently (see [2]).

The Kirchhoff equations for a rigid body in a potential flow were studied from a geometric point of view in [3, 4]. In this paper, we show that this formalism can be extended to the case where the circulation around the body is not necessarily zero. The motion of a rigid body in a perfect fluid, even with circulation, can be viewed as a prime example of *geometric reduction theory* (see [5,6]). From this point of view, the body-fluid system first is defined as a dynamical system on an infinite-dimensional configuration space Q, consisting of two parts: one accounting for the position of the body, and the other consisting of maps taking the fluid labels to their respective positions in material space at a certain instant.

The degrees of freedom of the system on Q can then be reduced to a finite number by realizing that two distinct symmetry groups act on Q, and dividing out by these group actions. First, there is the group of volume preserving diffeomorphisms, which acts on the label space of the fluid and simply permutes the labels of the fluid particles. Secondly, the whole system (consisting of solid and fluid) is invariant under global translations and rotations. Dividing out by these symmetry groups naturally leads to the Kirchhoff equations with an additional lift force proportional to the circulation (equation 25 below).

We limit ourselves to the case of a rigid body moving in a potential flow with circulation but no external vorticity, as this case is not overly complicated but already exhibits many of the interesting features present in geometric reduction theory. A significant feature of our analysis is that the lift force experienced by the body turns out to be nothing but the velocity vector of the body contracted with a certain curvature tensor (see equation 20). In this way, we provide an alternative geometric description of what is known in the classical literature on fluid dynamics as the Kutta-Joukowski theorem.

The layout of the paper is as follows. After describing the

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problem setting and recalling some well-known facts from potential flow theory and rigid body dynamics, we describe the geometric approach to perfect fluid dynamics. We then apply this theory to the case at hand, *i.e.* the dynamics of a rigid circular body in a potential flow with circulation. The bulk of the paper is devoted to reducing the dynamics with respect to the symmetry groups described above. In the process, we use a certain connection on the unreduced phase space and calculate its curvature. In the final sections, the equations of motion are derived and we discuss the physical significance of this curvature. The paper ends with an outlook on possible generalizations of this approach.

PROBLEM SETTING

We consider a rigid body of cylindrical shape moving in an inviscid, incompressible fluid. The body – considered to be uniform and neutrally-buoyant (the body weight is balanced by the force of buoyancy) – may be represented by a disc in $\mathbb{R}^2$ and for the sake of convenience we assume that the fluid fills the complement of the body in $\mathbb{R}^2$. This assumption can easily be relaxed, for example to the case where the fluid moves in a bounded container, or on a two-dimensional surface different from $\mathbb{R}^2$ (such as the 2-sphere). The reference configuration of the fluid will be denoted by F_0, and that of the body by B_0. The space taken by the fluid at a generic time t will be denoted by F. Note however that as time progresses, the position of the body changes and hence so does its complement F.

Rigid body kinematics. Introduce an orthonormal inertial frame $\{\mathbf{e}_{1,2,3}\}$ where $\{\mathbf{e}_1, \mathbf{e}_2\}$ span the plane of motion and $\mathbf{e}_3$ is the unit normal to this plane. The configuration of the submerged rigid body can then be described by a rotation β about $\mathbf{e}_3$ and a translation $\mathbf{r} = x_o\mathbf{e}_1 + y_o\mathbf{e}_2$ of a point O (often chosen to coincide with the mass center) in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ directions (see figure). The angular and translational velocities expressed relative to the inertial frame are of the form $\dot\beta\,\mathbf{e}_3$ and $\mathbf{v} = v_x\,\mathbf{e}_1 + v_y\,\mathbf{e}_2$ where $v_x = \dot x_o$, $v_y = \dot y_o$ (the dot denotes derivative with respect to time t). It is convenient for the following development to introduce a moving frame $\{\mathbf{b}_{1,2,3}\}$ attached to the body. The point transformation from the body to the inertial frame can be represented as

$$\mathbf{x} = R\mathbf{X} + \mathbf{r}, \qquad R = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}, \tag{1}$$

where $\mathbf{x} = x\,\mathbf{e}_1 + y\,\mathbf{e}_2$ and $\mathbf{X} = X\,\mathbf{b}_1 + Y\,\mathbf{b}_2$, while vectors transform as $\mathbf{v} = R\mathbf{V}$. The angular and translational velocities expressed in the body frame take the form $\Omega = \Omega\,\mathbf{b}_3$ (where $\Omega = \dot\beta$) and $\mathbf{V} = V_x\mathbf{b}_1 + V_y\mathbf{b}_2$ (where $V_x = \dot x_o\cos\beta + \dot y_o\sin\beta$ and $V_y = -\dot x_o\sin\beta + \dot y_o\cos\beta$). Note that the orientation and position (β, x_o, y_o) form an element of $SE(2)$, the group of rigid body motions in $\mathbb{R}^2$. The velocity in the body-frame $\xi = (\Omega, V_1, V_2)^T$, where $()^T$ denotes the transpose operation, is an element of the vector space $\mathfrak{se}(2)$ which is the space of infinitesimal rotations and translations in $\mathbb{R}^2$ and is referred to as the Lie algebra of $SE(2)$.

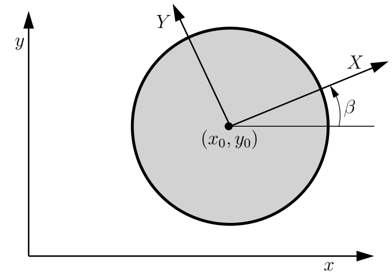

Figure 1. Orientation of the rigid body.

Fluid velocity. The fluid velocity $\mathbf{u}$ can be written using the Helmholtz-Hodge decomposition as follows

$$\mathbf{u} = \nabla\Phi \;+\; \mathbf{u}_{\rm v}, \tag{2}$$

where $\mathbf{u}_{\rm v}$ is a divergence-free vector field and can be written as $\mathbf{u}_{\rm v} = \nabla\times\Psi + \mathbf{u}_\Gamma$. The vector potential Ψ satisfies $\Delta\Psi = -\omega$ subject to the boundary conditions $(\nabla\times\Psi)\cdot\mathbf{n} = 0$ on $\partial\mathcal{B}$ and $\nabla\times\Psi = 0$ at infinity. Here, $\omega = \nabla\times\mathbf{u}_{\rm v}$ is the vorticity in the fluid domain, which implies that Ψ is zero in the absence of vorticity. For planar flows, the vector potential $\Psi = \psi\mathbf{e}_3$, where ψ is referred to as the stream function.

The harmonic vector field $\mathbf{u}_\Gamma$ is non-zero only when there is a net circulatory flow around the body; it satisfies $\nabla\cdot\mathbf{u}_\Gamma = 0$ and $\nabla\times\mathbf{u}_\Gamma = 0$ (i.e., $\Delta\mathbf{u}_\Gamma = 0$) and the boundary conditions $\mathbf{u}_\Gamma\cdot\mathbf{n} = 0$ on $\partial\mathcal{B}$ and $\mathbf{u}_\Gamma = 0$ at infinity. Note that, in three dimensional flows, one does not need the harmonic vector field $\mathbf{u}_\Gamma$.[1] For the planar problem considered here, the effect of having a net circulation Γ around the body is equivalent to placing a point vortex of strength Γ at the center of mass of the body such that

$$\mathbf{u}_\Gamma = \nabla\times(\psi_\Gamma\,\mathbf{e}_3), \qquad \psi_\Gamma = \frac{\Gamma}{4\pi}\log(X^2+Y^2), \tag{3}$$

where ψ_Γ is expressed in body coordinates (X, Y).

The potential function Φ is harmonic, that is, it is the solution to Laplace's equation $\Delta\Phi = 0$, subject to impermeable boundary conditions on $\partial\mathcal{B}$ ($\nabla\Phi\cdot\mathbf{n}$ = normal velocity of the boundary) and the velocity is required to vanish at infinity. Namely, one has

$$\nabla^2\Phi = 0 \quad \text{and} \quad \left.\frac{\partial\Phi}{\partial n}\right|_{\partial\mathcal{B}} = (\Omega\times\mathbf{X} + \mathbf{V})\cdot\mathbf{n}. \tag{4}$$

[1] In three dimensions, any closed curve in the exterior of a *bounded* body is contractible, so the harmonic vector field $\mathbf{u}_\Gamma$ may be set to zero. This result is due to *Poincaré Lemma* which can be alternatively stated as follows: a closed one-form on a (sub)-manifold with trivial first cohomology is globally exact.

Physically, $\Phi(X,Y)$ represents the irrotational motion of the fluid generated by moving the body. By linearity of Laplace's equation, one can write, following Kirchhoff (see [7]),

$$\Phi = \Omega\Phi_\Omega + V_x\Phi_x + V_y\Phi_y, \tag{5}$$

where $\Phi_\Omega, \Phi_x, \Phi_y$ are called velocity potentials and are solutions to Laplace's equation subject to the boundary conditions on $\partial\mathcal{B}$

$$\frac{\partial\Phi_\Omega}{\partial n} = \mathbf{X}\times\mathbf{n}\cdot\mathbf{b}_3\,, \quad \frac{\partial\Phi_x}{\partial n} = \mathbf{n}\cdot\mathbf{b}_1\,, \quad \frac{\partial\Phi_y}{\partial n} = \mathbf{n}\cdot\mathbf{b}_2 \tag{6}$$

In the case of a cylindrical body, these elementary potentials take the following form in a coordinate system fixed to the body (see [7]):

$$\Phi_\Omega = 0, \quad \Phi_x = -\frac{Y}{X^2+Y^2}, \quad \text{and} \quad \Phi_y = -\frac{X}{X^2+Y^2}. \tag{7}$$

Kirchhoff equations. The equations governing the motion of the body in potential flow, that is, in the absence of external vorticity and circulation ($\mathbf{u}_v = 0$), are known as Kirchhoff equations and take the form:

$$\begin{aligned} \dot{\Pi} &= P_xV_y - P_yV_x, \\ \dot{P}_x = -P_y\Omega, &\qquad \dot{P}_y = P_x\Omega. \end{aligned} \tag{8}$$

Here, Π denotes the angular momentum of the solid-fluid system while (P_x, P_y) denotes the linear momenta of the system expressed in the body frame. They are given in terms of the velocity in body frame by

$$\Pi = \mathbb{I}\Omega, \quad P_x = (m_b+m_x)V_x, \quad \text{and} \quad P_y = (m_b+m_y)V_y,$$

where m_b and $\mathbb{I}$ are the mass of the cylinder and its moment of inertia along the axis, respectively. The quantities m_x and m_y are the added masses of the cylinder in the x and y direction; for a circular cylinder, $m_x = m_y = \pi\rho_{\mathcal{F}}$, with $\rho_{\mathcal{F}}$ the mass density of the fluid. Note that this implies that for a *circular* cylinder, $\dot{\Pi} = 0$.

One of the main objectives of this paper is to use the methods of geometric mechanics to derive the equations governing the motion of the body in potential flow and with non-zero circulation. The general case of a body of arbitrary geometry interacting with external vorticity will be addressed in a future publication.

GEOMETRIC FORMULATION

The Configuration Space. The configuration space Q for the fluid-rigid body system consists of pairs (φ, g), where $\varphi \in \mathrm{Emb}(F_0, \mathbb{R}^2)$ is an embedding of F_0 in $\mathbb{R}^2$, and g is an element of $SE(2)$ such that the following conditions are fulfilled:

(i) φ approaches the identity at infinity;
(ii) φ represents an incompressible fluid: $\varphi^*\eta = \eta_0$ (η is the Euclidian volume form on $\mathbb{R}^2$ and η_0 is its restriction to F_0);
(iii) $\varphi(\partial F_0) = g(\partial B_0)$ as sets, where g is interpreted as an embedding of B_0 into $\mathbb{R}^2$.

Condition (ii) simply means that the Jacobian of φ is unity, whereas condition (iii) is a rewriting of the slip boundary condition in (4). For future reference, we also introduce the group of volume-preserving diffeomorphisms of F_0, denoted by $\mathrm{Diff}_{\mathrm{vol}}$, and consisting of diffeomorphisms $\phi : F_0 \to F_0$ such that $\phi^*\eta_0 = \eta_0$.

Let (φ, g) be an element of Q. Any tangent vector to Q can be represented as a pair $(\varphi, \dot{\varphi}; g, \dot{g})$, where $(g, \dot{g})$ is an element of $T_gSE(2)$ and $\dot{\varphi}$ is a map from F_0 to TF such that $\dot{\varphi}(x) \in T_{\varphi(x)}F$. The vector field $\dot{\varphi}$ represents the *Lagrangian velocity field* of the fluid. The *Eulerian velocity field* $\mathbf{u}$ is then defined as $\mathbf{u} = \dot{\varphi}\circ\varphi^{-1}$. Note that $\mathbf{u}$ is a vector field on F, in contrast to $\dot{\varphi}$ (which is a vector field along the map φ).

Geometric Fluid Mechanics. Following [9, 10], we introduce the vorticity field as an exact two-form on F_0:

Definition 1. *The vorticity field μ of $\mathbf{u}$ is defined as $\mu = \mathrm{d}(\varphi^*\mathbf{u}^\flat)$.*

From a geometric point of view, the vorticity takes values in the space $\Omega^1(F_0)/\mathrm{d}\Omega^0(F_0)$ of one-forms up exact forms. This space is isomorphic to $\mathrm{d}\Omega^1(F_0)\times\mathbb{R}\times\cdots\times\mathbb{R}$, where the number of factors of $\mathbb{R}$ is equal to the rank of the first homology group of F_0 (in our case, the rank is one). This isomorphism is explicitly given by (see [9])

$$[\alpha] \mapsto (\mathrm{d}\alpha, \Gamma), \quad \text{where} \quad \Gamma = \int_{\mathcal{C}} \alpha, \tag{9}$$

and $\mathcal{C}$ is a closed contour encircling the rigid body exactly once. Physically speaking, $\mathrm{d}\alpha$ represents the external vorticity of the fluid, while Γ is the circulation around the rigid body.

Particle relabeling Symmetry. In the absence of external vorticity, the action of the particle relabelling group $\mathrm{Diff}_{\mathrm{vol}}$ on the fluid does not change the circulation around the rigid body. This is a consequence of Kelvin's theorem, but can easily be proved directly.

Let (φ_t, g_t) be a curve in Q describing the motion of the solid-fluid system, and consider an arbitrary element ϕ of $\mathrm{Diff}_{\mathrm{vol}}$. The transformed motion is then given by (φ_t', g_t), where $\varphi_t' = \varphi_t\circ\phi$, and the difference in circulation between the original and the transformed motion is given by

$$\Gamma' - \Gamma = \int_{\phi(C)} \mathbf{u}'\,\mathrm{d}\mathbf{l} - \int_C \mathbf{u}\,\mathrm{d}\mathbf{l} = \int_A \nabla\times\mathbf{u}\,\mathrm{d}\mathbf{A} \tag{10}$$

where $\mathcal{C}$ is a curve encircling the body, A is the region bounded by $\mathcal{C}$ and $\phi(\mathcal{C})$ and $\mathbf{u}'$ is the transformed Eulerian velocity given by

$$\mathbf{u}' = (\dot{\varphi} \circ \phi) \circ (\varphi \circ \phi)^{-1} = \dot{\varphi} \circ \varphi^{-1} = \mathbf{u}. \tag{11}$$

If the external vorticity is zero, the right-hand side of (10) vanishes and hence Γ' is equal to Γ.

Mathematically speaking, the vorticity μ can be interpreted as an element of the dual of the Lie algebra of $\mathrm{Diff}_{\mathrm{vol}}$ (see [9,10]). The fact that arbitrary diffeomorphism do not change the circulation, as we have just shown, then translates to the following key theorem.

Theorem 2. *Let* $\mu = \mathrm{d}(\varphi^* \mathbf{u}_V^\flat)$ *be the vorticity representing a given amount of circulation* Γ. *Then the isotropy subgroup* $(\mathrm{Diff}_{\mathrm{vol}})_\mu$ *of diffeomorphisms leaving* μ *invariant coincides with the whole of* $\mathrm{Diff}_{\mathrm{vol}}$.

The group of rigid motions. We recall some basic facts about Lie groups and algebras in the context of the special Euclidian group $SE(2)$.[2] The group $SE(2)$ can be parametrized by pairs $(R, \mathbf{r})$ as done in (1), where $R \in SO(2)$ describes the orientation of the body, while $\mathbf{r} \in \mathbb{R}^2$ fixes the location of the center of mass. The group law in $SE(2)$ is given by

$$(R_1, \mathbf{r}_1) \cdot (R_2, \mathbf{r}_2) = (R_1 R_2, R_1 \mathbf{r}_2 + \mathbf{r}_1).$$

An element $(R, \mathbf{r})$ of $SE(2)$ can alternatively be written as the following 3-by-3 matrix:

$$(R, \mathbf{r}) \mapsto \begin{pmatrix} R & \mathbf{r} \\ 0 & 1 \end{pmatrix}.$$

The composition operation is then simply given by the multiplication of the corresponding matrices. For more information on this group and its relevance to mechanics, see [6, 8].

The Lie algebra of $SE(2)$ is denoted by $\mathfrak{se}(2)$. Its elements are matrices of the form

$$\begin{pmatrix} A & \mathbf{b} \\ 0 & 0 \end{pmatrix},$$

where A is an antisymmetric matrix and $\mathbf{b} \in \mathbb{R}^2$. Note that $\mathfrak{se}(2)$ is isomorphic to $\mathbb{R}^3$ by means of the following mapping: $(A, \mathbf{b}) \mapsto (\Omega, \mathbf{b})$, where Ω is the non-zero lower-left entry in the antisymmetric matrix A.

The dual of $\mathfrak{se}(2)$ is denoted by $\mathfrak{se}(2)^*$. The duality pairing between $\mathfrak{se}(2)$ and $\mathfrak{se}(2)^*$ is given by (a multiple of) the Killing form. Under this correspondence $\mathfrak{se}(2)^*$ can be identified with $\mathbb{R}^3$ and the duality pairing is then just the Euclidian inner product on $\mathbb{R}^3$.

Below, we will need to use a left-invariant basis for one-forms on $SE(2)$. This basis is constructed as follows. We first define the following basis of $\mathfrak{se}(2)$:

$$\mathbf{e}_\Omega = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and let $\{\mathbf{e}_\Omega^*, \mathbf{e}_x^*, \mathbf{e}_y^*\}$ be the corresponding dual basis of $\mathfrak{se}(2)^*$. Denote by θ the left-invariant *Maurer-Cartan form* on $SE(2)$, given by $\theta_g(v_g) = TL_{g^{-1}}(v_g)$. This is an $\mathfrak{se}(2)$-valued one-form; its components, given by

$$\theta_x = \langle \mathbf{e}_x^*, \theta \rangle, \quad \theta_y = \langle \mathbf{e}_y^*, \theta \rangle, \quad \text{and} \quad \theta_\Omega = \langle \mathbf{e}_\Omega^*, \theta \rangle \tag{12}$$

form a left-invariant basis of one-forms on $SE(2)$. Explicitly, the left invariant forms defined in (12) are locally given by the following expressions:

$$\begin{cases} \theta_\Omega = \mathrm{d}\beta \\ \theta_x = \quad \cos\beta \, \mathrm{d}x_0 + \sin\beta \, \mathrm{d}y_0 \\ \theta_y = -\sin\beta \, \mathrm{d}x_0 + \cos\beta \, \mathrm{d}y_0, \end{cases}$$

where $\{x_0, y_0, \beta\}$ are coordinates on $SE(2)$.

THE MECHANICAL CONNECTION

The configuration space Q of a rigid body moving in a fluid consists of pairs (φ, g), where $\varphi \in \mathrm{Emb}(F_0, M)$ and $g \in SE(2)$. The manifold Q is fibered over $SE(2)$ by mapping (φ, g) to g, but more is true: this projection makes Q into the total space of a principal fibre bundle over $SE(2)$. The structure group of this principal fiber bundle is the group $\mathrm{Diff}_{\mathrm{vol}}$ of volume-preserving diffeomorphisms of F_0.

Indeed, recall that this group acts on the right on $\mathrm{Emb}(F_0, M)$ (and hence also on Q) by putting $\varphi \cdot \phi = \varphi \circ \phi$. If we define the projection $\pi : Q \to SE(2)$ by $\pi(\varphi, g) = g$, then it is clear that the orbits of $\mathrm{Diff}_{\mathrm{vol}}$ in Q coincide with the fibers of π.

Definition of the mechanical connection

As shown above, the configuration space Q is the total space of a principal fiber bundle over $SE(2)$. Furthermore, Q is equipped with a principal fiber bundle connection, defined as a $\mathrm{Diff}_{\mathrm{vol}}$-equivariant one-form $\mathcal{A} : TQ \to \mathfrak{X}_{\mathrm{vol}}$ given by

$$\mathcal{A}_{(\varphi, g)}(\dot{\varphi}, \dot{g}) = \varphi^* \mathbf{u}_{\mathrm{v}},$$

where $\mathbf{u}_{\mathrm{v}}$ is the divergence-free part of the Helmholtz-Hodge decomposition (see [11]) of the Eulerian velocity field.

[2] This section can be safely skipped by readers familiar with Lie groups.

It can be shown that this connection is the *mechanical connection* associated to the kinetic energy of the fluid-rigid body system.

Curvature of the mechanical connection

In the subsequent analysis, we will need the curvature of the mechanical connection, paired with the circulation (3) in the appropriate sense. Before dealing with this specific case, let us first review some of the general theory of connections on a principal fibre bundle.

The curvature of a principal fiber bundle connection $\mathcal{A}$ is the two-form $\mathcal{B}$ on Q defined as follows: for $u_q, v_q \in T_qQ$, $\mathcal{B}(u_q, v_q) = \mathrm{d}\mathcal{A}(\mathbf{h}(u_q), \mathbf{h}(v_q))$, where the map $\mathbf{h}$ projects the tangent vectors u_q, v_q onto their horizontal parts.

It can be seen from the above formula that $\mathcal{B}$ is a two-form taking values in $\mathfrak{g}$. By pairing $\mathcal{B}$ with an element μ of $\mathfrak{g}^*$, we obtain a regular form on Q. Generally, this form is only invariant with respect to G_μ, but keeping theorem 2 in mind, we may assume for now that $G_\mu = G$. In this case, the paired form $\langle \mu, \mathcal{B} \rangle$ drops to a form on Q/G given by the following formula:

$$\mathcal{B}_\mu(\dot{g}, \dot{h}) = \left\langle \mu, \mathcal{B}((\dot{g})^H_q, (\dot{h})^H_q) \right\rangle,$$

where the superscript 'H' denotes the horizontal lift of an element of $T(Q/G)$ to TQ.

For the solid-fluid system, G is $\mathrm{Diff}_{\mathrm{vol}}$, the relevant connection is the mechanical connection, and $\mathcal{B}_\mu$ is defined on $SE(2)$. In the case we are considering here, μ is given by $\mathrm{d}\varphi^* \mathbf{u}_V^\flat$, with $\mathbf{u}_V$ as in (3).

Montgomery [12] offers the following general formula for $\mathcal{B}_\mu$:

$$\mathcal{B}_\mu(\dot{g}_1, \dot{g}_2) = \int_F (\nabla \times \mathbf{u}_V) \cdot (\nabla\Phi_1 \times \nabla\Phi_2) \mathrm{d}x - \int_{\partial F} \mathbf{u}_V \cdot (\mathbf{n} \times (\nabla\Phi_1 \times \nabla\Phi_2)) \mathrm{d}l, \quad (13)$$

where Φ_1 and Φ_2 are the solutions of the Neumann problem (4) associated to $\dot{g}_1$ and $\dot{g}_2$, respectively, and $\mathbf{u}_V$ is the velocity field given by (3). It should be noted that Montgomery's formula is valid for arbitrary vorticity fields, and is not limited to the case of circulation only.

Explicit calculation of the curvature

Once the solution of the Neumann problem (4) for arbitrary boundary conditions is known, the curvature can be calculated explicitly.

Proposition 3. *The μ-component of the curvature is a left $SE(2)$-invariant 2-form on $SE(2)$ given by*

$$\mathcal{B}_\mu = \Gamma \theta_x \wedge \theta_y, \quad (14)$$

where Γ is the circulation around the body.

Proof: The velocity potential (5) is left $SE(2)$-invariant, in the sense that the solutions of (4) for $(g, \dot{g})$ and $(hg, h\dot{g})$ coincide (for an arbitrary element h of $SE(2)$), and the same holds for $\mathcal{B}_\mu$. As shown in (7), the velocity potential associated to an infinitesimal rotation is identically zero and it follows therefore that $\mathcal{B}_\mu$ is proportional to $\theta_x \wedge \theta_y$. In the remainder of this proof, we determine the constant of proportionality by calculating $\mathcal{B}_\mu$ evaluated on $\mathbf{e}_x$ and $\mathbf{e}_y$.

The vector product of $\nabla\Phi_x$ and $\nabla\Phi_y$ is given by

$$\nabla\Phi_x \times \nabla\Phi_y = \frac{1}{(X^2 + Y^2)^2} \mathbf{k},$$

where $\mathbf{k}$ is the unit vector perpendicular to the plane of motion.

The first term in (13) is always zero as the integration is over the fluid domain, while the support of the vorticity function $\nabla \times \mathbf{u}_V$, where $\mathbf{u}_V$ is given by (3), is contained in the body.

The second term can be rewritten as follows:

$$-\int_{\partial F} \mathbf{u}_V \cdot (\mathbf{n} \times (\nabla\Phi_1 \times \nabla\Phi_2)) \mathrm{d}l = -\int_{\partial F} \mathbf{u}_V \cdot (\mathbf{n} \times \mathbf{k}) \mathrm{d}l = \int_{\partial F} \mathbf{u}_V \cdot \mathbf{t} \mathrm{d}l = \Gamma,$$

where $\mathbf{n}$ and $\mathbf{t}$ are the normal and the tangent vector field to the boundary, respectively. ◇

The expression for $\mathcal{B}_\mu$ can therefore be simplified to

$$\mathcal{B}_\mu = \Gamma \mathrm{d}x_0 \wedge \mathrm{d}y_0. \quad (15)$$

REDUCTION: THE DIFFEOMORPHISM GROUP

The group $\mathrm{Diff}_{\mathrm{vol}}$ of volume-preserving diffeomorphisms of F_0 acts on Q, and hence on T^*Q by the cotangent lifted action. Below, we show that this action leaves the kinetic energy invariant. Furthermore, the reduced phase space has a very simple form: it is the cotangent bundle to $SE(2)$, but with the canonical symplectic form shifted by a certain *magnetic two-form* B_μ proportional to the circulation. This two-form manifests itself through the classical *Kutta-Joukowski lift force* on the rigid body (see [13]).

The reduced Hamiltonian

For the sake of completeness, we review in this section some of the theory of rigid bodies moving in a potential flow, in particular the introduction of added masses and moments of inertia. For more information, see [7, 13].

The kinetic energy for the solid-fluid system on Q is given by $T = T_{\mathrm{fluid}} + T_{\mathrm{body}}$, where T_{fluid} is the kinetic energy of the fluid in spatial representation:

$$T_{\text{fluid}} = \frac{1}{2}\int_F \|\mathbf{u}\|^2\, d\mathbf{x}, \tag{16}$$

and T_{body} is the kinetic energy of the body given by

$$T_{\text{body}}(g,\dot{g}) = \frac{1}{2}(\mathbb{I}\Omega^2 + m\mathbf{V}^2),$$

which may be rewritten as

$$T_{\text{body}} = \frac{1}{2}\xi^T \mathbb{M}_b \xi, \qquad \mathbb{M}_b = \begin{pmatrix} \mathbb{I} & 0 & 0 \\ 0 & m_b & 0 \\ 0 & 0 & m_b \end{pmatrix}$$

Recall from (11) that the Eulerian velocity $\mathbf{u}$ is invariant under the right action of Diff_{vol}: as a result, so is the kinetic energy (16) of the fluid. It follows that T_{fluid} drops to a function on $TSE(2)$, given explicitly by

$$T_{\text{fluid}} = \frac{\rho_{\mathcal{F}}}{2}\int_F \|\nabla\Phi\|^2\, d\mathbf{x} = \frac{1}{2}\xi^T \mathbb{M}_f \xi,$$

where $\mathbb{M}_f$ is the matrix of *added masses and moments of inertia* induced by the fluid:

$$\mathbb{M}_f = \pi\rho_{\mathcal{F}} \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{pmatrix},$$

where $\mathbf{I}$ is the 2-by-2 identity matrix. For more details about this derivation, see [3, 7]. In case of a body of arbitrary shape, $\mathbb{M}_f$ generally has a more complicated form.

The kinetic energy of the body-fluid system takes the following convenient form:

$$T = \frac{1}{2}\xi^T \mathbb{M}\xi, \quad \text{where} \quad \mathbb{M} = \mathbb{M}_f + \mathbb{M}_b.$$

This is a quadratic form on $\mathfrak{se}(2)$ and hence, by left-invariant extension, also on $TSE(2)$. It follows that T induces a left-invariant metric on $SE(2)$, explicitly given by $\langle (g,\dot{g}_1),(g,\dot{g}_2)\rangle = \xi_1^T \mathbb{M}\xi_2$, where $\xi_i = g^{-1}\dot{g}_i$, $i = 1,2$. Note that $\mathbb{M}$ is again a diagonal matrix with entries $\mathbb{I}$, $m_b + m_x$, and $m_b + m_x$. For notational sake, we put $m := m_b + m_x$.

Using this metric to identify $TSE(2)$ and $T^*SE(2)$, the kinetic energy T induces a Hamiltonian H on $T^*SE(2)$ given by

$$H(g,\alpha_g) = \begin{pmatrix} \Pi & P_x & P_y \end{pmatrix} \mathbb{M} \begin{pmatrix} \Pi \\ P_x \\ P_y \end{pmatrix} \tag{17}$$

where $\begin{pmatrix} \Pi & P_x & P_y \end{pmatrix}$ is the element of $\mathfrak{se}(2)^*$ obtained by left translating $\alpha_g \in T_g^*SE(2)$ to the identity. It follows at once that H is left $SE(2)$-invariant.

The reduced phase space

Using symplectic reduction (see [5] and the references therein), this symmetry can be divided out, yielding a system with a phase space of lower dimension. Generally, this reduced phase space is of the form $J^{-1}(\mu)/G_\mu$, where J is the *momentum map* associated to the symmetry.

In the case of Diff_{vol} acting on the solid-fluid system, the momentum map is simply the vorticity of the system (see [9]), but it is not necessary to know the explicit form of the momentum map in order to proceed with symplectic reduction. If, as in this case, the unreduced phase space is a cotangent bundle and the isotropy group G_μ of a fixed element μ is the whole of G, then the reduced phase space is another cotangent bundle but with a non-canonical symplectic structure, as described in the following theorem.

Theorem 4 (See [5], theorem 2.2.3). *Assume μ is a regular value of J. Then there is a symplectic diffeomorphism between $J^{-1}(\mu)/G$ and $T^*(Q/G)$, the latter with symplectic form $\Omega_{\mathcal{B}}$, defined as*

$$\Omega_{\mathcal{B}} = \Omega_{\text{can}} - B_\mu, \tag{18}$$

where B_μ is the magnetic two-form described below.

The *magnetic two-form B_μ* in this description is a form on $T^*(Q/G)$, given by $B_\mu = \pi_{Q/G}^*\mathcal{B}_\mu$, where $\pi_{Q/G} : T^*(Q/G) \to Q/G$ is the cotangent bundle projection. This is essentially a consequence of theorem 2.1.13 in [5]. The reason behind this particular terminology will become clear in a moment.

It should be noted that the explicit expression for B_μ as the μ-component of the curvature $\mathcal{B}$ is a direct consequence of the assumption that the isotropy group G_μ is the whole of G. In general, additional terms arise in the expression for B_μ; see [5] for more information. From now on, we will no longer make any notational distinction between B_μ and $\mathcal{B}_\mu$.

Putting the results of this and the previous section together, we have the following theorem.

Theorem 5. *The motion of a rigid cylinder in a potential flow with circulation Γ is a Hamiltonian system on $SE(2)$, with Hamiltonian H given by (17) and shifted symplectic structure $\Omega_{\mathcal{B}}$ given by (18).*

Explicitly, the trajectories described by a rigid body moving in a potential flow with circulation are given by the integral curves of the vector field X_H, satisfying

$$i_{X_H}\Omega_{\mathcal{B}} = \mathrm{d}H. \tag{19}$$

The lift force on the rigid body At this stage, the physical relevance of $\mathcal{B}_\mu$ can be made more explicit by rewriting equation (19) as $i_{X_H}\Omega_{\mathrm{can}} = \mathrm{d}H + i_{X_H}\mathcal{B}_\mu$.

This is a canonical Hamiltonian system under the influence of a *gyroscopic force* $i_{X_H}\mathcal{B}_\mu$. In inertial coordinates, this force is given by

$$i_{X_H}\mathcal{B}_\mu = \Gamma\mathbf{v}\times\mathbf{e}_3, \tag{20}$$

where $\mathbf{v}$ is the translational velocity of the rigid body. Hence, the curvature $\mathcal{B}_\mu$ induces a force proportional to the circulation and at right angles to the velocity of the rigid body. This is nothing but the classical Kutta-Joukowski lift force (see [13]) on a rigid body with circulation.

The effect of this force will be made more clear below once we divide out the rigid-body symmetry to obtain the final equations of motion (25). For now, note that this situation is very reminiscent of the geometric description of a charged particle in a magnetic field (see [6, 8]), where the magnetic field can either be brought in by modifying the symplectic structure to include the magnetic field, or alternatively by adding the Lorentz force as a gyroscopic force to the right-hand side of the Hamiltonian equations.

REDUCTION: THE GROUP $SE(2)$

The reduced Hamiltonian (17) as well as the shifted symplectic structure (18) are both invariant under the left action of $SE(2)$ on itself. This is a consequence of the fact that the solid-fluid system is invariant under global translations and rotations of the rigid body and the fluid simultaneously.

This residual symmetry allows us to reduce the original system on $T^*SE(2)$ even further, down to a system on the dual Lie algebra $\mathfrak{se}(2)^*$. This would be a straightforward application of Lie-Poisson reduction (see [6]), if it weren't for the magnetic term in the symplectic structure.

In particular, the symplectic structure on $T^*SE(2)$ induces a Poisson structure on $\mathfrak{se}(2)^*$. This is demonstrated in [5], where the authors develop a reduction theory for this kind of systems. We quote from that reference:

Theorem 6 (Theorem 7.2.1 in [5]). *The Poisson reduced space for the left cotangent lifted action of G on $(T^*G, \Omega - \pi_G^* B_\mu)$ is $\mathfrak{g}^*$ with Poisson bracket given by*

$$\{f,g\}_{\mathcal{B}}(\mu) = -\left\langle \mu, \left[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right]\right\rangle - \mathcal{B}_\mu(e)\left(\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right) \tag{21}$$

for $f, g \in C^\infty(\mathfrak{g}^*)$.

The theorem in [5] is proved for right actions, whereas the action of $SE(2)$ here is from the left. However, the same proof continues to hold, *mutatis mutandis*.

Notice that the first term in (21) is the Lie-Poisson bracket on $\mathfrak{se}(2)^*$, given by

$$\{F,G\}_{\mathfrak{se}(2)^*} = (\nabla F)^T \Lambda \nabla G, \quad \text{where} \quad \Lambda = \begin{pmatrix} 0 & -P_y & P_x \\ P_y & 0 & 0 \\ -P_x & 0 & 0 \end{pmatrix} \tag{22}$$

and $F(\Pi, P_x, P_y)$ and $G(\Pi, P_x, P_y)$ are arbitrary functions on $\mathfrak{se}(2)^*$.

The second term in (21) is due to the magnetic two-form. The entire Poisson bracket is then given by

$$\{F,G\}_{\mathcal{B}} = \{F,G\}_{\mathfrak{se}(2)^*} - \Gamma\left(\frac{\partial F}{\partial P_x}\frac{\partial G}{\partial P_y} - \frac{\partial F}{\partial P_y}\frac{\partial G}{\partial P_x}\right). \tag{23}$$

The equations of motion

The Hamiltonian H in (17) is explicitly $SE(2)$-invariant and induces the following Hamiltonian function on $\mathfrak{se}(2)^*$:

$$H(\Pi, P_x, P_y) = \frac{1}{2}\left(\frac{\Pi^2}{\mathbb{I}} + \frac{P_x^2}{m} + \frac{P_y^2}{m}\right),$$

where $(\mathbb{I}, m, m)$ are the diagonal elements of the matrix $\mathbb{M}$. It follows from this and (23) that the equations of motion for the rigid body are given by

$$\begin{cases} \dot{\Pi} = 0 \\ \dot{P}_x = -\Pi P_y/\mathbb{I} - \Gamma P_y/m \\ \dot{P}_y = \quad \Pi P_x/\mathbb{I} + \Gamma P_x/m, \end{cases} \tag{24}$$

where $\Pi = \mathbb{I}\Omega$, $P_x = mV_x$, and $P_y = mV_y$. These equations were first derived by Chaplygin and Lamb, and were the focus (among other things) of recent work by Borisov and Mamaev [1], and Kanso and Oskouei [14].

EXAMPLES

The equations of motion (24) can be solved explicitly. Since $\dot{\Pi} = 0$, β is a constant. Without loss of generality, we may hence take $\beta = 0$. From this, it follows that the linear momentum in inertial frame, with components (p_x, p_y), equals the momentum (P_x, P_y) in the body frame.

If the body has an initial velocity U in the positive x-direction at $t = 0$, then its center of mass traces out the following trajectory: $x = \frac{U}{\Gamma}\sin\Gamma t$ and $y = \frac{U}{\Gamma}(\cos\Gamma t - 1)$, which is a circle with center $(0, \frac{U}{\Gamma})$ and radius $\frac{U}{\Gamma}$. As Γ goes to zero, the path of the rigid body becomes a straight line. This is similar to the behavior of a charged particle in a magnetic field, with now the circulation Γ playing the role of the magnetic field strength B.

A more interesting case is obtained when the rigid body is dragged along with a constant velocity U along the x-axis. To this end, we add to (24) an external force whose components in the body frame are denoted by (F_x, F_y). The equations of motion for the rigid body subject to circulation and to this force then become

$$\begin{cases} \dot{\Pi} = 0 \\ \dot{P}_x = -\Pi P_y/m_\Omega - \Gamma P_y/m + F_x \\ \dot{P}_y = \quad \Pi P_x/m_\Omega + \Gamma P_x/m + F_y. \end{cases} \tag{25}$$

Adding external forces to a system like (24) is relatively straightforward and proceeds in the same way as (for example) for the rigid body; see [6] for more details.

For the same reasons as above, we may set $\beta = 0$. By choosing the applied force such that $F_x = \Gamma P_y/m$, we ensure that $\dot{P}_x = 0$, or $x = Ut$. The remaining equation of motion can then be integrated to give $y = \frac{\Gamma U}{2m} t^2$, *i.e.* the system traces out a parabola. Again, when Γ goes to zero, this parabola turns into a straight line. Moreover, in that case the force needed to keep P_x constant vanishes.

CONCLUSIONS AND OUTLOOK

In this paper we provided a geometric underpinning for the dynamics of a rigid body in a flow with nonzero circulation. We showed that the equations of motion can be obtained through successive reductions, and in the process we obtained a geometric interpretation for the lift force on the body in terms of connections and curvatures.

The geometric concepts that we used are not limited to this specific case, and can conceivably be extended to cover more general situations. From a geometric point of view, there is no reason to restrict ourselves to a planar flow or to a body of circular shape: the same concepts and methods could reasonably be extended to the dynamics of fully three-dimensional, non-symmetric bodies.

Another possible direction in which this method could be generalised deals with the specification of external vorticity. One specific case which is of special interest to us is the case where *point vortices* are present in the flow. In this way, we hope to shed more light on recent work of Shashikanth *et al.* [2] and Borisov *et al.* [15] on the dynamics of a circular disc interacting with point vortices. A detailed analysis of this system will be the subject of a forthcoming paper.

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