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Covariant and dynamical reduction for principal bundle field theories

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Abstract Reduction for field theories with symmetry can be done either covariantly—that is, on spacetime—or dynamically—that is, after spacetime is split into space and time. The purpose of this article is to show that these two reduction procedures are, in an appropriate sense, equivalent for a class of field theories whose fields take values in a principal bundle. One can think of this class of field theories as including examples such as a "sea of rigid bodies" with and appropriate interbody coupling potential.

Keywords Variational calculus · Symmetries · Reduction · Euler–Poincare equations

1 Introduction

There are basically two different geometric approaches available to study evolution problems for fields defined by a variational principle. For simplicity, we consider spacetimes of the form $M \times \mathbb{R}$, where M is the "space", an n-dimensional manifold, and $\mathbb{R}$ is the time. The first approach, called the *dynamical approach*, considers the infinite dimensional manifold of all sections (or local sections) $Q = \Gamma(E)$ as the configuration space, where $E \to M$ is the bundle the sections of which are the fields. In this setting, the Lagrangian is a smooth function $\mathcal{L}\colon TQ \to \mathbb{R}$ (or $\mathcal{L}\colon TQ \times \mathbb{R} \to \mathbb{R}$ if time dependend Lagrangians are under consideration). In this case, the structure of the infinite dimensional manifold Q must by taken into account. The variational principle used in this dynamical approach is the standard Hamilton principle; that is, one makes stationary the *time integral* of $\mathcal{L}$. This formulation uses, as its main ingredient, the infinite dimensional manifold TQ.

A second approach, called the *covariant or jet approach* considers a Lagrangian density $\mathfrak{L}$ that is of the form $\mathfrak{L} = L\mu\colon J^1(E \times \mathbb{R}) \to \Lambda^{n+1}(M \times \mathbb{R})$. The notation is explained as

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follows: μ is a fixed volume form on $M \times \mathbb{R}$, so it is a section of the bundle $\Lambda^{n+1}(M \times \mathbb{R})$ of $n + 1$-forms on $M \times \mathbb{R}$. Also, $L \in C^\infty(J^1(E \times \mathbb{R}))$ where $J^1(E \times \mathbb{R})$ denotes the first jet bundle of the bundle $E \times \mathbb{R} \to M \times \mathbb{R}$. In this approach the variational principle used requires that the *spacetime integral* of $\mathfrak{L}$ is stationary. This formulation uses, as its main ingredient, the finite dimensional jet bundle.

The Euler–Lagrange equations of the two preceding variational principles are shown to give equivalent solutions. However, in any case the techniques (and the philosophy of the two variational formulations) are quite different in the two frameworks. Actually, both present advantages and disadvantages to be considered when one is trying to solve a problem or to study a specific property of a theory. In this article, the spacetime is chosen to be $M \times \mathbb{R}$ for simplicity. We could replace it by a spacetime $(n + 1)$-manifold and in that case, to connect the covariant theory to the dynamical one, slicings must be introduced, as in, for example, [11].

When a variational Lagrangian theory, either in covariant or dynamical formulation has a group of symmetries, one can bring reduction theory to bear. These reduction techniques developed in the dynamical framework have been studied thoroughly in the literature (see for example [14] and the references therein cited). In the jet formulation setting, the geometric constructions needed for reduction have been presented more recently (see for example [3,5,7]). There are several points where the reduction process in the dynamical and covariant approaches are quite different. The main one is the presence of a compatibility condition (in addition to the Euler–Lagrange equations) for the reconstruction of solutions of the original problem from solutions of the reduced problem in the jet formulation. This compatibility condition, interestingly, does not appear in the dynamical approach. Another interesting issue is the different formulation of the Hamiltonian picture of the evolutions problem in both settings. The dynamical approach defined in the cotangent bundle T^*Q, and the jet approach defined in the dual jet bundle $(J^1(E \times \mathbb{R}))^*$, requires different geometrical objects such as Poisson brackets, symplectic or multisymplectic forms, etc. The reduction process in the dynamical setting is a broad and active field of study. The jet approach has been much less studied [3]. In any case, it seems that the brackets defined in the dual jet construction are only defined for a special family of forms, called Poisson forms, whereas the canonical Poisson bracket in T^*Q is given for any pair of functions.

The goal of this article is to show the equivalence of dynamical and covariant reduction for both the Lagrangian and Hamiltonian settings. A clear deduction of the reduced equations in one setting starting from the analogous equations in the other setting gives a better understanding of the reduction principles involved. In addition, the differences of both approaches are analyzed and, in particular, the role of the compatibility condition and the definition of the Poisson bracket are clarified through the equivalence between the reduction processes in both frameworks.

In order to be specific, we confine ourselves to the case where the configuration bundle is a principal bundle $\pi : P \to M$. Moreover, we assume that the structure group G is the group of symmetries itself. This is the setting in which covariant reduction leads to interesting covariant Euler–Poincaré and Lie–Poisson formulations. The understanding of this crucial case is expected to shed light on more general cases.

The organization of the article is as follows. Section 2 gives a quick review of both Lagrangian and Hamiltonian reduction for field theories in the covariant, or jet formalism. Section 3 studies the special form that the objects given in Sect. 2 take when the configuration bundle is sliced. Section 4, provides a review of the dynamical approach to reduction and Sect. 5 gives the formulation of the dynamical problem induced by a Lagrangian or Hamiltonian defined in the jet framework. Section 6 shows the equivalence of the Euler–Poincaré and

Lie–Poisson equation of the jet formalism and the equations of the Lagrangian and Hamiltonian reduction on the dynamical side. Also, at the end of this section the influence of the topology of the manifold M when reconstruction is considered. Finally, Sect. 7 gives a simple example.

2 Covariant reduction

2.1 Covariant Lagrangian reduction

For this section we refer the reader to [6,7] for a complete description of the results herein mentioned.

Let $\bar{M}$ be an $n+1$-manifold. In later sections, we will assume that $\bar{M}$ is sliced; that is, it has the form $\bar{M} = M \times \mathbb{R}$ where M is an n-manifold. Generally, to distinguish non-sliced spaces from spatial slices, we shall use an overbar.

Let $\bar{\pi}\colon \bar{P} \to \bar{M}$ be a principal G-bundle and let $L\colon J^1\bar{P} \to \mathbb{R}$ be a first order Lagrangian function. Assuming that $\bar{M}$ is oriented with a distinguished volume form $\bar{\mathbf{v}}$, we have a variational problem on the set of local sections of $\bar{P}$. The group G acts naturally on $J^1\bar{P}$ by setting

$$j^1_x\bar{s}\cdot g \mapsto j^1_x(R_g\circ\bar{s}),$$

for any local section $\bar{s}$ of $\bar{\pi}$, where $R_g\colon \bar{P} \to \bar{P}$ stands for the right action of g in $\bar{P}$. We now suppose that L is invariant under this action of G. The variational principle defined by $\mathfrak{L} = L\bar{\mathbf{v}}$ on $\bar{M}$ drops to the quotient space $(J^1\bar{P})/G$. This quotient is an affine bundle on $\bar{M}$ called the bundle of connections $\bar{C} \to \bar{M}$ of $\bar{\pi}$. The sections of this bundle can be identified with principal connections in $\bar{P}$, and the model vector bundle is $T^*\bar{M}\otimes\tilde{\mathfrak{g}} \to M$, where $\tilde{\mathfrak{g}}$ is the adjoint bundle of $\bar{P}$ (see for example [4,12]).

If we denote by $l\colon \bar{C} \to \bar{M}$ the dropped Lagrangian, the induced variational problem has constraints on the set of possible variations. Indeed, given a section $\bar{\sigma}$ of $\bar{C}$ that is induced from a section $\bar{s}$ of $\bar{P}$, the possible variations $\delta\bar{\sigma}$ are the projections of the one-jet lifts of variations $\delta\bar{s}$. As is shown in the following Proposition, given the section $\bar{\sigma}$, these possible variations have the form $\nabla^{\bar{\sigma}}\bar{\eta}$, that is, the covariant derivative with respect to the connection defined by $\bar{\sigma}$ of any section $\bar{\eta}$ of the adjoint bundle $\tilde{\mathfrak{g}} \to \bar{M}$. The following Proposition (see [3,7] for the proof) contains additional information on these variations that is important in the sequel.

Proposition 2.1 *Let $\bar{\eta}$ be a section of $\tilde{\mathfrak{g}} \cong (\bar{P}\times\mathfrak{g})/G$. Then $\bar{\eta}$ naturally defines a G-invariant vertical vector field $\bar{X}$ in $\bar{P}$ (called a gauge vector field) whose value at a point $p\in\bar{P}$ is*

$$\bar{X}_p = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} R_{\exp(\epsilon B)}(p),$$

where $\bar{\eta}(x) = [p, B]_G$, and where $B\in\mathfrak{g}$ and $x = \bar{\pi}(p)$. Let $\bar{X}_{\bar{C}}$ be the induced vector field in the bundle of connections, i.e., the projection of the one-jet lift $\bar{X}^{(1)}$ of $\bar{X}$, from $J^1\bar{P}$ to $(J^1\bar{P})/G = \bar{C}$ Then

$$\bar{X}_{\bar{C}}|_{\bar{\sigma}(\bar{M})} = -\nabla^{\bar{\sigma}}\bar{\eta}, \tag{2.1}$$

for any section $\bar{\sigma}$ of $\bar{C} \to \bar{M}$.

Note that formula (2.1) makes sense because the covariant derivative of $\bar{\eta}$ is a one form on $\bar{M}$ taking values in $\tilde{\mathfrak{g}}$; that is, it is a section of $T^*\bar{M}\otimes\tilde{\mathfrak{g}} \to \bar{M}$. Since this vector bundle underlies the affine bundle $\bar{C}$, the section may be viewed as a vertical vector field along the section $\bar{\sigma}$.

One proves that the equations obtained by the constrained variational problem, namely that of varying the integral of $l(\bar{\sigma})$ with variations subject to the constraints (2.1) and with compact support, are

$$\operatorname{div}^{\bar{\sigma}} \frac{\delta l}{\delta \bar{\sigma}} = 0,$$

which are called the covariant Euler–Poincaré equations. Here, $\delta l/\delta\bar{\sigma}$ is the vertical derivative of l, that is, the map

$$\frac{\delta l}{\delta \bar{\sigma}}(\bar{\tau}_x) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} l(\bar{\sigma}(x) + \epsilon\bar{\tau}_x)$$

for any $\bar{\tau}_x \in T_x^*\bar{M} \otimes (\tilde{\mathfrak{g}})_x$, $x \in \bar{M}$. Hence, $\delta l/\delta\bar{\sigma}$ can be seen as a section of the vector bundle $(T^*\bar{M} \otimes \tilde{\mathfrak{g}})^* = T\bar{M} \otimes \tilde{\mathfrak{g}}^*$, that is, a $\tilde{\mathfrak{g}}$-valued vector field on $\bar{M}$. Moreover, $\operatorname{div}^{\bar{\sigma}}$ is the divergence operator with respect to the volume form $\bar{\mathbf{v}}$ and the connection $\bar{\sigma}$.

If a preferred principal connection $\bar{\mathcal{A}}$ is fixed, the Euler–Poincaré equation above can be rewritten in a more classical fashion as

$$\operatorname{div}^{\bar{\mathcal{A}}} \frac{\delta l}{\delta \bar{\sigma}} + \operatorname{ad}_{\bar{\sigma}^{\bar{\mathcal{A}}}} \frac{\delta l}{\delta \bar{\sigma}} = 0 \tag{2.2}$$

where now $\bar{\sigma}^{\bar{\mathcal{A}}} = \bar{\sigma} - \bar{\mathcal{A}}$ is a section of $T^*\bar{M} \otimes \tilde{\mathfrak{g}}$ and ad^* stands for the coadjoint operator in $\tilde{\mathfrak{g}}^*$.

Not any solution of the reduced Euler–Poincaré equations comes from a solution of the original Euler–Lagrange equations of L. An extra equation, a compatibility condition, must be imposed. This condition simply reads

$$\operatorname{Curv}(\bar{\sigma}) = 0,$$

that is, the critical connection $\bar{\sigma}$ must be flat. Then the reconstruction process, namely the recovery of critical solutions of the unreduced problem, is simple. The integral leaves of $\bar{\sigma}$ are, at least locally, critical sections of the original variational problem. In this point, one has to take into account the topology of $\bar{M}$ for, if $\bar{M}$ is not simply connected, the holonomy of $\bar{\sigma}$ may be not trivial. See Sect. 6.3 below for details.

2.2 Covariant Hamiltonian reduction

We follow [3,11]. The Hamiltonian covariant framework for field theories is formulated in the bundle $(J^1\bar{P})^*$, the affine dual bundle of the affine bundle $J^1\bar{P} \to \bar{M}$. We still assume that the configuration bundle of the problem is a principal G-bundle $\bar{P} \to \bar{M}$. The manifold $(J^1\bar{P})^*$ is endowed with a canonical $(n+1)$-form Θ, $n+1 = \dim\bar{M}$, the differential of which $\Omega = -d\Theta$ is a multisymplectic form in $(J^1\bar{P})^*$. Given a Lagrangian $L\colon J^1\bar{P} \to \mathbb{R}$ the Legendre transformation $\mathbb{F}L\colon J^1\bar{P} \to (J^1\bar{P})^*$ is defined as

$$\mathbb{F}L(j_x^1\bar{s})(j_x^1\bar{s}') = L(j_x^1 s) + \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L\left(j_x^1\bar{s} + \epsilon(j_x^1\bar{s}' - j_x^1\bar{s})\right),$$

that is, the first order vertical Taylor expansion of L. The Legendre transform gives the Poincaré–Cartan form in $J^1\bar{P}$ (cf. [10,11]) as the pull-back $(\mathbb{F}L)^*\Theta$.

A useful tool for the Hamiltonian approach is the polysymplectic bundle, defined to be $\bar{\Pi} = T\bar{M} \otimes_{\bar{P}} V^*\bar{P}$, where $V\bar{P} = \ker\bar{\pi}_* \subset T\bar{P}$ is the vertical bundle and $V^*\bar{P}$ its dual. The dual jet bundle fibers over the polysymplectic bundle through the projection

$$\rho\colon (J^1\bar{P})^* \to \bar{\Pi};\quad \phi \mapsto \vec{\phi} \tag{2.3}$$

where $\vec{\phi}$ is the linear morphism associated with the affine morphism $\phi \in (J^1\bar{P})^*$. Actually, ρ is a (real) line vector bundle. A Hamiltonian system is, by definition, a section $\mathfrak{s}$ of this bundle. Solutions of a given Hamiltonian system $\mathfrak{s}$ are sections $\bar{p}$ of $\bar{\Pi} \to \bar{M}$ such that

$$\bar{p}^*(\mathrm{i}_Y \mathfrak{s}^*\Omega) = 0, \tag{2.4}$$

for any vertical vector field Y in $\bar{\Pi}$. Note that $\mathfrak{s}^*\Omega$ is a multisymplectic form in Π though it is not canonical: it depends on the Hamiltonian system $\mathfrak{s}$. Moreover, if one fixes a connection $\bar{\mathcal{A}}$ in $\bar{P}$, a section $\mathfrak{s}_{\bar{\mathcal{A}}}$ of $(J^1\bar{P})^* \to \bar{\Pi}$ is naturally defined and any Hamiltonian system $\mathfrak{s}$ is decomposed as $\mathfrak{s} = \mathfrak{s}_{\bar{\mathcal{A}}} + H\bar{\mathbf{v}}$, where $\bar{\mathbf{v}}$ is a fixed volume form on $\bar{M}$. The function $H\colon \bar{\Pi} \to \mathbb{R}$ is called the Hamiltonian.

A bracket can be defined in this context. We say that a horizontal form F in $(J^1\bar{P})^*$ is Poisson if there is a vertical multivector $\mathcal{X}_F$ field such that

$$\mathrm{i}_{\mathcal{X}_F}\Omega = dF. \tag{2.5}$$

Given two Poisson forms F_1 and F_2 we define their Poisson bracket to be

$$\{F_1, F_2\} = (-1)^{rs}\mathrm{i}_{\mathcal{X}_{F_1}}\mathrm{i}_{\mathcal{X}_{F_2}}\Omega \tag{2.6}$$

where r, s are the degrees of F_1 and F_2, respectively. It is easy to check that any function on $(J^1\bar{P})^*$ is Poisson and that higher order Poisson forms are necessarily projectable to $\bar{\Pi}$.

Moreover, Poisson n-forms can be characterized as follows. For any $\bar{\pi}$-vertical vector field X in $\bar{P}$, we define

$$\theta_X\colon \bar{\Pi} \to \Lambda^n\bar{M};\quad z \mapsto \mathrm{i}_{\langle z, X\rangle}\bar{\mathbf{v}}.$$

Then one can prove that any Poisson n-form is of the the type

$$F = \theta_X + \bar{\pi}^*_{\bar{P}\bar{\Pi}}\omega + Z \tag{2.7}$$

where X is a vertical vector field, ω is an n-form in $\bar{P}$ and Z is a closed form on $\bar{\Pi}$.

Using this Poisson multibracket, the characterization of critical sections $\bar{p}\colon \bar{M} \to \bar{\Pi}$ of a Hamiltonian system is given by the formula

$$\{F, H\}\bar{\mathbf{v}} \circ \bar{p} = d(\bar{p}^*F) - \bar{p}^*(d^{\bar{\mathcal{A}}}F), \tag{2.8}$$

for all Poisson n-forms F in $\bar{\Pi}$, where $d^{\bar{\mathcal{A}}}$ stands for the covariant derivative defined in $\bar{\Pi} = T\bar{M} \otimes_{\bar{M}} V^*\bar{P}$ with respect to $\bar{\mathcal{A}}$ and any linear connection in $\bar{M}$.

We are interested in the Hamiltonian formulation induced by a Lagrangian $L\colon J^1\bar{P} \to \mathbb{R}$. We define a new Legendre transformation $\widehat{\mathbb{F}L}\colon J^1\bar{P} \to \bar{\Pi}$ given by

$$\widehat{\mathbb{F}L}(j^1_x\bar{s})(\bar{\omega}) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(j^1_x\bar{s} + \epsilon\bar{\omega})$$

which satisfies the relation $\rho \circ \mathbb{F}L = \widehat{\mathbb{F}L}$ with respect the fibration (2.3). The space $\bar{R} = \mathbb{F}L(J^1\bar{P}) \subset (J^1\bar{P})^*$ is called the primary constraint and it is a submanifold when L is assumed to be quasiregular (which is the common case in field theories). When L is hyperregular, that is, the Legendre transformation $\widehat{\mathbb{F}L}$ is a diffeomorphism, we define a Hamiltonian system $\mathfrak{s}\colon \bar{\Pi} \to (J^1\bar{P})^*$ as $\mathfrak{s} = \mathbb{F}L \circ (\widehat{\mathbb{F}L})^{-1}$. When L is only quasiregular, the associated

Hamiltonian system will be a section of the real line bundle $\rho^{-1}(\rho(\bar{R})) \to \rho(\bar{R}) \subset \bar{\Pi}$. In both cases, the equations for the Hamiltonian system defined by L are as in (2.4).

If H is invariant under the natural (right) action of G in $\bar{\Pi}$, then we have a reduced Hamiltonian $h\colon \bar{\Pi}/G \to \mathbb{R}$. One can identify

$$\bar{\Pi}/G = T\bar{M} \otimes \tilde{\mathfrak{g}}^*$$

and the set of projectable n-Poisson forms are those in (2.7) with X a gauge vector field (that is, a section of $\tilde{\mathfrak{g}} \to \bar{M}$) and Z a G-invariant closed form. The Poisson multibracket projects to the quotient $\bar{\Pi}/G$ and reads

$$\{F, h\}_{+}(\bar{\mu}) = \left\langle \bar{\mu}, \left[X, \frac{\delta h}{\delta \bar{\mu}}(\bar{\mu}) \right] \right\rangle, \tag{2.9}$$

for any $h \in C^{\infty}(T\bar{M} \otimes \tilde{\mathfrak{g}}^*)$, where $\bar{\mu} \in \bar{\Pi}/G$ and $[\cdot,\cdot]$ is the fiberwise Lie bracket of $\tilde{\mathfrak{g}}$. If one deals with left invariant structures, the useful bracket would be $\{\cdot,\cdot\}_{-} = -\{\cdot,\cdot\}_{+}$. The bracket (2.9) is called the Lie–Poisson covariant bracket and defines an equation for sections $\bar{\mu}$ of $\bar{\Pi}/G \to \bar{M}$ by

$$\{F, h\}_{+}\bar{\mathbf{v}} = d(\bar{\mu}^* F) - \bar{\mu}^*(d^{\bar{\mathcal{A}}} F), \tag{2.10}$$

for any projectable Poisson n-form where now $d^{\bar{\mathcal{A}}}$ denotes the covariant derivative in $T^*\bar{M} \otimes \tilde{\mathfrak{g}}^*$ induced by the connection $\bar{\mathcal{A}}$ and any linear connection in $\bar{M}$. This condition leads to the so-called covariant Lie–Poisson equation

$$\operatorname{div}^{\bar{\mathcal{A}}} \bar{\mu} = \operatorname{ad}^*_{\frac{\delta h}{\delta \bar{\mu}}} \bar{\mu}. \tag{2.11}$$

3 Covariant formulation induced by a slicing $\bar{M} = M \times I$

3.1 Sliced covariant Euler–Poincaré

We now consider the specific case in which the manifold $\bar{M}$ is sliced; that is, $\bar{M} = M \times I$, where I is a closed real interval $[a, b] \subset \mathbb{R}$ or the entire real line $\mathbb{R}$. These foliations mainly represent space–time decompositions arising in many physical problems. In fact, such a decomposition is the framework where time evolution variational problems take place. In the following, we will coordinatize I by the variable t and assume that the volume form $\bar{\mathbf{v}}$ in $\bar{M}$ is written as $\mathbf{v} \wedge dt$.

Similarly, we decompose the principal bundle $\bar{P} = P \times I$ where the factor I trivially projects onto itself. Obviously, $\pi : P \to M$ is a principal G-bundle. We now write the Euler–Poincaré equations for G-invariant Lagrangians when the slicing is taken into account.

First, we decompose the jet bundle of $\bar{P}$ as

$$J^1\bar{P} = J^1 P \times VP \times I. \tag{3.1}$$

This identification is given by

$$j^1_{(x,t)}\bar{s} \mapsto \left(j^1_x(\bar{s}|_{M_t}), (\bar{s})_* \left(\frac{\partial}{\partial t} \right) - \frac{\partial}{\partial t}, t \right), \tag{3.2}$$

where $\bar{s}$ is any local section of $\bar{\pi}$, $\partial/\partial t$ is the natural vector field of I, both in $\bar{P}$ and $\bar{M}$, and M_t stands for the slice $M \times \{t\}$ (see [11]). Throughout the article, we will write $s(t)$ (or directly s if it is clear in which slice M_t we are) instead of $\bar{s}|_{M_t}$.

Second, if we quotient by G we have $(J^1\bar{P})/G = \bar{C}$, $(J^1P)/G = C$ and $(VP)/G = \tilde{\mathfrak{g}}$, the last identification being $[B_p^*]_G \mapsto [p, B]_G$, where $B \in \mathfrak{g}$, $p \in P$ and B_p^* is the infinitesimal generator of the curve $R_{exp(\epsilon B)}(p)$. For simplicity, we denote by $\tilde{\mathfrak{g}}$ the adjoint bundles of both $\bar{P} \to \bar{M}$ and $P \to M$. Then the quotient of the identities (3.1, 3.2) reads

$$\bar{C} = C \times \tilde{\mathfrak{g}} \times I; \quad \bar{\sigma}_{(x,t)} \mapsto ((\bar{\sigma}|_{M_t})_x, \xi, t), \tag{3.3}$$

where $\xi = -[p, \bar{\sigma}(\partial/\partial t)_p]_G$, $\bar{\sigma}(\partial/\partial t)_p$ being the value of the connection form $\bar{\sigma}$ applied to $\partial/\partial t \in T_p\bar{P}$. As before, we will write $\sigma(t)$ (or directly σ) instead of $\bar{\sigma}|_{M_t}$.

We consider again that a G invariant Lagrangian L is given. Let l be the dropped or reduced Lagrangian. According to the identification (3.3), given a section $\bar{\sigma} = (\sigma, \xi, t)$, the vertical derivative $\delta l/\delta\bar{\sigma}$ splits in two terms as

$$\frac{\delta l}{\delta \bar{\sigma}} = \frac{\delta l}{\delta \sigma} + \frac{\delta l}{\delta \xi}\frac{\partial}{\partial t}, \tag{3.4}$$

where the vertical derivatives $\delta l/\delta\sigma$ and $\delta l/\delta\xi$ are defined as usual for any t. They can be understood as time dependent sections of $TM \otimes \tilde{\mathfrak{g}}^*$ and $\tilde{\mathfrak{g}}^*$, respectively. We suppose that a connection $\bar{\mathcal{A}}$ on $\bar{P}$ has been fixed and assume that $\partial/\partial t$ is a horizontal vector field in $\bar{P}$. We write $\mathcal{A} = \mathcal{A}(t)$ the restriction of $\bar{\mathcal{A}}$ on the slice $\pi^{-1}(M \times t)$.

With all the previous identifications and notations, one transforms the Euler–Poincaré equation as follows.

Proposition 3.1 *A section $\bar{\sigma} = (\sigma, \xi, t)$ of $\bar{C} = C \times \tilde{\mathfrak{g}} \times I$ satisfies the Euler–Poincaré equation if and only if it satisfies*

$$\operatorname{div}^{\mathcal{A}}\frac{\delta l}{\delta \sigma} + \frac{d}{dt}\frac{\delta l}{\delta \xi} + \operatorname{ad}^*_{\sigma^{\mathcal{A}}}\frac{\delta l}{\delta \sigma} + \operatorname{ad}^*_{\xi}\frac{\delta l}{\delta \xi} = 0. \tag{3.5}$$

This is equivalent to

$$\operatorname{div}^{\sigma}\frac{\delta l}{\delta \sigma} + \frac{d}{dt}\frac{\delta l}{\delta \xi} + \operatorname{ad}^*_{\xi}\frac{\delta l}{\delta \xi} = 0. \tag{3.6}$$

Note that the left hand side of (3.5) is a section of $\tilde{\mathfrak{g}}^* \times I \to M \times I$, that is, a time dependent section of the coadjoint bundle.

Proof From (3.4), as the vector field $\partial/\partial t$ is horizontal with respect to $\bar{\mathcal{A}}$, we can write

$$\operatorname{div}^{\bar{\mathcal{A}}}\frac{\delta l}{\delta \bar{\sigma}} = \operatorname{div}^{\bar{\mathcal{A}}}\left(\frac{\delta l}{\delta \sigma} + \frac{\delta l}{\delta t}\frac{\partial}{\partial t}\right) = \operatorname{div}^{\mathcal{A}}\frac{\delta l}{\delta \sigma} + \frac{d}{dt}\frac{\delta l}{\delta \xi}.$$

Moreover $\bar{\sigma}^{\bar{\mathcal{A}}} = \sigma^{\mathcal{A}} + \xi dt$, and again form (3.4)

$$\operatorname{ad}^*_{\bar{\sigma}^{\bar{\mathcal{A}}}}\frac{\delta l}{\delta \bar{\sigma}} = \operatorname{ad}^*_{\bar{\sigma}^{\bar{\mathcal{A}}}}\left(\frac{\delta l}{\delta \sigma} + \frac{\delta l}{\delta t}\frac{\partial}{\partial t}\right) = \operatorname{ad}^*_{\sigma^{\mathcal{A}}}\frac{\delta l}{\delta \sigma} + \operatorname{ad}^*_{\xi}\frac{\delta l}{\delta t}.$$

One then obtains (3.5) directly from (2.2). □

In addition, the compatibility condition for $\bar{\sigma}$ can be split as follows:

Proposition 3.2 *Let $\bar{\sigma} = (\sigma, \xi, t)$ be a connection on $\bar{P} = P \times I$. Then $\bar{\sigma}$ is flat if and only if*

$$\operatorname{Curv}(\sigma) = 0, \quad \textit{and} \quad \dot{\sigma} = -\nabla^{\sigma}\xi, \tag{3.7}$$

for all $t \in I$.

Note that, as C is an affine bundle modelled on $T^*M \otimes \tilde{\mathfrak{g}}$, the derivative $\dot{\sigma}$ of a family of sections $\sigma = \sigma(t)$ of the bundle $C \to M$ with respect to t is a 1-form taking values in $\tilde{\mathfrak{g}}$, as is $\nabla^\sigma \xi$, the covariant derivative of ξ with respect to the connection σ.

Proof If we regard $\bar{\sigma}$ as a connection form, the identification (3.3) can be rewritten in the language of forms as $\bar{\sigma} = \sigma - \tilde{\xi} dt$, where, $\tilde{\xi} \in C^\infty(\bar{P}, \mathfrak{g})$ is given by $\xi_{(x,t)} = [\bar{p}, \tilde{\xi}(p)]_G$, $\bar{\pi}(\bar{p}) = (x,t)$. The curvature of $\bar{\sigma}$ is then $\mathrm{d}\bar{\sigma} + \frac{1}{2}[\bar{\sigma}, \bar{\sigma}]$ where recall that the differential d is taken with respect to the variables of the whole manifold $\bar{P} = P \times I$. In fact, it is convenient to write it as $\mathrm{d} = \mathrm{d}_x + \mathrm{d}_t$. Expanding the curvature we find

$$\begin{aligned} \mathrm{d}\bar{\sigma} + \frac{1}{2}[\bar{\sigma}, \bar{\sigma}] &= \mathrm{d}_x \sigma + \frac{1}{2}[\sigma, \sigma] - \mathrm{d}_x(\tilde{\xi} dt) - [\sigma, \tilde{\xi} dt] - \dot{\sigma} \wedge dt \\ &= \mathrm{Curv}(\sigma) - \mathrm{d}_x(\tilde{\xi} dt) - [\sigma, \tilde{\xi} dt] - \dot{\sigma} \wedge dt. \end{aligned} \tag{3.8}$$

Note that the part $\mathrm{d}_x(\tilde{\xi} dt) + [\sigma, \tilde{\xi} dt]$ is just the covariant derivative of ξ expressed in P with respec to σ. When one views this as a form on M taking values in $\tilde{\mathfrak{g}}$, one has $\nabla^\sigma \xi$. The proof is completed by taking into account that (3.8) vanishes if and only its summands with dt and the other summands vanish separately. □

Remark The condition (3.7) can be seen as an evolution equation. It is interesting to point out that, given any arbitrary time dependent section $\xi(t)$ of $\tilde{\mathfrak{g}}$, $t \in I = [a, b]$, and an initial flat connection σ_a, the condition $\dot{\sigma} = -\nabla^\sigma \xi$ is an affine ODE in the affine space of all connections. The solution $\sigma(t)$ with $\sigma(a) = \sigma_a$ will be a flat connection for any $t \in I$. Indeed, if we understand ξ as a (time dependent) gauge vector field on P, then $-\nabla^\sigma \xi$ is the induced gauge vector field in the space of connections (see Proposition 2.1 above). However, gauge transformations (and hence, infinitesimal gauge transformations) leave the set of flat connections invariant. Then $\mathrm{Curv}(\sigma) = 0$, for all $t \in I$, and the first condition in (3.7) becomes redundant.

3.2 Sliced covariant Lie–Poisson Equations

Given the slicing $\bar{P} = P \times I$, the polysymplectic bundle $\bar{\Pi} \to M \times I$ can be decomposed as

$$\bar{\Pi} = \big((TM \times \mathbb{R}) \otimes V^*P\big) \times I = \Pi \times V^*P \times I.$$

where Π is the polysymplectic bundle of $P \to M$. Moreover, the constitutive function θ of formula (2.7) needed for the definition of the Poisson n-forms F, also given in Eq. 2.7 takes the form

$$\begin{aligned} \theta_X \colon \Pi \times V^*P \times I &\to \wedge^{n-1} T^* \bar{M} \\ (p, \nu, t) &\mapsto (\mathrm{i}_{\langle p, X \rangle} \mathbf{v}) \wedge dt + \langle \nu, X \rangle \, \mathbf{v} \end{aligned} \tag{3.9}$$

for any vertical vector field X in $\bar{P} = P \times I$ now seen as a time dependent vertical vector field in P.

If we consider the quotient by the G-action, we have

$$(\bar{\Pi})/G = (\Pi)/G \times \tilde{\mathfrak{g}}^* \times I = (TM \otimes \tilde{\mathfrak{g}}^*) \times \tilde{\mathfrak{g}}^* \times I.$$

The points $\bar{\mu}$ in $\bar{\Pi}/G$ are fiberwise decomposed as $\bar{\mu} = (\mu, \nu, t)$, with μ in $\Pi/G = TM \otimes \tilde{\mathfrak{g}}^*$ and $\nu \in \tilde{\mathfrak{g}}^*$.

We fix a principal connection $\bar{\mathcal{A}}$ in $\bar{P}$. We assume, as in Sect. 3.1, the horizontality of $\partial/\partial t$ in $\bar{P}$. We also have a G-invariant Hamiltonian $H \in C^\infty(\bar{\Pi})$ the dropped (or reduced

Hamiltonian) of which will be denoted by $h \in C^\infty(\bar{\Pi}/G)$. Moreover, G-invariant Poisson n-forms will be (neglecting the irrelevant term Z) as in (2.7) with X being a time dependent G-invariant vector field, that is, a time dependent section of $\mathfrak{g}$, and ω any n-form in $\bar{M} = M \times I$. We can therefore give the adapted expression of the the reduced bracket defined in (2.9) as

$$\{F, h\}_+ = \left\langle \mu, \left[X, \frac{\delta h}{\delta \mu}(\mu, \nu, t) \right] \right\rangle + \left\langle \nu, \left[X, \frac{\delta h}{\delta \nu}(\mu, \nu, t) \right] \right\rangle. \tag{3.10}$$

In this sliced framework, the Lie–Poisson equation (2.11) takes the form

$$\operatorname{div}^{\mathcal{A}} \mu + \frac{d\nu}{dt} = \operatorname{ad}^*_{\frac{\delta h}{\delta \mu}} \mu + \operatorname{ad}^*_{\frac{\delta h}{\delta \nu}} \nu \tag{3.11}$$

for sections (μ, ν, t) of $(TM \otimes \tilde{\mathfrak{g}}^*) \times \tilde{\mathfrak{g}}^* \times I \to M \times I$. Note that, along any section (μ, ν, t), the functional derivatives $\delta h/\delta \mu$ and $\delta h/\delta \nu$ are sections of $T^*M \otimes \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}$, respectively thus making sense the pairings and brackets appearing in the right-hand-side of (3.10) and the coadjoint operator in (3.11).

4 Dynamical reduction

4.1 Lagrangian reduction

We refer the reader to [9] for a full description of Lagrange–Poincaré reduction. Let Q be a smooth manifold on which a Lie group G acts properly and freely. In this article we consider right actions. If the action is left, the formulation below is equivalent up to some signs in the final equations. The quotient space Q/G is then a smooth manifold and the projection $\pi_Q \colon Q \to Q/G$ a principal G-bundle.

Let $\mathcal{L} \colon TQ \to \mathbb{R}$ be a first order Lagrangian on Q defining a variational problem on the set of curves $q : I \to Q$, for certain interval $I = [a, b] \subset \mathbb{R}$. If we suppose that $\mathcal{L}$ is invariant under the natural action of G in TQ, it drops to the quotient and gives a function

$$\ell \colon (TQ)/G \to \mathbb{R},$$

defining a constrained variational problem for curves in $(TQ)/G$. Given any principal connection ϑ in $Q \to Q/G$, we obtain an identification

$$\begin{aligned} (TQ)/G &\longrightarrow T(Q/G) \oplus \widetilde{\mathfrak{G}} \\ [q, \dot{q}]_G &\mapsto (([q]_G, (\pi_Q)_*(\dot{q})), [q, \vartheta(\dot{q})]_G), \end{aligned} \tag{4.1}$$

where $\widetilde{\mathfrak{G}}$ is the adjoint bundle of the principal bundle $p \colon Q \to Q/G$. If we call $(x, \dot{x})$ and v the variables of $T(Q/G)$ and $\widetilde{\mathfrak{G}}$, respectively, the identification (4.1) gives

$$\begin{aligned} \ell \colon T(Q/G) \oplus \widetilde{\mathfrak{G}} \to \mathbb{R} \\ \ell = \ell(x, \dot{x}, v). \end{aligned}$$

The variational equations in $T(Q/G) \oplus \widetilde{\mathfrak{G}}$ are written in two sets, the so-called vertical and horizontal Lagrange–Poincaré equations. The notion of verticality or horizontality comes when one takes vertical or horizontal variations in Q with respect to the connection ϑ. These equations are, respectively

$$\begin{aligned} &\frac{D}{dt}\frac{\delta \ell}{\delta v} + \mathrm{ad}_v^* \frac{\delta \ell}{\delta v} = 0 \\ &\frac{\partial \ell}{\partial x} + \frac{D}{dt}\frac{\delta \ell}{\delta \dot{x}} = \left\langle \frac{\delta \ell}{\delta v}, \mathrm{i}_{\dot{x}}\, \Xi \right\rangle. \end{aligned} \tag{4.2}$$

The vertical derivatives $\delta\ell/\delta v$ and $\delta\ell/\delta\dot{x}$ above are defined in the natural way and are seen as curves in $\widetilde{\mathfrak{G}}^*$ and in $T^*(Q/G)$, respectively. The time derivative D/dt is computed with respect to the connection ϑ for $\delta\ell/\delta v$ and with respect to a chosen linear connection for $\delta\ell/\delta\dot{x}$. The term $\partial\ell/\partial x$ is computed as follows: given a tangent vector $\eta \in T_x(Q/G)$, let $x(\epsilon)$ be a curve such that $x(0) = x$ and $dx(\epsilon)/d\epsilon|_{\epsilon=0} = \eta$ and let $\dot{x}(\epsilon),\ v(\epsilon)$ be its horizontal lift to $T(Q/G)$ and $\widetilde{\mathfrak{G}}$, respectively, by means of the linear connection mentioned before and the connection ϑ. Then

$$\frac{\partial \ell}{\partial x}(\eta) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \ell(x(\epsilon), \dot{x}(\epsilon), v(\epsilon)). \tag{4.3}$$

Finally, Ξ is the curvature of ϑ seen as a 2-form on Q/G taking values in the adjoint bundle $\widetilde{\mathfrak{G}}$. Hence, the coupling $\langle \delta\ell/\delta v, \mathrm{i}_{\dot{x}}\, \Xi\rangle$ yields an element of $T^*(Q/G)$, as the left hand side of the horizontal Lagrange–Poincaré equation.

Time dependent variational problems behave in a similar way. In this case, the Lagrangian $\mathcal{L}$ and the reduced Lagrangian ℓ are defined in $TQ \times \mathbb{R}$ and $(TQ)/G \times \mathbb{R}$, respectively, but the variational principle and the equations for critical reduced or unreduced solutions remain unchanged.

4.2 Cotangent Poisson reduction

The Hamiltonian picture of reduction fits into the well known theory of symplectic or Poisson reduction (see, for instance, [13]). We now have the cotangent bundle $T^*Q \to Q$ and a Hamiltonian $\mathcal{H}\colon T^*Q \to \mathbb{R}$ invariant under the natural action of G in T^*Q. The identification given in (4.1) by means of the fixed connection ϑ in $Q \to Q/G$ induces the identification

$$(T^*Q)/G = T^*(Q/G) \oplus \widetilde{\mathfrak{G}}^* \tag{4.4}$$

just by duality. The manifold T^*Q is canonically Poisson and it is easy to check that the Poisson bracket of two G-invariant functions in T^*Q is also G-invariant. Hence, we have a natural bracket in $(T^*Q)/G$. The Hamiltonian also drops to a reduced Hamiltonian

$$\hbar\colon T^*(Q/G) \oplus \widetilde{\mathfrak{G}}^* \to \mathbb{R}.$$

If one denotes the points in $T^*(Q/G) \oplus \widetilde{\mathfrak{G}}^*$ as $(x, y; \mu)$, the explicit expression of the reduced Poisson bracket reads (see [8])

$$\{f, h\} = \frac{\partial f}{\partial x}\frac{\delta h}{\delta y} - \frac{\delta f}{\delta y}\frac{\partial h}{\partial x} + \left\langle \mu, \Xi\left(\frac{\delta f}{\delta y}, \frac{\delta h}{\delta y}\right)\right\rangle + \left\langle \mu, \left[\frac{\delta h}{\delta \mu}, \frac{\delta f}{\delta \mu}\right]\right\rangle, \tag{4.5}$$

for any pair of functions $f, h \in C^\infty(T^*(Q/G) \oplus \widetilde{\mathfrak{G}}^*)$, where the vertical derivatives $\delta/\delta y$ and $\delta/\delta\mu$ are defined as usual and $\partial/\partial x$ is defined by means of ϑ and a linear connection on Q/G as in formula (4.3). The Poisson equation

$$\{f, \hbar\} = \dot{f}, \qquad \forall f \in C^\infty(T^*(Q/G) \oplus \widetilde{\mathfrak{G}}^*)$$

defined by this bracket gives the Hamilton–Poincaré equations

$$\frac{Dy}{dt} = -\frac{\partial \hbar}{\partial x} - \langle \mu, \mathrm{i}_{\dot{x}}\Theta\rangle, \qquad \dot{x} = \frac{\delta \hbar}{\delta y}, \qquad \frac{D\mu}{dt} = \mathrm{ad}^*_{\frac{\delta \hbar}{\delta \mu}}\mu. \tag{4.6}$$

The first two equations can be thought as the horizontal part of the Hamilton–Poincaré equations and the third one as the vertical part.

As in the Lagrangian setting, dynamical Hamiltonian problems for time dependent Hamiltonians $\mathcal{H}\colon T^*Q\times\mathbb{R}$ do not present any substantial difference. The reduced Hamiltonian $\hbar\colon (T^*Q)/G\times\mathbb{R}$ defines the same Hamilton–Poincaré equations. On the other hand, the characterization of the solutions by means of the bracket is applied for time dependent functions f and reads

$$\{f,\hbar\}=\dot f-\partial f/\partial t.$$

5 Dynamical formulation induced by a slicing $\bar{M}=M\times I$

5.1 Lagrangian picture

The goal of this section is to show how the covariant Lagrangian setting induces Lagrangian dynamics in an appropriate infinite dimensional space of fields. In order to do this, we start by going back to the sliced situation of Sect. 3.1. The variational problem defined by the Lagrangian $L\colon J^1\bar{P}\to\mathbb{R}$ can be seen as a time evolution problem on sections of $P\to M$. Indeed, we consider the set of global sections of $\pi: P\to M$ as the configuration space Q. Because P is a principal bundle, it has a global section if and only if it is trivializable, so to ensure that Q is nontrivial, we assume that P is trivializable, although not a preferred trivialization needs to be chosen. If P is not trivial, one could consider Q as the set of local sections, and the results described in this article would be basically the same. Nevertheless, for the sake of simplicity we will assume the triviality of P.

The set Q can be endowed with the structure of an infinite dimensional manifold. Accordingly, for any $s\in Q$, the tangent space T_sQ is the set of π-vertical vector field X along s, in other words, TQ is the set of sections X of the bundle $VP\to M$. Recalling the identification (3.1)–(3.2), we define a Lagrangian $\mathcal{L}\colon TQ\times I\to\mathbb{R}$ as

$$\mathcal{L}(s,X;t)=\int_M L(j^1s,X,t)\mathbf{v},\tag{5.1}$$

for any $t\in I$, any $s\in Q$ and any X vertical vector field along s. Given a curve $s(t)$ in Q, we see that the action defined by it, namely

$$\int_I\mathcal{L}(s,\dot s;t)dt=\int_I\int_M L(j^1s,\dot s,t)\mathbf{v}\wedge dt=\int_{\bar M}L(j^1\bar s)\bar{\mathbf{v}}$$

coincides with the action defined by the section $\bar s=(s(t),t)$ in the covariant setting. Hence, a curve $s(t)$ in Q is critical for $\mathcal{L}$ if and only if the section $(s(t),t)$ is critical for $L\bar{\mathbf{v}}$. The Euler–Lagrange equations are thus equivalent for both approaches. Moreover, if L is G invariant, the new Lagrangian $\mathcal{L}$ is also G invariant under the natural action of G in Q given by $s\cdot g=R_g\circ s,\ s\in Q, g\in G$.

We now explore the manifold Q/G. Geometrically, a class $[s]_G\in Q/G$ is a G-invariant foliation of P by sections of π. This could be understood as the integral leaves of certain flat connection on P. Hence, Q/G can be viewed as the set of flat connections with trivial holonomy. We leave the holonomy problem for Sect. 6.3 (i.e., we assume now that M is simply connected) and put

$$Q/G=\{\sigma\text{ section of }C\mid\operatorname{Curv}\sigma=0\}.\tag{5.2}$$

This is a submanifold of the affine space $\mathfrak{A}$ of all connections of P. Recall that the vector space modelling $\mathfrak{A}$ is the space $\Omega^1(M, \tilde{\mathfrak{g}})$ of 1-forms on M taking values in $\tilde{\mathfrak{g}}$. For any flat connection σ, a tangent vector in $T_\sigma(Q/G)$ is then an element of $\Omega^1(M, \tilde{\mathfrak{g}})$ preserving the flatness condition. These elements are precisely the gauge vector fields in C (see [7]) and, taking into account Proposition 2.1, we can write

$$T_\sigma(Q/G) = \{-\nabla^\sigma \xi \mid \xi \in \Gamma(\tilde{\mathfrak{g}})\}, \tag{5.3}$$

for any $\sigma \in Q/G$.

We now relate $T(Q/G)$ and TQ. An element of $T_s Q$, $s \in Q$, is a π-vertical vector field along s. It univocally defines a G-invariant vector field along the full P, a gauge vector field, and hence a section ξ of the adjoint bundle $\tilde{\mathfrak{g}}$. In these terms, the differential of the projector $\pi_Q \colon Q \to Q/G$ is

$$(\pi_Q)_* \colon TQ \longrightarrow T(Q/G); \quad (s, \xi) \mapsto (\sigma = [s]_G, -\nabla^\sigma \xi). \tag{5.4}$$

The adjoint bundle $\widetilde{\mathfrak{G}}$ of the bundle $Q \to Q/G$ has also a geometrical interpretation. First of all, let $\widetilde{\mathfrak{G}}_\sigma$ denote the fiber of this adjoint bundle over the point $\sigma \in Q/G$. Given an element of this fiber, $v \in \widetilde{\mathfrak{G}}_\sigma$, it can be written as $[s, B]_G$, with $B \in \mathfrak{g}$ once an integral leaf $s \in Q$ of σ is chosen. The cosets $[s(x), B]_G$, $x \in M$, thus describe a section η_v of the adjoint bundle $\tilde{\mathfrak{g}}$ of $P \to M$. As a simple computation shows, $\nabla^\sigma \eta_v = 0$. That is, we have a mapping

$$\widetilde{\mathfrak{G}}_\sigma \longrightarrow \{\eta \in \Gamma(\tilde{\mathfrak{g}}) \mid \nabla^\sigma \eta = 0\}; \quad v \mapsto \eta_v \tag{5.5}$$

In fact, this mapping is a bijection.

For the reduction process, we choose a principal connection ϑ in the principal bundle $Q \to Q/G$. We then have the identification (4.1). It is not difficult to check that the dropped Lagrangian $\ell \colon ((TQ)/G) \times I \to \mathbb{R}$ is related with the reduced Lagrangian of the covariant setting by

$$\begin{aligned} \ell \colon T(Q/G) \times \widetilde{\mathfrak{G}} \times I &\longrightarrow \mathbb{R} \\ (\sigma, -\nabla^\sigma \xi; v; t) &\longrightarrow \int_M l(\sigma, \xi^h + \eta_v, t)\mathbf{v}, \end{aligned} \tag{5.6}$$

where, η_v is as in the identification (5.5) and, ξ^h is the only section of $\tilde{\mathfrak{g}}$ such that $(\pi_Q)_*\xi^h = -\nabla^\sigma \xi$ and $\vartheta(\xi^h) = 0$, that is, the horizontal lift of $-\nabla^\sigma \xi \in T_\sigma(Q/G)$ with respect to ϑ.

Remark An inspection of the definitions of Q/G and $T(Q/G)$ in (5.2) and (5.3) shows that the compatibility conditions (3.7) found in the covariant approach to reduction is recovered from the beginning of the dynamical approach in the very definitions of the objects, where the reduced variational principle is going to be defined. This is consistent with the fact that the dynamical approach does not have any compatibility condition for the reconstruction process and hence the compatibility conditions of the covariant framework must appear as something intrinsic to the phase spaces involved in the dynamical setting.

5.2 Hamiltonian picture

We follow the notation in Sects. 2.2 and 3.2, where the Hamiltonian is defined by a G-invariant quasiregular Lagrangian, and hence it is restricted to the primary constraint manifold $\bar{R}$. For the sake of simplicity, we will assume hyperregularity so that $\rho(\bar{R}) = \bar{\Pi}$. We can thus state the result in the full polysymplectic manifold and $H \colon \bar{\Pi} = \Pi \times V^*P \times I \to \mathbb{R}$. The non hyperregular case would need simple adaptations of the statements to $\rho(\bar{R}) \subset \bar{\Pi}$.

For a section $s \in Q$, the space T_s^*Q is defined to be the set of 1-forms along s restricted to vertical vectors. In other words, T^*Q is the set of sections of $V^*P \to M$. We define the Hamiltonian $\mathcal{H}\colon T^*Q \times I \to \mathbb{R}$ as

$$\mathcal{H}(s, \nu; t) = \int_M H(\widehat{\mathbb{F}L}(j^1 s), \nu, t)\mathbf{v}, \tag{5.7}$$

where $t \in I$ and ν is a 1-form along s for vertical vectors. The Hamiltonian $\mathcal{H}$ is G invariant because L and H are. Taking into account the identification (4.4), we also have a reduced Hamiltonian

$$\hbar\colon T^*(Q/G) \times \widetilde{\mathfrak{G}}^* \times I \to \mathbb{R}.$$

The canonical symplectic structure $\boldsymbol{\Omega}$ on T^*Q is related with the multisymplectic form $\mathfrak{s}^*\Omega = -d\mathfrak{s}^*\Theta$ (see formula (2.4) above) in $\bar{\Pi}$ as follows.

Proposition 5.1 *The canonical symplectic 2-form $\boldsymbol{\Omega}$ in T^*Q can be given as*

$$\boldsymbol{\Omega}(V, W) = \int_M (\widehat{\mathbb{F}L}(j^1 s), \nu)^* \mathrm{i}_{\mathcal{V}} \mathrm{i}_{\mathcal{W}}(\mathfrak{s}^*\Omega) \tag{5.8}$$

for any $V, W \in T_{(s,\nu)}(T^*Q)$, $(s, \nu) \in T^*Q$, where $\mathcal{V}$ and $\mathcal{W}$ are vertical vector fields along the section $(\widehat{\mathbb{F}L}(j^1 s), \nu)$ of $\Pi \times V^*P \to M$ whose projection onto V^*P are V and W, respectively.

Proof The canonical 1-form $\boldsymbol{\Theta}$ in T^*Q is defined as

$$\boldsymbol{\Theta}(V) = \int_M (\hat{\mathbf{F}L}(j^1 s), \nu)^* \left\langle \nu, T\pi_{Q,T^*Q} V \right\rangle \mathbf{v}$$

for any $V \in T_{(s,\nu)}(T^*Q)$, the map $\pi : T^*Q \to Q$ being the natural cotangent projection. If M can be covered by a single coordinate domain, then we put $\nu = \nu_A dy^A$ and we can write

$$\boldsymbol{\Theta} = \int_M \nu_A dy^A \otimes \mathbf{v}$$

and hence

$$\boldsymbol{\Omega} = -d\boldsymbol{\Theta} = \int_M dy^A \wedge d\nu_A \otimes \mathbf{v}.$$

On the other hand, the local expression of the multisymplectic form $\mathfrak{s}^*\Omega$ is

$$\mathfrak{s}^*\Omega = dy^A \wedge d\pi_A^i \wedge \mathbf{v}_i \wedge dt + dy^A \wedge d\nu_A \wedge \mathbf{v} + d(H + \Gamma) \wedge \mathbf{v} \wedge dt$$

where Γ stands for the coefficients of the chosen connection used to define the Hamiltonian H. Hence

$$\mathrm{i}_{\mathcal{V}} \mathrm{i}_{\mathcal{W}}(\mathfrak{s}^*\Omega) = \mathrm{i}_{\mathcal{V}} \mathrm{i}_{\mathcal{W}}(dy^A \wedge d\pi_A^i \wedge \mathbf{v}_i \wedge dt + dy^A \wedge d\nu_A \wedge \mathbf{v})$$

which clearly gives the same as in $\boldsymbol{\Omega}(V, W)$ when pulled-back by $(\widehat{\mathbb{F}L}(j^1 s), \nu)$ and integrated along M when the projections of $\mathcal{V}$ and $\mathcal{W}$ coincide with V and W. In case M needs more than one coordinate domain, we prove the result using a partition of the unity. □

Moreover, horizontal Poisson n-forms F in $\bar{\Pi} = \Pi \times V^*P \times I$ define a special set of functions in T^*Q. Recall that (see (2.7) above) the forms F are of the type $F = \theta_X + \pi_{\bar{P}\Pi}^* \omega + Z$, with θ_X as in (3.9) and X being a time dependent vertical vector field in P. We define the function

$$\mathcal{F}\colon T^*Q \to \mathbb{R}; \quad (s,\nu) \mapsto \int_M (\widehat{\mathbb{F}L}(j^1 s), \nu)^* F. \tag{5.9}$$

Proposition 5.2 *The functions $\mathcal{F}$ obtained as in (5.9) from horizontal Poisson n-forms in $\bar{\Pi} = \Pi \times V^*P \times I$ are affine in the variable ν.*

Proof If we forget the term Z in F (which gives a constant when $\mathcal{F}$ is constructed as in (5.9)), a simple computation from (3.9) shows that

$$\mathcal{F}(s,\nu) = \int_M \langle \nu, X\rangle\, \mathbf{v} + \int_M s^*\omega \tag{5.10}$$

which is evidently affine in ν. □

Remark Note that the functions in T^*Q defined by Poisson n-forms in $\bar{\Pi}$ are restricted since they must be affine in the variable ν. However, in the variable s, the functions obtained in (5.10) are quite general, since ω is an arbitrary form in $P \times I$. In any case, this small set of functions has arbitrary first derivatives at any point and so are enough to defined the Poisson bracket of *any* two functions on the cotangent bundle, and hence are enough to define Hamiltonian dynamics of a given Hamiltonian in T^*Q. See for example [1].

6 Equivalence between the sliced and dynamical formulations

6.1 Lagrangian formulation

We explore the equivalence between the sliced covariant Euler–Poincaré and the dynamical Lagrange–Poincaré formulations described above. Some few remarks are in due.

First, note that Lagrange–Poincaré (4.2) consists of two equations whilst the covariant Euler–Poincaré (3.5) is written in terms of one single equation.

Second, an essential point is the nature of the objects appearing in all these equations. On the one hand, (3.5) is an equation in $\tilde{\mathfrak{g}}^*$ satisfied for all $t \in [a,b]$. On the other hand, the vertical Lagrange–Poincaré equation (the first in (4.2)), is an identity in $\widetilde{\mathfrak{G}}^*$, whereas the horizontal one (the second in (4.2)) is in $T^*(Q/G)$, for all $t \in [a,b]$.

We couple the covariant Euler–Poincaré equation (3.5) or (3.6) with an time-dependent arbitrary section η of the adjoint bundle $\tilde{\mathfrak{g}}$. In order to get the two sets of equations in (4.2) from the single Eq. (3.6) we consider a convenient decomposition of the space $\Gamma(\tilde{\mathfrak{g}})$. Indeed, given $\sigma \in Q/G$ and $s \in Q$ such that $\pi_Q(s) = \sigma$, we identify $T_sQ = \Gamma(\tilde{\mathfrak{g}})$ as in (5.4) and consider the horizontal–vertical decomposition induced by the connection ϑ

$$\Gamma(\tilde{\mathfrak{g}}) = \{\eta \in \Gamma(\tilde{\mathfrak{g}}) | \nabla^\sigma \eta = 0\} \oplus \{\eta \in \Gamma(\tilde{\mathfrak{g}}) | \vartheta(\eta) = 0\} \tag{6.1}$$

Then, the Euler–Poincaré equation (3.6) will vanish if and only if its couplings with η in the two subspaces of (6.1) vanish, for any $t \in [a,b]$. This double condition gives, as we will see now, the two Lagrange–Poincaré equations.

6.1.1 The vertical equation

We couple the covariant Euler–Poincaré equation (3.6) with $\eta(t)$ such that $\nabla^{\sigma(t)}\eta(t) = 0,\ \forall t \in I$. We drop the variable t for convenience. Integrating along M, we have

$$\int_M \left\langle \operatorname{div}^\sigma \frac{\delta l}{\delta \sigma}, \eta \right\rangle \mathbf{v} + \int_M \left\langle \frac{d}{dt}\frac{\delta l}{\delta \xi}, \eta \right\rangle \mathbf{v} + \int_M \left\langle \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi}, \eta \right\rangle \mathbf{v} = 0. \tag{6.2}$$

The first summand identically vanishes as

$$\operatorname{div}\left\langle \frac{\delta l}{\delta \sigma}, \eta \right\rangle = \left\langle \operatorname{div}^\sigma \frac{\delta l}{\delta \sigma}, \eta \right\rangle + \left\langle \frac{\delta l}{\delta \sigma}, \nabla^\sigma \eta \right\rangle, \tag{6.3}$$

together with Stokes Theorem and $\nabla^\sigma \eta = 0$.

We now need the following result induced from the vertical–horizontal decomposition (6.1)

Proposition 6.1 *We have that*

$$\int_M \left\langle \frac{\delta l}{\delta \xi}, \eta \right\rangle \mathbf{v} = \left\langle \frac{\delta \ell}{\delta \dot\sigma}, -\nabla^\sigma \eta \right\rangle + \left\langle \frac{\delta \ell}{\delta v}, \vartheta(\eta) \right\rangle, \tag{6.4}$$

for any η *section of* $\tilde{\mathfrak{g}}$.

Proof For any point $(\sigma, \dot\sigma = -\nabla^\sigma \xi, v) \in T(Q/G) \times \widetilde{\mathfrak{G}}$, from the Eq. (5.6), we have

$$\begin{aligned}
&\left\langle \frac{\delta \ell}{\delta \dot\sigma}, -\nabla^\sigma \eta \right\rangle + \left\langle \frac{\delta \ell}{\delta v}, \vartheta(\eta) \right\rangle \\
&= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \ell(\sigma, -\nabla^\sigma \xi - \epsilon \nabla^\sigma \eta, v) + \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \ell(\sigma, \dot\sigma, v + \epsilon \vartheta(\eta)) \\
&= \int_M \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} l(\sigma, \xi^h + \epsilon \eta^h + \eta_v)\mathbf{v} + \int_M \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} l(\sigma, \xi^h + \eta_v + \epsilon \eta_{\vartheta(\eta)})\mathbf{v} \\
&= \int_M \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} l(\sigma, \xi^h + \eta_v + \epsilon(\eta^h + \eta_{\vartheta(\eta)}))\mathbf{v} \\
&= \int_M \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} l(\sigma, \xi + \epsilon \eta)(v) = \int_M \left\langle \frac{\delta l}{\delta \xi}, \eta \right\rangle \mathbf{v}.
\end{aligned}$$

□

We now work with the second term in (6.2). From the previous Proposition and $\nabla^\sigma \eta = 0$ we have

$$\begin{aligned}
\int_M \left\langle \frac{d}{dt}\frac{\delta l}{\delta \xi}, \eta \right\rangle \mathbf{v} &= \frac{d}{dt}\int_M \left\langle \frac{\delta l}{\delta \xi}, \eta \right\rangle \mathbf{v} - \int_M \left\langle \frac{\delta l}{\delta \xi}, \dot\eta \right\rangle \mathbf{v} \\
&= \frac{d}{dt}\left\langle \frac{\delta \ell}{\delta v}, \vartheta(\eta) \right\rangle + \left\langle \frac{\delta \ell}{\delta \dot\sigma}, \nabla^\sigma \dot\eta \right\rangle - \left\langle \frac{\delta \ell}{\delta v}, \vartheta(\dot\eta) \right\rangle \\
&= \left\langle \frac{D}{dt}\frac{\delta \ell}{\delta v}, \eta \right\rangle + \left\langle \frac{\delta \ell}{\delta v}, \frac{D}{dt}\eta \right\rangle + \left\langle \frac{\delta \ell}{\delta \dot\sigma}, \nabla^\sigma \dot\eta \right\rangle - \left\langle \frac{\delta \ell}{\delta v}, \vartheta(\dot\eta) \right\rangle
\end{aligned}$$

From the definition of the covariant derivative with respect to ϑ, the second and fourth terms in the last line above cancel. Moreover, it is easy to check that in general

$$\frac{d}{dt}(\nabla^\sigma \eta) = \nabla^\sigma \dot{\eta} + [\dot{\sigma}, \eta]. \tag{6.5}$$

In our case, η is vertical ($\nabla^\sigma \eta = 0$) and we have

$$\nabla^\sigma \dot{\eta} = -[\dot{\sigma}, \eta] = [\nabla^\sigma \xi, \eta] = \nabla^\sigma [\xi, \eta]$$

and then

$$\int_M \left\langle \frac{d}{dt}\frac{\delta l}{\delta \sigma}, \eta \right\rangle \mathbf{v} = \left\langle \frac{D}{dt}\frac{\delta \ell}{\delta v}, \eta \right\rangle + \left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma [\xi, \eta] \right\rangle. \tag{6.6}$$

Finally, using again Proposition (6.1), the third term in (6.2) reads

$$\begin{aligned}\int_M \left\langle \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}, \eta \right\rangle \mathbf{v} &= \int_M \left\langle \frac{\delta l}{\delta \xi}, [\xi, \eta] \right\rangle \mathbf{v} \\ &= -\left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma [\xi, \eta] \right\rangle + \left\langle \frac{\delta \ell}{\delta v}, [\eta_v, \eta] \right\rangle \\ &= -\left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma [\xi, \eta] \right\rangle + \left\langle \mathrm{ad}^*_v \frac{\delta \ell}{\delta v}, \eta \right\rangle. \end{aligned} \tag{6.7}$$

Collecting (6.6) and (6.7), we have that (6.2) equals

$$\left\langle \frac{d}{dt}\frac{\delta \ell}{\delta v} + \mathrm{ad}^*_v \frac{\delta \ell}{\delta v}, \eta \right\rangle = 0$$

for all vertical η. We conclude that the covariant Euler–Poincaré equation restricted to the first subspace of variations in the decomposition (6.1) is equivalent to the vertical dynamical Lagrange–Poincaré equation.

6.1.2 *The horizontal equation*

We now consider a variation $\eta = \eta(t)$ of $s = s(t)$ such that $\vartheta(\eta) = 0$, for all $t \in [a, b]$. The variation of the curve $\sigma = \sigma(t) = \pi_Q(s(t))$ in Q/G is $-\nabla^\sigma \eta$ and its horizontal lift to Q is again η. Coupling the covariant Euler–Poincaré equation (3.6) with η, integrating on M and taking into account the identity (6.3) one obtains

$$-\int_M \left\langle \frac{\delta l}{\delta \sigma}, \nabla^\sigma \eta \right\rangle \mathbf{v} + \int_M \left\langle \frac{d}{dt}\frac{\delta l}{\delta \xi}, \eta \right\rangle \mathbf{v} + \int_M \left\langle \frac{\delta l}{\delta \xi}, [\xi, \eta] \right\rangle \mathbf{v} = 0. \tag{6.8}$$

Taking into account Proposition 6.1, the expansion of the second term in (6.8) is

$$\begin{aligned}\int_M \left\langle \frac{d}{dt}\frac{\delta l}{\delta \xi}, \eta \right\rangle \mathbf{v} &= \frac{d}{dt}\int_M \left\langle \frac{\delta l}{\delta \xi}, \eta \right\rangle \mathbf{v} - \int_M \left\langle \frac{\delta l}{\delta \xi}, \dot{\eta} \right\rangle \mathbf{v} \\ &= -\frac{d}{dt}\left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma \eta \right\rangle + \left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma \dot{\eta} \right\rangle - \left\langle \frac{\delta \ell}{\delta v}, \vartheta(\dot{\eta}) \right\rangle \\ &= -\left\langle \frac{D}{dt}\frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma \eta \right\rangle - \left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \frac{D}{dt}(\nabla^\sigma \eta) \right\rangle + \left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma \dot{\eta} \right\rangle + \left\langle \frac{\delta \ell}{\delta v}, \eta(\vartheta(\xi)) \right\rangle,\end{aligned}$$

where, taking derivatives in the relation $0 = \vartheta(\eta(t))$, we have $0 = \vartheta(\dot{\eta}) + \dot{\vartheta}(\eta) = \vartheta(\dot{\eta}) + i_\eta d\vartheta(\xi) = \vartheta(\dot{\eta}) + \eta(\vartheta(\xi))$, as $\xi = \dot{s}$. The third term in (6.8), using again Proposition 6.1 reads

$$\int_M \left\langle \frac{\delta l}{\delta \xi}, [\xi, \eta] \right\rangle \mathbf{v} = -\left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma [\xi, \eta] \right\rangle + \left\langle \frac{\delta \ell}{\delta v}, \vartheta([\xi, \eta]) \right\rangle. \tag{6.9}$$

Therefore, Eq. (6.8) can be rewritten as

$$\begin{aligned} -\int_M \left\langle \frac{\delta l}{\delta \sigma}, \nabla^\sigma \eta \right\rangle - \left\langle \frac{D}{dt} \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma \eta \right\rangle - \left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \frac{D}{dt} (\nabla^\sigma \eta) \right\rangle \\ + \left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma \dot{\eta} \right\rangle + \left\langle \frac{\delta \ell}{\delta v}, \eta(\vartheta(\xi)) \right\rangle - \left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma [\xi, \eta] \right\rangle + \left\langle \frac{\delta \ell}{\delta v}, \vartheta([\xi, \eta]) \right\rangle = 0. \end{aligned} \tag{6.10}$$

We now manipulate the horizontal equation of (4.2) of the dynamical formulation. This equation being defined in $T(Q/G)$, we couple it with the projected variation $-\nabla^\sigma \eta \in T_\sigma(Q/G)$. We have

$$\left\langle \frac{\partial \ell}{\partial \sigma}, -\nabla^\sigma \eta \right\rangle - \left\langle \frac{D}{dt} \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma \eta \right\rangle - \left\langle \frac{\delta \ell}{\delta v}, \Xi(-\nabla^\sigma \xi, -\nabla^\sigma \eta) \right\rangle = 0.$$

We compute the curvature in Q in the point s though we understand its value in $\tilde{\mathfrak{G}}$ so that it can be coupled with $\delta \ell / \delta v$. Taking into account that η is horizontal, we have

$$\Xi(\xi, \eta) = d\vartheta(\xi, \eta) = -\eta(\vartheta(\xi)) - \vartheta([\xi, \eta]).$$

The horizontal Eq. (4.2) then reads

$$\left\langle \frac{\partial \ell}{\partial \sigma}, -\nabla^\sigma \eta \right\rangle - \left\langle \frac{D}{dt} \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma \eta \right\rangle + \left\langle \frac{\delta \ell}{\delta v}, \eta(\vartheta(\xi)) + \vartheta([\xi, \eta]) \right\rangle = 0. \tag{6.11}$$

The expansion of the first term in (6.11) is

$$\left\langle \frac{\partial \ell}{\partial \sigma}, -\nabla^\sigma \eta \right\rangle = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \ell(\gamma(\epsilon), -\nabla^{\gamma(\epsilon)} \rho(\epsilon), w(\epsilon)),$$

where $\gamma(\epsilon)$ is any curve such that $\gamma(0) = \sigma$, $\gamma'(0) = -\nabla^\sigma \eta$; the curve $\nabla^\gamma \rho(\epsilon)$ in $T(Q/G)$ is the parallel transport of $\nabla^\sigma \xi = \nabla^\sigma \xi^h$ along γ with respect the chosen linear connection in Q/G; and $w(\epsilon)$ is the horizontal lift with $w(0) = v$ of the curve γ to $\widetilde{\mathfrak{G}}$ with respect to ϑ. Note that the prime denotes the derivative with respect to ϵ at $\epsilon = 0$. From (5.6) and Proposition 6.4 we have

$$\begin{aligned} -\left\langle \frac{\partial \ell}{\partial \sigma}, \nabla^\sigma \eta \right\rangle &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_M l(\gamma(\epsilon), \rho(\epsilon)^h + \eta_{w(\epsilon)}) \mathbf{v} \\ &= \int_M \left\langle \frac{\delta l}{\delta \sigma}, \gamma' \right\rangle \mathbf{v} + \int_M \left\langle \frac{\delta l}{\delta \xi}, (\rho')^h + (\eta_w)' \right\rangle \mathbf{v} \\ &= -\int_M \left\langle \frac{\delta l}{\delta \sigma}, \nabla^\sigma \eta \right\rangle \mathbf{v} - \left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma ((\rho')^h + (\eta_w)') \right\rangle + \left\langle \frac{\delta \ell}{\delta v}, \vartheta(w') \right\rangle \\ &= -\int_M \left\langle \frac{\delta l}{\delta \sigma}, \nabla^\sigma \eta \right\rangle \mathbf{v} - \left\langle \frac{\delta \ell}{\delta \dot{\sigma}}, \nabla^\sigma (\rho') \right\rangle, \end{aligned}$$

where $\vartheta(w') = 0$ as w is a horizontal lift. Therefore, (6.11) can be expressed as

$$-\int_M \left\langle \frac{\delta l}{\delta \sigma}, \nabla^\sigma \eta \right\rangle \mathbf{v} - \left\langle \frac{\delta \ell}{\delta \dot\sigma}, \nabla^\sigma(\rho') \right\rangle \left\langle \frac{D}{dt}\frac{\delta \ell}{\delta \dot\sigma}, \nabla^\sigma \eta \right\rangle$$
$$+ \left\langle \frac{\delta \ell}{\delta v}, \eta(\vartheta(\xi)) + \vartheta([\xi, \eta]) \right\rangle = 0. \tag{6.12}$$

We finish by comparing (6.10) and (6.12). For that, we need the following remarks. As $\nabla^\gamma \rho$ is the parallel transport along γ of $\nabla^\sigma \xi$, the derivative $D/d\epsilon(\nabla^\gamma \rho)$ vanishes. Moreover, the linear connection assumed to be torsionless in Q/G,

$$-\frac{D}{dt}\nabla^\sigma \eta = -\frac{D}{dt}\nabla^\sigma \eta + \frac{D}{d\epsilon}\nabla^\gamma \rho$$

gives the Lie bracket of $\nabla^\sigma \eta$ and $\nabla^\gamma \rho$ which, at $\epsilon = 0$ is $-\nabla^\sigma[\xi, \eta]$. On the other hand, from (6.5),

$$\nabla^\sigma \dot\eta = (\nabla^\sigma \eta)^{\cdot} + [\nabla \xi, \eta], \qquad \nabla^\sigma \rho' = (\nabla^\gamma \rho)' + [\nabla^\sigma \eta, \xi]$$

and then

$$\nabla^\sigma \dot\eta - \nabla^\sigma \rho' = (\nabla^\sigma \eta)^{\cdot} - (\nabla^\gamma \rho)' + \nabla[\xi, \eta] = 2\nabla[\xi, \eta],$$

where the definition of the Lie Bracket in Q/G has been used.

6.2 Hamiltonian formulation

For the equivalence between the covariant and dynamical formulations in the Hamiltonian picture, we could proceed as in the Lagrangian picture above, that is, we would start with Eq. (3.11) and proceed in a similar way as in Sect. 6.1 to get the corresponding reduced equations in the dynamical framework (4.6). In order to show a different approach, we are going to proceed differently in this section. More precisely, we will explore below the link of the dynamical and covariant Hamiltonian descriptions through their Poisson brackets when reduction is performed.

We consider the covariant Poisson bracket (2.6) for a Hamiltonian H and a horizontal Poisson n-forms F. Recall that

$$F = (\mathrm{i}_{\langle p, X\rangle} \mathbf{v}) \wedge dt + \langle \nu, X\rangle\, \mathbf{v} + \pi^*_{P\Pi}\omega$$

in any point $(p, \varrho, t) \in \Pi \times V^*P \times I$, where X and ω are any time dependent vertical vector field and horizontal n-form in P, respectively. The form F induces a function $\mathcal{F}$ in $T^*Q \times I$ defined in (5.10) and the Hamiltonian H induces a Hamiltonian $\mathcal{H}$ in $T^*Q \times I$ defined in (5.7).

Proposition 6.2 *For any Hamiltonian H and any horizontal Poisson n-form F in $\bar\Pi = \Pi \times V^*P \times I$, we have*

$$\int_M (\widehat{\mathbb{F}L}(j^1 s), \nu)^*\{F, H\}(t)\mathbf{v} = \{\mathcal{F}, \mathcal{H}\}(s, \nu; t) \qquad \forall (s, \nu; t) \in T^*Q \times I \tag{6.13}$$

where the bracket in the integral is the covariant bracket (2.6) and the bracket in the right hand side is the canonical bracket in the cotangent bundle T^*Q.

Proof Given the horizontal Poisson n-form F, let $X_{\mathcal{F}}$ be the Hamiltonian vector field in T^*Q defined by the function $\mathcal{F}$ through the Hamilton equation, i.e., $\mathrm{i}_{X_{\mathcal{F}}}\mathbf{\Omega} = d\mathcal{F}$. Since F is a Poisson n-form, there exist a vector field $\mathcal{X}_F$ in $\Pi \times V^*P \times I$ such that Eq. (2.5) is satisfied. We claim that the projection of $\mathcal{X}_F \in \mathfrak{X}(\Pi \times V^*P \times I)$ onto V^*P gives the vector field X_F. Indeed, for any vector V in $T_{(s,\nu)}(T^*Q)$, we have

$$\int_M (\widehat{\mathbb{F}L}(j^1 s), \nu)^* \mathrm{i}_{\mathcal{V}} \mathrm{i}_{\mathcal{X}_F} \mathfrak{s}^*\Omega = \int_M (\widehat{\mathbb{F}L}(j^1 s), \nu)^* \mathrm{i}_{\mathcal{V}} dF = \mathrm{i}_V d\mathcal{F} = \mathrm{i}_V \mathrm{i}_{X_{\mathcal{F}}} \mathbf{\Omega}$$

and, according to Proposition 5.1, our claim is established. Hence,

$$\begin{aligned}\{\mathcal{F}, \mathcal{H}\}(s, \nu) &= \mathrm{i}_{X_{\mathcal{F}}} d\mathcal{H}(s, \nu) = \int (\widehat{\mathbb{F}L}(j^1 s), \nu)^* (\mathrm{i}_{\mathcal{X}_F} dH)\mathbf{v} \\ &= \int_M (\widehat{\mathbb{F}L}(j^1 s), \nu)^* \mathrm{i}_{\mathcal{X}_F} \mathrm{i}_{\mathcal{X}_H} (\mathfrak{s}^*\Omega)\mathbf{v} = \int_M (\widehat{\mathbb{F}L}(j^1 s), \nu)^* \{F, H\}\mathbf{v},\end{aligned}$$

and we finish the proof. □

Corollary 6.3 *A curve $(s(t), \nu(t))$ is a solution of the Hamilton equation in $T^*Q \times I$ if and only if the section $(\widehat{\mathbb{F}L}(j^1 s(t)), \nu(t), t)$ of $\bar{\Pi} = \Pi \times V^*P \times I$ is a solution of the Hamilton equation* (2.8).

Proof It is a consequence of the identity (6.13) for any horizontal Poisson n-form, integrating (2.8) along M and taking into account the evolution equation $\{\mathcal{F}, \mathcal{H}\} = d\mathcal{F}/dt$ determines the solution of the Hamilton equation in T^*Q even when only functions $\mathcal{F}\colon T^*Q \to \mathbb{R}$ linear in the momentum ν are considered. □

The reduction process is now quite simple. First, it is easy to see that if H is a G-invariant Hamiltonian in $\bar{\Pi}$, the induced Hamiltonian $\mathcal{H}\colon T^*Q \times I \to \mathbb{R}$ is also G-invariant. Hence, the reduced Hamiltonians $h\colon (\bar{\Pi}/G) \to \mathbb{R}$ and $\hbar\colon ((T^*Q)/G) \times I \to \mathbb{R}$ are related. Indeed, recall that, given a connection in $Q \to Q/G$, we have the identification $(T^*Q)/G = T^*(Q/G) \times \widetilde{\mathfrak{G}}^*$. Then, for any $(\sigma, \tau; \nu; t) \in T^*(Q/G) \times \widetilde{\mathfrak{G}}^* \times I$, we easily deduce

$$\hbar(\sigma, \tau; \nu; t) = \int_M h(\widehat{\mathbb{F}l}(\sigma), \tau^h + \nu, t)\mathbf{v}$$

where $\widehat{\mathbb{F}l}\colon C \to TM \otimes \tilde{\mathfrak{g}}^*$ is the vertical derivative of $l\colon C \times \tilde{\mathfrak{g}} \to \mathbb{R}$ with respect to the fiber $C \to M$, and τ^h denotes the horizontal lift with respect to the connection in $Q \to Q/G$ seen as a section of $\tilde{\mathfrak{g}}^*$ (a G-invariant vertical 1-form in P).

Similarly, any G-invariant Poisson n-form F induces a G-invariant function $\mathcal{F}\colon T^*Q \to \mathbb{R}$. We call f the projected form in $\bar{\Pi}/G = (\Pi/G) \times \tilde{\mathfrak{g}}^* \times I$ and f the projected function in $(T^*Q)/G$, respectively. Taking this notation into account, we have the following result.

Theorem 6.4 *Let f a horizontal Poisson n-form in $\bar{\Pi}/G$ and $h\colon (\Pi/G) \times \tilde{\mathfrak{g}}^* \times I \to \mathbb{R}$ a G-invariant Hamiltonian. We have*

$$\int_M (\widehat{\mathbb{F}l}(\sigma), \tau^h + \nu)^* \{f, h\}(t)\mathbf{v} = \{f, \hbar\}(\sigma, \mu; \nu; t), \qquad (\sigma, \mu; \nu; t) \in T^*(Q/G) \times \widetilde{\mathfrak{G}}^* \times I, \tag{6.14}$$

where the bracket in the integral is the reduced bracket given in (3.10) and the bracket in the right hand side is the reduced Poisson bracket (4.5).

Proof One first considers the G-invariant horizontal Poisson $(n-1)$-form F and Hamiltonian H defining f and h, respectively. Then the result is a consequence of (6.13) once one takes into account that the projections $\bar{\Pi} \to (\bar{\Pi}/G)$ and $T^*Q \to (T^*Q)/G$ are Poisson with respect to their brackets. □

Corollary 6.5 *A curve* $((\sigma(t), \tau(t)), \nu(t))$ *in* $(T^*Q)/G = T^*(Q/G) \times \widetilde{\mathfrak{G}}^*$ *is a solution of the Hamilton–Poincaré equation* (4.6) *if and only if the section* $(\hat{\mathbb{F}l}(\sigma(t)), \tau + \nu(t), t)$ *of* $(\Pi \times V^*P \times I)/G = (TM \otimes \tilde{\mathfrak{g}}^*) \times \tilde{\mathfrak{g}}^* \times I$ *is a solution of the Lie–Poisson equation* (3.11).

6.3 Holonomy

The equations defined by the reduced and the unreduced variational principles are basically equivalent in both the covariant and dynamical approaches. Starting with a solution of the unreduced equations, its projection to the reduced phase space gives automatically a solution of the reduced equations. The inverse problem, that is, the so-called reconstruction process, consists of obtaining one solution (or all possible solutions) of the unreduced problem starting from a given solution of the reduced equations. The covariant setting shows the need of a compatibility condition, namely flatness of the connection which, on the other hand, does not appear in the dynamic framework. It is known that the integral leaves of this connection will be sections of the bundle when M is simply connected, and then the reconstruction gives all the desired solutions of the unreduced problem. This topological assumption has been assumed all along the article. However, when M is not simply connected some spurious solutions of the reduced equations, that is, solutions not coming from solutions of the unreduced problem, may occur. As any manifold is locally simply connected, this extra solutions are obtained through global topological considerations.

Moreover, in the dynamical setting, the definition of Q/G as the set of flat connections in Sect. 3.1 also opens the possibility of extra solutions when M is not simply connected. In any case, the reduced covariant and dynamical approaches are still equivalent as both depend on the topological considerations of M in the same way.

It is interesting to point out that these extra solutions may be quite relevant and so should not be neglected. They provide solutions with discontinuities such as the *defects* in many field theories, or *phases* in other gauge theories. Their importance is nowadays increasing.

We briefly give an extremely simple example showing the existence of these solutions. Consider $M = S^1$ the unitary circle and $P = S^1 \times \mathbb{R} \to S^1$ as the principal bundle. Consider the Lagrangian

$$L\colon J^1P = T^*S^1 \otimes T\mathbb{R} \to \mathbb{R}; \quad (g(\phi), g'(\phi)) \mapsto (g'(\phi))^2$$

where the sections of $P \to S^1$ are written as $s(\phi) = (\phi, g(\phi))$, $\phi \in S^1$ and g' is the derivative with respect to ϕ. The Euler–Lagrange equation is simply

$$g''(\phi) = 0, \tag{6.15}$$

which gives $g(\phi)$ constant along S^1.

The Lagrangian is invariant under translations in the fiber $\mathbb{R}$. The reduced phase space is $T^*S^1 \otimes \mathbb{R} = T^*S^1$ and the reduced Lagrangian $l(\alpha) = |\alpha|^2$. It is not difficult to check that the Euler–Poincaré equation is

$$df = 0, \tag{6.16}$$

where $\alpha = f(\phi)d\phi$. Note that $d\phi$ and f are globally defined. Globally there are solutions $f =$ constant that are not induced from global solutions g. For example $f = 1$ locally defines g as a determination of the angle ϕ, which has a discontinuity when considered globally.

7 Example

For simplification, we confine ourselves to the Lagrangian picture of the results above. Let $\bar{P} = \bar{M} \times G \to \bar{M}$ be a trivial principal G-bundle over a Riemannian manifold $\bar{M} = \mathbb{R} \times M$ with metric $\bar{g} = dt^2 + g,\ t \in \mathbb{R}$, where g is a time independent metric on M. We assume that M is simply connected and compact. It is also supposed that the structure group G is equipped with a right invariant metric h. For the definition of the variational problem on $\bar{P}$, we will identify sections of this bundle as time dependent mappings $\varphi\colon M \to G$. The Lagrangian is given by

$$\begin{aligned} L\colon J^1\bar{P} &= J^1(M,G) \times_G TG \to \mathbb{R} \\ (j^1_x\varphi, V) &\mapsto \|(d\varphi)_x\|^2_{g,h} + \|V\|^2_h\,, \end{aligned}$$

$x \in M$, where the norms are computed using g and h, or only h, respectively. Note that V is understood as the derivative $\partial\varphi/\partial t$. This Lagrangian is G-invariant and for $G = SO(3)$ it generalizes the Lagrangian of the rigid body dynamics, which is given when M is a single point. In fact, L is a harmonic Lagrangian (a sigma model Lagrangian in the field theoretical language) and can be intuitively seen as a model of a sea of rigid bodies, in an appropriate sense. The dropped Lagragian is then

$$\begin{aligned} l\colon C \times \mathfrak{g} &\to \mathbb{R} \\ (\sigma,\xi) &\mapsto \|\sigma\|^2_g + \|\xi\|^2 \end{aligned}$$

where the metric on $\mathfrak{g}$ is the restriction of h on T_eG. For the definition of l one makes use of the fact that the projection $J^1(M,G) \times_G TG \to C \times \mathfrak{g}$ is given by $(j^1_x\varphi, V) \mapsto (\sigma = ((L_{\varphi^{-1}})_* d\varphi)_x, \xi = (L_{\varphi^{-1}})_* V_x)$. Note that, in this case, $C \simeq T^*M \otimes \mathfrak{g}$. The Euler–Poincaré equations are then

$$\mathrm{div}\sigma^\sharp + \frac{d}{dt}\xi^\sharp + \langle\sigma^\sharp, [\sigma,\cdot]\rangle + \langle\xi^\sharp, [\xi,\cdot]\rangle = 0 \tag{7.1}$$

where $^\sharp$ stands for the musical isomorphisms $T^*M \to TM,\ \mathfrak{g} \to \mathfrak{g}^*$ induced by the metrics g and h, that is, $\sigma^\sharp$ is a section of $TM \otimes \mathfrak{g}^*$ and $\xi^\sharp$ of $\mathfrak{g}^*$. The reconstruction condition is finally

$$d\sigma + [\sigma,\sigma] = 0, \qquad \dot{\sigma} = -d\xi - [\sigma,\xi].$$

On the other hand, for the dynamical framework, we have

$$Q = \{\varphi\colon M \to G \mid \varphi \in C^\infty\}$$

and

$$Q/G = \{\sigma \in \Omega^1(M,\mathfrak{g}) \mid d\sigma + [\sigma,\sigma] = 0\}.$$

The Lagrangian induced by L reads

$$\begin{aligned}\mathcal{L}\colon TQ &\to \mathbb{R}\\ (\varphi,\dot{\varphi}) &\mapsto \int_M \left(\|d\varphi\|^2_{g,h} + \|\dot{\varphi}\|^2_h\right)\mathbf{v}_g,\end{aligned}$$

where $\mathbf{v}_g$ is the Riemannian volume form defined by g on M. On the other hand, the reduced Lagrangian is

$$\begin{aligned}\ell\colon T(Q/G)\times\tilde{\mathfrak{G}} &\to \mathbb{R}\\ (\sigma,-\nabla^\sigma\xi;\eta) &\mapsto \int_M \left(\|\sigma\|^2_g + \|\xi^h+\eta\|^2\right)\mathbf{v}_g,\end{aligned}$$

where we make use of a connection ϑ in $Q\to Q/G$. The Lagrange–Poincaré equations are

$$\begin{aligned}\int_M \left(\frac{D}{dt}(\xi^h+\eta)^\sharp + \langle(\xi^h+\eta)^\sharp,[\xi^h+\eta,\cdot]\rangle\right)\mathbf{v}_g &= 0\\ \frac{\partial\ell}{\partial\sigma} + \int_M \frac{D}{dt}(\xi^h+\eta)^\sharp\mathbf{v}_g - \left\langle\int_M(\xi^h+\eta)^\sharp\mathbf{v}_g, i_\xi\,\Xi\right\rangle &= 0,\end{aligned} \tag{7.2}$$

where for the determination of $\partial\ell/\partial\sigma$ the connection ϑ is applied. Therefore, according to the results of the previous section, the two sets of Eq. (7.1) and (7.2) are equivalent. In any case, with this particular Lagrangian, the equivalence can be also done directly.

8 Conclusions and future work

In many field theories, such as electromagnetism and gravity, the equivalence of the covariant and dynamical equations are directly seen to be equivalent (see [11] and references therein); this also holds for the case of field theories where the fields take values in principal bundles. In [3] it is shown (amongst other things) how to covariantly reduce Maxwell's equations. The main result of this article is to carry out reduction, both covariantly and dynamically for principle bundle theories and to show that, under some compatibility conditions, that these reduced descriptions are equivalent, both in the Lagrangian and the Hamiltonian settings.

In the future we hope to explore a number of issues related to those treated in this article. First of all, the link between covariant reduced equations and their reduced dynamical counterparts needs to be explored for a wider class of field theories. A key example, where this is not too well understood (except for a "hand" calculation) is the case of relativistic fluids; see [2]. Of course there are many other examples as well. These sorts of examples also have the feature that their symmetry groups are much larger (e.g., such as gauge symmetries), and so greater care is needed in carrying out the reduction.

A second thing that needs additional work is to use more general slicings, associated to a general lapse and shift. In this article we only considered the simple situation of a trivially sliced spacetime, but it was already complicated enough to illustrate the main points.

Finally, the issue of defects needs additional work and in particular, it would be interesting to link the ideas in the present article to defects in liquid crystals and nonlinear elasticity, for example.

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