Theor. Comput. Fluid Dyn. (2008) 22: 37–64
DOI 10.1007/s00162-007-0065-y

ORIGINAL ARTICLE

Banavara N. Shashikanth · Artan Sheshmani · Scott David Kelly · Jerrold E. Marsden

Hamiltonian structure for a neutrally buoyant rigid body interacting with N vortex rings of arbitrary shape: the case of arbitrary smooth body shape

Received: 19 December 2006 / Accepted: 14 August 2007 / Published online: 21 September 2007

Abstract We present a (noncanonical) Hamiltonian model for the interaction of a neutrally buoyant, arbitrarily shaped smooth rigid body with N thin closed vortex filaments of arbitrary shape in an infinite ideal fluid in Euclidean three-space. The rings are modeled without cores and, as geometrical objects, viewed as N smooth closed curves in space. The velocity field associated with each ring in the absence of the body is given by the Biot–Savart law with the infinite self-induced velocity assumed to be regularized in some appropriate way. In the presence of the moving rigid body, the velocity field of each ring is modified by the addition of potential fields associated with the image vorticity and with the irrotational flow induced by the motion of the body. The equations of motion for this dynamically coupled body-rings model are obtained using conservation of linear and angular momenta. These equations are shown to possess a Hamiltonian structure when written on an appropriately defined Poisson product manifold equipped with a Poisson bracket which is the sum of the Lie–Poisson bracket from rigid body mechanics and the canonical bracket on the phase space of the vortex filaments. The Hamiltonian function is the total kinetic energy of the system with the self-induced kinetic energy regularized. The Hamiltonian structure is independent of the shape of the body, (and hence) the explicit form of the image field, and the method of regularization, provided the self-induced velocity and kinetic energy are regularized in way that satisfies certain reasonable consistency conditions.

Keywords Ideal hydrodynamics · Hydrodynamical interaction · Vortex ring · Vorticity · Hamiltonian structure · Poisson bracket · Lie–Poisson bracket

PACS 47.15.ki, 47.10.Df, 47.32.cb

Communicated by D.D. Holm.

B. N. Shashikanth (✉)
Department of Mechanical Engineering, New Mexico State University, MSC 3450, Las Cruces, NM 88003, USA
E-mail: shashi@nmsu.edu

A. Sheshmani
Department of Mathematics, University of Illinois at Urbana-Champaign, MC-382, Urbana, IL 61801, USA

S. D. Kelly
Department of Mechanical Engineering and Science, University of North Carolina at Charlotte,
9201 University City Boulevard, Charlotte, NC 28223-0001, USA

J. E. Marsden
Department of Control and Dynamical Systems, California Institute of Technology, CDS 107-81, Pasadena, CA 91125, USA

Contents

1 Introduction

The classical approach to the mechanics of bodies in fluids and of vortex rings has a rich history, for which [19] and [32] are representative references. The geometric approach to fluid mechanics also has a rich history, going back at least as far as Poincaré in the late 1800s and [3] and [13], providing many interesting insights and new results, some of which are described in [4,5,27], and references therein. In this paper, we make use of both the classical view and the geometric view of fluid mechanics to establish the governing equations and the Hamiltonian structure for the dynamic interaction of a system of vortex rings with a neutrally buoyant rigid body.

The Hamiltonian structure of a system of vortex filaments, modeled as curves supporting a delta distribution of vorticity, was given in [28], and the Poisson geometry of the filament equation and its relation to the Hasimoto transformation was studied further in [20]. Using the ideas of discrete symmetries and fixed point subspaces, [35], in their study of leapfrogging vortex rings, showed that for closed filaments modeled as curves—which we term *vortex rings* throughout the present paper—a special Hamiltonian structure is derivable in the case of circular coaxial rings (using discrete reduction) from that for rings of arbitrary shape. A Poisson formulation for the dynamics of a passive particle advected by the flow due to a single vortex filament was given in [8] and for a circular vortex ring interacting with a stationary rigid sphere was given in [9]. The Hamiltonian structure of the system comprising a rigid circular cylinder dynamically interacting with N point vortices in the plane was presented in [36], and this structure was used to analyze stability for certain special configurations of this system. This Hamiltonian structure was shown to hold for a rigid cylinder of arbitrary smooth shape in [33].

In the current paper we present a model for the dynamic interaction of a 3D rigid body of arbitrary smooth shape with N arbitrarily shaped and arbitrarily oriented vortex rings identified with N closed curves in $\mathbb{R}^3$, as in Fig. 1.

The interaction of such rings in the absence of solid boundaries is given by the Biot–Savart law. The velocity field of each ring in the presence of the solid boundary introduced by the moving body is modified by the addition of a potential field associated with the image vorticity inside the body. The well-known singularity in the self-induced velocity field of each ring, in the corresponding curve's binormal direction, is assumed to be regularized in some appropriate way. The actual method of regularization is not a concern for us in this paper.

We will derive the equations of motion for this system and show that they possess a noncanonical Hamiltonian structure. The Hamiltonian is the total kinetic energy of the body-rings system with the self-induced kinetic energy regularized. Our approach in obtaining the Hamiltonian structure is similar in spirit to that used in [33] for the 2D problem with point vortices. However, because of the extra spatial dimension and the additional feature of having vortex filaments that are curved, considerably more groundwork needs to be established before proving the main result in the present context. In particular, we shall need to exploit certain properties of the vector potential in $\mathbb{R}^3$ (see, for example, [14]), which is the analog of the stream function in $\mathbb{R}^2$. We demonstrate these properties explicitly for our problem configuration. The Hamiltonian structure we obtain does not depend on the shape of the smooth body, so the explicit form of the image field does not have to be known. Moreover, this structure is independent of the specific methods by which the self-induced velocity field and the self-induced kinetic energy are regularized, provided these regularization are done in a way that satisfies some consistency conditions. In particular, the regularized self-induced velocity field should be the Hamiltonian vector field, relative to the Hamiltonian structure, for the real-valued function which is the regularized self-induced kinetic energy.

The outline of the paper is as follows. In Sect. 2, the problem setting is defined and the Lie–Poisson equations of the coupled body-rings system are obtained following a traditional momentum balance analysis, the details of which are given in Appendix A. In Sect. 3, the evolution equations for the N curves representing the N rings in the presence of the moving body are presented. Following this the combined equations of motion for the whole body-rings system are written. In Sect. 4, properties of the vector potential that will be needed in the sequel are proven through a series of propositions. In Sect. 5, it is proven that the equations of motion

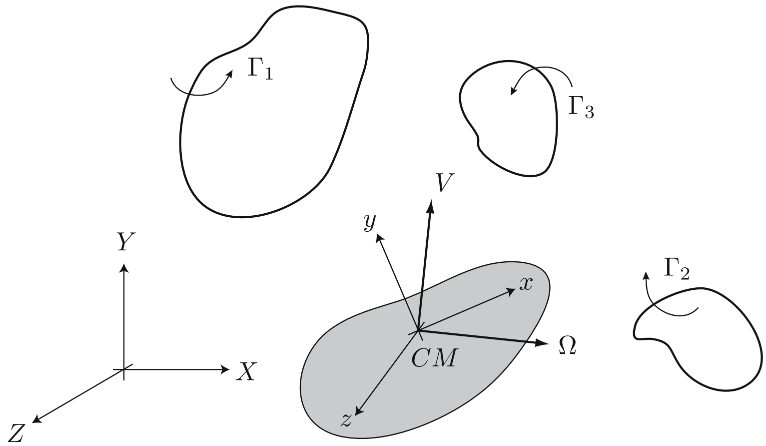

Fig. 1 Three vortex rings and a neutrally buoyant rigid body of arbitrary shape dynamically interacting with each other in an incompressible fluid

for the system possess the proposed Hamiltonian structure on an appropriately defined Poisson manifold. In Sect. 6, we discuss our results and some directions for future research.

2 Equations of motion for the body-rings system

In this section, we derive the equations of motion governing the dynamic interaction of a free rigid body with N vortex rings.

Problem setting and assumptions. We consider a rigid body of arbitrary smooth shape immersed in an ideal (inviscid, incompressible) fluid. The fluid extends to infinity in all directions away from the body. The body and fluid have uniform and equal density, taken (without loss of generality) to be unity. We require the normal velocity of the fluid on the body's surface to match the normal velocity of the surface (the "free-slip" condition) and require the fluid to be at rest infinitely far away. The fluid vorticity field is confined to a compact region around the body and assumed to be a delta distribution supported on N vortex rings of arbitrary shape, these rings intersecting neither each other nor the body. As stated before, the N rings can be viewed as N arbitrary smooth closed curves in $\mathbb{R}^3$.

The body is free to move under the instantaneous pressure field induced on its surface by the fluid. The motion of the body in turn induces a motion of the fluid. The velocity field of the fluid consists, according to the Hodge decomposition, of two parts: (i) the irrotational Kirchhoff potential field which satisfies the normal velocity matching condition on the body surface and (ii) the divergence-free, rotational field due to the N vortex rings in the presence of the body with zero normal velocity on the body surface. This second part further decomposes into two components: the velocity field due to the rings in the absence of the body and the velocity field due to the image vorticity inside the body. The normal components of these latter two velocity fields cancel one another on the body surface.

The equations of motion. In the absence of any external forces or moments on the body-rings system, the system's linear and angular momentum are conserved in time. However, as is well known (see [19] and [32]), the integral representing the linear or the angular momentum of an unbounded fluid may not be convergent. To avoid dealing with this integral and to deal directly with the vorticity field, we make use of some key vector integral relations [see (2.1) and (2.4) below], written for a bounded domain in $\mathbb{R}^3$, which allow us to express the fluid momenta in terms of the moments of vorticity and the moments of circulation around the body.[1] For applications of these integral relations to problems in fluid mechanics, and a history, see [38]. Obviously, these relations by themselves do not get rid of the momentum divergence problem. The volume integrals on the right-hand sides of these relations are well defined for a compact distribution of vorticity, such as N rings, but

[1] If the fluid domain is simply connected, then the circulation around the body is zero, but the moments of circulation need not be.

the boundary terms could still diverge as the domain of integration is extended to infinity. For our purposes, however, what is needed is really the time rate of change of the fluid momentum; using these relations we will show, during the derivation of the equations, that the time rate of change of the terms that diverge spatially do not contribute to the equations of motion. These observations regarding the fluid momentum are of course not new—see, for example, [39]—and the same ideas were also used in the planar problem by [36].

Notation. Before we begin the derivation of the equations of motion, we pause to introduce some of our notation for the domains used and for distinguishing among vectors, vector fields, and tensors in spatially fixed and body-fixed frames. Additional such notation will be defined later in the paper. The moving rigid body, a compact subset of $\mathbb{R}^3$, is denoted by B and its boundary, a smooth, two-dimensional manifold, is denoted by ∂B. Let D denote the externally unbounded domain occupied by the fluid and let $\widetilde{D}$ denote a bounded subset of D with $\partial\widetilde{D} = \partial B \cup S$, where S is an (imaginary) smooth, compact, two-dimensional manifold containing the body and rings in its interior. In addition, the following convention is adopted. Any vector/vector field/tensor/differential form that is denoted by the same (English or Greek) letter in both the spatially fixed and body-fixed frames is distinguished in the spatially fixed frame by an overbar. For notational consistency, this convention is followed everywhere, even when writing equations and relations that are actually valid in any frame of reference [such as (2.1), (2.2), (2.4) and (2.8) below].

In the region $\widetilde{D} \subset \mathbb{R}^3$, consider the following vector identity (see, for example, [38] and [32]) which holds in general for any (smooth) divergence-free vector field $\bar{\mathbf{a}}$ on any bounded subset of $\mathbb{R}^3$:

$$\int_{\widetilde{D}} \bar{\mathbf{a}} d\overline{V} = \frac{1}{2}\left\{\int_{\widetilde{D}} (\mathbf{r} \times (\overline{\nabla} \times \bar{\mathbf{a}}))\, d\overline{V} + \int_{\partial\widetilde{D}} \mathbf{r} \times (\bar{\mathbf{n}} \times \bar{\mathbf{a}})\, d\bar{A}\right\}, \tag{2.1}$$

where $\mathbf{r}$ is the position vector with respect to the spatially fixed frame and $\bar{\mathbf{n}}$ is the unit inward normal vector on the boundary $\partial\widetilde{D}$ of $\widetilde{D}$. Letting $\bar{\mathbf{a}} = \bar{\mathbf{u}}$, the divergence-free velocity field of the fluid occupying $\widetilde{D}$, and defining the vorticity field

$$\bar{\omega} = \overline{\nabla} \times \bar{\mathbf{u}}, \tag{2.2}$$

it is easily seen that (2.1) expresses the linear momentum in terms of the moment of vorticity and moment of circulation around the boundary.

The motion equations for the system follow from Newton's second law applied to the linear momentum of the fluid in the domain $\widetilde{D}$, which written in a spatially fixed frame yields

$$\begin{aligned}\bar{\mathbf{F}}_{\text{ext}} &= \frac{d}{dt}\int_{\widetilde{D}} \bar{\mathbf{u}}\, d\overline{V} - \int_S \bar{\mathbf{u}}(\bar{\mathbf{u}}\cdot\bar{\mathbf{n}}) d\bar{A} \\ &= \int_{\partial B} p_B \bar{\mathbf{n}} d\bar{A} + \int_S p_s \bar{\mathbf{n}} d\bar{A}, \\ &= -m_B \frac{d\overline{\mathbf{U}}}{dt} + \int_S p_s \bar{\mathbf{n}} d\bar{A},\end{aligned}$$

where p_B is the pressure field on the surface of the body, p_s the pressure field on the imaginary surface S, m_B is the mass of the body (equal to its volume as per the unit density assumption) and $\overline{\mathbf{U}}$ is the velocity of the center (of mass) of the body in the spatially fixed frame. Using (2.1) along with (2.2), we see that the preceding equation leads to

$$\begin{aligned} m_B\frac{d\overline{\mathbf{U}}}{dt} + \frac{d}{dt}\left(\frac{1}{2}\left\{\int_{\widetilde{D}} (\mathbf{r}\times\bar{\omega}) d\overline{V} + \int_{\partial B} \mathbf{r}\times(\bar{\mathbf{n}}\times\bar{\mathbf{u}}) d\bar{A}\right)\right\} \\ = \int_S \bar{\mathbf{u}}(\bar{\mathbf{u}}\cdot\bar{\mathbf{n}}) d\bar{A} - \frac{d}{dt}\left(\frac{1}{2}\int_S \mathbf{r}\times(\bar{\mathbf{n}}\times\bar{\mathbf{u}}) d\bar{A}\right) + \int_S p_s\bar{\mathbf{n}} d\bar{A}. \end{aligned} \tag{2.3}$$

Analogous to (2.1), the identity for the angular momentum of the fluid or, in general, the first moment of any (smooth) divergence-free vector field $\mathbf{a}$ in $\widetilde{D}$ is

$$\int_{\widetilde{D}} \mathbf{r} \times \bar{\mathbf{a}}\, d\bar{A} = -\tfrac{1}{2} \int_{\widetilde{D}} (r^2 \overline{\nabla} \times \bar{\mathbf{a}}) d\overline{V} - \tfrac{1}{2} \int_{\partial \widetilde{D}} r^2 (\bar{\mathbf{n}} \times \bar{\mathbf{a}}) d\bar{A}, \tag{2.4}$$

where $r = \|\mathbf{r}\|$.

Using (2.2), (2.4) and balance of angular momentum of the fluid in the domain $\widetilde{D}$ about the origin of the spatially fixed frame gives

$$\begin{aligned} &\frac{d}{dt}(m_B \bar{\mathbf{b}} \times \overline{\mathbf{U}} + \overline{\mathbf{I}\Omega}) - \frac{1}{2}\frac{d}{dt}\left(\int_{\widetilde{D}} (r^2 \bar{\omega}) d\overline{V} + \int_{\partial B} r^2 (\bar{\mathbf{n}} \times \bar{\mathbf{u}}) d\bar{A} \right) \\ &= \int_S \mathbf{r} \times \bar{\mathbf{u}} (\bar{\mathbf{u}} \cdot \bar{\mathbf{n}}) d\bar{A} + \frac{1}{2}\frac{d}{dt} \int_S r^2 (\bar{\mathbf{n}} \times \bar{\mathbf{u}}) d\bar{A} + \int_S p_s (\mathbf{r} \times \bar{\mathbf{n}}) d\bar{A}, \end{aligned} \tag{2.5}$$

where $\bar{\mathbf{b}}$ is the position vector of the center (of mass) of the body and $\overline{\mathbf{I}\Omega}$ is the body angular momentum, both referred to the spatially fixed frame. $\mathbf{I}$ is the moment of inertia tensor and Ω the angular velocity of the body in the body-fixed frame.

Equations (2.3) and (2.5) are the starting points for the derivation of the equations for the system. Note that, as presented, they are still in a general form and as such apply to any (smooth) body shape and any (compact) distribution of vorticity in $\widetilde{D}$. The derivation essentially consists of the following steps. Additional details of these steps can be found in Appendix A:

1. Use the Hodge decomposition to split $\bar{\mathbf{u}}$ in (2.3) and (2.5) as:

$$\bar{\mathbf{u}} = \overline{\nabla} \Phi_B + \bar{\mathbf{u}}_V, \tag{2.6}$$

   where

$$\overline{\nabla}^2 \Phi_B = 0 \quad \text{and} \quad \overline{\nabla} \cdot \bar{\mathbf{u}}_V = 0 \tag{2.7}$$

   in the infinite fluid domain D and

$$\overline{\nabla} \Phi_B \cdot \bar{\mathbf{n}} = \overline{\mathbf{V}}_B \cdot \bar{\mathbf{n}} \quad \text{and} \quad \bar{\mathbf{u}}_V \cdot \bar{\mathbf{n}} = 0 \tag{2.8}$$

   on the body surface ∂B. Moreover, at infinity $\overline{\nabla} \Phi_B \to 0$ and $\bar{\mathbf{u}}_V \to 0$. In the above, $\overline{\mathbf{V}}_B$ represents the velocity of points on the body.
2. Show that as the imaginary external bounding surface S goes to infinity—or, in other words, as $\widetilde{D} \to D$—all terms on the right-hand sides of (2.3) and (2.5) go to zero.
3. Use the special geometry of the vorticity field $\bar{\omega}$ to express the integrals of the moments of $\bar{\omega}$ in (2.3) and (2.5) in terms of the rings variables.
4. Transform all quantities in (2.3) and (2.5) to a body-fixed frame with origin at the center (of mass) of the body.

Following these steps the equations of motion for the rigid body system are

$$\left(\frac{d}{dt} + \mathbf{\Omega} \times \right) \mathbf{L} = 0, \tag{2.9}$$

$$\left(\frac{d}{dt} + \mathbf{\Omega} \times \right) \mathcal{A} + \mathbf{U} \times \mathbf{L} = 0, \tag{2.10}$$

where

$$\begin{pmatrix} \mathbf{L} \\ \mathcal{A} \end{pmatrix} = M \begin{pmatrix} \mathbf{U} \\ \Omega \end{pmatrix} + \begin{pmatrix} \mathbf{P} \\ \Pi \end{pmatrix}. \tag{2.11}$$

In the above, M is the 6×6 symmetric mass matrix consisting of the mass and inertia terms of the body plus added mass terms, $(\mathbf{U}, \Omega) \equiv \mathbf{U}_B$ are the linear velocity of the body's center of mass and the body's angular velocity in the body-fixed frame and

$$\mathbf{P} = \frac{1}{2}\sum_{i=1}^{N}\Gamma_i \left(\oint_{C_i} (\mathbf{l}_i(s_i) \times \mathbf{t}_i(s_i))ds_i\right) + \frac{1}{2}\int_{\partial B} \mathbf{l} \times (\mathbf{n} \times \mathbf{u}_V)dA, \tag{2.12}$$

$$\Pi = -\frac{1}{2}\sum_{i=1}^{N}\Gamma_i \oint_{C_i} l_i^2(s_i)\mathbf{t}_i(s_i)ds_i - \frac{1}{2}\int_{\partial B} l^2(\mathbf{n} \times \mathbf{u}_V)dA, \tag{2.13}$$

where Γ_i is the strength of the ith ring, $\mathbf{l}$ denotes the position vector of points in the body-fixed frame with $l = \|\mathbf{l}\|$, $\mathbf{l}_i$ is $\mathbf{l}$ for points on the ith ring with $l_i = \|\mathbf{l}_i\|$, C_i is the parameterized curve denoting the ith ring in the body-fixed frame, $\mathbf{t}_i$ is the unit tangent vector field on the ith ring and s_i is the arc-length parameter for the ith ring.

Image vorticity and velocity. Before deriving the evolution equations for the rings to be coupled with the rigid body equations (2.9), we discuss the velocity field $\mathbf{u}_I$ due to image vorticity. As mentioned previously, the image vorticity is introduced inside the body to ensure that the velocity field $\mathbf{u}_V$ due to the total vorticity inside and outside the body has zero normal velocity on the body surface, thus ensuring that the second of the boundary conditions in (2.8) is satisfied. The divergence-free velocity field $\mathbf{u}_V$ can then be viewed as the sum

$$\mathbf{u}_V = \mathbf{u}_R + \mathbf{u}_I \ \text{ in } \ D, \tag{2.14}$$

where $\mathbf{u}_R$ is the divergence-free velocity field due to the N rings in the absence of the body for which an explicit expression, valid in all of $\mathbb{R}^3$, can be obtained using the Biot–Savart law, as will be discussed in the next paragraph. With $\mathbf{u}_R$ known, a unique $\mathbf{u}_I$ can be obtained as the solution to the following boundary value problem:

$$\nabla \times \mathbf{u}_I = 0 \ \text{ in } \ D, \tag{2.15}$$

$$\mathbf{u}_R \cdot \mathbf{n} = -\mathbf{u}_I \cdot \mathbf{n} \ \text{ on } \ \partial B. \tag{2.16}$$

The equation (2.15) holds since in an inviscid framework the introduction of a body in the field of the N rings cannot generate new vorticity. Since $\mathbf{u}_I$ is also divergence-free, equations (2.15) and (2.16) can be viewed, equivalently, as the definition of a Neumann problem for Φ_I, the velocity potential of $\mathbf{u}_I$. We note that the uniqueness of the velocity field $\mathbf{u}_I$ obtained in this way depends essentially on the three-dimensional nature of our problem. In two dimensions, a boundary condition of the form (2.16) is insufficient to specify the circulation that can exist around an interior boundary, adding a term to the velocity potential.

There is no known explicit formula for $\mathbf{u}_I$ for general smooth body shapes but in all cases it is obtained as the solution of the Neumann problem defined above. For special, simple body geometries like the sphere, $\mathbf{u}_I$ can be obtained following the techniques of [22] or [18]. In general, the field $\mathbf{u}_I$ depends on (i) the strength of the rings, (ii) the shape and position of the rings relative to the body and (iii) the shape of the body. Thus, we can write (in the body-fixed frame)

$$\mathbf{u}_I = \mathbf{u}_I(\mathbf{l}; C_1, \ldots, C_N, \Gamma_1, \ldots, \Gamma_N, \partial B). \tag{2.17}$$

Velocity field of the N rings in the absence of the body. The velocity field due to the N rings in the absence of the body is given by the Biot–Savart law or, equivalently, by inverting (2.2) in $\mathbb{R}^3$ and applying it to the vorticity distribution supported by the N curves. At points not on any ring, the velocity field due to the rings is given by

$$\mathbf{u}_R(q) = \sum_{i=1}^{N}\frac{\Gamma_i}{4\pi}\oint_{C_i}\frac{\mathbf{t}_i(s_i) \times (\mathbf{l}(q) - \mathbf{l}_i(s_i))\,ds_i}{|\mathbf{l}(q) - \mathbf{l}_i(s_i)|^3}, \qquad q \notin C_i (i = 1, \ldots, N). \tag{2.18}$$

As the field point approaches any ring, the velocity blows up. This singular nature of the field is essentially due to the lack of ring cores coupled with the curvature of the rings. Expansion of $\mathbf{u}_R$ in the vicinity of each

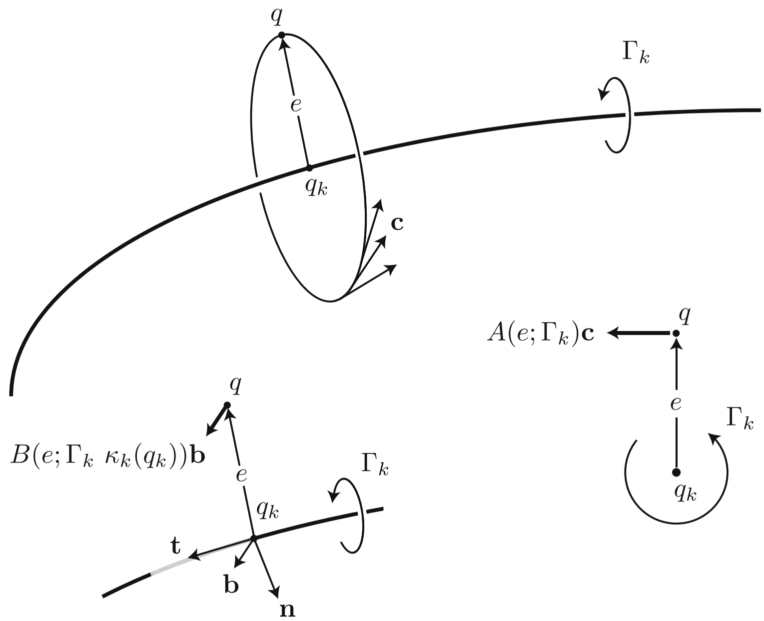

Fig. 2 First-order components of the velocity field $\mathbf{u}_R$ at the point q close to the kth ring

ring shows that (see [32] and [37] for more details) there are two singular terms. One is a point vortex type of singularity, with the velocity blowing up in circumferential directions on small circles perpendicular to the ring. The other singularity, due to the curvature of the ring, has the velocity blowing up in the binormal direction. More explicitly, $\mathbf{u}_R$ in the close vicinity of the kth ring can be expanded as (see, for example, [32] or [17])

$$\mathbf{u}_R(q) \sim A(e;\Gamma_k)\mathbf{c} + B(e;\Gamma_k,\kappa_k(q_k))\mathbf{b} + O(1), \tag{2.19}$$

where e is the minimum distance from the ring to q and also the radius of the circle to which $\mathbf{c}$ is the unit tangent vector field, $A(e;\Gamma_k) = \Gamma_k/2\pi e = O(1/e)$ is the magnitude of the point vortex velocity, $B(e;\Gamma_k,\kappa_k(q_k)) = O(\log e)$, κ_k represents the (principal) curvature field on the kth ring and q_k is the point on the ring to which q is closest, as depicted in Fig. 2.

Whereas $A(e;\Gamma_k)\mathbf{c}$ does not contribute to the self-induced velocity of the ring, the presence of $B(e;\Gamma_k,\kappa_k)\mathbf{b}$ results in an infinite self-induced velocity in the binormal direction. This latter term, thus, has to be regularized in some way. When this is done, the velocity field due to the N rings in the absence of the body, *at points on a ring*, becomes:

$$\mathbf{u}_R(q_k) = \sum_{i,i\neq k}^{N} \frac{\Gamma_i}{4\pi} \oint_{C_i} \frac{\mathbf{t}_i(s_i) \times (\mathbf{l}(q_k) - \mathbf{l}_i(s_i))\, ds_i}{|\mathbf{l}(q_k) - \mathbf{l}_i(s_i)|^3} + \mathbf{u}_{SI}(q_k),\ \ q_k \in C_k \tag{2.20}$$

in the spatially fixed frame. A simple though somewhat crude fix to regularize this infinite self-induced velocity field and to obtain $\mathbf{u}_{SI}$ is to use the *local induction approximation* (see references cited above for the history of this method). Using this approximation (see, for example, [35]), one gets

$$\mathbf{u}_{SI}(q_k) = \mathbf{u}_{LI}(q_k) = -\frac{\Gamma_k \kappa_k(s_k(q_k)) \log(c_k)}{4\pi} \mathbf{b}_k(s_k(q)),\ \ q_k \in C_k, \tag{2.21}$$

where $\mathbf{b}_k$ represents the unit binormal on the kth ring and c_k is a cut-off parameter, assumed to be constant.

The specific choice of the regularization method is not important for the present paper as long as it satisfies a certain consistency condition in the Hamiltonian framework of our model. This is discussed in more detail at a later stage.

Far-field decay rates of $\mathbf{u}_R$, $\mathbf{u}_I$ and $\nabla\Phi_B$. For future reference, it is useful to note the far-field decay rates of the different components of the velocity field. Expanding (2.18) for large $l = \| \mathbf{l} \|$, we show straightforwardly that

$$\mathbf{u}_R \sim \sum_{i=1}^{N} \frac{\Gamma_i}{4\pi} \left(\frac{1}{l^3} \oint_{C_i} \mathbf{l}_i \times \mathbf{t}_i ds_i - \frac{3\mathbf{l}}{l^5} \times \oint_{C_i} \langle \mathbf{l}, \mathbf{l}_i \rangle \mathbf{t}_i ds_i \right) + O\left(\frac{1}{l^4}\right), \tag{2.22}$$

where we have also used the fact that $\oint_{C_i} \mathbf{t}_i ds_i = 0$ (see [37] for a more detailed exposition of far-field decay rates of vortical structures in $\mathbb{R}^3$). The decay rates for $\mathbf{u}_I$ and $\nabla\Phi_B$ are given by the analysis of potential fields in [6] (Chap. 2). It is demonstrated therein that provided $\int_{\partial B} \mathbf{n} \cdot \nabla\phi dA = 0$, the potential function ϕ decays at least as fast as $1/l^2$. It is obvious from the boundary conditions (2.8) and (2.16) and the fact that both $\mathbf{u}_R$ and the body velocity field are divergence-free in B that both Φ_I and Φ_B satisfy this property. Hence, both $\mathbf{u}_I$ and $\nabla\Phi_B$ decay as $1/l^3$. To summarize, in the far field

$$\mathbf{u}_R = O\left(\frac{1}{l^3}\right), \quad \mathbf{u}_I = O\left(\frac{1}{l^3}\right), \quad \nabla\Phi_B = O\left(\frac{1}{l^3}\right). \tag{2.23}$$

Velocity field of the N rings in the presence of the body. The velocity field of the N rings in the presence of the body is given by (2.14), with $\mathbf{u}_R$ given by (2.18) and $\mathbf{u}_I$ given by (2.17).

3 Evolution equations for the rings

The evolution equations for the rings follow from a fundamental law of inviscid vortex motion, namely that singular distributions of vorticity (for example rings, point vortices, and vortex sheets) are convected by the fluid flow (see [32] (Chaps. 1, 2) for more details). In a spatially fixed frame, these equations are

$$\frac{\partial \overline{C}_i}{\partial t} = \left(\sum_{j, j \neq i}^{N} \bar{\mathbf{u}}_{V,j} + \bar{\mathbf{u}}_{I,i} + \overline{\nabla}\Phi_B \right)^n_{|\overline{C}_i} + \left(\bar{\mathbf{u}}_{SI,i}\right)^n, \quad i = 1, \dots, N,$$

where $\bar{\mathbf{u}}_{V,j}$ is $\bar{\mathbf{u}}_V$ due to the jth ring alone, and so on. The superscript n indicates that only the non-parallel component of each vector field in parentheses contributes to changes in the curve shape. The non-parallel component on C_i of any vector field $\mathbf{X}$ is given by

$$(\mathbf{X})^n = \mathbf{t}_i \times (\mathbf{X} \times \mathbf{t}_i).$$

Transferred to the body-fixed frame, the equations are

$$\frac{\partial C_i}{\partial t} + (\mathbf{U} + \Omega \times \mathbf{l}_i)^n_{|C_i} = \left(\sum_{j, j \neq i}^{N} \mathbf{u}_{V,j} + \mathbf{u}_{I,i} + \nabla\Phi_B \right)^n_{|C_i} + \left(\mathbf{u}_{SI,i}\right)^n, \quad i = 1, \dots, N. \tag{3.1}$$

In the above, Φ_B is the potential function generating the Kirchhoff field induced by the motion of the body, which has a linear decomposition in terms of the components of the velocity $\mathbf{U}$ of the body center (of mass) (see [29]):

$$\Phi_B = \phi_x u + \phi_y v + \phi_z w + \phi_\lambda \lambda + \phi_\chi \chi + \phi_\varsigma \varsigma, \tag{3.2}$$

where $\mathbf{U} \equiv (u, v, w)$ and $\Omega \equiv (\lambda, \chi, \varsigma)$.

The final coupled equations of motion for the dynamically interacting system comprising a rigid body of arbitrary shape and N vortex rings of arbitrary shape are thus given, in a body-fixed frame, by (2.9), (2.10) and (3.1):

$$\left(\frac{d}{dt} + \mathbf{\Omega}\times\right) \mathbf{L} = 0 \tag{3.3}$$

$$\left(\frac{d}{dt} + \mathbf{\Omega}\times\right) \mathcal{A} + \mathbf{U} \times \mathbf{L} = 0, \tag{3.4}$$

$$\frac{\partial C_i}{\partial t} + (\mathbf{U} + \Omega \times \mathbf{l}_i)^n_{|C_i} = \left(\sum_{j, j \neq i}^{N} \mathbf{u}_{V,j} + \mathbf{u}_{I,i} + \nabla\Phi_B \right)^n_{|C_i} + \left(\mathbf{u}_{SI,i}\right)^n, \quad i = 1, \dots, N. \tag{3.5}$$

Equations (3.3), (3.4) and (3.5) constitute a set of coupled ordinary and partial differential equations whose solutions give the simultaneous evolution of the body velocities $\mathbf{U}$, Ω and the ring shapes as described by the curves C_i. The velocity field at the location of the ith ring due to jth ring is given by

$$\mathbf{u}_{V,j}{}_{|C_i} = \frac{\Gamma_j}{4\pi} \oint_{C_j} \frac{\mathbf{t}_j \times (\mathbf{l}_i - \mathbf{l}_j) ds_j}{|\mathbf{l}_i - \mathbf{l}_j|^3} + \mathbf{u}_{I,j}{}_{|C_i} \equiv \mathbf{u}_{R,j}{}_{|C_i} + \mathbf{u}_{I,j}{}_{|C_i}, \tag{3.6}$$

where $\mathbf{u}_{I,j}$ is obtained from the solution of the Neumann problem defined by (2.15) and (2.16) for the jth ring and, similarly, Φ_B is the solution of the Neumann problem defined by (2.7) and (2.8). As mentioned earlier, only for bodies of simple geometry can one expect to find an exact representation of these potential fields. For non-simple (but still smooth) body geometries, these fields could be generated numerically simultaneously with their solutions using standard Runge–Kutta solvers or, more appropriately, variational or symplectic integrators for (3.3), (3.4) and (3.5) coupled with Poisson solvers for the fields $\mathbf{u}_{I,j}$ and $\nabla\Phi_B$. In addition, of course, the self-induced field $\mathbf{u}_{SI,i}$ would have to be updated as per the method of regularization employed.

4 The vector potential and its properties

To demonstrate the Hamiltonian structure of (3.3), (3.4) and (3.5), we need to establish some preliminaries. First, we introduce the vector potential $\mathbf{A}$ in $\mathbb{R}^3$ defined as

$$\mathbf{u} = \nabla \times \mathbf{A}.$$

The existence of $\mathbf{A}$ for a given smooth, divergence-free $\mathbf{u}$ in *unbounded* $\mathbb{R}^3$ can be posed as a standard problem in linear elliptic partial differential equations by imposing the condition that $\nabla \cdot \mathbf{A} = 0$. Since $\omega = \nabla \times \mathbf{u}$, the problem can be stated as: given ω, find a solution $\mathbf{A}$ to the Poisson equation

$$\omega = -\nabla^2 \mathbf{A}. \tag{4.1}$$

For examples of solutions to this classical equation see, for example, Chap. 1 in [32].

Solutions to this equation are not unique, being defined modulo gradient fields of harmonic functions, but uniqueness is not an issue for us. The general problem of existence of solutions of (4.1) depends on using elliptic theory in appropriate function spaces. Those that are appropriate in this context are the Nirenberg–Walker spaces, which are weighted Sobolev spaces that build in the correct asymptotic fall-off conditions at infinity; see [30]. It was shown in [10] that these spaces are appropriate for fluid mechanics.

Existence of A in bounded domains. In a bounded subset of $\mathbb{R}^3$ like our domain D, [14,7,2] and others have shown that given a *smooth* $\mathbf{u}$, the vector potential $\mathbf{A}$ exists and is unique provided the body has certain topological features and, more importantly, $\mathbf{u}$ satisfies the boundary condition $\int_{\partial B} \mathbf{u} \cdot \mathbf{n} = 0$. The latter two papers prove these results in a function space setting.

Referring to equations (2.18), (2.20) and (2.21) and the accompanying discussion, we find that the velocity field due to the N rings in the absence of the body obviously does not satisfy the smoothness properties required everywhere in D for the existence of a vector potential. In particular, at all points on the rings, the velocity field has singularities as discussed previously. It is nevertheless possible to demonstrate the existence of an $\mathbf{A}$, but with singularities at the ring locations, by invoking the results of the above authors provided this $\mathbf{A}$ is split as described below. Corresponding to the splitting

$$\mathbf{u} = \mathbf{u}_R + \mathbf{u}_I + \nabla\Phi_B,$$

where, it may be recalled, both $\mathbf{u}_I$ and $\nabla\Phi_B$ are divergence-free gradient fields in D, we split $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{A}_R + \mathbf{A}_I + \mathbf{A}_B.$$

Existence of $\mathbf{A}$ then follows immediately from the existence of $\mathbf{A}_R$, $\mathbf{A}_I$ and $\mathbf{A}_B$ as discussed below.

Existence of $\mathbf{A}_R$. This is actually known and requires no proof. Indeed, the vector potential (see, for example, [35])

$$\mathbf{A}_R(q) = \sum_{i=1}^{N} \frac{\Gamma_i}{4\pi} \oint_{C_i} \frac{\mathbf{t}_i(s_i)}{|\mathbf{l}-\mathbf{l}_i|} ds_i, \qquad q \notin C_i (i = 1, \ldots, N), \tag{4.2}$$

provides the correct $\mathbf{u}_R$, given by equation (2.18), at every point in D not on the rings. The singularities of $\mathbf{A}_R$ are also consistent with the singularities of $\mathbf{u}_R$ in the sense that

$$\lim_{q \to q_k} \mathbf{u}_R = \lim_{q \to q_k} \nabla \times \mathbf{A}_R,$$

for $q \to q_k$ along a prescribed path.

Existence of $\mathbf{A}_I$ *and* $\mathbf{A}_B$. The existence problem for $\mathbf{A}_I$ and $\mathbf{A}_B$ is as follows: given a smooth $\mathbf{u}_I$ and a smooth $\nabla \Phi_B$ in D, find an $\mathbf{A}_I$ and $\mathbf{A}_B$, respectively, such that

$$\nabla \times \mathbf{A}_B = \nabla \Phi_B, \quad \nabla \times \mathbf{A}_I = \mathbf{u}_I$$

everywhere in D. This can be transformed to a standard Poisson (Laplace) equation like (4.1) by imposing a divergence-free condition on $\mathbf{A}_I$ and $\mathbf{A}_B$ and the problem can be redefined as: find a divergence-free $\mathbf{A}_I$ and $\mathbf{A}_B$, respectively, in the bounded domain D such that

$$\nabla^2 \mathbf{A}_B = 0, \quad \nabla^2 \mathbf{A}_I = 0$$

everywhere in D, and such that the boundary conditions

$$(\nabla \times \mathbf{A}_B) \cdot \mathbf{n} = \mathbf{V}_B \cdot \mathbf{n}, \quad (\nabla \times \mathbf{A}_I) \cdot \mathbf{n} = -\mathbf{u}_R \cdot \mathbf{n}$$

are satisfied on ∂B.

The above boundary conditions are equivalent to determining the tangential components of $\mathbf{A}_B$, $\mathbf{A}_I$ on ∂B modulo gradient fields (with respect to the surface gradient operator). Viewing these tangential components as Dirichlet boundary conditions in D, the problem can then be solved for each of the three Cartesian components of $\mathbf{A}_B$ and $\mathbf{A}_I$. The ambiguity in the choice of the surface gradient field does not cause a problem since the vector potential can be modified by a corresponding gradient field in D which leaves the divergence and the curl of the vector potential invariant (see Proposition 4.1 below). We note that this solution may not give the correct tangential velocity on ∂B, since the boundary conditions do not determine the normal components of $\mathbf{A}_B$ and $\mathbf{A}_I$, but this does not matter for the results proved in this paper.

For more details and discussions of these existence issues, the reader is referred again to the cited references. For the rest of this paper, we will assume that given a $\mathbf{u}_R$, $\mathbf{u}_I$ and $\nabla \Phi_B$, an $\mathbf{A}_R$, $\mathbf{A}_I$, and $\mathbf{A}_B$ exist.

Properties of the vector potential. Certain properties of the vector potential $\mathbf{A}$—in particular, properties of the components $\mathbf{A}_V := \mathbf{A}_R + \mathbf{A}_I$ and $\mathbf{A}_B$—will now be demonstrated. These are important for demonstrating, in the next section, the Hamiltonian structure of our equations.

The vector potential $\mathbf{A}_V$ has the following property, noted and proved by [14] for the case of a smooth $\mathbf{u}_V$. The result also holds for the non-smooth $\mathbf{u}_V$ in our problem. For the sake of completeness, we provide our own coordinate version of the proof below.

Proposition 4.1 *The vector potential* $\mathbf{A}_V$ *can be adjusted by the addition of a gradient field so that* $\mathbf{A}_V$ *is normal at all points on the body boundary* ∂B.

Proof Pick a family of orthogonal curves, parameterized by s_1 and s_2, that cover ∂B smoothly. At each point on ∂B, pick a local frame with basis vectors given by the unit normal $\mathbf{n}$ together with $\mathbf{s}_1$ and $\mathbf{s}_2$. Write $\mathbf{A}_V$ with respect to this local frame as

$$\mathbf{A}_V^B = A_1(s_1, s_2)\mathbf{s}_1(s_1, s_2) + A_2(s_1, s_2)\mathbf{s}_2(s_1, s_2) + A_3(s_1, s_2)\mathbf{n}(s_1, s_2).$$

The boundary condition (2.8) for $\mathbf{u}_V$ written with respect to a local frame becomes

$$(\nabla_B \times \mathbf{A}_V^B) \cdot \mathbf{n} = 0. \tag{4.3}$$

We seek solutions to (4.3) of the form

$$(A_1, A_2) = \nabla_B^{\parallel} f$$

for some $f : \partial B \to \mathbb{R}$. Now consider a harmonic extension $\hat{f} : D \to \mathbb{R}^3$ of f. Define $\mathbf{A}_V^{'}$ in D as follows:

$$\mathbf{A}_V^{'} = \mathbf{A}_V - \nabla \hat{f}.$$

It is easily verified that $\mathbf{u}_V = \nabla \times \mathbf{A}_V^{'}$ and $\nabla \cdot \mathbf{A}_V^{'} = \nabla \cdot \mathbf{A}_V$ and, written in a local frame at a point on ∂B,

$$\begin{aligned}(\mathbf{A}_V^{'})^B &= \mathbf{A}_V^B - \nabla_B \hat{f},\\ &= \mathbf{A}_V^B - \nabla_B^{\parallel} \hat{f} - \nabla_B^{\perp} \hat{f},\\ &= \mathbf{A}_V^B - \nabla_B^{\parallel} f - \nabla_B^{\perp} \hat{f},\\ &= (A_3 - \nabla_B^{\perp} \hat{f})\mathbf{n} = A_3^{'}\mathbf{n}.\end{aligned}$$

Note that the above proposition is the 3D analog of the 2D version stating that the stream function ψ_V is constant along ∂B (a curve in 2D) and can be chosen to be zero.

Definition of exterior tube domains. For much of what is proved later, we need to define the following domain in $\mathbb{R}^3$. Envelop each ring with a smooth tube of circular cross-section whose boundary is diffeomorphic to the torus $\mathbb{T}^2$. Denote (as a set) the tube with cross-sectional diameter d enveloping the kth ring by T_k^d. Define the domain

$$D^d := D \backslash (T_1^d \cup \cdots \cup T_N^d).$$

The boundary of this domain is given by

$$\partial D^d = \partial B \cup \partial T_1^d \cup \cdots \cup \partial T_N^d,$$

and we have the obvious identification

$$D = \lim_{d \to 0} D^d. \tag{4.4}$$

Similarly, corresponding to $\widetilde{D}$, define

$$\widetilde{D}^d := \widetilde{D} \backslash (T_1^d \cup \cdots \cup T_N^d)$$

with boundary

$$\partial \widetilde{D}^d = \partial B \cup S \cup \partial T_1^d \cup \cdots \cup \partial T_N^d,$$

and we have the identification

$$\widetilde{D} = \lim_{d \to 0} \widetilde{D}^d. \tag{4.5}$$

It follows that

$$D = \lim_{\substack{S \to \infty \\ d \to 0}} \widetilde{D}^d. \tag{4.6}$$

Proposition 4.2 *In the domain $D \subset \mathbb{R}^3$ occupied by the fluid, the following holds:*

$$\sum_{i=1}^{N} \Gamma_i \oint_{C_i} \langle \mathbf{A}_B, \mathbf{t}_i \rangle ds_i = - \int_{\partial B} \langle \mathbf{A}_B, \mathbf{n} \times \mathbf{u}_V \rangle dA.$$

Proof In the domain $\widetilde{D}^d$, using integration by parts and Stokes' theorem, we have

$$\begin{aligned}
\int_{\widetilde{D}^d} \langle \mathbf{u}_V, \nabla\Phi_B\rangle \; dV &= -\int_{\partial\widetilde{D}^d} \Phi_B \mathbf{u}_V \cdot \mathbf{n}\, dA, \\
&= -\int_{\partial B} \Phi_B \mathbf{u}_V \cdot \mathbf{n}\, dA - \int_{S} \Phi_B \mathbf{u}_V \cdot \mathbf{n}\, dA - \int_{\partial T_1^d\cup\cdots\cup\partial T_N^d} \Phi_B \mathbf{u}_V \cdot \mathbf{n}\, dA, \\
&= -\int_{S} \Phi_B \mathbf{u}_V \cdot \mathbf{n}\, dA - \int_{\partial T_1^d\cup\cdots\cup\partial T_N^d} \Phi_B \mathbf{u}_V \cdot \mathbf{n}\, dA. \text{ [using (2.8)]}
\end{aligned}$$

Since $\mathbf{u}_V = \mathbf{u}_R + \mathbf{u}_I$ and $\mathbf{u}_I$ is bounded everywhere in D, the singularities of $\mathbf{u}_V$ are the same as of those of $\mathbf{u}_R$. Substituting (2.19) for $\mathbf{u}_V$ and taking the double limit $d \to 0$, $S \to \infty$, we obtain

$$\begin{aligned}
\int_{D} \langle \mathbf{u}_V, \nabla\Phi_B\rangle \; dV &= \lim_{\substack{S\to\infty \\ d\to 0}} \int_{\widetilde{D}^d} \langle \mathbf{u}_V, \nabla\Phi_B\rangle \; dV, \\
&= \lim_{S\to\infty}\left(-\int_{S} \Phi_B \mathbf{u}_V \cdot \mathbf{n}\, dA\right) + \lim_{d\to 0}\left(-\int_{\partial T_1^d\cup\cdots\cup\partial T_N^d} \Phi_B \mathbf{u}_V \cdot \mathbf{n}\, dA\right), \\
&= \lim_{d\to 0}\left(-\int_{\partial T_1^d\cup\cdots\cup\partial T_N^d} \Phi_B \left(A\left(\frac{d}{2};\Gamma_k\right)\mathbf{c} + B\left(\frac{d}{2};\Gamma_k,\kappa_k\right)\mathbf{b} + O(1)\right)\cdot \mathbf{n}\, dA\right) \\
&\qquad \text{[sufficiently rapid decay of } \mathbf{u}_{\mathrm{V}} \text{ and } \Phi_{\mathrm{B}} \text{ at infinity and (2.19)]}, \\
&= \lim_{d\to 0}\left(-\int_{\partial T_1^d\cup\cdots\cup\partial T_N^d} \Phi_B \left(B\left(\frac{d}{2};\Gamma_k,\kappa_k\right)\mathbf{b} + O(1)\right)\cdot \mathbf{n}\, dA\right) \\
&\qquad \text{[since } \mathbf{c}\cdot\mathbf{n} = 0 \text{ always]}, \\
&= \lim_{d\to 0}\left(O(d\log d) + O(d)\right) \quad \left[\text{since } B = O(\log d),\ \Phi_B = O(1) \text{ and } \int_{\partial T_k^d} dA = O(d)\right], \\
&= 0.
\end{aligned}$$

Integrating by parts in a different way, by first writing $\nabla\Phi_B = \nabla \times \mathbf{A}_B$, we obtain

$$\begin{aligned}
\int_{\widetilde{D}^d} \langle \mathbf{u}_V, \nabla\Phi_B\rangle \; dV &= \int_{\widetilde{D}^d} \langle \mathbf{A}_B, \omega\rangle\, dV + \int_{\partial\widetilde{D}^d} \langle \mathbf{A}_B, \mathbf{n}\times\mathbf{u}_V\rangle \; dA, \\
&= \int_{\partial\widetilde{D}^d} \langle \mathbf{A}_B, \mathbf{n}\times\mathbf{u}_V\rangle \; dA
\end{aligned}$$

since $\omega = \nabla \times \mathbf{u}_V = 0$ in $\widetilde{D}^d$ for all $d > 0$. Using (2.19) again, we find

$$\begin{aligned}
\lim_{d\to 0}\int_{\partial T_k^d} \langle \mathbf{A}_B, \mathbf{n}\times\mathbf{u}_V\rangle \; dA &= \lim_{d\to 0}\oint_{C_k^b}\int_0^{2\pi} \frac{\Gamma_k}{\pi d}\langle \mathbf{A}_B, \mathbf{n}\times\mathbf{c}\rangle \left(\frac{d}{2}\right) d\theta\, ds_k \\
&\qquad \text{[all other terms vanish as in the previous result and } \mathbf{A}_{\mathrm{B}} = \mathrm{O}(1)], \\
&= \Gamma_k \oint_{C_k^b} \langle \mathbf{A}_B, \mathbf{t}_k\rangle\, ds_k.
\end{aligned}$$

Gathering all the above results, we obtain

$$\sum_{i=1}^{N} \Gamma_i \oint_{C_i} \langle \mathbf{A}_B, \mathbf{t}_i \rangle \, ds_i + \int_{\partial B} \langle \mathbf{A}_B, \mathbf{n} \times \mathbf{u}_V \rangle dA + \int_{S} \langle \mathbf{A}_B, \mathbf{n} \times \mathbf{u}_V \rangle dA = 0.$$

Using the far-field decay rates of $\mathbf{u}_V$ and $\mathbf{A}_B$, one sees that

$$\lim_{S \to \infty} \int_{S} \langle \mathbf{A}_B, \mathbf{n} \times \mathbf{u}_V \rangle \ dA = 0,$$

proving the proposition.

Note: The signs of the boundary integrals in the above Proposition are determined by our convention of *inward* pointing unit normals.

Another important property of $\mathbf{A}_V$ that we will use later is a Green's-function-type reciprocity property. First we make note of the order of magnitude of $\mathbf{A}_{V,k}$ in the vicinity of the kth ring. Consistent with (2.19), it can be seen that

$$\mathbf{A}_{V,k} = O(\log e). \tag{4.7}$$

For this, write $\mathbf{A}_{V,k} = \mathbf{A}_{R,k} + \mathbf{A}_{I,k}$, note that $\mathbf{A}_{I,k} = O(1)$ everywhere in D and then use the integral expression for $\mathbf{A}_{R,k}$ given by (4.2).

Proposition 4.3 *The following relation holds for any two vortex rings, C_1 and C_2, in D:*

$$\Gamma_1 \oint_{C_1} \langle \mathbf{A}_{V,2}, \mathbf{t}_1 \rangle ds_1 = \Gamma_2 \oint_{C_2} \langle \mathbf{A}_{V,1}, \mathbf{t}_2 \rangle ds_2, \tag{4.8}$$

where $\mathbf{A}_{V,1}, \mathbf{A}_{V,2}$ are the vector potentials associated with the velocity fields (in the presence of the body) $\mathbf{u}_{V,1}, \mathbf{u}_{V,2}$ of the two rings, respectively.

Proof The proof can be constructed similarly to that of Proposition 4.2 using integration by parts on $\int_{\widetilde{D}^d} \langle \mathbf{u}_{V,1}, \mathbf{u}_{V,2} \rangle \, dV$ and expanding in two different ways. First, expand it as

$$\begin{aligned}
\int_{\widetilde{D}^d} \left\langle \mathbf{u}_{V,1}, \mathbf{u}_{V,2} \right\rangle \, dV &= \int_{\partial \widetilde{D}^d} \left\langle \mathbf{A}_{V,2}, \mathbf{n} \times \mathbf{u}_{V,1} \right\rangle \, dA, \\
&= \int_{\partial B \cup S \cup \partial T_1^d \cup \partial T_2^d} \left\langle \mathbf{A}_{V,2}, \mathbf{n} \times \mathbf{u}_{V,1} \right\rangle \, dA, \\
&= \int_{S \cup \partial T_1^d \cup \partial T_2^d} \left\langle \mathbf{A}_{V,2}, \mathbf{n} \times \mathbf{u}_{V,1} \right\rangle \, dA,
\end{aligned}$$

using Proposition 4.1 to show that the integrals over the body surface and ∂T_1^d are zero. In applying Proposition 4.1, we note that boundary condition (2.8) is also satisfied by each individual ring, i.e., $\int_{\partial B} \mathbf{u}_{V,1} \cdot \mathbf{n} = 0$, etc. Next, expand as

$$\int_{\widetilde{D}^d} \left\langle \mathbf{u}_{V,1}, \mathbf{u}_{V,2} \right\rangle \, dV = \int_{S \cup \partial T_1^d \cup \partial T_2^d} \left\langle \mathbf{A}_{V,1}, \mathbf{n} \times \mathbf{u}_{V,2} \right\rangle \, dA.$$

Using (4.7) and the near-ring expansion and farfield decay rates of $\mathbf{u}_{V,1}$, $\mathbf{u}_{V,2}$—equations (2.19) and (2.23), respectively—, we find the result follows in the limit $d \to 0$, $S \to \infty$

Corollary 4.4 *If C_a and C_b denote two different positions of the same ring (in the presence of the body) then*

$$\oint_{C_a} \langle \mathbf{A}_{I,b}, \mathbf{t}_a \rangle ds_a = \oint_{C_b} \langle \mathbf{A}_{I,a}, \mathbf{t}_b \rangle ds_b. \tag{4.9}$$

Proof Put $\Gamma_1 = \Gamma_2$ in (4.8). Write $\mathbf{A}_{V,a} = \mathbf{A}_{R,a} + \mathbf{A}_{I,a}$ etc. and then using (4.2) note that the Biot–Savart terms cancel on each side of (4.8) to give (4.9).

Next, we establish an important property of the vector potential $\mathbf{A}_B$.

Just as the velocity potential Φ_B can be written as a linear combination of the body velocities, as in (3.2), we can write, relative to a body-fixed frame,

$$\mathbf{A}_B = [A]\mathbf{U}_B, \tag{4.10}$$

where $\mathbf{U}_B = (u, v, w, \lambda, \chi, \varsigma)$ are the body center-of-mass velocities and the body angular velocities in the (body-fixed frame's) x, y and z directions, respectively, and $[A]$ is a 3×6 matrix:

$$[A] := \begin{pmatrix} A_{ux} & A_{vx} & A_{wx} & A_{\lambda x} & A_{\chi x} & A_{\varsigma x} \\ A_{uy} & A_{vy} & A_{wy} & A_{\lambda y} & A_{\chi y} & A_{\varsigma y} \\ A_{uz} & A_{vz} & A_{wz} & A_{\lambda z} & A_{\chi z} & A_{\varsigma z} \end{pmatrix} \equiv \left(\mathbf{A}_u \; \mathbf{A}_v \; \mathbf{A}_w \; \mathbf{A}_\lambda \; \mathbf{A}_\chi \; \mathbf{A}_\varsigma \right), \tag{4.11}$$

where $\nabla \times \mathbf{A}_u$ is the fluid (potential) velocity field induced by the motion of the body with unit speed in the x-direction, etc. The vector potentials $\mathbf{A}_u, \mathbf{A}_v, \mathbf{A}_w, \mathbf{A}_\lambda, \mathbf{A}_\chi, \mathbf{A}_\varsigma$ satisfy $\nabla \times \mathbf{A}_u = \nabla \phi_x$, $\nabla \times \mathbf{A}_v = \nabla \phi_y$, etc. in D, where ϕ_x, ϕ_y, etc. are the components of Φ_B in the analogous linear decomposition of Φ_B, and also satisfy the boundary conditions on ∂B:

$$(\nabla \times \mathbf{A}_u) \cdot \mathbf{n} = \mathbf{i} \cdot \mathbf{n}, \quad (\nabla \times \mathbf{A}_v) \cdot \mathbf{n} = \mathbf{j} \cdot \mathbf{n}, \quad (\nabla \times \mathbf{A}_w) \cdot \mathbf{n} = \mathbf{k} \cdot \mathbf{n}, \tag{4.12}$$

$$(\nabla \times \mathbf{A}_\lambda) \cdot \mathbf{n} = (\mathbf{i} \times \mathbf{l}) \cdot \mathbf{n}, \quad (\nabla \times \mathbf{A}_\chi) \cdot \mathbf{n} = (\mathbf{j} \times \mathbf{l}) \cdot \mathbf{n}, \quad (\nabla \times \mathbf{A}_\varsigma) \cdot \mathbf{n} = (\mathbf{k} \times \mathbf{l}) \cdot \mathbf{n}, \tag{4.13}$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the x, y and z axes, respectively, and $\mathbf{l}$ is the position vector of points on ∂B in the body-fixed frame. We can thus write

$$\nabla \times \mathbf{A}_B = [(\nabla \times \mathbf{A}_u)^T, (\nabla \times \mathbf{A}_v)^T, \ldots, (\nabla \times \mathbf{A}_\varsigma)^T]\mathbf{U}_B,$$

and it can easily be checked that the boundary condition $(\nabla \times \mathbf{A}_B) \cdot \mathbf{n} = \mathbf{V}_B \cdot \mathbf{n}$ is satisfied on ∂B.

Proposition 4.5 *Restricted to ∂B, the vector potentials $\mathbf{A}_u, \mathbf{A}_v, \mathbf{A}_w, \mathbf{A}_\lambda, \mathbf{A}_\chi, \mathbf{A}_\varsigma$ can be expressed as*

$$\begin{aligned} \mathbf{A}_u &= \frac{1}{2}(\mathbf{i} \times \mathbf{l}) + \nabla_B^{\parallel} f, \\ \mathbf{A}_v &= \frac{1}{2}(\mathbf{j} \times \mathbf{l}) + \nabla_B^{\parallel} g, \\ \mathbf{A}_w &= \frac{1}{2}(\mathbf{k} \times \mathbf{l}) + \nabla_B^{\parallel} h, \\ \mathbf{A}_\lambda &= -\frac{1}{2} l^2 \mathbf{i} + \nabla_B^{\parallel} i, \\ \mathbf{A}_\chi &= -\frac{1}{2} l^2 \mathbf{j} + \nabla_B^{\parallel} j, \\ \mathbf{A}_\varsigma &= -\frac{1}{2} l^2 \mathbf{k} + \nabla_B^{\parallel} k, \end{aligned}$$

for $f, g, \ldots, k : \partial B \to \mathbb{R}$.

Proof This is simply a matter of verifying that the above expressions satisfy the boundary conditions (4.12) and (4.13).

Proposition 4.6 *Assuming that there exists a neighborhood of the body in which there is no vorticity,*[2] *it is the case for any smooth function* $f : \partial B \to \mathbb{R}$ *that*

$$\int_{\partial B} \langle \nabla_B^{\|} f, \boldsymbol{n} \times \mathbf{u}_V \rangle dA = 0.$$

Proof Since there is no vorticity on ∂B,

$$\int_{\partial B} \langle \nabla_B^{\|} f, \mathbf{n} \times \mathbf{u}_V \rangle dA = \int_{\partial B} \langle \mathbf{n}, \mathbf{u}_V \times \nabla_B^{\|} f \rangle dA,$$
$$= \int_{\partial B} \langle \mathbf{n}, \nabla_B^{\|} \varphi \times \nabla_B^{\|} f \rangle dA,$$

where $\varphi : U \to \mathbb{R}$ is defined on the neighborhood of the body which is vorticity-free. Rewriting the integral in terms of differential forms and using Stokes' theorem for a boundaryless manifold, we find that it follows that

$$\int_{\partial B} i^*(\mathbf{d}\varphi \wedge \mathbf{d} f) = \int_{\partial B} \mathbf{d}_B \varphi \wedge \mathbf{d}_B f = \int_{\partial B} \mathbf{d}_B(\varphi \wedge \mathbf{d}_B f) = 0,$$

where the first integrand is the pullback of the two-form $\mathbf{d}\varphi \wedge \mathbf{d} f$ under the inclusion map $i : \partial B \to U$.

5 Hamiltonian structure of the equations

In this section it will be shown that (3.3), (3.4) and (3.5), taken together, possess a Hamiltonian structure relative to the kinetic energy Hamiltonian of the system and an appropriate Poisson bracket.

The kinetic energy. Consider the kinetic energy of the body-rings system:

$$K = \frac{1}{2} \int_D \langle \mathbf{u}, \mathbf{u} \rangle \, dV + \frac{1}{2} \langle (\mathbf{U}, \Omega), M_b(\mathbf{U}, \Omega) \rangle ,$$

where M_b is the mass matrix for the rigid body. Using the Hodge and Kirchhoff decompositions, we can write

$$\begin{aligned} K &= \frac{1}{2} \int_D \langle \mathbf{u}_V + \nabla \Phi_B, \mathbf{u}_V + \nabla \Phi_B \rangle \, dV + \frac{1}{2} \langle (\mathbf{U}, \Omega), M_b(\mathbf{U}, \Omega) \rangle , \\ &= \frac{1}{2} \int_D \langle \mathbf{u}_V, \mathbf{u}_V \rangle + \frac{1}{2} \int_D \langle \nabla \Phi_B, \nabla \Phi_B \rangle \, dV + \frac{1}{2} \langle (\mathbf{U}, \Omega), M_b(\mathbf{U}, \Omega) \rangle \\ &\qquad [L^2 - \text{orthogonality of } \mathbf{u}_\mathrm{V} \text{ and } \Phi_\mathrm{B}], \\ &= \frac{1}{2} \int_D \langle \mathbf{u}_V, \mathbf{u}_V \rangle + \frac{1}{2} \langle (\mathbf{U}, \Omega), M(\mathbf{U}, \Omega) \rangle \ [\text{Kirchhoff decomposition } (3.2)], \end{aligned}$$

where $\mathbf{U}$ is the velocity of the center (of mass) of the body and Ω is the angular velocity of the body in the body-fixed frame. The symmetric effective mass matrix M incorporates both the actual translational and rotational inertias of the body (given by M_b) and the *added inertias* accounting for the resistance of the fluid to body translation and rotation. This identification of the fluid effects manifest in $\nabla \Phi_B$ with amendments to the body's inertia follows [19] and other classical references, and hinges on the form (3.2) of Φ_B.

[2] This assumption holds in our problem since our phase space excludes the intersection points.

Increasing the domain D to infinity does not cause K to diverge, as per the far-field decay rates (2.23), but $\mathbf{u}_V$ has singularities at the ring locations, and—referring to (2.19)—the point-vortex-like term results in the divergence of K.

Phase space of the system. Before defining the Hamiltonian function, we define the system's phase space to be

$$P := P_b \times P_R \equiv \mathfrak{se}(3)^* \times (\mathcal{S}\backslash\Delta), \tag{5.1}$$

where P_b is the reduced phase space of the body in the body-fixed frame and is identified with $\mathfrak{se}(3)^*$, the dual of the Lie algebra corresponding to the Lie group of rigid body translations and rotations in $\mathbb{R}^3$. P_R is the phase space of the rings in the body-fixed frame, identified with the space $\mathcal{S}$ of N (smooth) closed curves in $\mathbb{R}^3\backslash B$ minus Δ, the intersection set of the curves. Note that the phase space excludes all intersections of the rings amongst themselves and with the body.

Kinetic energy regularization and the Hamiltonian function. The kinetic energy K is a real-valued function on P when $(\mathbf{U}, \Omega)$ is expressed in terms of $\mathbf{L}$ and $\mathbf{P}$ according to (2.11). Note that the integral over the fluid, given the body shape and ring strengths, is a function on $\mathcal{S}$ after substituting for $\mathbf{u}_V$ from (3.6), (2.18) and (2.17). To regularize the kinetic energy, we first write

$$\mathbf{u}_V = \sum_{i=1}^{N} \mathbf{u}_{V,i},$$

and, correspondingly,

$$\mathbf{A}_V = \sum_{i=1}^{N} \mathbf{A}_{V,i},$$

where $\mathbf{u}_{V,i}$ is $\mathbf{u}_V$ due to the ith ring alone and $\mathbf{A}_{V,i}$ is its corresponding vector potential.

Expressing the fluid integral in the domain D^d defined earlier:

$$\int_{D^d} \langle \mathbf{u}_V, \mathbf{u}_V\rangle \, dV = \sum_{i=1}^{N} \int_{D^d} \langle \mathbf{u}_{V,i}, \sum_{j\neq i}^{N} \mathbf{u}_{V,j}\rangle \, dV + \sum_{k=1}^{N} \int_{D^d} \langle \mathbf{u}_{V,k}, \mathbf{u}_{V,k}\rangle \, dV.$$

The final sum of integrals on the right-hand side of this equation causes the divergence of K. Using (i) integration by parts, (ii) equations (2.19) and (4.7), (iii) Proposition 4.1 and (iv) the procedure outlined in the proof of Proposition 4.2, one sees that

$$\begin{aligned}\int_{D} \langle \mathbf{u}_V, \mathbf{u}_V\rangle \, dV &= \lim_{d\to 0} \int_{D^d} \langle \mathbf{u}_V, \mathbf{u}_V\rangle \, dV,\\ &= \sum_{i=1}^{N} \oint_{C_j} \langle \mathbf{A}_{V,i}, \sum_{j\neq i}^{N} \mathbf{t}_j\rangle ds_j + \lim_{d\to 0} \sum_{k=1}^{N} \int_{\partial T_k^d} \langle \mathbf{A}_{V,k}, \mathbf{n}\times \mathbf{u}_{V,k}\rangle \, dA.\end{aligned}$$

Splitting the integral in the last sum into four integrals using $\mathbf{A}_{V,k} = \mathbf{A}_{R,k} + \mathbf{A}_{I,k}$ and $\mathbf{u}_{V,k} = \mathbf{u}_{R,k} + \mathbf{u}_{I,k}$, one finds that two of the integrals go to zero, one goes to a non-zero limit and the fourth blows up as $\log d$ as $d \longrightarrow 0$. A finite approximation to this set of terms has to be made to regularize K. Since these singular terms isolate the effect of the self-induced field in the absence of the body, they represent the self-induced kinetic energy. It should be noted that for the case of parallel, rectilinear vortex filaments (or point vortices in a plane if one takes a perpendicular slice), the self-induced kinetic energy term is simply dropped since there is no self-induced velocity.

The Hamiltonian function for the system, $H : P \to \mathbb{R}$, is the regularized K written in terms of the phase space variables:

$$H(\mathbf{L}, \mathcal{A}, \mathfrak{s}) = \frac{1}{2} \sum_{i=1}^{N} \left(\oint_{C_j} \left\langle \mathbf{A}_{V,i}, \sum_{j \neq i}^{N} \Gamma_j \mathbf{t}_j \right\rangle ds_j + \oint_{C_i} \langle \mathbf{A}_{I,i}, \Gamma_i \mathbf{t}_i \rangle ds_i \right)$$
$$+ H_{SI} + \frac{1}{2} \left\langle (\mathbf{L}, \mathcal{A}) - (\mathbf{P}, \Pi), M^{-1}((\mathbf{L}, \mathcal{A}) - (\mathbf{P}, \Pi)) \right\rangle, \tag{5.2}$$

where $\mathfrak{s} \equiv (C_1, \ldots, C_N) \in \mathcal{S}$ and

$$H_{SI} = \mathrm{Reg} \left(\lim_{d \to 0} \sum_{k=1}^{N} \int_{\partial T_k^d} \langle \mathbf{A}_{R,k}, \mathbf{n} \times \mathbf{u}_{R,k} \rangle \, dA \right).$$

Note that $\mathbf{P}$ and Π as defined by equations (2.12) and (2.13) are also functions on $\mathcal{S}$ (given a body shape and vortex strengths).

Functional derivatives. Functional derivatives are defined using the pairing between $T_p P$ and $T_p^* P$. Since $P = \mathfrak{se}(3)^* \times \mathcal{S} \backslash \Delta$ and $\mathfrak{se}(3)^*$ is identified, in the usual way, with $\mathbb{R}^{6*} (\equiv \mathbb{R}^{3*} \times \mathbb{R}^{3*})$, which in turn is identified with $\mathbb{R}^6$ using the standard Euclidean pairing, it follows that

$$T_p P \equiv \mathbb{R}^6 \times \mathfrak{X}_{\text{rings}},$$
$$T_p^* P \equiv \mathbb{R}^6 \times \mathfrak{X}_{\text{rings}}^*,$$

where $\mathfrak{X}_{\text{rings}}$ is the vector space of smooth vector fields on N curves in $\mathbb{R}^3$. Identify elements of $\mathfrak{X}_{\text{rings}}^*$ with one-forms on the N curves in $\mathbb{R}^3$. Pair $\mathbb{R}^{3*}$ and $\mathbb{R}^3$ using the standard Euclidean pairing. Pair $\mathfrak{X}_{\text{rings}}$ and $\mathfrak{X}_{\text{rings}}^*$ as follows:

$$\langle X, \alpha \rangle_{\mathcal{S}} = \sum_{i=1}^{N} \oint \gamma_\alpha \left(u_X(s_i) \right) \, ds_i = \sum_{i=1}^{N} \oint v_\alpha^\flat \left(u_X(s_i) \right) \, ds_i ,$$

where γ_α is the one-form in $\mathbb{R}^3$ (identifiable with the vector field v_α using the standard inner product on $\mathbb{R}^3$) corresponding to $\alpha \in \mathfrak{X}_{\text{rings}}^*$ and u_X is the vector field in $\mathbb{R}^3$ corresponding to $X \in \mathfrak{X}_{\text{rings}}$. This allows us to define the pairing between $T_p P$ and $T_p^* P$ as:

$$\langle (u, X), (v, \alpha) \rangle = \langle u, v \rangle_{\mathbb{R}^6} + \langle X, \alpha \rangle_{\mathcal{S}} , \tag{5.3}$$

where the first pairing on the right is the standard Euclidean inner product on $\mathbb{R}^6$.

Now for $p \equiv (\mathbf{L}, \mathcal{A}, \mathfrak{s})$, consider variations $\delta p \equiv (\delta \mathbf{L}, \delta \mathcal{A}, \delta \mathfrak{s}) \equiv (\delta \mathbf{L}, \delta \mathcal{A}, \delta C_1, \ldots, \delta C_N)$. The functional derivatives $\delta F / \delta C_i$ are defined using the fluid pairing:

$$\left\langle \delta \mathfrak{s}, \frac{\delta F}{\delta \mathfrak{s}} \right\rangle = \sum_{i=1}^{N} \oint_i \left\langle \delta C_i, \frac{\delta F}{\delta C_i} \right\rangle ds_i := \textstyle\sum_{i=1}^{N} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(F \left(\mathbf{L}, \mathcal{A}, C_i + \epsilon \delta C_i \right) - F \left(\mathbf{L}, \mathcal{A}, C_i \right) \right).$$

The functional derivative $\partial F / \partial \mu$ for $\mu \in \mathfrak{se}(3)^*$ is defined similarly using the standard Euclidean pairings. For $\mu \equiv (\mathbf{L}, \mathcal{A}) \in \mathfrak{se}(3)^*$ and $\delta \mu \equiv (\delta \mathbf{L}, \delta \mathcal{A})$,

$$\left\langle \frac{\partial F}{\partial \mu}, \delta \mu \right\rangle := \left\langle \frac{\partial F}{\partial \mathbf{L}}, \delta \mathbf{L} \right\rangle_{\mathbb{R}^3} + \left\langle \frac{\partial F}{\partial \mathcal{A}}, \delta \mathcal{A} \right\rangle_{\mathbb{R}^3} .$$

Poisson brackets. Consider the following Poisson bracket on $P \equiv P_b \times P_v = \mathbb{R}^{6*} \times \mathcal{S} \backslash \Delta$. For $F, G : P \to \mathbb{R}$,

$$\{F, G\}_P = \left\{F|_{P_b}, G|_{P_b}\right\} + \left\{F|_{P_v}, G|_{P_v}\right\}, \tag{5.4}$$

where the first bracket is a Lie–Poisson bracket. Recall that the Lie–Poisson bracket on $\mathfrak{se}(3)^* \equiv \mathbb{R}^{3*} \times \mathbb{R}^{3*}$ is given as follows (see, for instance, [27]):

$$\begin{aligned}\left\{\tilde{F}, \tilde{G}\right\}_{\pm}(\mu) &= \pm\left\langle \mu, \left[\frac{\partial \tilde{F}}{\delta \mu}, \frac{\partial \tilde{G}}{\partial \mu}\right]\right\rangle, \\ &= \mp\left\langle \frac{\partial \tilde{F}}{\partial \mu}, \mathrm{ad}^*_{\partial \tilde{G}/\partial \mu}\, \mu\right\rangle,\end{aligned} \tag{5.5}$$

for $\tilde{F}, \tilde{G} : \mathfrak{se}(3)^* \to \mathbb{R}$ and $\mu \in \mathfrak{se}(3)^*$.

The second bracket is the canonical bracket associated with the symplectic form on the phase space of rings/filaments in $\mathbb{R}^3$ derived by [28]. It was shown in that paper that the phase space of rings/filaments is a coadjoint orbit of the dual of the Lie algebra of the group of volume-preserving diffeomorphisms of $\mathbb{R}^3$. Thus, there is a natural symplectic structure on this phase space, namely the Kirillov–Kostant–Soureau symplectic structure on coadjoint orbits of the dual of the Lie algebra. For N rings in $\mathbb{R}^3$, the vorticity two-form in $\mathbb{R}^3$ is

$$\omega(x, y, z) = \sum_{i=1}^{N} \Gamma_i \mathbf{i}_{\mathbf{t}_i} dx \wedge dy \wedge dz\; \delta_i(x, y, z), \tag{5.6}$$

where δ_i is the delta function defined relative to the ith ring. In other words, the vorticity two-form is the sum of the contractions of the standard volume form in $\mathbb{R}^3$ with the tangent vector fields on the rings. The symplectic form is then given by

$$\Omega(\mathcal{L}_u\omega, \mathcal{L}_v\omega) = \textstyle\sum_{i=1}^{N} \Gamma_i \oint_i \mathbf{t}_i \cdot (u(s_i) \times v(s_i)) ds_i = \sum_{i=1}^{N} \Gamma_i \oint_i v(s_i) \cdot (\mathbf{t}_i \times u(s_i)) ds_i, \tag{5.7}$$

where $\mathcal{L}_u\omega, \mathcal{L}_v\omega$ are identified with tangent vectors to the coadjoint orbit and u, v are vector fields on the rings. Note that only the vector component normal to the curves contributes to the symplectic form (see [35] for more on this). The above symplectic form can obviously also be viewed as a symplectic form on $\mathcal{S}$, the space of closed curves defined earlier, by identifying tangent vectors to $\mathcal{S}$ with the vector fields u, v etc. on the rings. Thus, for functions $\hat{F}, \hat{G} : \mathcal{S} \to \mathbb{R}$, the bracket associated with this symplectic form is ([27]):

$$\left\{\hat{F}, \hat{G}\right\}(\mathfrak{s}) = \Omega(X_{\hat{F}}(\mathfrak{s}), X_{\hat{G}}(\mathfrak{s})), \;\; \mathfrak{s} \in \mathcal{S}, \tag{5.8}$$

where $X_{\hat{F}}, X_{\hat{G}} : \mathcal{S} \to T\mathcal{S}$ are the Hamiltonian vector fields corresponding to $\hat{F}, \hat{G}$.

To obtain this bracket, note that from the coordinate-free version of Hamilton's equations on a symplectic manifold ([27]), $\mathbf{i}_{X_{\hat{F}}}\Omega = D\hat{F}$, the definition of the functional derivative and (5.7), it follows (as was implicit in [35]) that:

$$\mathbf{t}_i \times u_{\hat{F}} = \left(\tfrac{\delta \hat{F}}{\delta C_i}\right)^{\sharp}, \tag{5.9}$$

which holds at each point of the ith ring. In the above, $\left(\tfrac{\delta \hat{F}}{\delta C_i}\right)^{\sharp}$ is the vector field on the ring associated with the one-form $\left(\tfrac{\delta \hat{F}}{\delta C_i}\right)$ on the ring using the standard metric on $\mathbb{R}^3$, and $u_{\hat{F}}$ is the vector field on the ring associated with the element at $\mathfrak{s}$ of the Hamiltonian vector field $X_{\hat{F}}$. Using this relation and the fact that $\mathbf{n}_i$, $\mathbf{t}_i$ and $\mathbf{b}_i$ form an orthogonal system at each point of the curve, we have

Proposition 5.1 *The Poisson bracket* (5.8) *is given by*

$$\begin{aligned}\left\{\hat{F}, \hat{G}\right\}(\mathfrak{s}) &= \sum_{i=1}^{N} \Gamma_i \oint_i \left(\frac{\delta \hat{F}}{\delta C_i}(\mathbf{n}_i)\frac{\delta \hat{G}}{\delta C_i}(\mathbf{b}_i) - \frac{\delta \hat{F}}{\delta C_i}(\mathbf{b}_i)\frac{\delta \hat{G}}{\delta C_i}(\mathbf{n}_i)\right) ds_i, \\ &\equiv \sum_{i=1}^{N} \Gamma_i \oint_i \left\langle \left(\frac{\delta \hat{F}}{\delta C_i}\right)^{\sharp} \times \left(\frac{\delta \hat{G}}{\delta C_i}\right)^{\sharp}, \mathbf{t}_i \right\rangle ds_i.\end{aligned} \tag{5.10}$$

Proof Use (5.9), $\mathbf{n}_i \times \mathbf{b}_i = \mathbf{t}_i$, etc. and the vector product identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

to show that (5.10) leads to (5.7).

Hamiltonian vector field. We now obtain the Hamiltonian vector field X_H corresponding to the Hamiltonian H given by equation (5.2) relative to the Poisson bracket (5.4) and show that it provides the same equations of motion as those from the momentum balance analysis, i.e., equations (3.3), (3.4) and (3.1).

Hamilton's equations of motion are determined by requiring the Poisson bracket form of the equations:

$$\dot{F} = \{F, H\}_P$$

(see, for instance, [27]). From this we obtain the Hamiltonian vector field X_H as

$$\left\langle X_H, \frac{\delta F}{\delta p} \right\rangle = \{F, H\}_P$$

using the pairing (5.3) and the Poisson bracket (5.4). Define variation in the Hamiltonian (5.2) in the C_i-direction as

$$\delta_{C_i} H := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(H\left(\mathbf{L}, \mathcal{A}, C_i + \epsilon\delta C_i\right) - H\left(\mathbf{L}, \mathcal{A}, C_i\right) \right),$$

where $\epsilon\delta C_i = \epsilon\delta \mathbf{l}_i$. Then

$$\begin{aligned}
\delta_{C_i} H &= \frac{1}{2}\delta_{C_i} \sum_{i=1}^{N} \left(\oint_{C_j} \langle \mathbf{A}_{V,i}, \sum_{j \neq i}^{N} \Gamma_j \mathbf{t}_j \rangle ds_j + \oint_{C_i} \langle \mathbf{A}_{I,i}, \Gamma_i \mathbf{t}_i \rangle ds_i \right) \\
&\quad + \delta_{C_i} H_{SI} - \left\langle \delta_{C_i}(\mathbf{P}, \Pi), M^{-1}((\mathbf{L}, \mathcal{A}) - (\mathbf{P}, \Pi)) \right\rangle \\
&\quad \text{[using the fact that M, and hence M}^{-1}\text{, is symmetric],} \\
&= \frac{1}{2}\delta_{C_i} \oint_{C_j} \langle \mathbf{A}_{V,i}, \sum_{j \neq i}^{N} \Gamma_j \mathbf{t}_j \rangle ds_j + \frac{1}{2}\delta_{C_i} \sum_{j \neq i}^{N} \oint_{C_i} \langle \mathbf{A}_{V,j}, \Gamma_i \mathbf{t}_i \rangle ds_i \\
&\quad + \frac{1}{2}\delta_{C_i} \oint_{C_i} \langle \mathbf{A}_{I,i}, \Gamma_i \mathbf{t}_i \rangle ds_i + \delta_{C_i} H_{SI} - \left\langle \delta_{C_i}(\mathbf{P}, \Pi), (\mathbf{U}, \Omega) \right\rangle, \\
&= \delta_{C_i} \sum_{j \neq i}^{N} \oint_{C_i} \langle \mathbf{A}_{V,j}, \Gamma_i \mathbf{t}_i \rangle ds_i + \frac{1}{2}\delta_{C_i} \oint_{C_i} \langle \mathbf{A}_{I,i}, \Gamma_i \mathbf{t}_i \rangle ds_i + \delta_{C_i} H_{SI} - \left\langle \delta_{C_i}(\mathbf{P}, \Pi), (\mathbf{U}, \Omega) \right\rangle \\
&\quad \text{[using reciprocity relation (4.8)].}
\end{aligned}$$

Now recall the definitions of $\mathbf{P}$ and Π from (2.12) and (2.13).

Proposition 5.2 *The body integrals in* $\mathbf{P}$ *and* Π *can be written as*

$$\begin{aligned}
\frac{1}{2}\int_{\partial B} \boldsymbol{l} \times (\boldsymbol{n} \times \mathbf{u}_V) dA &= \mathbf{i} \int_{\partial B} \langle \mathbf{A}_u, \boldsymbol{n} \times \mathbf{u}_V \rangle dA + \mathbf{j} \int_{\partial B} \langle \mathbf{A}_v, \boldsymbol{n} \times \mathbf{u}_V \rangle dA \\
&\quad + \mathbf{k} \int_{\partial B} \langle \mathbf{A}_w, \boldsymbol{n} \times \mathbf{u}_V \rangle dA, \qquad (5.11) \\
-\frac{1}{2}\int_{\partial B} l^2 (\boldsymbol{n} \times \mathbf{u}_V) dA &= \mathbf{i} \int_{\partial B} \langle \mathbf{A}_\lambda, \boldsymbol{n} \times \mathbf{u}_V \rangle dA + \mathbf{j} \int_{\partial B} \langle \mathbf{A}_\chi, \boldsymbol{n} \times \mathbf{u}_V \rangle dA \\
&\quad + \mathbf{k} \int_{\partial B} \langle \mathbf{A}_\varsigma, \boldsymbol{n} \times \mathbf{u}_V \rangle dA. \qquad (5.12)
\end{aligned}$$

Proof This follows from Propositions 4.5 and 4.6.

Using Propositions 5.2 and 4.2 (applied to each of $\mathbf{A}_u$, $\mathbf{A}_v$, etc.), one can write the C_i-variation in H as:

$$\delta_{C_i} H - \delta_{C_i} H_{SI} = \delta_{C_i} \sum_{j \neq i}^{N} \oint_{C_i} \langle \mathbf{A}_{V,j}, \Gamma_i \mathbf{t}_i \rangle ds_i + \frac{1}{2} \delta_{C_i} \oint_{C_i} \langle \mathbf{A}_{I,i}, \Gamma_i \mathbf{t}_i \rangle ds_i$$
$$+ \left\langle \left(\delta_{C_i} \int_{C_i} \langle \mathbf{A}_u, \Gamma_i \mathbf{t}_i \rangle ds_i, \delta_{C_i} \int_{C_i} \langle \mathbf{A}_v, \Gamma_i \mathbf{t}_i \rangle ds_i, \cdots, \delta_{C_i} \int_{C_i} \left\langle \mathbf{A}_\varsigma, \Gamma_i \mathbf{t}_i \right\rangle ds_i \right), (\mathbf{U}, \Omega) \right\rangle$$
$$- \left\langle \left(\delta_{C_i} \left(\frac{\Gamma_i}{2} \oint_{C_i} (\mathbf{l}_i(s_i) \times \mathbf{t}_i(s_i)) ds_i \right), \delta_{C_i} \left(-\frac{\Gamma_i}{2} \oint_{C_i} l_i^2(s_i) \mathbf{t}_i(s_i) ds_i \right) \right), (\mathbf{U}, \Omega) \right\rangle.$$

We are now ready to state our main theorem.

Theorem 5.3 *Equations* (3.3)*,* (3.4) *and* (3.1) *form a Hamiltonian system for the Hamiltonian function H, given by (5.2), on the Poisson manifold P, defined by (5.1), equipped with the Poisson bracket (5.4).*

Proof We first evaluate the variations in each of the terms of $\delta_{C_i} H$. The first sum of terms gives

$$\delta_{C_i} \sum_{j \neq i}^{N} \oint_{C_i} \langle \mathbf{A}_{V,j}, \Gamma_i \mathbf{t}_i \rangle ds_i = \Gamma_i \sum_{j \neq i}^{N} \left(\oint_{C_i} \langle \mathbf{A}_{V,j}, \mathbf{D_l t}_i \cdot \delta \mathbf{l}_i \rangle ds_i + \oint_{C_i} \langle \mathbf{D_l A}_{V,j} \cdot \delta \mathbf{l}_i, \mathbf{t}_i \rangle ds_i \right)$$
$$= \Gamma_i \sum_{j \neq i}^{N} \left(\oint_{C_i} \langle \left(-\mathbf{D_l A}_{V,j} \right) \cdot \mathbf{t}_i, \delta \mathbf{l}_i \rangle ds_i + \oint_{C_i} \langle (\mathbf{D_l A}_{V,j})^T \cdot \mathbf{t}_i, \delta \mathbf{l}_i \rangle ds_i \right)$$

where we have made the identifications

$$\mathbf{D_l t}_i \cdot \delta \mathbf{l}_i \equiv \delta_{C_i} \mathbf{t}_i \equiv \left(\delta \frac{dx_i}{ds_i}, \delta \frac{dy_i}{ds_i}, \delta \frac{dz_i}{ds_i} \right) \equiv \left(\frac{d\delta x_i}{ds_i}, \frac{d\delta y_i}{ds_i}, \frac{d\delta z_i}{ds_i} \right)$$

and have used integration by parts. Thus, we get

$$\delta_{C_i} \sum_{j \neq i}^{N} \oint_{C_i} \langle \mathbf{A}_{V,j}, \Gamma_i \mathbf{t}_i \rangle ds_i = \Gamma_i \sum_{j \neq i}^{N} \left(\oint_{C_i} \langle \left(\left(-\mathbf{D_l A}_{V,j} \right) + (\mathbf{D_l A}_{V,j})^T \right) \cdot \mathbf{t}_i, \delta \mathbf{l}_i \rangle ds_i \right),$$
$$= \Gamma_i \sum_{j \neq i}^{N} \oint_{C_i} \langle - \left(\nabla \times \mathbf{A}_{V,j} \right) \times \mathbf{t}_i, \delta \mathbf{l}_i \rangle ds_i.$$

Proceeding in the same way yields

$$\frac{1}{2} \delta_{C_i} \oint_{C_i} \langle \mathbf{A}_{I,i}, \Gamma_i \mathbf{t}_i \rangle ds_i = \frac{1}{2} \cdot 2 \cdot \oint_{C_i} \langle - \left(\nabla \times \mathbf{A}_{I,j} \right) \times \mathbf{t}_i, \delta \mathbf{l}_i \rangle ds_i.$$

To explain the factor of two on the right it is useful to think of the variation in the above term to consists of two parts: one due to the variation in the curve position in the field of the old $\mathbf{A}_{I,i}$ and the other due to the variation in $\mathbf{A}_{I,i}$ at the old curve position. And by (4.9), these variations are equal.

In a similar manner, it can be shown that

$$\delta_{C_i} \int_{C_i} \langle \mathbf{A}_u, \Gamma_i \mathbf{t}_i \rangle ds_i = \Gamma_i \oint_{C_i} \langle \Big((-\mathbf{D}_\mathbf{l} \mathbf{A}_u) + (\mathbf{D}_\mathbf{l} \mathbf{A}_u)^T \Big) \cdot \mathbf{t}_i, \delta \mathbf{l}_i \rangle ds_i,$$
$$= \Gamma_i \oint_{C_i} \langle - (\nabla \times \mathbf{A}_u) \times \mathbf{t}_i, \delta \mathbf{l}_i \rangle ds_i$$

and so on. Next, we find that

$$\left\langle \delta_{C_i} \left(\frac{\Gamma_i}{2} \oint_{C_i} (\mathbf{l}_i(s_i) \times \mathbf{t}_i(s_i)) ds_i \right), \mathbf{U} \right\rangle = \frac{\Gamma_i}{2} \left\langle \oint_{C_i} (\mathbf{D}_\mathbf{l} \mathbf{l}_i \cdot \delta \mathbf{l}_i \times \mathbf{t}_i + \mathbf{l}_i \times \mathbf{D}_\mathbf{l} \mathbf{t}_i \cdot \delta \mathbf{l}_i)\, ds_i, \mathbf{U} \right\rangle,$$
$$= \frac{\Gamma_i}{2} \oint_{C_i} (\langle \mathbf{D}_\mathbf{l} \mathbf{l}_i \cdot \delta \mathbf{l}_i, \mathbf{t}_i \times \mathbf{U} \rangle + \langle \mathbf{D}_\mathbf{l} \mathbf{t}_i \cdot \delta \mathbf{l}_i, \mathbf{U} \times \mathbf{l}_i \rangle)\, ds_i,$$
$$= \frac{\Gamma_i}{2} \oint_{C_i} (\langle \delta \mathbf{l}_i, \mathbf{t}_i \times \mathbf{U}_B \rangle + \langle \delta \mathbf{l}_i, -\mathbf{D}_\mathbf{l}\, (\mathbf{U} \times \mathbf{l}_i) \cdot \mathbf{t}_i \rangle)\, ds_i,$$
$$= \Gamma_i \oint_{C_i} \langle \delta \mathbf{l}_i, \mathbf{t}_i \times \mathbf{U} \rangle ds_i$$

and, in like manner,

$$\left\langle \delta_{C_i} \left(\frac{\Gamma_i}{2} \oint_{C_i} (l_i^2(s_i) \mathbf{t}_i(s_i)) ds_i \right), \Omega \right\rangle = \frac{\Gamma_i}{2} \oint_{C_i} \big\langle \mathbf{D}_\mathbf{l} (l_i^2 \mathbf{t}_i) \cdot \delta \mathbf{l}_i, \Omega \big\rangle\, ds_i,$$
$$= \Gamma_i \oint_{C_i} \left\langle \frac{l_i^2}{2} \mathbf{D}_\mathbf{l} \mathbf{t}_i \cdot \delta \mathbf{l}_i + \langle \mathbf{l}_i, \delta \mathbf{l}_i \rangle \mathbf{t}_i, \Omega \right\rangle ds_i,$$
$$= \Gamma_i \oint_{C_i} \left(-\langle \delta \mathbf{l}_i, \mathbf{D}_\mathbf{l} (\frac{l_i^2}{2} \Omega) \cdot \mathbf{t}_i \rangle + \langle \langle \mathbf{l}_i, \delta \mathbf{l}_i \rangle \mathbf{t}_i, \Omega \rangle \right) ds_i,$$
$$= \Gamma_i \oint_{C_i} (-\langle \delta \mathbf{l}_i, \Omega \rangle \langle \mathbf{l}_i, \mathbf{t}_i \rangle + \langle \mathbf{l}_i, \delta \mathbf{l}_i \rangle \langle \mathbf{t}_i, \Omega \rangle)\, ds_i,$$
$$= \Gamma_i \oint_{C_i} \langle \delta \mathbf{l}_i, \mathbf{t}_i \times (\mathbf{l}_i \times \Omega) \rangle ds_i.$$

Thus the functional derivative of H with respect to C_i satisfies

$$\left(\frac{\delta H}{\delta C_i} \right)^\sharp = \left(- \sum_{j \neq i}^{N} \big(\nabla \times \mathbf{A}_{V,j} \big) \times \mathbf{t}_i - \big(\nabla \times \mathbf{A}_{I,i} \big) \times \mathbf{t}_i + \left(\frac{\delta H_{SI}}{\delta C_i} \right)^\sharp - (\nabla \times \mathbf{A}_B) \times \mathbf{t}_i + (\mathbf{U} + \Omega \times \mathbf{l}_i) \times \mathbf{t}_i \right)_{|C_i}.$$

Using (5.10), one sees that

$$\frac{\partial C_i}{\partial t} = \left(\frac{\delta H}{\delta C_i} \right)^\sharp \times \mathbf{t}_i\ |_{C_i},$$
$$= \left(\sum_{j \neq i}^{N} \big(\nabla \times \mathbf{A}_{V,j} \big) + \nabla \times \mathbf{A}_{I,i} + \nabla \times \mathbf{A}_B - (\mathbf{U} + \Omega \times \mathbf{l}_i) \right)^n_{|C_i} + \left(\frac{\delta H_{SI}}{\delta C_i} \right)^\sharp \times \mathbf{t}_i\ |_{C_i},$$

which is the same as (3.1) assuming that the following consistency condition relative to the Hamiltonian structure is satisfied by the self-induced field:

$$\mathbf{u}^n_{SI} = \left(\frac{\delta H_{SI}}{\delta C_i}\right)^\sharp \times \mathbf{t}_i \tag{5.13}$$

everywhere on the ith ring. In other words, the Hamiltonian vector field corresponding to the regularized self-induced kinetic energy term should be equal to $\mathbf{u}_{SI}$.

The above equation is indeed satisfied if, for example, the local induction approximation is used, as shown in [35]. Alternatively, one could regularize only one of either the self-induced kinetic energy or the velocity and prescribe the regularization of the other in a manner such that (5.13) is satisfied.

To prove the Lie–Poisson part, it can be shown by matrix manipulation that the partial derivatives of (5.2) with respect to $\mathbf{L}$ and $\mathcal{A}$ are

$$\frac{\partial H}{\partial \mathbf{L}} = \mathbf{U},$$
$$\frac{\partial H}{\partial \mathcal{A}} = \Omega.$$

The ad* operator on $\mathfrak{se}^*(3)$ is given by ([27])

$$\mathrm{ad}^*_{(m_1,m_2)}(n_1, n_2) = (n_1 \times m_1 + n_2 \times m_2, n_2 \times m_1).$$

It then follows that

$$\mathrm{ad}^*_{\left(\frac{\partial H}{\partial \mathcal{A}}, \frac{\partial H}{\partial \mathbf{L}}\right)}(\mathcal{A}, \mathbf{L}) = (\mathbf{L} \times \mathbf{U} + \mathcal{A} \times \Omega, \mathbf{L} \times \Omega).$$

Using the negative Lie–Poisson bracket (5.5), one obtains (3.3) and (3.4).

6 Conclusions and future directions

We have demonstrated that the system comprising a free rigid body of arbitrary smooth shape interacting dynamically with N closed vortex filaments of arbitrary shape in an infinite ideal fluid possesses a Hamiltonian structure, provided that the regularization of the system's divergent kinetic energy is undertaken in a particular way. This paper may be viewed as an extension of earlier work by the authors in [36,35] and [33] and is written in a similar spirit. We note that only a small fraction of the traditional engineering literature addressing fluid-structure interactions deals with problems—like ours—in which boundary motions are determined dynamically rather than prespecified, despite the importance of such problems in areas like aeroelasticity.

The results of the present paper suggest several avenues for future study. We briefly survey two categories of these below.

Extensions. In this paper, vortex rings are modeled as flow singularities along closed curves; it would be natural to reformulate our problem with vortex rings endowed with finite cores. Even in the absence of a rigid body—or any boundary on the fluid domain—the system comprising N arbitrarily shaped rings with core structure has not been examined thoroughly in a Hamiltonian context. The ideas put forth in [21] regarding the Hamiltonian structure underpinning the vorticity dynamics in a freely evolving bubble may be useful in considering such an extension.

Expanding our treatment to accommodate deformability of the body would be of basic mathematical interest but could also lead to the improved practical modeling of problems in aquatic locomotion. A variety of marine animals are understood to propel themselves through the shedding of coherent vortex structures; dynamic models and control strategies for such mechanisms promise to impact the development of agile and efficient biomimetic robotic vehicles. There has been quite a bit of theoretical research in the past decade related to the existence and uniqueness of solutions and functional analytic aspects of such coupled systems (for cases including deformable and rigid bodies in inviscid and viscous frameworks, see, for example, [12] and [11] and references therein). However, the dynamics and control of such coupled systems, especially with a focus on vortical structures, remains a fairly open area for theoretical research. In an inviscid framework involving rigid bodies, the dynamics of coupled systems in the plane for some special configurations has been studied by [34]. Some control models for coupled systems, in a similar framework, have been investigated by

[23] and [15]. In the former, the control input is a force on a rigid cylinder in the presence of a nearby point vortex and in the latter the control input is the instantaneous shape of an articulated three–link mechanism in a potential flow. Motion planning and control of a deformable body by shape changes in the presence of vortical structures is a promising direction for future research. We note, however, that the successful modeling of real flow phenomena—particularly at high Reynolds number—may require the proper treatment of viscous effects.

Another extension, that has in fact been pursued by three of us for a few years now, is to place and understand the Hamiltonian structure of models like the one in this paper in a more general geometric mechanics framework of symmetry and reduction [24–26]. In particular, we want to understand clearly the roles that the fluid particle relabeling symmetry and the rigid body rotational and translational symmetries play in obtaining the Poisson brackets of this model, which are on a symmetry reduced space, from a more general canonical 'fluids + rigid body' Poisson bracket structure on an unreduced space. Work with similar objectives in mind, but on the Lagrangian side, was initiated in the theses of [16] and [31]. For a history of related work for fluids without moving rigid boundaries, on both the Lagrangian and Hamiltonian sides, see [27].

Specific cases. The realization of explicit equations of motion for the system described in the present paper is particularly direct when the body is taken to be spherical, the rings taken to be circular initially, and the body and rings are aligned along a common axis of symmetry such that the circular shape of the rings is preserved. The equations governing the evolution of the rings, and thus the equations of motion for the system overall, reduce to a system of ordinary differential equations which can be integrated easily using standard ODE solvers. We are currently examining this system numerically in parallel with laboratory experiments being performed at NMSU, described in [1], anticipating the comparison of our inviscid model's dynamics with those of a real viscous flow. Emerging numerical techniques like immersed boundary methods and immersed interface methods for moving boundary problems will, furthermore, allow us to study a model for this system based on the Navier–Stokes equations, providing another basis for comparison and model validation.

Appendix A: Derivation of the Lie–Poisson equations

Following the steps outlined in Sect. 2.1, beginning with (2.3) and (2.5), we now derive the Lie–Poisson equations for the system. Using the Hodge decomposition (2.6), we write (2.3) and (2.5) as:

$$m_B\frac{d\overline{\mathbf{U}}}{dt}+\frac{d}{dt}\left(\frac{1}{2}\left\{\int_{\widetilde{D}}(\mathbf{r}\times\bar{\omega})d\overline{V}+\int_{\partial B}\mathbf{r}\times(\bar{\mathbf{n}}\times\overline{\nabla}\Phi_B)d\bar{A}+\int_{\partial B}\mathbf{r}\times(\bar{\mathbf{n}}\times\bar{\mathbf{u}}_V)d\bar{A}\right)\right\}=\bar{\mathbf{F}}_S$$

$$\frac{d}{dt}(m_B\bar{\mathbf{b}}\times\overline{\mathbf{U}}+\overline{\mathbf{I}\Omega})-\frac{1}{2}\frac{d}{dt}\left(\int_{\widetilde{D}}(r^2\bar{\omega})d\overline{V}+\int_{\partial B}r^2(\bar{\mathbf{n}}\times\overline{\nabla}\Phi_B)d\bar{A}+\int_{\partial B}r^2(\bar{\mathbf{n}}\times\bar{\mathbf{u}}_V)d\bar{A}\right)=\bar{\mathbf{M}}_S,$$

where

$$\bar{\mathbf{F}}_S=\int_S\bar{\mathbf{u}}(\bar{\mathbf{u}}\cdot\bar{\mathbf{n}})d\bar{A}-\frac{d}{dt}\left(\frac{1}{2}\int_S\mathbf{r}\times(\bar{\mathbf{n}}\times\bar{\mathbf{u}})d\bar{A}\right)+\int_S p_s\bar{\mathbf{n}}d\bar{A},\tag{A.1}$$

$$\bar{\mathbf{M}}_S=\int_S\mathbf{r}\times\bar{\mathbf{u}}(\bar{\mathbf{u}}\cdot\bar{\mathbf{n}})d\bar{A}+\frac{1}{2}\frac{d}{dt}\int_S r^2(\bar{\mathbf{n}}\times\bar{\mathbf{u}})d\bar{A}+\int_S p_s(\mathbf{r}\times\bar{\mathbf{n}})d\bar{A}.\tag{A.2}$$

Substituting the vorticity two-form (5.6) into the equations for the linear and angular momenta, one obtains:

$$m_B\frac{d\overline{\mathbf{U}}}{dt}+\frac{1}{2}\left\{\frac{d}{dt}\oint_{\partial B}\mathbf{r}\times(\bar{\mathbf{n}}\times\overline{\nabla}\Phi_B)d\bar{A}+\frac{d}{dt}\sum\Gamma_i\oint_{\overline{C}_i}(\mathbf{r}_i\times\bar{\mathbf{t}}_i)ds_i+\frac{d}{dt}\oint_{\partial B}\mathbf{r}\times(\bar{\mathbf{n}}\times\bar{\mathbf{u}}_V)d\bar{A}\right\}=\bar{\mathbf{F}}_S,$$

$$\frac{d}{dt}(m_B\bar{\mathbf{b}}\times\overline{\mathbf{U}}+\overline{\mathbf{I}\Omega})-\frac{1}{2}\frac{d}{dt}\oint_{\partial B}r^2(\bar{\mathbf{n}}\times\overline{\nabla}\Phi_B)d\bar{A}-\frac{1}{2}\frac{d}{dt}\sum\Gamma_i\oint_{\overline{C}_i}r_i^2\bar{\mathbf{t}}_i ds_i-\frac{1}{2}\frac{d}{dt}\oint_{\partial B}r^2(\bar{\mathbf{n}}\times\bar{\mathbf{u}}_V)dA=\bar{\mathbf{M}}_S,$$

where $\overline{C}_i$ denotes the arc-length parameterized ith curve in the spatially-fixed frame.

The terms $\bar{\mathbf{F}}_S$ and $\bar{\mathbf{M}}_S$ as the boundary S goes to infinity. The terms $\bar{\mathbf{F}}_S$ and $\bar{\mathbf{M}}_S$ are now evaluated in the limit as S goes to infinity. Since S, by definition, always traverses a region of irrotational flow, there exists a single-valued potential function Φ such that $\bar{\mathbf{u}} = \overline{\nabla}\Phi$ on S. Moreover, the unsteady Bernoulli's equation

$$\frac{\partial \Phi}{\partial t} + \left(\frac{|\nabla \Phi|^2}{2}\right) + p_s = f(t),$$

with the density of the fluid taken to be unity, is applicable on S.

The first term on the right in each of (A.1) and (A.2) goes to zero as $S \to \infty$ using the far-field decay rates (2.23). To evaluate the remaining terms, we use the above form of Bernoulli's equation together with the following vector identities, all derivable from Stokes' theorem using the relations (2.1) and (2.4) and harmonic extension arguments:

$$\frac{1}{2}\int_S \mathbf{r} \times (\bar{\mathbf{n}} \times \overline{\nabla}\Phi) d\bar{A} = -\int_S \Phi \bar{\mathbf{n}} d\bar{A}, \tag{A.3}$$

$$\frac{1}{2}\int_S r^2 (\bar{\mathbf{n}} \times \overline{\nabla}\Phi) d\bar{A} = -\int_S \Phi (\bar{\mathbf{n}} \times \mathbf{r}) d\bar{A}. \tag{A.4}$$

Note, in particular, the special cases of these identities obtained by setting $\Phi = 1$. Considering $\bar{\mathbf{F}}_S$ first,

$$\begin{aligned}
-\frac{d}{dt}\left(\frac{1}{2}\int_S \mathbf{r} \times (\bar{\mathbf{n}} \times \bar{\mathbf{u}}) d\bar{A}\right) + \int_S p_s \bar{\mathbf{n}} d\bar{A}, &= -\frac{d}{dt}\left(\frac{1}{2}\int_S \mathbf{r} \times (\bar{\mathbf{n}} \times \overline{\nabla}\Phi) d\bar{A}\right) + \int_S \left(f(t) - \left(\frac{\partial \Phi}{\partial t} + \left(\frac{|\overline{\nabla}\Phi|^2}{2}\right)\right)\right) \bar{\mathbf{n}} d\bar{A} \\
&= \frac{d}{dt}\left(\int_S \Phi \bar{\mathbf{n}} d\bar{A}\right) + \int_S \left(f(t) - \left(\frac{\partial \Phi}{\partial t} + \left(\frac{|\overline{\nabla}\Phi|^2}{2}\right)\right)\right) \bar{\mathbf{n}} d\bar{A} \\
&= \int_S -\left(\frac{|\overline{\nabla}\Phi|^2}{2}\right) \bar{\mathbf{n}} d\bar{A} \quad (\text{for } S \text{ fixed in time}), \\
&= O(1/|\mathbf{r}|^5).
\end{aligned}$$

The remaining two terms in $\mathbf{M}_S$ can be similarly shown to go to zero as $S \to \infty$:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\int_S r^2 (\bar{\mathbf{n}} \times \bar{\mathbf{u}}) d\bar{A}\right) + \int_S p_s (\mathbf{r} \times \bar{\mathbf{n}}) d\bar{A}, &= \frac{d}{dt}\left(\frac{1}{2}\int_S r^2 (\bar{\mathbf{n}} \times \overline{\nabla}\Phi) d\bar{A}\right) + \int_S \left(f(t) - \left(\frac{\partial \Phi}{\partial t} + \left(\frac{|\overline{\nabla}\Phi|^2}{2}\right)\right)\right) (\mathbf{r} \times \bar{\mathbf{n}}) d\bar{A} \\
&= -\frac{d}{dt}\left(\int_S \Phi (\bar{\mathbf{n}} \times \mathbf{r}) d\bar{A}\right) + \int_S \left(f(t) - \left(\frac{\partial \Phi}{\partial t} + \left(\frac{|\overline{\nabla}\Phi|^2}{2}\right)\right)\right) (\mathbf{r} \times \bar{\mathbf{n}}) d\bar{A} \\
&= \int_S -\left(\frac{|\overline{\nabla}\Phi|^2}{2}\right) (\mathbf{r} \times \bar{\mathbf{n}}) d\bar{A} \quad (\text{for } S \text{ fixed in time}), \\
&= O(1/|\mathbf{r}|^4).
\end{aligned}$$

Body-fixed frame. The equations of motion are now written with respect to the instantaneous body-fixed frame, which is translating and rotating with origin fixed at the body center (of mass). Position vectors in the two frames are related by

$$\mathbf{r} = R(t)\mathbf{l} + \bar{\mathbf{b}}(t) = R(t)\left(\mathbf{l} + \mathbf{b}(t)\right), \tag{A.5}$$

where $\mathbf{l}$ is the position vector of a point with respect to the instantaneous body frame, $\bar{\mathbf{b}}(t) \in \mathbb{R}^3$ is the position vector of the origin of the body-frame with respect to the origin of the inertial frame and $R(t) \in SO(3)$. Vectors of the same norm are related by:

$$\bar{\mathbf{w}} = R(t)\mathbf{w}$$

and time derivatives of such vectors are related by:

$$\frac{d}{dt}\bar{\mathbf{w}} = R(t)\tfrac{d\mathbf{w}}{dt} + R(t)(\Omega \times \mathbf{w}).$$

Use is also made of the relations

$$R(t)^{-1}\dot{R(t)}\mathbf{w} = \Omega \times \mathbf{w},$$
$$R(t)(\mathbf{w}_1 \times \mathbf{w}_2) = R(t)\mathbf{w}_1 \times R(t)\mathbf{w}_2$$

and the vector identities

$$\int_{\partial B} \mathbf{l} \times (\mathbf{n} \times \nabla\Phi_B) dA = -2 \int_{\partial B} \Phi_B \mathbf{n} dA,$$
$$\frac{1}{2}\int_{\partial B} l^2 (\mathbf{n} \times \nabla\Phi_B) dA = -\int_{\partial B} \Phi_B (\mathbf{n} \times \mathbf{l}) dA,$$
$$\oint_{C_i} \mathbf{t}_i ds_i = 0,$$
$$\int_{\partial B} \mathbf{n} \times \nabla\Phi_B \, dA = 0,$$
$$\oint_{\partial B} \mathbf{n} \times \mathbf{u}_V dA = 0.$$

The first two vector identities are the same as (A.3) and (A.4) now written in the body-fixed frame. The third identity is obvious. The fourth and fifth are again proved by applying Stokes' theorem in $\widetilde{D}$, for example

$$\int_{\partial B \cup S} \mathbf{n} \times \mathbf{u}_V dA = \sum \Gamma_i \oint_{C_i} \mathbf{t}_i ds_i = 0,$$

and then showing that the outer integral goes to zero as $\widetilde{D} \to D$ using the far-field estimates (2.23).

Using the above relations, we see that the linear momentum equation in the body-fixed frame becomes

$$m_B \frac{d\mathbf{U}}{dt} + \frac{1}{2}\left\{ \frac{d}{dt}\int_{\partial B} \mathbf{l} \times (\mathbf{n} \times \nabla\Phi_B) dA + \frac{d}{dt}\sum \Gamma_i \oint_{C_i} (\mathbf{l}_i \times \mathbf{t}_i) ds_i + \frac{d}{dt}\int_{\partial B} \mathbf{l} \times (\mathbf{n} \times \mathbf{u}_V) dA \right\}$$
$$+\Omega \times \left(m_B \mathbf{U} + \frac{1}{2}\left\{ \int_{\partial B} \mathbf{l} \times (\mathbf{n} \times \nabla\Phi_B) dA + \sum \Gamma_i \oint_{C_i} (\mathbf{l}_i \times \mathbf{t}_i) ds_i + \int_{\partial B} \mathbf{l} \times (\mathbf{n} \times \mathbf{u}_V) dA \right\} \right) = 0, \quad \text{(A.6)}$$

where $\overline{\mathbf{U}} = R(t)\mathbf{U}$. Note that

$$\frac{d}{dt}(R(t)\mathbf{b}) = R(t)\mathbf{U}.$$

To see how the angular momentum transforms, we need to establish the following vector identities.

Proposition A.1 *For any gradient vector field ∇f defined on ∂B, the following is true:*

$$\int_{\partial B} \langle \mathbf{b}, \mathbf{l} \rangle\, \mathbf{n} \times \nabla f \; dA = -\int_{\partial B} \langle \mathbf{b}, \mathbf{n} \times \nabla f \rangle\, \mathbf{l}\, dA. \tag{A.7}$$

Proof The strategy is to show that each side of the above relation is equal to the same integral. Let $\hat{f}$ be the harmonic extension of f into B. The left side is

$$\begin{aligned}\int_{\partial B} \langle \mathbf{b}, \mathbf{l} \rangle\, \mathbf{n} \times \nabla f \; dA &= \int_{\partial B} \mathbf{n} \times \langle \mathbf{b}, \mathbf{l} \rangle\, \nabla f \; dA,\\ &= -\int_{B} \nabla \times \langle \mathbf{b}, \mathbf{l} \rangle\, \nabla \hat{f} \; dV,\\ &= -\int_{B} \nabla \langle \mathbf{b}, \mathbf{l} \rangle \times \nabla \hat{f} \; dV,\\ &= -\int_{B} \mathbf{b} \times \nabla \hat{f} \; dV,\end{aligned}$$

since $\mathbf{b}$ is a constant vector and $\mathbf{l}$ is the position vector. The right side of (A.7) is

$$\begin{aligned}-\int_{\partial B} \langle \mathbf{b}, \mathbf{n} \times \nabla f \rangle\, \mathbf{l}\, dA &= -\int_{\partial B} \langle \mathbf{n}, \nabla f \times \mathbf{b} \rangle\, \mathbf{l}\, dA,\\ &= \int_{B} \nabla \cdot (\mathbf{l}(\nabla \hat{f} \times \mathbf{b}))\, dV \quad \text{(with slight abuse of notation)},\\ &= \int_{B} \nabla \hat{f} \times \mathbf{b}\, dV.\end{aligned}$$

Proposition A.2 *For curves parameterized by arc-length,*

$$\oint_{C_i} \langle \mathbf{b}, \mathbf{l}_i \rangle\, \boldsymbol{t}_i ds_i = -\oint_{C_i} \langle \mathbf{b}, \mathbf{t}_i \rangle\, \boldsymbol{l}_i ds_i. \tag{A.8}$$

Proof Since $d\mathbf{l}_i/ds_i = \mathbf{t}_i$ and $\mathbf{b}$ is constant, the result follows simply from

$$\oint_{C_i} \frac{d}{ds_i} \left(\langle \mathbf{b}, \mathbf{l}_i \rangle\, \mathbf{l}_i \right) ds_i = 0.$$

To transform the angular momentum equation, consider first the transformation of the fluid terms. Note that in the derivation below, we again assume that there is a neighborhood of the body in which the field $\mathbf{u}_V$ is potential:

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\partial B} r^2 (\bar{\mathbf{n}} \times \overline{\nabla} \Phi_B) d\bar{A} + \sum \Gamma_i \oint_{\overline{C}_i} r_i^2 \bar{\mathbf{t}}_i ds_i + \int_{\partial B} r^2 (\bar{\mathbf{n}} \times \bar{\mathbf{u}}_V) dA \right)\\ &= \frac{d}{dt}\left(R(t) \left(\int_{\partial B} (l^2 + 2\langle \mathbf{b}, \mathbf{l} \rangle + b^2)(\mathbf{n} \times \nabla \Phi_B + \mathbf{n} \times \mathbf{u}_V) dA + \sum \Gamma_i \oint_{C_i} (l_i^2 + 2\langle \mathbf{b}, \mathbf{l}_i \rangle + b^2) \mathbf{t}_i ds_i \right) \right),\end{aligned}$$

$$= \frac{d}{dt}\left(R(t)\left(\int_{\partial B}(l^2 + 2\langle \mathbf{b}, \mathbf{l}\rangle)(\mathbf{n}\times\nabla\Phi_B + \mathbf{n}\times\mathbf{u}_V)dA + \sum \Gamma_i \oint_{C_i}(l_i^2 + 2\langle \mathbf{b}, \mathbf{l}_i\rangle)\mathbf{t}_i ds_i\right)\right),$$

$$= \frac{d}{dt}\left(R(t)\left(\int_{\partial B} l^2(\mathbf{n}\times\nabla\Phi_B + \mathbf{n}\times\mathbf{u}_V)dA + \sum \Gamma_i \oint_{C_i} l_i^2\mathbf{t}_i ds_i - \mathbf{b}\times\int_{\partial B}\mathbf{l}\times(\mathbf{n}\times\nabla\Phi_B + \mathbf{n}\times\mathbf{u}_V)dA\right.\right.$$
$$\left.\left. -\mathbf{b}\times\sum\Gamma_i\oint_{C_i}\mathbf{l}_i\times\mathbf{t}_i ds_i\right)\right)$$

where we have used (A.7), (A.8) and the triple product identity. Thus, the preceding expressions become

$$= R(t)\left(\frac{d}{dt} + \Omega\times\right)\left(\int_{\partial B} l^2(\mathbf{n}\times\nabla\Phi_B + \mathbf{n}\times\mathbf{u}_V)dA + \sum\Gamma_i\oint_{C_i} l_i^2\mathbf{t}_i ds_i\right)$$
$$-R(t)\left(\mathbf{b}\times\left(\frac{d}{dt} + \Omega\times\right)\right)\left(\int_{\partial B}\mathbf{l}\times(\mathbf{n}\times\nabla\Phi_B + \mathbf{n}\times\mathbf{u}_V)dA + \sum\Gamma_i\oint_{C_i}\mathbf{l}_i\times\mathbf{t}_i ds_i\right)$$
$$-R(t)\left(\mathbf{U}\times\left(\int_{\partial B}\mathbf{l}\times(\mathbf{n}\times\nabla\Phi_B + \mathbf{n}\times\mathbf{u}_V)dA + \sum\Gamma_i\oint_{C_i}\mathbf{l}_i\times\mathbf{t}_i ds_i\right)\right).$$

The body terms transform as

$$\frac{d}{dt}\left(m_B\bar{\mathbf{b}}\times\overline{\mathbf{U}} + \overline{\mathbf{I}\Omega}\right) = R(t)\left(\mathbf{b}\times\left(\frac{d}{dt} + \Omega\times\right)\mathbf{U} + \left(\frac{d}{dt} + \Omega\times\right)\mathbf{I}\Omega\right).$$

Combined with (A.6), it is straightforward to see that the above equations and (A.6) give (2.9) and (2.10).

Acknowledgements Research of SDK was partially supported by the NSF grants CMMI 04-49319 and ECCS 05-01407 and the research of JEM was partially supported by NSF grant IIS-0413078 and the California Institute of Technology.

References

1. Allen, J.J., Jouanne, Y., Shashikanth, B.N.: Vortex interaction with a moving sphere. J. Fluid Mech. **587**, 337–346
2. Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three dimensional non-smooth domains. Math. Methods Appl. Sci. **21**, 823–864 (1998)
3. Arnold, V.I.: Sur la géométrie differentielle des groupes de Lie de dimenson infinie et ses applications à l'hydrodynamique des fluids parfaits. Ann. Inst. Fourier Grenoble **16**, 319–361 (1966)
4. Arnold, V.I., Khesin, B.: Topological Methods in Hydrodynamics. Appl. Math. Sciences, vol. 125 Springer, Heidelberg (1998)
5. Arnold, V.I., Khesin, B.: Topological methods in hydrodynamics. Ann. Rev. Fluid Mech. **24**, 145–166 (1992)
6. Batchelor, G.K.: An Introduction to Fluid Dynamics. Cambridge University Press, London (1967)
7. Bendali, A., Dominguez, J.M., Gallic, S.: A variational approache for the vector potential formulation of the Stokes and Navier-Stokes problems in three-dimensional domains. J. Math. Anal. Appl. **107**, 537–560 (1985)
8. Blackmore, D., Knio, O.: Hamiltonian structure for vortex filament flows. ZAMM **81**, 45–48 (2001)
9. Blackmore, D., Knio, O.: Hamiltonian formulation of the dynamics of interaction between a vortex ring and a rigid sphere. In: Proceedings of the European Congress on Computational Methods in Applied Sciences and Engineering. ECCOMAS 2000, Barcelona, Spain, 11–14 September (2000)
10. Cantor, M.: Perfect fluid flows over R^n with asymptotic conditions. J. Funct. Anal. **18**, 73–84 (1975)
11. Coutand, D., Shkoller, S.: Motion of an elastic solid inside an incompressible visoucs fluid. Arch. Ration. Mech. Anal. **176**(1), 25–102 (2005)
12. Desjardins, B., Esteban, M.J.: Existence of weak solutions for the motion of a rigid body in a visocus fluid. Arch. Ration. Mech. Anal. **146**(1), 59–71 (1999)

13. Ebin, D.G., Marsden, J.E.: Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. Math. **92**, 102–163 (1970)
14. Hirasaki, G.J., Hellums, J.D.: A general formulation of the boundary conditions on the vector potential in three-dimensional hydrodynamics. Quart. Appl. Math. **26**(3), 331–342 (1967)
15. Kanso, E., Marsden, J.E., Rowley, C.W., Melli-Huber, J.B.: Locomotion of articulated bodies in a perfect fluid. J. Nonlinear Sci. **15**, 255–289 (2005)
16. Kelly, S.D.: The mechanics and control of robotic locomotion with applications to aquatic vehicles. Ph.D. Thesis, Institute of Technology, California (1998)
17. Klein, R., Majda, A.J., Damodaran, K.: Simplified equations for the interaction of nearly parallel vortex filaments. J. Fluid Mech. **288**, 201–248 (1995)
18. Knio, O.M., Ting, L.: Vortical flow outside a body and sound generation. SIAM J. Appl. Math. **57**(4), 972–981 (1997)
19. Lamb, H.: Hydrodynamics. Dover, NY, sixth edition (1932)
20. Langer, J., Perline, R.: Poisson geometry of the filament equation. J. Nonlinear Sci. **1**, 71–94 (1991)
21. Lewis, D., Marsden, J.E., Montgomery, R., Ratiu, T.S.: The Hamiltonian structure for dynamic free boundary problems. Phys. D **18**, 391–404 (1986)
22. Lighthill, M.J.: The image system of a vortex element in a rigid body. Proc. Camb. Phil. Soc. **52**, 317–321 (1956)
23. Ma, Z., Shashikanth, B.N.: Dynamics and control of the system of a 2D rigid circular cylinder and point vortices. In: Proceedings of the American Control Conference. Minneapolis, Minnesota, USA, 14–16 June (2006)
24. Marsden, J.E.: Lectures on Mechanics, vol. 174. London Math. Soc. Lecture Note Ser., vol. 174. Cambridge University Press, London (1992)
25. Marsden, J.E., Montgomery, R., Ratiu, T.: Gauged Lie–Poisson structures. Contempl. Math. **28**, 101–114 (1984)
26. Marsden, J.E., Montgomery, R., Ratiu, T.S.: Reduction, symmetry and phases in mechanics, vol. 436. Memoirs of the AMS, vol. 436. Am. Math. Soc. Providence (1990)
27. Marsden, J.E., Ratiu, T.S.: Introduction to Mechanics and Symmetry, vol. 17. Texts in Applied Mathematics, vol. 17; 1994, 2nd edn. Springer, Heidelberg (1999)
28. Marsden, J.E., Weinstein, A.: Coadjoint orbits, vortices and Clebsch variables for incompressible fluids. Phys. D **7**, 305–323 (1983)
29. Milne-Thomson, L.M.: Theoretical Hydrodynamics, 5th edn. Dover, New York (1996)
30. Nirenberg, L., Walker, H.F.: The null spaces of elliptic partial differential operators in $\mathbb{R}^n$. J. Math. Anal. Appl. **42**, 271–301 (1973)
31. Radford, J.: Symmetry, reduction and swimming in a perfect fluid. Ph.D. Thesis, California Institute of Technology (2003)
32. Saffman, P.G.: Vortex Dynamics. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge Univ. Press., London (1992)
33. Shashikanth, B.N.: Poisson brackets for the dynamically interacting system of a 2D rigid boundary and N point vortices: The case of arbitrary smooth cylinder shapes. Reg. Chaotic Dyn. **10**(1), 1–14 (2005)
34. Shashikanth, B.N.: Symmetric pairs of point vortices interacting with a neutrally buoyant 2D circular cylinder. Phys. Fluids **18**, 127103 (2006)
35. Shashikanth, B.N., Marsden, J.E.: Leapfrogging vortexrings: Hamiltonian structure, geometricphases and disete reduction. Fluid Dyn. Res. **33**, 333–356 (2003)
36. Shashikanth, B.N., Marsden, J.E., Burdick, J.W., Kelly, S.D.: The Hamiltonian structure of a 2D rigid circular cylinder interacting dynamically with N point vortices. Phys. Fluids 14(3), 1214–1227 (2002) (see also Erratum. Phys. of Fluids 14(11), 4099)
37. Ting, L., Klein, R.: Viscous Vortical Flows. Lecture Notes in Physics, vol. 374. Springer, Berlin (1991)
38. Truesdell, C.: The Kinematics of Vorticity. Indiana University Press, Bloomington (1954)
39. Vladimirov, V.A.: Vortical momentum of bounded ideal incompressible fluid flows. J. Appl. Mech. Tech. Phys. 20, 157–163 (1979) (Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No.2, 53–61, (1979))