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HIGHER–ORDER IMPLICIT FUNCTION THEOREMS AND DEGENERATE NONLINEAR BOUNDARY-VALUE PROBLEMS

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This was sometime a paradox, but now the time gives it proof,
William Shakespeare: Hamlet, Prince of Denmark

Abstract. The first part of this paper considers the problem of solving an equation of the form $F(x, y) = 0$, for $y = \varphi(x)$ as a function of x, where $F : X \times Y \to Z$ is a smooth nonlinear mapping between Banach spaces. The focus is on the case in which the mapping F is degenerate at some point (x^*, y^*) with respect to y, i.e., when $F'_y(x^*, y^*)$, the derivative of F with respect to y, is not invertible and, hence, the classical Implicit Function Theorem is not applicable. We present pth-order generalizations of the Implicit Function Theorem for this case. The second part of the paper uses these pth-order implicit function theorems to derive sufficient conditions for the existence of a solution of degenerate nonlinear boundary-value problems for second-order ordinary differential equations in cases close to resonance. The last part of the paper presents a modified perturbation method for solving degenerate second-order boundary value problems with a small parameter. The results of this paper are based on the constructions of p-regularity theory, whose basic concepts and main results are given in the paper *Factor–analysis of nonlinear mappings: p–regularity theory* by Tret'yakov and Marsden (Communications on Pure and Applied Analysis, **2** (2003), 425–445).

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1. **Introduction.** Suppose that $F : X \times Y \to Z$ is a given smooth mapping, where X, Y and Z are Banach spaces and let (x^*, y^*) be a given point in $X \times Y$ that satisfies $F(x^*, y^*) = 0$. This paper considers the problem of the existence of a locally defined mapping $\varphi : Y \to X$, written as $y = \varphi(x)$, which is a solution of the equation $F(x, y) = 0$ near the given solution (x^*, y^*); that is $F(x, \varphi(x)) = 0$ and $y^* = \varphi(x^*)$. We are interested in the case when the mapping F is *degenerate (nonregular)* at (x^*, y^*); that is, when $F'_y(x^*, y^*)$, the derivative of F with respect to y, is not onto, and, hence, the classical Implicit Function Theorem can not be applied to guarantee the (local) existence of a solution $\varphi(x)$. The importance of consideration of this problem follows from the need of solving various nonlinear problems, many of which, as was shown in [17], are, by their nature, singular (degenerate).

The first goal of the paper, carried out in Section 3, is to establish pth-order generalizations of the Implicit Function Theorem. An example of application of these results is the existence and uniqueness question for the following nonlinear boundary-value problem (BVP): consider the nth-order ordinary differential equation:

$$y^{(n)}(t) + y^{(n-1)}(t) + \cdots + y'(t) + y(t) + g(y(t)) = x(t) \tag{1}$$

with the boundary conditions of the general form

$$p_i(y(0), y'(0), \ldots, y^{(n-1)}(0), y(\pi), y'(\pi), \cdots, y^{(n-1)}(\pi)) = 0, \quad i = 1, \ldots n,$$

under appropriate assumptions on $g(\cdot)$, p_i and $x(t)$. Another example of application of the higher-order implicit function theorems is ordinary differential equations with a small parameter in cases that are close to resonance; that is, *small divisor problems*.

While the results can be extended to nth-order BVPs as in (1), this paper focuses on second-order BVPs. Specifically, the paper applies pth-order generalizations of the implicit function theorem to obtain new sufficient conditions for the existence of a solution of degenerate BVPs for second-order ordinary differential equations. This then yields the existence of a periodic solution of second-order BVPs as well as a new asymptotic formula for this solution. Moreover, the results establish the existence of a nontrivial periodic solution for degenerate homogeneous second-order BVPs. This latter result can be viewed as a modification of the Poincaré–Andronov–Hopf Theorem.

The nonlinear BVP analyzed in this paper has the form

$$y''(t) + y(t) + g(y(t)) = x(t) \tag{2}$$

with boundary conditions

$$y(0) = y(\pi) = 0, \tag{3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth mapping. In this problem, $x(\cdot) \in C[0, \pi]$ and g are given and a solution $y(\cdot) \in C^2[0, \pi]$ is sought.

Specifically, assume that for some $p \in \mathbb{N}$, g is a C^{p+1} mapping, and

$$x(0) = x(\pi) = 0, \quad g(0) = g'(0) = \ldots = g^{(p-1)}(0) = 0. \tag{4}$$

Under assumptions (4), the homogeneous BVP associated with (2)–(3), namely

$$y''(t) + y(t) + g(y(t)) = 0, \quad y(0) = y(\pi) = 0,$$

has the trivial solution $y(t) \equiv 0$. We are then concerned with the existence of a nontrivial solution of the nonhomogeneous BVP (2)–(3).

Introduce the notation

$$F(x,y) = y'' + y + g(y) - x, \tag{5}$$

and regard F as a mapping

$$F : X \times Y \to Z,$$

where

$$X = \{x \in C[0,\pi] \mid x(0) = x(\pi) = 0\}, \quad Y = \{y \in C^2[0,\pi] \mid y(0) = y(\pi) = 0\},$$

$Z = C[0,\pi]$, and, as above, for a positive integer p, g is a C^{p+1} mapping from $\mathbb{R}$ to $\mathbb{R}$ satisfying $g(0) = g'(0) = \ldots = g^{(p-1)}(0) = 0$.

It follows from the classical "Omega lemma" (see, for instance, [1] for an exposition) that F is of class C^{p+1}. Then we can rewrite equation (2) as

$$F(x,y) = 0. \tag{6}$$

Our assumptions (3)–(4) imply that $(0,0)$ is a solution of (6); i.e., $F(0,0) = 0$. Without loss of generality, we may restrict our attention to some neighborhood $U \times V \subset X \times Y$ of the point $(0,0)$. Then the problem of existence of a solution of the nonhomogeneous BVP (2)–(3) for $x(t) \in U$ is equivalent to the problem of existence of an implicit function $\varphi(x) : U \to Y$ such that $y = \varphi(x)$ and

$$F(x,y) = y'' + y + g(y) - x = 0, \tag{7}$$

for all $x \in U$, and where $y(0) = y(\pi) = 0$.

In the case when $F(0,0) = 0$ and the mapping F is regular at $(0,0)$; i.e., when its derivative with respect to y, denoted $F'_y(0,0)$, is invertible, the classical Implicit Function Theorem guarantees the existence of a smooth mapping φ defined on a neighborhood of $x^* = 0$ such that $F(x,\varphi(x)) = 0$ and $\varphi(0) = 0$. In our case, the operator $F'_y(0,0) = (\cdot)'' + (\cdot) + g'(0)$ is given by

$$F'_y(0,0)\xi = \xi'' + \xi.$$

However, with $z(t) = \sin t$, $z(\cdot) \in Z$, *the boundary-value problem*

$$y''(t) + y(t) = \sin t, \qquad y(0) = y(\pi) = 0,$$

does not have a solution. To see this, assume there is a solution; simply multiply the equation by $\sin t$ and integrate each side from 0 to π; the left side, after integration by parts, gives zero and the right hand side is nonzero. Nonsurjectivity of the linearization $F'_y(0,0)$ can alternatively be readily seen using the Fredholm alternative—note that the adjoint of the operator $F'_y(0,0)$, which is the same operator in this case, has a nontrivial kernel. In any case, the operator $F'_y(0,0)$ is not surjective and so *the classical Implicit Function Theorem cannot be applied* to guarantee the existence of an implicit function $y = \varphi(x)$ satisfying (7).

In general, we call a nonhomogeneous BVP similar to (2)–(3) ***degenerate*** when the associated mapping F defined by the linearization around a known solution (as in (5) in the above example) is degenerate (nonregular) in the sense that the operator $F'_y(0,0)$ is not surjective.

To overcome the problem of nonregularity of the mapping F associated with the BVP (2)–(3), pth-order generalizations of the Implicit Function Theorem are established in the first part of the paper (Section 3). Then in the second part of the paper (Section 4), these pth-order implicit function theorems are used to obtain sufficient conditions for existence of a solution of nonlinear BVP (2)–(3) as well as a BVP with boundary conditions $y(0) = y(2\pi) = 0$. In the last part of the paper (Section 5), a modified perturbation method for solving degenerate second-order boundary value problems with a small parameter is proposed. The classical perturbation method is not applicable to such problems because it requires solving a problem that does not have a solution at the first step of the method. The modified perturbation method is based on the sufficient conditions for existence of a solution of degenerate nonlinear BVPs presented in the second part of the paper. The modified perturbation method uses an additional term in the perturbation expansion that is a multiple of a negative power of the perturbation parameter. Two realizations of the modified perturbation method are presented. The first realization is similar in spirit to the classical perturbation method, while the second realization consists of constructing an equivalent system of equations obtained by projection of the original problem onto subspaces. The form of these subspaces depends on the structure of the BVP.

In summary, the main contribution of the paper consists of pth-order implicit function theorems for the degenerate (nonregular) case, and the application of these theorems to obtain sufficient conditions for the existence of a solution of degenerate boundary value problems and an accompanying new modified perturbation method for solving degenerate boundary value problems with a small parameter. The results obtained in this paper are based on the constructions of p-regularity theory, whose basic concepts and main results are described in [10, 11] and [17]. It is interesting to note that one of the basic results of p–regularity theory, namely the theorem about the structure of the zero set of a nonregular mapping satisfying a special higher-order regularity condition, was simultaneously obtained in [7] and [14]. In [8], it was noted that the theorem in [7] was a powerful generalization of Morse Lemma. In this paper, the result is extended to generalized implicit function theorems for nonregular mappings.

Comparing our results with existing ones, we would like to note that the first result relevant to the implicit function theorem for nonregular mappings, the distance estimate to the zero set of the nonregular mapping, was proposed in [13, 14]. The first generalization of the implicit function theorem applicable to nonregular mappings was introduced in [16]. Then some other generalizations of the Implicit Function Theorem for 2-regular mappings were obtained in [2], [3], and [12], and for p-regular mappings in [4], [5], and [6]. The implicit function theorems given in the present paper are applicable to p-regular mappings for any $p \geq 2$ and reduce to the classical Implicit Function Theorem in the regular case. The main focus of the paper is on the application of the p-order implicit function theorems to degenerate nonlinear boundary-value problems for second-order ordinary differential equations. The difference and novelty of the present development is in providing an integrated theoretical approach to the analysis and solution of degenerate BVPs on the basis of p-regularity theory. There are several modifications of the perturbation method applicable to some degenerate boundary-value problems; for example, [18]. However, there is no uniform approach to their underlying mathematical infrastructure. The modified perturbation method proposed in the paper is based on the new sufficient conditions for existence of a solution of a degenerate nonlinear BVP. The method in this paper gives a new approach to the numerical solution of degenerate BVPs and can be extended to the boundary value problems for partial differential equations. This paper can be considered as a development of p-regularity theory and extension of the results presented in [4, 5, 6, 10, 11, 16] and [17].

The organization of the paper is as follows. The next section recalls the main definitions and concepts of p-regularity theory. Section 3 formulates and proves the pth-order implicit function theorems. Section 4 presents new sufficient conditions for the existence of solutions of BVPs, obtained using the pth-order implicit function theorems. Finally, in Section 5 the new existence result is used to obtain a modified perturbation method.

Notation. Let $\mathcal{H}(S_1, S_2)$ denote the ***Hausdorff distance*** between two sets S_1 and S_2:

$$\mathcal{H}(S_1, S_2) = \max\left\{\sup_{x\in S_1} \operatorname{dist}(x, S_2), \sup_{y\in S_2} \operatorname{dist}(y, S_1)\right\}.$$

Let $\mathcal{L}(X, Y)$ be the space of all continuous linear operators from X to Y and for a given linear operator $\Lambda : X \to Y$, we denote its kernel and image by $\operatorname{Ker}\Lambda = \{x \in X \mid \Lambda x = 0\}$ and $\operatorname{Im}\Lambda = \{y \in Y \mid y = \Lambda x \text{ for some } x \in X\}$.

Let p be a natural number and let $B : X \times X \times \ldots \times X$ (with p copies of X) $\to Y$ be a continuous symmetric p-multilinear mapping. The p-form associated to B is the map $B[\cdot]^p : X \to Y$ defined by

$$B[x]^p = B(x, x, \ldots, x),$$

for $x \in X$. Alternatively, we may simply view $B[\cdot]^p$ as a homogeneous polynomial $Q : X \to Y$ of degree p, i.e., $Q(\alpha x) = \alpha^p Q(x)$. The space of continuous homogeneous polynomials $Q : X \to Y$ of degree p will be denoted by $\mathcal{Q}^p(X, Y)$.

For a differentiable mapping $F : X \times Y \to Z$, its (Fréchet) derivative with respect to y at a point $(x, y) \in X \times Y$ will be denoted $F'_y(x, y) : Y \to Z$. If F is of class C^p, we let $F^{(p)}_{y\ldots y}(x, y)$ be the pth derivative of F with respect to y at the point (x, y) (a symmetric multilinear map of p copies of Y to Z) and the associated p-form, called the ***p*th–*order mapping***, is defined by

$$F^{(p)}_{y\ldots y}(x, y)[h]^p = F^{(p)}_{y\ldots y}(x, y)(h, h, \ldots, h).$$

The p-kernel of this pth-order mapping is defined by

$$\operatorname{Ker}^p F^{(p)}(x, y) = \{h \in X \times Y \mid F^{(p)}(x, y)\, [h]^p = 0\,\}.$$

2. **The p-factor operator.** Consider a nonlinear mapping $F : X \times Y \to Z$, where X, Y and Z are Banach spaces. Assume that for some point $(x^*, y^*) \in X \times Y$,

$$\operatorname{Im} F'_y(x^*, y^*) \neq Z.$$

For the purpose of describing nonlinear problems, the concept of p-regularity was introduced by [13, 14, 16] using the notion of a p-factor operator.

The p-factor operator is constructed under the *assumption* that the space Z is decomposed into the direct sum

$$Z = Z_1 \oplus \ldots \oplus Z_p,$$

where $Z_1 = \operatorname{cl}(\operatorname{Im} F'_y(x^*, y^*))$, the closure of the image of the first partial derivative of F with respect to y evaluated at (x^*, y^*), and the remaining spaces are defined as follows. Let W_2 be a closed complementary subspace to Z_1 (we are *assuming* that such a closed complement exists) and let $P_{W_2} : Z \to W_2$ be the projection operator onto W_2 along Z_1. Let Z_2 be the closed linear span of the image of the quadratic map $P_{W_2} F''_{yy}(x^*, y^*)[\cdot]^2$. More generally, define inductively,

$$Z_i = \operatorname{cl}(\operatorname{span}\ \operatorname{Im} P_{W_i} F^{(i)}_{y\ldots y}(x^*, y^*)[\cdot]^i) \subseteq W_i, \quad i = 2, \ldots, p-1,$$

where W_i is a choice of closed complementary subspace for $(Z_1 \oplus \ldots \oplus Z_{i-1})$ with respect to Z, $i = 2, \ldots, p$, and $P_{W_i} : Z \to W_i$ is the projection operator onto W_i along $(Z_1 \oplus \ldots \oplus Z_{i-1})$ with respect to Z, $i = 2, \ldots, p$. Finally, let $Z_p = W_p$.

Define the following mappings (see [16])

$$f_i(x,y) : X \times Y \to Z_i, \quad f_i(x,y) = P_{Z_i} F(x,y), \quad i = 1, \ldots, p, \tag{8}$$

where $P_{Z_i} : Z \to Z_i$ is the projection operator onto Z_i along $(Z_1 \oplus \ldots \oplus Z_{i-1} \oplus Z_{i+1} \oplus \ldots \oplus Z_p)$ with respect to Z, $i = 1, \ldots, p$.

Then the mapping F can be represented as

$$F(x,y) = f_1(x,y) + \ldots + f_p(x,y).$$

We sometimes write this representation of F as $(f_1(x,y), \ldots, f_p(x,y))$ and Z as $Z_1 \times Z_2 \times \ldots Z_p$.

The p-factor operator, given in the following definition, plays a central role in p-regularity theory.

Definition 2.1. The linear operator $\Psi_p(h) \in \mathcal{L}(Y, Z_1 \times \ldots \times Z_p)$, for $h \in Y$, is defined by

$$\Psi_p(h) = \left(f'_{1y}(x^*, y^*), f''_{2\,yy}(x^*, y^*)[h], \ldots, \frac{1}{(p-1)!} f^{(p)}_{p\ \ y \ldots y}(x^*, y^*)[h]^{p-1} \right), \tag{9}$$

and is called the p-***factor operator***.

We also introduce the corresponding inverse multivalued operator Ψ_p^{-1} that is defined by

$$\begin{aligned} &\{\Psi_p(h)\}^{-1}(z) \\ &= \left\{ \xi \in Y \;\middle|\; \left(f'_{1y}(x^*, y^*)[\xi], \ldots, \frac{1}{(p-1)!} f^{(p)}_{p\ \ y \ldots y}(x^*, y^*)[h^{p-1}, \xi] \right) = (z_1, \ldots, z_p) \right\}, \end{aligned}$$

where $z_i \in Z_i$, $i = 1, \ldots, p$.

Definition 2.2. The mapping $F(x,y)$ is called p-***regular*** at the point (x^*, y^*) with respect to the vector h if

$$\operatorname{Im} \Psi_p(h) = Z_1 \times Z_2 \times \ldots Z_p.$$

Definition 2.3. The mapping $F(x,y)$ is called ***uniformly*** p-***regular*** over the set M if

$$\sup_{h \in M} \|\{\Psi_p(\bar{h})\}^{-1}\| < \infty, \quad \bar{h} = \frac{h}{\|h\|}, \quad h \neq 0,$$

where

$$\|\{\Psi_p(\bar{h})\}^{-1}\| = \sup_{\|z\|=1} \inf\{\|y\| \mid \Psi_p(h)[y] = z\}.$$

3. **Implicit function theorem–the degenerate case.** Consider the equation

$$F(x,y) = 0,$$

where $F \in C^{p+1}(X \times Y, Z)$, and where X, Y and Z are Banach spaces. Assume that for some (x^*, y^*), we have $F(x^*, y^*) = 0$ and that we are in the degenerate (nonregular) case; that is,

$$\operatorname{Im} F'_y(x^*, y^*) \neq Z.$$

This section establishes p-order implicit function theorems for the mapping $F(x,y)$.

Introduce the mapping $\Phi_p : Y \to Z_1 \times \ldots \times Z_p$: defined by

$$\Phi_p = \left(f'_{1\,y}(x^*, y^*), \frac{1}{2} f''_{2\,yy}(x^*, y^*), \ldots, \frac{1}{p!} f^{(p)}_{p\,y\ldots y}(x^*, y^*) \right),$$

where

$$\Phi_p[y]^p = \left(f'_{1\,y}(x^*, y^*)[y], \frac{1}{2} f''_{2\,yy}(x^*, y^*)[y]^2, \ldots, \frac{1}{p!} f^{(p)}_{p\,y\ldots y}(x^*, y^*)[y]^p \right).$$

Under the assumption that $Z_1 \oplus \ldots \oplus Z_p = Z$, we also introduce the corresponding inverse multivalued operator Φ_p^{-1}:

$$\Phi_p^{-1}(z) = \left\{ \eta \in Y \;\middle|\; \left(f'_{1y}(x^*, y^*)[\eta], \ldots, \frac{1}{p!} f^{(p)}_{p\,y\ldots y}(x^*, y^*)[\eta]^p \right) = (z_1, z_2, \ldots, z_p) \right\},$$

where $z_i \in Z_i$, $i = 1, \ldots, p$.

The main result in this section is Theorem 3.2; to prove it, we will need the following theorem of [9].

Theorem 3.1 (Multivalued Contraction Mapping Theorem)**.** *Let W be Banach space, $w_0 \in W$, and $\Lambda : B_{r_1}(w_0) \to 2^W$ be a multivalued mapping defined for some ball $B_{r_1}(w_0) \subset W$. Assume that $\Lambda(w) \neq \emptyset$ for any $w \in B_{r_1}(w_0)$. Assume also that there exists a number $\alpha \in (0, 1)$ such that*

1) $\mathcal{H}(\Lambda(w_1), \Lambda(w_2)) \leq \alpha \, \|w_1 - w_2\|$, *for all $w_1, w_2 \in B_{r_1}(w_0)$;*
2) $\mathrm{dist}(w_0, \Lambda(w_0)) < (1 - \alpha) r_1$.

Then for any r_2 such that

$$\mathrm{dist}(w_0, \Lambda(w_0)) < r_2 < (1 - \alpha) r_1$$

there exists $\bar{w} \in B_{r_3}(w_0)$ with $r_3 = r_2/(1 - \alpha)$ such that

$$\bar{w} \in \Lambda(\bar{w}). \tag{10}$$

Moreover, among the points $\bar{w}$ satisfying (10), there exists a point such that

$$\|\bar{w} - w_0\| \leq \frac{2}{1 - \alpha} \, \mathrm{dist}(w_0, \Lambda(w_0)).$$

Theorem 3.2 (The pth-order Implicit Function Theorem)**.** *Let X, Y and Z be Banach spaces, $N(x^*)$ and $N(y^*)$ be sufficiently small neighborhoods of $x^* \in X$ and $y^* \in Y$ respectively, $F \in C^{p+1}(X \times Y)$, and $F(x^*, y^*) = 0$. Let the mappings $f_i(x, y)$, $i = 1, \ldots, p$, introduced in equation (8), satisfy the following conditions:*

1) ***singularity condition**:*

$$f^{(r)}_{i\,\underbrace{x\ldots x}_{q}\,\underbrace{y\ldots y}_{r-q}}(x^*, y^*) = 0, \quad r = 1, \ldots, i - 1, \;\; q = 0, \ldots, r - 1, \;\; i = 1, \ldots, p,$$

$$f^{(i)}_{i\,\underbrace{x\ldots x}_{q}\,\underbrace{y\ldots y}_{i-q}}(x^*, y^*) = 0, \quad q = 1, \ldots, i - 1, \;\; i = 1, \ldots, p;$$

2) *p**-factor-approximation**: for all $y_1, y_2 \in (N(y^*) - y^*)$,*

$$\left\| f_i(x, y^* + y_1) - f_i(x, y^* + y_2) - \frac{1}{i!} f^{(i)}_{i\,y\ldots y}(x^*, y^*)[y_1]^i + \frac{1}{i!} f^{(i)}_{i\,y\ldots y}(x^*, y^*)[y_2]^i \right\|$$
$$\leq \varepsilon \left(\|y_1\|^{i-1} + \|y_2\|^{i-1} \right) \|y_1 - y_2\|, \qquad i = 1, \ldots, p,$$

where $\varepsilon > 0$ is sufficiently small;

3) ***Banach condition***: *there exists a nonempty set* $\Gamma(x^*) \subset N(x^*)$ *in* X *such that for any sufficiently small* γ, *we have* $\Gamma(x^*) \cap B_\gamma(x^*) \neq \{x^*\}$. *Moreover, for* $x \in \Gamma(x^*)$, *there exists* $h(x)$ *such that*

$$\Phi_p[h(x)]^p = -F(x, y^*), \quad \|h(x)\| \leq c_1 \sum_{r=1}^{p} \|f_r(x, y^*)\|_{Z_r}^{1/r}, \tag{11}$$

where $0 < c_1 < \infty$ *is a constant;*

4) ***uniform*** p***-regularity condition*** *of the mapping* $F(x, y)$ *over the set* $\Phi_p^{-1}(-F(x, y^*))$. *Moreover, assume that for any sufficiently small* γ *such that* $B_\gamma(x^*) \subset N(x^*)$ *the intersection of* $\Gamma(x^*)$ *and* $B_\gamma(x^*)$ *is not empty.*

Then there exists a constant $k > 0$, *a sufficiently small* δ, *and a mapping* $\varphi : \Gamma(x^*) \cap B_\delta(x^*) \to N(y^*)$ *such that the following hold for* $x \in \Gamma(x^*) \cap B_\delta(x^*)$:

$$\begin{gathered} \varphi(x^*) = y^*; \\ F(x, \varphi(x)) = 0, \\ \|\varphi(x) - y^*\|_Y \leq k \sum_{r=1}^{p} \|f_r(x, y^*)\|_{Z_r}^{1/r}. \end{gathered}$$

Proof. By the assumptions of the theorem, for any sufficiently small $\varepsilon > 0$, the Banach condition (11) holds for $x \in \Gamma(x^*)$. Then by the definition of $\Phi_p[h(x)]^p$, we can rewrite (11) as

$$\begin{array}{lcl} f'_{1y}(x^*, y^*)[h(x)] & = & -f_1(x, y^*) \\ \frac{1}{2} f''_{2yy}(x^*, y^*)[h(x)]^2 & = & -f_2(x, y^*) \\ & \vdots & \\ \frac{1}{p!} f^{(p)}_{py\ldots y}(x^*, y^*)[h(x)]^p & = & -f_p(x, y^*) \end{array} \tag{12}$$

Introduce the multivalued mapping $\Lambda(y) = y - \{\Psi_p(h)\}^{-1} F(x, y^* + h + y)$, where $y \in (N(y^*) - y^*)$ with $h = h(x)$, $x \in \Gamma(x^*)$. Taking into account the definition of F and (12), rewrite $\Lambda(y)$ as follows:

$$\Lambda(y) = y - \{\Psi_p(h)\}^{-1} \begin{pmatrix} f_1(x, y^* + h + y) - f_1(x, y^*) - f'_{1y}(x^*, y^*)h \\ \vdots \\ f_i(x, y^* + h + y) - f_i(x, y^*) - \frac{1}{i!} f^{(i)}_{i\,y\ldots y}(x^*, y^*)[h]^i \\ \vdots \\ f_p(x, y^* + h + y) - f_p(x, y^*) - \frac{1}{p!} f^{(p)}_{p\,y\ldots y}(x^*, y^*)[h]^p \end{pmatrix}.$$

Next, it will be verified that all conditions of Theorem 3.1 are satisfied for $\Lambda(y)$ with some $B_{r_1}(0) \subset (N(y^*) - y^*)$ and $w_0 = 0$. To do this, choose r_1 small enough so that $\|y\| = o(\|h\|)$ for all $y \in B_{r_1}(0)$. Now it will be shown that condition 1) of Theorem 3.1 holds for all $y_1, y_2 \in B_{r_1}(0)$, that is

$$\mathcal{H}(\Lambda(y_1), \Lambda(y_2)) \leq \alpha \|y_1 - y_2\|, \quad 0 \leq \alpha < 1.$$

By definition of $\Lambda(y)$, we have

$$\begin{aligned}\mathcal{H}(\Lambda(y_1),\Lambda(y_2)) &= \inf\{\|z_1-z_2\| \mid z_i\in\Lambda(y_i),\ i=1,2\}\\ &= \inf\{\|z_1-z_2\| \mid \Psi_p(h)z_i=\Psi_p(h)y_i-F(x,y^*+h+y_i),\\ &\qquad i=1,2\}\\ &= \inf\{\|z\| \mid \Psi_p(h)z=\Psi_p(h)(y_1-y_2)-F(x,y^*+h+y_1)\\ &\qquad +F(x,y^*+h+y_2)\}.\end{aligned} \tag{13}$$

One can show that under conditions of the theorem, the version of the Banach Open Mapping Theorem in [9] shows that there exists $\tilde{C}\geq 0$ such that the following holds for the operator $\Psi_p(h)$:

$$\|\Psi_p(h)^{-1}w\|\leq \tilde{C}\left\|\begin{array}{c}\{f'_{1y}(x^*,y^*)\}^{-1}w_1\\ \vdots\\ \left\{\frac{1}{(p-1)!}f_p^{(p)}(x^*,y^*)[h]^{p-1}\right\}^{-1}w_p\end{array}\right\|, \tag{14}$$

where

$$w=(w_1,\dots,w_p),\quad w_k\in\operatorname{Im}\left(\frac{1}{(k-1)!}f_k^{(k)}(x^*,y^*)[h]^{k-1}\right),\quad k=1,\dots,p.$$

Then by using (14), we get from (13) that

$$\begin{aligned}&\mathcal{H}(\Lambda(y_1),\Lambda(y_2))\\ &\leq \tilde{C}\left\|\begin{array}{c}\{f'_{1y}(x^*,y^*)\}^{-1}\left(f'_{1y}(x^*,y^*)(y_1-y_2)-f_1(x,y_1^*(h))+f_1(x,y_2^*(h))\right)\\ \vdots\\ \left\{\frac{1}{(p-1)!}f_p^{(p)\,*}[h]^{p-1}\right\}^{-1}\left(\frac{1}{(p-1)!}f_p^{(p)\,*}[h]^{p-1}(y_1-y_2)-f_p(x,y_1^*(h))+f_p(x,y_2^*(h))\right)\end{array}\right\|\\ &\leq \tilde{C}\left[\left\|\{f'_{1y}(x^*,y^*)\}^{-1}\right\|\left\|\left(f'_{1y}(x^*,y^*)(y_1-y_2)-f_1(x,y_1^*(h))+f_1(x,y_2^*(h))\right)\right\|+\dots\right.\\ &\quad+\left\|\left\{\frac{1}{(p-1)!}f_p^{(p)\,*}[h]^{p-1}\right\}^{-1}\right\|\\ &\quad\left.\times\left\|\left(\frac{1}{(p-1)!}f_p^{(p)\,*}[h]^{p-1}(y_1-y_2)-f_p(x,y_1^*(h))+f_p(x,y_2^*(h))\right)\right\|\right],\end{aligned}$$

where $f_p^{(p)\,*}=f^{(p)}_{p\,y\dots y}(x^*,y^*)$, $y_1^*(h)=y^*+h+y_1$, $y_2^*(h)=y^*+h+y_2$, $x\in\Gamma(x^*)$ and $y_1,y_2\in(N(y^*)-y^*)$. By the uniform p-regularity condition it follows that there exists $\bar{C}$ such that

$$\left\|\left\{\frac{1}{(i-1)!}f^{(i)}_{i\,y\dots y}(x^*,y^*)[h]^{i-1}\right\}^{-1}\right\|\leq\frac{\bar{C}}{\|h\|^{i-1}},\quad i=1,\dots p. \tag{15}$$

Then for some $C>0$,

$$\begin{aligned}&\mathcal{H}(\Lambda(y_1),\Lambda(y_2))\\ &\leq C\left\|f'_{1y}(x^*,y^*)(y_1-y_2)-f_1(x,y^*+h+y_1)+f_1(x,y^*+h+y_2)\right\|+\dots\\ &\quad+\frac{C}{(p-1)!\|h\|^{p-1}}\left\|f^{(p)}_{py\dots y}(x^*,y^*)[h]^{p-1}(y_1-y_2)-f_p(x,y^*+h+y_1)\right.\\ &\quad\left.+f_p(x,y^*+h+y_2)\right\|.\end{aligned} \tag{16}$$

By the p-factor-approximation condition, for the first terms in (16) corresponding to $i = 1$, we obtain the estimate with some $C_1 > 0$:

$$C \left\| f'_{1y}(x^*, y^*)(y_1 - y_2) - f_1(x, y^* + h + y_1) + f_1(x, y^* + h + y_2) \right\| \leq C_1 \varepsilon \|y_1 - y_2\|.$$

Next we will estimate the norm of the term in (16) corresponding to $i = 2$, that is

$$\frac{C}{\|h\|} \left\| f''_{2y}(x^*, y^*)[h, y_1 - y_2] - f_2(x, y^* + h + y_1) + f_2(x, y^* + h + y_2) \right\|. \tag{17}$$

To estimate (17), we will use the following identity

$$\begin{aligned} & f''_{2y}(x^*, y^*)[h, y_1 - y_2] \\ & = \frac{1}{2} f''_{2y}(x^*, y^*)[h + y_1]^2 - \frac{1}{2} f''_{2y}(x^*, y^*)[h + y_2]^2 - \frac{1}{2} f''_{2y}(x^*, y^*)[y_1 - y_2, y_1 + y_2]. \end{aligned} \tag{18}$$

By the p-factor approximation condition with $i = 2$, we obtain

$$\begin{aligned} & \Bigg\| \quad f_2(x, y^* + h + y_1) - f_2(x, y^* + h + y_2) \\ & \qquad - \frac{1}{2} f_2{}''_{yy}(x^*, y^*)[y_1 + h]^2 + \frac{1}{2} f''_{2\,yy}(x^*, y^*)[y_2 + h]^2 \Bigg\| \\ & \leq \varepsilon \left(\|y_1 + h\| + \|y_2 + h\| \right) \|y_1 - y_2\|. \end{aligned} \tag{19}$$

By substituting (18) into (17) and using (19) as well as the assumption that $\|y_1\| = o(\|h\|)$ and $\|y_2\| = o(\|h\|)$, we get with some $C_2 > 0$

$$\begin{aligned} & \frac{C}{\|h\|} \left\| f''_{2y}(x^*, y^*)[h, y_1 - y_2] - f_2(x, y^* + h + y_1) + f_2(x, y^* + h + y_2) \right\| \\ & \leq \frac{C\varepsilon}{\|h\|} \left(\|y_1 + h\| + \|y_2 + h\| \right) \|y_1 - y_2\| + \frac{C}{2\|h\|} \left\| f''_{2y}(x^*, y^*)[y_1 - y_2, y_1 + y_2] \right\| \\ & \leq C\varepsilon \|y_1 - y_2\| + \frac{\bar{C}}{\|h\|} \|y_1 + y_2\| \, \|y_1 - y_2\| \\ & \leq C_2 \varepsilon \|y_1 - y_2\|. \end{aligned}$$

We show how to estimate the norm of the other terms in (16) that have the form

$$\begin{aligned} A_r = & \frac{C}{(r-1)! \|h\|^{r-1}} \left\| f^{(r)}_{ry\ldots y}(x^*, y^*)[h^{r-1}, y_1 - y_2] - f_i(x, y^* + h + y_1) \right. \\ & \left. + f_i(x, y^* + h + y_2) \right\|. \end{aligned} \tag{20}$$

To estimate (20), we are using the identity:

$$\begin{aligned} & \frac{1}{(r-1)!} f^{(r)}_{ry\ldots y}(x^*, y^*)[h^{r-1}, y_1 - y_2] \\ & = \frac{1}{r!} f^{(r)}_{ry\ldots y}(x^*, y^*)[h + y_1]^r - \frac{1}{r!} f^{(r)}_{ry\ldots y}(x^*, y^*)[h + y_2]^r \\ & - \frac{1}{r!} \sum_{k=2}^{r} \left(C_r^k f^{(r)}_{ry\ldots y}(x^*, y^*)[h^{r-k}, y_1^k] - C_r^k f^{(r)}_{ry\ldots y}(x^*, y^*)[h^{r-k}, y_2^k] \right), \end{aligned} \tag{21}$$

where

$$C_r^k = \frac{r!}{(r-k)!\, k!}.$$

By using the p-factor approximation condition with $i = r$ and (21), we see that

$$A_r \leq \varepsilon C_r \|y_1 - y_2\|.$$

Combining the estimates for the norms of all terms in (16), we get:

$$\mathcal{H}(\Lambda(y_1), \Lambda(y_2)) \leq \varepsilon(C_1 + C_2 + \ldots + C_r)\|y_1 - y_2\| < \alpha\|y_1 - y_2\|, \quad 0 \leq \alpha < 1,$$

for $x \in \Gamma(x^*)$. Since the last inequality holds for $y_1, y_2 \in B_{r_1}(0)$ with some sufficiently small ε, then $\varepsilon(C_1 + C_2 + \ldots + C_r) \leq \alpha < 1$. We can assure the last inequality by making ε small enough. Hence, condition 1) of Theorem 3.1 holds.

Verification of condition 2) of Theorem 3.1 is done by estimating the norm of $\|\Lambda(0)\|$. By using the definition of $\Lambda(0)$, the p-factor-approximation condition and (15), it follows that there is a constant $C > 0$ such that

$$\|\Lambda(0)\| = \|\{\Psi_p(h)\}^{-1} F(x, y^* + h)\| \leq C\|h\|.$$

Then by using the Banach condition, we get

$$\|\Lambda(0)\| \leq \bar{C} \sum_{r=1}^{p} \|f_r(x, y^*)\|_{Z_r}^{1/r} < (1 - \alpha) r_1.$$

The last inequality holds for $x \in \hat{\Gamma} = \Gamma(x^*) \cap B_\delta(x^*)$ where $B_\delta(x^*) \subset N(x^*)$ and δ is sufficiently small.

By Theorem 3.1, for $x \in \hat{\Gamma}$, there exists $\theta(x)$ such that

$$\theta(x) \in \Lambda(\theta(x)) \tag{22}$$

and

$$\|\theta(x)\| \leq \frac{2}{1 - \alpha}\|\Lambda(0)\| \leq \hat{C}\|h(x)\|.$$

The inclusion (22) is equivalent to

$$0 \in \{\Psi_p(h)\}^{-1} \left(F(x, y^* + h + \theta(x)\right).$$

Hence,

$$F(x, y^* + h + \theta(x)) = 0.$$

Let $\varphi(x) = y^* + h(x) + \theta(x)$. Then $F(x, \varphi(x)) = 0$ and

$$\|\varphi(x) - y^*\| \leq \|h(x)\| + \|\theta(x)\| \leq \|h(x)\| + C\varepsilon\|h(x)\| \leq \tilde{C}\|h(x)\|.$$

By the Banach condition (11), we see that

$$\|\varphi(x) - y^*\| \leq \tilde{C}\|h(x)\| \leq k \sum_{r=1}^{p} \|f_r(x, y^*)\|_{Z_r}^{1/r}$$

for all $x \in \hat{\Gamma}$. By substituting x^* into the last inequality we get $\varphi(x^*) = y^*$, which completes the proof. □

As will be shown in Section 4, Theorem 3.2 is not applicable to some boundary-value problems for nonlinear ordinary-differential equations. For example, consider the problem in which one modifies the boundary conditions in problem (2). Namely, consider (3) and, instead, the following BVP:

$$y''(t) + y(t) + g(y(t)) = x(t), \quad y(0) = y(2\pi) = 0 \tag{23}$$

with corresponding modifications of the definitions of functions x, y, and g. We will show in Section 4 that Theorem 3.2 is applicable to derive a result about existence of the solution of problem (2)–(3), but it is not applicable to analyze problem (23).

However, as will be explained in Section 4, the following theorem is applicable to derive an existence theorem for problem (23).

Theorem 3.3 (Existence of an Implicit Function in the case of a Nontrivial Kernel)**.** *Let X, Y and Z be Banach spaces, $F \in C^{p+1}(X \times Y)$, and $F(x^*, y^*) = 0$. Let mappings $f_i(x, y)$, $i = 1, \dots, p$, be given as in (8) and let Ψ_p be defined by (9). Assume that there exists an element $\bar{h} \in \bigcap_{r=1}^{p} \mathrm{Ker}^r f_{r\,y}^{(r)}(x^*, y^*)$, $\|\bar{h}\| = 1$, such that $\mathrm{Im}\Psi_p(\bar{h}) = Z$.*

Then for a sufficiently small $\varepsilon > 0$, $\nu > 0$, and $\delta = \varepsilon\nu^p$, there exists a mapping $\varphi(x) : B_\delta(x^*) \to B_\varepsilon(y^*)$, and constants $k > 0$ and $c_1 > 0$ such that the following hold:

a) $\varphi(x^*) = y^*$;

b) $F(x, \varphi(x)) = 0$ for all $x \in B_\delta(x^*)$;

c) $\varphi(x) = y^* + h(x) + \bar{y}(x)$, where $h(x)$ is defined by

$$h(x) = \gamma(x)\bar{h}$$

and $\gamma(x)$ satisfies

$$c_1\|x - x^*\|^{1/p} \le \|\gamma(x)\| \le \nu.$$

Moreover,

$$\|\bar{y}(x)\|_Y \le k \sum_{r=1}^{p} \frac{\|f_r(x, y^* + h(x)\|_{Z_r}}{\|\gamma(x)\|^{(r-1)}}, \quad x \in B_\delta(x^*), \quad \gamma(x) \neq 0. \tag{24}$$

A slight modification of Theorem 3.3 was derived in [10] and [15].

Remark 1. Estimate (24) can be replaced by the following one:

$$\|\bar{y}(x)\|_Y \le K \sum_{r=1}^{p} \|f_r(x, y^* + h(x))\|_{Z_r}^{1/r}, \quad x \in B_\delta(x^*), \quad x \neq x^*,$$

where $K > 0$ is a constant.

Theorem 3.4. *Let $F(x, y) \in C^{p+1}(X \times Y)$, $F : X \times Y \to Z$, where X, Y and Z are Banach spaces. Assume that $F(x^*, y^*) = 0$ and F is p-regular with respect to y along $h \in \bigcap_{k=1}^{p} \mathrm{Ker}^k\, f_k^{(k)}(x^*, y^*)$, $h = (\bar{x}, 0)$, $\bar{x} \neq 0$; that is*

$$\left\{ f_1'(x^*, y^*) + f_2''(x^*, y^*)[h] + \cdots + f_p^{(p)}(x^*, y^*)[h]^{p-1} \right\} \cdot (\{0\} \times Y) = Z.$$

Then for $x = x^* + t\bar{x}$, $t \in [0, \varepsilon)$, $\varepsilon > 0$ there exists $y = y(x)$ such that

$$F(x, y(x)) = 0$$

and

$$\|y(x) - y^*\| \le C\|F(x, y^*)\|^{1/p},$$

where $C > 0$ is a constant (independent of x).

Proof. The proof is similar to the proof of Theorem 3.2 but with the mapping Λ defined as

$$\Lambda(\xi) = \xi - \left\{ f_1'(x^*, y^*) + \cdots + \frac{1}{(p-1)!} f_p^{(p)}(x^*, y^*)[h]^{p-1} \right\}_Y^{-1} \cdot F(x, y^* + \xi).$$

□

The following theorem is a modification of the preceding one.

Theorem 3.5. *Let $F(x,y) \in C^{p+1}(X \times Y)$, $F : X \times Y \to Z$, where X, Y and Z are Banach spaces. Assume that $F(x^*, y^*) = 0$ and F is p-regular with respect to y along $h \in \bigcap_{k=1}^{p} \operatorname{Ker}^k f_k^{(k)}(x^*, y^*)$, $h = (h_x, h_y)$, $\|h_x\|_X \neq 0$, $\|h_y\|_Y \neq 0$, that is*

$$\left\{ f_1'(x^*, y^*) + f_2''(x^*, y^*)[h] + \cdots + f_p^{(p)}(x^*, y^*)[h]^{p-1} \right\} \cdot (\{0\} \times Y) = Z.$$

Then for $x = x^* + th_x$, $t \in [0, \varepsilon)$, $\varepsilon > 0$, there exists $y = y(x)$ such that

$$F(x, y(x)) = 0$$

and

$$\|y(x) - y^*\| \leq C \|F(x, y^* + th_y)\|^{1/p},$$

where $C > 0$ is a constant (independent of x).

Proof. The proof is similar to the proof of Theorem 3.2 but with the mapping Λ defined as

$$\Lambda(\xi) = \xi - \left\{ f_1'(x^*, y^*) + \cdots + \frac{1}{(p-1)!} f_p^{(p)}(x^*, y^*)[h]^{p-1} \right\}_Y^{-1} \cdot F(x, y^* + th_y + \xi).$$

□

4. **Existence: the degenerate boundary-value problem.** This section illustrates the application of the degenerate implicit function theorems from the preceding section to the specific boundary-value problems given in the introduction. Of course the method applies to other degenerate boundary value problems as well, but to keep the exposition concrete, we focus on two particular ones.

4.1. **The first existence theorem.** This subsection is concerned with the problem of existence of a nontrivial solution of BVP (2)–(3) under the assumption (4). Define the mapping

$$F(x,y) = y'' + y + g(y) - x, \tag{25}$$

where F is a C^{p+1} mapping from $X \times Y$ to Z, where, as before,

$$X = \{x \in C[0,\pi] \mid x(0) = x(\pi) = 0\}, \quad Y = \{y \in C^2[0,\pi] \mid y(0) = y(\pi) = 0\},$$

and $Z = C[0,\pi]$. Recall that g is a C^{p+1} mapping from $\mathbb{R}$ to $\mathbb{R}$ satisfying $g(0) = g'(0) = \ldots = g^{(p-1)}(0) = 0$ for some $p \in \mathbb{N}$.

Then we can rewrite equation (2) as

$$F(x,y) = 0.$$

Having a trivial solution means, as before, that $F(0,0) = 0$.

The problem of existence of a solution of the nonhomogeneous BVP for a given x is equivalent to the problem of existence of an implicit function $y = \varphi(x)$ such that (7) holds:

$$F(x,y) = y'' + y + g(y) - x = 0, \qquad y(0) = y(\pi) = 0.$$

Without loss of generality, we restrict our attention to some neighborhood of the point $(x^*(t), y^*(t)) = (0,0)$, $t \in [0,\pi]$.

As we showed in the introduction, the operator $F_y'(0,0)$ is not surjective. Hence, the classical Implicit Function Theorem cannot be applied to guarantee existence of an implicit function $y = \phi(x)$ such that (7) holds. However, we can apply the pth-order Implicit Function Theorem 3.2 to derive conditions for the existence of the implicit function $y = \varphi(x)$, and, hence, for existence of the solution of BVP (2)–(3).

To apply Theorem 3.2, we will first introduce some auxiliary spaces and functions for the mapping $F(x,y)$ in accordance with Section 2.

The image of the operator $F'_y(0,0)$ is the set of all $z(t) \in Z$ such that there exists a ξ satisfying

$$\xi'' + \xi = z(t), \quad \xi(0) = \xi(\pi) = 0. \tag{26}$$

The general solution of (26) has the form:

$$\xi(t) = C_1 \cos t + C_2 \sin t - \sin t \int_0^t \cos\tau z(\tau) d\tau + \cos t \int_0^t \sin\tau z(\tau) d\tau, \quad C_1, C_2 \in \mathbb{R}.$$

By substituting the boundary conditions we get $C_1 = 0$ and

$$\int_0^\pi \sin\tau z(\tau) d\tau = 0.$$

Hence,

$$Z_1 = \operatorname{Im} F'_y(0,0) = \left\{ z(\cdot) \in Z \;\middle|\; \int_0^\pi \sin\tau z(\tau) d\tau = 0 \right\} \tag{27}$$

and as expected, $Z_1 \neq Z$.

The following boundary value problem defines the kernel of $F'_y(0,0)$:

$$\xi'' + \xi = 0, \quad \xi(0) = \xi(\pi) = 0.$$

This problem has the solution $\xi(t) = C \sin t$, C is a constant. Hence $\operatorname{Ker}(F'_y(0,0)) = \operatorname{span}(\sin t)$ and, as is easy to verify, $W_2 = \operatorname{span}(\sin t)$.

As in [11], the projector P_{W_2} can be defined as

$$P_{W_2} z = \frac{2}{\pi} \sin t \int_0^\pi \sin(\tau) z(\tau) d\tau, \quad z \in Z. \tag{28}$$

Using equation (4), we see that $g'(0) = 0$ and

$$F''_{yy}(0,0) = g''(0), \ldots, F^{(p)}_{y\ldots y}(0,0) = g^{(p)}(0). \tag{29}$$

Then

$$\begin{aligned} Z_2 &= \operatorname{span}\left(\operatorname{Im} P_{W_2} F''_{yy}(0,0)[\cdot]^2\right) \\ &= \operatorname{span}\Big\{ z(t) \mid \text{there exists } y \in Y \text{ such that} && (30) \\ &\qquad z(t) = \frac{2}{\pi} \sin t \int_0^\pi \sin\tau g''(0)[y(\tau)]^2 d\tau \Big\}. && (31) \end{aligned}$$

The other spaces $Z_3, \ldots, Z_p$ can be determined in a similar way and depend only on the mapping $g(y)$. Next, we define mappings $f_1(x,y), \ldots, f_p(x,y)$ as follows:

$$f_1(x,y) = F(x,y), \quad f_i(x,y) = P_{Z_i} F(x,y), \quad i = 2, \ldots, p. \tag{32}$$

Note that by (29), we have

$$f^{(i)}_{i\,y\ldots y}(0,0)[\cdot]^{i-1} = P_{Z_i} F^{(i)}_{y\ldots y}(0,0)[\cdot]^{i-1} = P_{Z_i} g^{(i)}(0)[\cdot]^{i-1}, \quad i = 2, \ldots, p. \tag{33}$$

In this case, the p-factor-operator has the following form:

$$\Psi_p(h) = (\cdot)'' + (\cdot) + P_{Z_2} g''(0)[h] + \cdots + \frac{1}{(p-1)!} P_{Z_p} g^{(p)}(0)[h]^{p-1}. \tag{34}$$

Note that Condition 1) of Theorem 3.2 holds for F because of the definition of mappings $f_i(x,y)$ and $g(y)$. Let us reformulate conditions 2)–4) of Theorem 3.2

for the mapping F defined by (25) and for $x^*(t) = 0$ and $y^*(t) = 0$. In this case, condition 2) (the p-factor-approximation condition) depends only on the properties of the mapping $g(y)$ and reduces to existence of a neighborhood $N(y^*)$ such that for a sufficiently small $\varepsilon > 0$,

$$\left\| P_{Z_k}\left(g(y_1) - g(y_2) - \frac{1}{k!}g^{(k)}(0)[y_1]^k + \frac{1}{k!}g^{(k)}(0)[y_2]^k\right)\right\| \\ \leq \varepsilon\left(\|y_1\|^{k-1} + \|y_2\|^{k-1}\right)\|y_1 - y_2\|, \quad k = 1, \dots, p, \quad y_1, y_2 \in N(y^*). \tag{35}$$

Condition 3) is equivalent to existence of a neighborhood $N(x^*)$ such that for some $x \in N(x^*)$ there is a function $h = h(x,t)$ and $c_1 > 0$ such that

$$h'' + h + \frac{1}{2}P_{Z_2}g''(0)[h]^2 + \cdots + \frac{1}{p!}P_{Z_p}g^{(p)}(0)[h]^p = x(t) \tag{36}$$

and

$$\|h(t)\| \leq c_1\|F(x,0)\|^{1/p}. \tag{37}$$

Condition 4) is equivalent to existence $c_2 > 0$ such that for $x \in N(x^*)$ defined in condition 3) and $M = \Phi_p^{-1}(-F(x,y^*))$,

$$\sup_{h\in M}\left\|\left\{(\cdot)'' + (\cdot) + P_{Z_2}g''(0)[\bar{h}] + \cdots + \frac{1}{(p-1)!}P_{Z_p}g^{(p)}(0)[\bar{h}]^{p-1}\right\}^{-1}\right\| \leq c_2, \tag{38}$$

where $\bar{h}(t) = h(t)/\|h(t)\|$.

Summarizing, we can formulate the following result, which follows from Theorem 3.2 for the mapping F defined in (25).

Theorem 4.1. *Let for the BVP (2)–(3) condition (4) hold, $x^*(t) = 0$, $y^*(t) = 0$, $t \in [0,\pi]$, and $F(x^*,y^*) = 0$. Assume that there exist neighborhoods $N(x^*)$ and $N(y^*)$ such that conditions (35)–(38) are satisfied. Then there exist $\sigma > 0$, such that $B_\sigma(x^*) \subset N(x^*)$ and for any $x(t) \in B_\sigma(x^*)$ there exists a solution $y = y(x(t),t)$ of the BVP such that*

$$\|y(x(t),t)\| \leq m\|x(t)\|^{1/p},$$

where $m > 0$ is an independent constant.

We illustrate application of Theorem 4.1 by the following examples.

Example 1. Consider the following ODE boundary value problem:

$$y''(t) + y(t) + y^2(t) = v\sin t, \qquad y(0) = y(\pi) = 0, \tag{39}$$

Here $g(y) = y^2$, $x(t) = v\sin t$, $F(x,y) = y'' + y + y^2 - v\sin t$, v is a constant and $F : X \times Y \to Z$, X, Y and Z were defined above. Let us verify that all conditions of Theorem 4.1 are satisfied for the mapping $F(x,y)$ with a sufficiently small $v > 0$ and $p = 2$. As is evident, $y^* \equiv 0$ is a solution of the homogeneous boundary problem corresponding to (39); thus, $F(x^*,y^*) = 0$.

For $p = 2$, condition (35) reduces to existence of a sufficiently small $\varepsilon > 0$ and a neighborhood $N(y^*)$ such that for all $y_1, y_2 \in N(y^*)$,

$$\left\|P_{Z_1}(y_1^2 - y_2^2)\right\| \leq \left\|y_1^2 - y_2^2\right\| \leq \varepsilon\|y_1 - y_2\|,$$

and

$$\left\|P_{Z_2}\left(y_1^2 - y_2^2 - y_1^2 + y_2^2\right)\right\| \leq \varepsilon\left(\|y_1\| + \|y_2\|\right)\|y_1 - y_2\|.$$

Both of the last two inequalities hold, so condition (35) is satisfied.

Note that for $p = 2$ we have $W_2 = Z_2$ and P_{Z_2} is defined by (28). Then condition (36) for this example reduces to existence of a neighborhood $N(x^*)$ such that for some $x \in N(x^*)$ there is a function $h = h(x(t)) \neq 0$ and $c_1 > 0$ such that

$$h'' + h + \frac{2}{\pi} \sin t \int_0^\pi \sin(\tau) h^2(\tau) d\tau = v \sin t. \tag{40}$$

Problem (40) has a solution

$$h(t) = \sqrt{\frac{3\pi v}{8}} \sin t \tag{41}$$

only for $v > 0$. Then condition (37) reduces to existence of a constant $c_1 > 0$ such that

$$\left\| \sqrt{\frac{3\pi v}{8}} \sin t \right\| \leq c_1 \|v \sin t\|^{1/2}.$$

The last inequality is equivalent to

$$\sqrt{\frac{3\pi}{8}} \leq c_1,$$

which holds, for example, with $c_1 = \sqrt{\frac{3\pi}{8}}$.

To verify (38), we observe that with $x = v \sin t$ and h defined by (41), the set $\Phi_2^{-1}(-F(x, y^*))$ is simply given by the point $\{h\}$. Then the operator $\Psi_2(\bar{h})$ is

$$\Psi_2(\bar{h}) = (\cdot)'' + (\cdot) + \frac{4}{\pi} \sin t \int_0^\pi \sin \tau \bar{h}(\tau)(\cdot) d\tau, \quad \bar{h}(t) = \sin t,$$

which is surjective and, hence, (38) holds.

Hence, all conditions of Theorem 4.1 are satisfied and there exists a solution $y(t)$ of the BVP (39) such that

$$\|y(t)\| \leq c \|v \sin t\|^{1/2} \leq c v^{1/2}, \quad c > 0.$$

Example 2. Consider

$$y''(t) + y(t) + y^k(t) = v \sin t, \qquad y(0) = y(\pi) = 0, \tag{42}$$

Here $g(y) = y^k$, $x(t) = v \sin t$, $F(x, y) = y'' + y + y^k - v \sin t$, $F : X \times Y \to Z$, and X, Y and Z were defined above. Similar to Example 1, we can verify that all conditions of Theorem 4.1 are satisfied for the mapping $F(x, y)$ with a sufficiently small $v \geq 0$, $p = k$, and even k. For an odd k, all the conditions of Theorem 4.1 are satisfied for the mapping $F(x, y)$ with a sufficiently small v (of any sign).

Example 3. Consider

$$y''(t) + y(t) + y^k(t) = v \sin^k t, \qquad y(0) = y(\pi) = 0, \tag{43}$$

Similar to Example 1, we can verify that all conditions of Theorem 4.1 are satisfied for the mapping $F(x, y) = y''(t) + y(t) + y^k(t) - \sin^k t$ with a sufficiently small $v \geq 0$ and $p = k$. We also observe that $y(t) = v^{1/k} \sin t$ is a solution to (43).

Reasoning similar to that in Example 3 also applies to the problem with the modified boundary conditions:

$$y(0) = y(2\pi) = 0.$$

We consider this case in the following section.

Recall that Theorem 4.1 follows from Theorem 3.2. Similarly, Theorem 3.2 can also be applied to the nonhomogeneous *van der Pol equation*:

$$y''(t) - \mu(1-y^2)y'(t) + y = v \sin t, \qquad y(0) = y(2\pi) = 0, \tag{44}$$

where μ is a real parameter. Introducing the mapping

$$F = y''(t) - \mu(1-y^2)y'(t) + y - v \sin t,$$

one can show that for $\mu = \sqrt{3}$ the operator $F'_y(0,0)$ is nonregular (degenerate), and $F''_{yy}(0,0) = 0$. Then one can verify that the conditions of Theorem 3.2 are satisfied for the mapping F with $p = 3$ and a sufficiently small v.

4.2. **The second existence theorem.** If we assume that equation (36) does not have a solution or has a trivial solution $h = 0$, then Theorem 4.1 is not applicable. We illustrate this situation in Example 4 below. In this subsection we derive another existence theorem, which is applicable to some other classes of the BVPs problems.

We modify the boundary conditions in problem (2)–(3) and consider the following BVP:

$$y''(t) + y(t) + g(y(t)) = x(t), \quad y(0) = y(2\pi) = 0 \tag{45}$$

with corresponding modifications of the definitions of functions x, y, and g. Now, we apply Theorem 3.3 to to derive conditions for the existence of the implicit function $y = \varphi(x)$, and, hence, for existence of the solution of BVP (45).

To apply Theorem 3.3, we introduce mapping $F : X \times Y \to Z$, defined in (25), where

$$\begin{aligned} X &= \{x \in C[0,2\pi] \mid x(0) = x(2\pi) = 0\}, \\ Y &= \{y \in C^2[0,2\pi] \mid y(0) = y(2\pi) = 0\}, \\ Z &= C[0,2\pi]. \end{aligned} \tag{46}$$

The problem of existence of a solution of the nonhomogeneous BVP for a given x is equivalent to the problem of existence of an implicit function $y = \varphi(x)$ such that

$$F(x,y) = y'' + y + g(y) - x = 0, \qquad y(0) = y(2\pi) = 0. \tag{47}$$

Without loss of generality, we again restrict our attention to some neighborhood of the point $(x^*(t), y^*(t)) = (0,0)$, $t \in [0,2\pi]$.

As in Section 4.1, we introduce some auxiliary spaces and functions for the mapping $F(x,y)$ in accordance with Section 2. Namely, for F defined by (47), we have

$$Z_1 = \operatorname{Im} F'_y(0,0) = \left\{ z(\cdot) \in Z \;\middle|\; \int_0^{2\pi} \sin\tau\, z(\tau) d\tau = 0 \right\}$$

and as expected, $Z_1 \neq Z$. Moreover, $\operatorname{Ker}(F'_y(0,0)) = \operatorname{span}(\sin t)$ and as is easy to verify that $W_2 = \operatorname{span}(\sin t)$.

Then, in accordance with [11] the projector P_{W_2} can be defined as

$$P_{W_2} z = \frac{1}{\pi} \sin t \int_0^{2\pi} \sin(\tau) z(\tau) d\tau, \quad z \in Z. \tag{48}$$

We assume that $g(0) = g'(0) = 0$ and then

$$F''_{yy}(0,0) = g''(0), \ldots, F^{(p)}_{y\ldots y}(0,0) = g^{(p)}(0).$$

Furthermore,

$$Z_2 = \operatorname{span} \left\{ z(t) \middle| \text{ there exists } y \in Y \text{ such that } z(t) = \frac{1}{\pi} \sin t \int_0^{2\pi} \sin\tau\, g''(0)[y(\tau)]^2 d\tau \right\}.$$

The other spaces $Z_3, \dots, Z_p$ can be determined in a similar way and depend only on the mapping $g(y)$.

The following is the reformulation of Theorem 3.3 for the mapping F defined by (47) and for $x^*(t) = 0$ and $y^*(t) = 0$.

Theorem 4.2 (The second existence theorem for BVPs)**.** *Let X, Y and Z be defined by (46), and for BVP (45) mapping F be defined by (47), $F \in C^{p+1}(X \times Y)$, $x^*(t) = 0$ and $y^*(t) = 0$, $t \in [0, 2\pi]$. Let mappings $f_1(x,y), \dots, f_p(x,y)$ be defined by (32), and the p-factor-operator Ψ_p be given by (34). Assume that there exists an element $\bar{h} \in \bigcap_{r=1}^{p} \operatorname{Ker}^r f_{r\,y}^{(r)}(x^*, y^*)$, $\|\bar{h}\| = 1$ such that $\operatorname{Im}\Psi_p(\bar{h}) = Z$.*

Then for a sufficiently small $\varepsilon > 0$, $\nu > 0$, and $\delta = \varepsilon\nu^p$, there exists a solution $y = y(x(t), t)$, of BVP (45) such that

$$y(x(t), t) = \nu\bar{h} + w(x(t), t), \quad \forall x \in B_\delta(x^*), \quad t \in [0, 2\pi], \tag{49}$$

where

$$\|w(x(t), t)\| = o(\nu). \tag{50}$$

Remark 2. As follows from Theorem 4.2, that for $x^*(t) = 0$, $t \in [0, 2\pi]$, in addition to the trivial solution $y^*(t) = 0$, $t \in [0, 2\pi]$, there is also a nontrivial solution $y(x^*, t)$ given by (49).

We illustrate the application of Theorem 4.2 with the following example. Theorem 4.2 applied to problem (51) can be viewed as a special variant of the Poincaré–Andronov–Hopf Theorem concerning the existence of a nontrivial solution of the boundary value problem (51).

Example 4. In this example, we modify problem (39), which is given in Example 1, and consider the boundary-value problem:

$$y''(t) + y(t) + y^2(t) = 0, \qquad y(0) = y(2\pi) = 0. \tag{51}$$

Note that Theorem 4.1 is not applicable to (51), since condition (38) requires $h \neq 0$, and, at the same time, condition (37) yields $h = 0$ in this example.

To apply Theorem 4.2, consider $\bar{h} = v \sin t$, $v \in \mathbb{R}$. It was shown above that $\operatorname{Ker} F'_y(0,0) = \operatorname{Ker} f'_{1y}(0,0) = \operatorname{span}(\sin t)$. Moreover,

$$\frac{1}{2\pi} \sin t \int_0^{2\pi} \sin(\tau) \sin^2(\tau) d\tau = 0,$$

and hence, $\bar{h} \in \bigcap_{r=1}^{2} \operatorname{Ker}^r f_{r\,y}^{(r)}(x^*, y^*)$. Moreover, for problem (51), the 2-factor-operator is given by

$$\Psi_2(h) = (\cdot)'' + (\cdot) + \frac{\sin t}{\pi} \int_0^{2\pi} \sin\tau\, h(\tau)(\cdot) d\tau, \quad h(t) = \sin t,$$

and $\operatorname{Im}\Psi_2(h) = Z$. Therefore, the conditions of Theorem 4.2 are satisfied, and, by Theorem 4.2, there is a sufficiently small $\nu > 0$ and a solution $y = y(0, t)$ of BVP (51) such that

$$y(0, t) = \nu \sin t + w(0, t),$$

where

$$\|w(0, t)\| = o(\nu).$$

5. **A BVP perturbation method.** This section develops a modification of the standard perturbation method for solving degenerate second-order boundary value problems with a small parameter.

5.1. **Introduction to the modified perturbation method.** We start this section with considering the following BVP:

$$y''(t) + y(t) + \mu y^2(t) = \sin t, \qquad y(0) = y(\pi) = 0, \tag{52}$$

where $\mu > 0$ is a parameter.

Note that BVP (52) is a specific case of the following problem with the parameter $a = 1$:

$$\ddot{y}(t) + a^2 y(t) + \mu y^2(t) = \sin t, \qquad y(0) = y(\pi) = 0. \tag{53}$$

A standard method for solving BVP (53) is the perturbation method in which one seeks a solution $y = y(t)$ of the form:

$$y(t) = y_0(t) + \mu y_1(t) + \mu^2 y_2(t) + \dots. \tag{54}$$

Substituting (54) into (53) and comparing the coefficients of the similar powers of μ gives the following problem to determine the function $y_0(t)$:

$$y_0''(t) + a^2 y_0(t) = \sin t, \qquad y_0(0) = y_0(\pi) = 0. \tag{55}$$

However, problem (55) does not have a solution with $a = 1$ and, hence, the perturbation method is not applicable to BVP (52).

In this section, we use the results derived in Section 4 to construct a modified perturbation method, which is applicable to BVP (52). The modified perturbation method has two realizations. In both realizations we use an additional term that is a multiple of $\mu^{-1/2}$. Moreover, we also use the fractional powers of μ in a series representation of y.

First realization of the modified perturbation method. Let us modify the perturbation method and look for a solution $y(t)$ of BVP (52) in the form:

$$y(t) = h(t) + y_0(t) + \mu^{1/2} y_1(t) + \mu y_2(t) + \mu^{3/2} y_3(t) + \dots, \tag{56}$$

where $h(t)$ is defined as a solution of equation (36) with $p = 2$. Namely, taking into account that $Z_2 = W_2$, and using (28) to define P_{Z_2} and (31) to define Z_2, we get the following equation to determine $h(t)$:

$$h''(t) + h(t) + \frac{2\mu \sin t}{\pi} \int_0^\pi \sin \tau h^2(\tau) d\tau = \sin t.$$

The last equation has a solution

$$h(t) = \sqrt{\frac{3\pi}{8\mu}} \sin t.$$

Substitution of (56) into (52) and comparing the coefficients of μ^0, gives the following equation that determines the function $y_0(t)$:

$$y_0''(t) + y_0(t) + \mu h^2(t) = \sin t$$

or

$$y_0''(t) + y_0(t) = \sin t - \frac{3\pi}{8} \sin^2 t.$$

The last equation has a solution

$$y_0(t) = \frac{\pi}{4} \cos t + C \sin t - \frac{t}{2} \cos t - \frac{3\pi}{16} - \frac{\pi}{16} \cos 2t,$$

where C is a constant.

Then in the equation that we obtained by substituting (56) into (52), we compare the coefficients of $\mu^{1/2}$ to define $y_1(t)$ and so on.

In this way, we get an approximate solution of the BVP (52):

$$y(t) = \frac{\sqrt{3\pi}}{2\sqrt{2}\mu^{1/2}}\sin t + y_0(t) + \mu^{1/2}y_1(t) + \dots.$$

Second realization of the modified perturbation method. First, we introduce the mapping F given by (25). In this example, F is 2-regular with the space Z_1 given by (27), Z_2 given by (31) and the projector $P_2 = P_{W_2}$ on $Z_2 = W_2$ defined by (28). Since $Z_1 \oplus Z_2 = Z$, we can consider the following system, which is equivalent to problem (52):

$$\begin{aligned} P_1(y''(t) + y(t) + \mu y^2(t)) &= P_1(\sin t), \\ P_2(y''(t) + y(t) + \mu y^2(t)) &= P_2(\sin t), \end{aligned} \tag{57}$$

where $y(0) = y(\pi) = 0$, and where P_i is the projector on Z_i, $i = 1, 2$.

We look for a solution $y(t)$ in the form:

$$y(t) = h(t) + y_0(t) + \mu^{1/2}y_1(t) + \mu y_2(t) + \mu^{3/2}y_3(t) + \dots, \tag{58}$$

where $y_i(t)$, $i = 0, 1\dots$ is defined as

$$y_i(t) = \tilde{y}_i(t) + \hat{y}_i(t), \quad P_1(\hat{y}_i) = 0,$$

and $\tilde{y}_i(t)$ and $\hat{y}_i(t)$ are defined below for every $i = 0, 1\dots$. Define $h(t)$ to be a solution of the following equation

$$P_2(\mu h^2(t)) = \sin t.$$

Using formula (28) for P_2 with $z(\tau) = h^2(\tau)$ we get:

$$\frac{2\mu \sin t}{\pi}\int_0^\pi \sin\tau h^2(\tau)d\tau = \sin t;$$

that is,

$$\int_0^\pi \sin\tau h^2(\tau)d\tau = \frac{\pi}{2\mu}. \tag{59}$$

Equation (59) has a solution

$$h(t) = \sqrt{\frac{3\pi}{8\mu}}\sin t.$$

Substituting (58) into the first equation of (57) and comparing the coefficients of μ^0 yields the following equation for $\tilde{y_0}$:

$$P_1\left(\tilde{y_0}''(t) + \tilde{y_0}(t) + \frac{3\pi}{8}\sin^2(t)\right) = P_1(\sin t).$$

The last equation gives

$$\tilde{y_0}''(t) + \tilde{y_0}(t) + \frac{3\pi}{8}\sin^2 t - \sin t = 0, \quad \tilde{y_0}(0) = \tilde{y_0}(\pi) = 0.$$

This problem has a solution

$$\tilde{y}_0(t) = \frac{\pi}{4}\cos t + C\sin t - \frac{t}{2}\cos t - \frac{3\pi}{16} - \frac{\pi}{16}\cos 2t, \tag{60}$$

where C is a constant. Now, by substituting (58) into the second equation of (57) we get the following equation for $\hat{y}_0$:

$$\frac{2\mu \sin t}{\pi}\int_0^\pi \sin\tau(h^2(\tau)+2h(\tau)\tilde{y}_0(\tau)+2h(\tau)\hat{y}_0(\tau))d\tau = \sin t.$$

The last equation yields the following one, corresponding to the coefficients of $\mu^{1/2}$:

$$\begin{aligned} 0 &= \int_0^\pi \sin(\tau)(\mu h(\tau)\tilde{y}_0(\tau)+\mu h(\tau)\hat{y}_0(\tau))d\tau \\ &= \int_0^\pi \sin^2(\tau)(\tilde{y}_0(\tau)+\hat{y}_0(\tau))d\tau. \end{aligned}$$

We will look for $\hat{y}_0$ in the form $\hat{y}_0 = A\sin t$ where A is a constant. Then

$$A\int_0^\pi \sin^2(\tau)\sin(\tau)d\tau = -\int_0^\pi \sin^2(\tau)(\tilde{y}_0(\tau))d\tau,$$

hence,

$$A = -\frac{3}{4}\int_0^\pi \sin^2(\tau)(\tilde{y}_0(\tau))d\tau.$$

By substituting $\tilde{y}_0(\tau)$ defined in (60) and integrating the last equation we get A and, hence, $\hat{y}_0$.

Then, we substitute $y(t)$ into the first equation of (57) and compare the coefficients of $\mu^{1/2}$ to define $\tilde{y}_1(t)$. After that we substitute $y(t)$ into the second equation of (57) to define $\hat{y}_1(t)$, and so on.

Eventually, we get an approximated solution of the BVP (52):

$$y(t) = \frac{\sqrt{3\pi}}{2\sqrt{2}\mu^{1/2}}\sin t + \tilde{y}_0(t) + \hat{y}(t) + \mu^{1/2}(\tilde{y}_1(t)+\hat{y}_1(t)) + \dots. \tag{61}$$

5.2. **Modified perturbation method.** Now we describe how to modify the perturbation method for solving BVP of the form:

$$y''(t)+y(t)+\mu g(y(t)) = \sin t, \qquad y(0)=y(\pi)=0, \tag{62}$$

where $\mu > 0$ is a small parameter and the function g satisfies the conditions:

$$g(0) = g'(0) = \ldots = g^{(p-1)}(0) = 0, \quad g^{(p)}(0) \neq 0, \tag{63}$$

for some $p \geq 1$. The realization described in this subsection is similar to the second realization of the modified perturbation method described in Section 5.1.

Again, if we try to apply the standard perturbation method to solve the BVP (62), then we are looking for a solution in the form:

$$y(t) = y_0(t) + \mu y_1(t) + \mu^2 y_2(t) + \dots. \tag{64}$$

Substituting (64) into (62) and comparing the coefficients of the similar powers of μ gives the following problem that defines the function $y_0(t)$:

$$y_0''(t) + y_0(t) = \sin t, \quad y_0(0) = y_0(\pi) = 0,$$

However, this problem does not have a solution.

To construct a modified perturbation method for the BVP (62), we introduce the mapping F as

$$F(x,y) = y''(t) + y(t) + \mu g(y(t)) - x.$$

For this mapping, the space Z_1 is defined by (27), $Z_2 = \ldots = Z_{p-1} = 0$ and

$$Z_p = \mathrm{span}(\mathrm{Im}\, P_{W_p} F^{(p)}_{y\ldots y}(0,0)[\cdot]^p),$$

where $P_p = P_{W_p}$ is projection operator onto W_p defined by

$$P_p z = \frac{2}{\pi} \sin t \int_0^{\pi} \sin \tau [z(\tau)] d\tau, \quad z \in \mathbb{Z}. \tag{65}$$

Under our assumptions, F is p-regular at $(0,0)$ and, hence, $Z_p = W_p$. Moreover, $Z = Z_1 \oplus Z_p$. Hence, the following system is equivalent to BVP (62):

$$\begin{aligned} P_1(y''(t) + y(t) + \mu g(y(t))) &= P_1(\sin t), \\ P_p(y''(t) + y(t) + \mu g(y(t))) &= P_p(\sin t), \end{aligned} \tag{66}$$

where $y(0) = y(\pi) = 0$, and where P_1 and P_p are projectors onto Z_1 and Z_p, respectively.

In the modified perturbation method, we look for a solution $y(t)$ of the BVP (62) in the form:

$$y(t) = h(t) + y_0(t) + \mu^{1/p} y_1(t) + \mu^{2/p} y_2(t) + \mu^{3/p} y_3(t) + \dots, \tag{67}$$

where $y_i(t)$, $i = 0, 1 \dots$, is defined as

$$y_i(t) = \tilde{y}_i(t) + \hat{y}_i(t), \quad P_1(\hat{y}_i) = 0,$$

and $\tilde{y}_i(t)$ and $\hat{y}_i(t)$ are defined below for every $i = 0, 1 \dots$. The function $h(t)$ is defined to be a solution of the equation

$$P_p \left(\frac{1}{p!} \mu g^{(p)}(0)[h(t)]^p \right) = \sin t.$$

Using formula (65) for P_p with $z(\tau) = h^p(\tau)$, we get the following equation for $h(t)$:

$$\frac{2\mu g^{(p)}(0) \sin t}{\pi p!} \int_0^{\pi} \sin \tau h^p(\tau) d\tau = \sin t.$$

Substituting (67) into the first equation of (66) and comparing the coefficients of μ^0, gives an equation for $\tilde{y_0}$. Then, substituting (67) into the second equation of (66) and comparing the coefficients of $\mu^{1/2}$, one obtains an equation that determines the function $\hat{y}_0$. After $\tilde{y_0}$ and $\hat{y}_0$ are determined, we substitute $y(t)$ into the first equation of (66) and compare the coefficients of $\mu^{1/2}$ to obtain $\tilde{y}_1(t)$. Subsequently, we substitute $y(t)$ into the second equation of (66) to obtain $\hat{y}_1(t)$, and so on. Proceeding this way, we get successively better approximate solutions of the BVP (66).

6. **Conclusions.**

The overall aim of this work has been to develop and apply p-regularity theory, the basic features of which were constructed in [10, 11, 17].

There are three major parts in the present paper. The first part (Section 3) considers the equation $F(x,y) = 0$, where F is a smooth nonlinear mapping between Banach spaces $X \times Y$ and Z. The main concern is the case when the mapping F is nonregular at some point (x^*, y^*) with respect to y, i.e., when the derivative $F_y(x^*, y^*)$ is not invertible and, hence, the classical Implicit Function Theorem is not applicable. The pth-order generalizations of the Implicit Function Theorem were proposed for this case. The second part of the paper (Section 4) applies these pth-order implicit function theorems to obtain sufficient conditions for the existence of a solution of degenerate second-order nonlinear boundary-value problems. The third part of the paper (Section 5) develops a modified perturbation method for solving degenerate second-order boundary value problems with a small parameter. The method is based on the methodology developed in the second part of the paper.

The development and applications of p-regularity theory given in this paper help point the way to other interesting applications that should be possible in future research. Specifically, it would be of interest to establish further links between p-regularity theory and bifurcation theory, singularity theory, as well as with algebraic geometry.

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