Dirac Structures and the Legendre Transformation for Implicit Lagrangian and Hamiltonian Systems

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Summary. This paper begins by recalling how a constraint distribution on a configuration manifold induces a Dirac structure together with an implicit Lagrangian system, a construction that is valid even for degenerate Lagrangians. In such degenerate cases, it is shown in this paper that an implicit Hamiltonian system can be constructed by using a generalized Legendre transformation, where the primary constraints are incorporated into a generalized Hamiltonian on the Pontryagin bundle. Some examples of degenerate Lagrangians for L–C circuits, nonholonomic systems, and point vortices illustrate the theory.

1 Introduction

In recent years, the theory of implicit Hamiltonian systems has been developed along with associated formulations of physical systems, such as L-C circuits and nonholonomic systems. This is a useful analytical tool, in which Dirac structures are employed to help understand how interconnected system elements are energetically related and are systematically incorporated into the Hamiltonian formalism; see, for instance, [9, 2, 8, 1]. The notion of Dirac structures, which was first developed in [3], is also relevant to Dirac's theory of constraints for degenerate Lagrangian systems. However, research has only just begun on the theory of implicit Lagrangian systems, and, in addition there is a need to understand how they are related to implicit Hamiltonian systems as well as with Dirac's theory of constraints.

Recently, the theory of implicit Lagrangian systems, namely, a Lagrangian analogue of implicit Hamiltonian systems, has been developed by [10, 11]. This theory, which also makes use of Dirac structures, has similar examples that can be systematically treated from the Lagrangian viewpoint, namely nonholonomic mechanical systems and degenerate Lagrangian systems, such as L-C circuits.

In the present paper, we investigate systems with degenerate Lagrangians and, following [10], we first show how to construct a Dirac structure on the cotangent

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bundle T^*Q induced from a constraint distribution on a configuration manifold Q. Second, we demonstrate how an implicit Lagrangian system can be constructed from the induced Dirac structure. Using this framework, we show how to construct an implicit Hamiltonian system from a given, possibly degenerate, Lagrangian. To do this, we make use of a generalized Legendre transformation for degenerate Lagrangians to define a Hamiltonian on a constraint momentum space $P \subset T^*Q$ and also define a generalized Hamiltonian on the Pontryagin bundle $TQ \oplus T^*Q$ by combining primary constraints in the sense of Dirac with the Hamiltonian. Thus, we show how degenerate Lagrangian systems that are useful in L-C circuits as well as in nonholonomic systems, can be represented in the context of both implicit Lagrangian systems and Hamiltonian systems. Lastly, we illustrate an example of degenerate Lagrangians for point vortices and the KdV equations.

2 Induced Dirac Structures

Dirac Structures. We begin by reviewing the definition of a Dirac structure on a vector space, following [3].

Let V be an n-dimensional vector space, V^* be its dual space, and let $\langle \cdot\,, \cdot \rangle$ be the natural paring between V^* and V. Define the symmetric paring $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $V \oplus V^*$ by

$$\langle\!\langle\, (v, \alpha), (\bar{v}, \bar{\alpha})\,\rangle\!\rangle = \langle \alpha, \bar{v} \rangle + \langle \bar{\alpha}, v \rangle,$$

for $(v, \alpha), (\bar{v}, \bar{\alpha}) \in V \oplus V^*$. A *Dirac structure* on V is a subspace $D \subset V \oplus V^*$ such that $D = D^{\perp}$, where $D^{\perp}$ is the orthogonal of D relative to the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Let M be a smooth differentiable manifold whose tangent bundle is denoted as TM and whose cotangent bundle is denoted as T^*M. Let $TM \oplus T^*M$ denote the Whitney sum bundle over M; that is, it is the bundle over the base M and with fiber over the point $x \in M$ equal to $T_xM \times T_x^*M$. An *(almost) Dirac structure* on M is a subbundle $D \subset TM \oplus T^*M$ that is a Dirac structure in the sense of vector spaces at each point $x \in M$.

In geometric mechanics, (almost) Dirac structures provide a simultaneous generalization of both two-forms (not necessarily closed, and possibly degenerate) as well as almost Poisson structures (that is brackets that need not satisfy the Jacobi identity). An *integrable Dirac structure*, which corresponds in geometric mechanics to assuming the two-form is closed or to assuming Jacobi's identity for the Poisson tensor, is one that satisfies

$$\langle \pounds_{X_1} \alpha_2, X_3 \rangle + \langle \pounds_{X_2} \alpha_3, X_1 \rangle + \langle \pounds_{X_3} \alpha_1, X_2 \rangle = 0,$$

for all pairs of vector fields and one-forms (X_1, α_1), (X_2, α_2), (X_3, α_3) that take values in D and where $\pounds_X$ denotes the Lie derivative along the vector field X on M.

Induced Dirac Structures. We now construct induced Dirac structure, an essential ingredient in the setting of implicit Lagrangian systems; see [10].

Let Q be an n-dimensional configuration manifold, whose kinematic constraints are given by a constraint distribution $\Delta_Q \subset TQ$, which is defined, at each $q \in Q$, by

$$\Delta_Q(q) = \{v \in T_qQ \mid \langle \omega^a(q), v \rangle = 0,\ a = 1, ..., m\},$$

where ω^a are m one-forms on Q. Define the distribution Δ_{T^*Q} on T^*Q by

$$\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q,$$

where $T\pi_Q : TT^*Q \to TQ$ is the tangent map of $\pi_Q : T^*Q \to Q$, while the annihilator of Δ_{T^*Q} can be defined for each $z = (q,p) \in T^*Q$, by

$$\Delta^{\circ}_{T^*Q}(z) = \{\alpha_z \in T^*_zT^*Q \mid \langle \alpha_z,\ w_z \rangle = 0 \ \text{ for all } w_z \in \Delta_{T^*Q}(z)\}.$$

Let Ω be the canonical symplectic structure on T^*Q and $\Omega^\flat : TT^*Q \to T^*T^*Q$ be the associated bundle map. Then, a Dirac structure D_{Δ_Q} on T^*Q induced from the constraint distribution Δ_Q can be defined for each $z = (q,p) \in T^*Q$, by

$$D_{\Delta_Q}(z) = \{\, (w_z, \alpha_z) \in T_zT^*Q \times T^*_zT^*Q \mid w_z \in \Delta_{T^*Q}(z), \\ \text{and } \ \alpha_z - \Omega^\flat(z) \cdot w_z \in \Delta^{\circ}_{T^*Q}(z) \,\}.$$

Local Representation. Let us choose local coordinates q^i on Q so that locally, Q is represented by an open set $U \subset \mathbb{R}^n$. The constraint set Δ_Q defines a subspace of TQ, which we denote by $\Delta(q) \subset \mathbb{R}^n$ at each point $q \in U$. If the dimension of the constraint space is $n - m$, then we can choose a basis $e_{m+1}(q), e_{m+2}(q), \ldots, e_n(q)$ of $\Delta(q)$.

The constraint sets can be also represented by the annihilator of $\Delta(q)$, which is denoted by $\Delta^\circ(q)$, spanned by such one-forms that we write as $\omega^1, \omega^2, \ldots, \omega^m$. Since the cotangent bundle projection $\pi_Q : T^*Q \to Q$ is locally denoted as $(q,p) \mapsto q$, its tangent map may be locally given by $T\pi_Q : (q,p,\dot{q},\dot{p}) \mapsto (q,\dot{q})$. So, we can locally represent Δ_{T^*Q} as

$$\Delta_{T^*Q} \cong \left\{ v_{(q,p)} = (q,p,\dot{q},\dot{p}) \mid q \in U, \dot{q} \in \Delta(q) \right\}.$$

Then, the annihilator of Δ_{T^*Q} is locally represented as

$$\Delta^{\circ}_{T^*Q} \cong \left\{ \alpha_{(q,p)} = (q,p,\alpha,w) \mid q \in U,\ \alpha \in \Delta^\circ(q) \text{ and } w = 0 \right\}.$$

Because of the local formula $\Omega^\flat(z) \cdot v_z = (q,p,-\dot{p},\dot{q})$, the condition $\alpha_z - \Omega^\flat(z) \cdot v_z \in \Delta^{\circ}_{T^*Q}$ reads

$$\alpha + \dot{p} \in \Delta^\circ(q), \quad \text{and} \quad w - \dot{q} = 0.$$

Thus, the induced Dirac structure is locally represented by

$$D_{\Delta_Q}(z) = \{((q,p,\dot{q},\dot{p}), (q,p,\alpha,w)) \mid \dot{q} \in \Delta(q),\ w = \dot{q},\ \alpha + \dot{p} \in \Delta^\circ(q)\}. \quad (1)$$

3 Implicit Lagrangian Systems

Dirac Differential Operator. Let $L : TQ \to \mathbb{R}$ be a Lagrangian (possibly degenerate). The differential of L is the map $\mathbf{d}L : TQ \to T^*TQ$, which is locally given, for each $(q, v) \in TQ$, by

$$\mathbf{d}L = \left(q, v, \frac{\partial L}{\partial q}, \frac{\partial L}{\partial v} \right).$$

Define the *Dirac differential* of a Lagrangian L, to be the map

$$\mathfrak{D}L : TQ \to T^*T^*Q$$

defined by

$$\mathfrak{D}L = \gamma_Q \circ \mathbf{d}L.$$

Here, the map $\gamma_Q : T^*TQ \to T^*T^*Q$ is the natural symplectomorphism (see [10]), which is defined by

$$\gamma_Q = \Omega^\flat \circ \kappa_Q^{-1},$$

where $\Omega^\flat : TT^*Q \to T^*T^*Q$ is the induced map from Ω and $\kappa_Q : TT^*Q \to T^*TQ$ is the natural symplectomorphism (see [7]). In coordinates, the symplectomorphism $\gamma_Q : T^*TQ \to T^*T^*Q$ is given by

$$(q, \delta q, \delta p, p) \mapsto (q, p, -\delta p, \delta q)$$

and hence the Dirac differential of L is locally given, at each $(q, v) \in TQ$, by

$$\mathfrak{D}L = \left(q, \frac{\partial L}{\partial v}, -\frac{\partial L}{\partial q}, v \right). \tag{2}$$

Implicit Lagrangian Systems. An *implicit Lagrangian system* is a triple (L, Δ_Q, X), which satisfies the condition

$$(X, \mathfrak{D}L) \in D_{\Delta_Q}, \tag{3}$$

where $X : \Delta_Q \oplus P \subset TQ \oplus T^*Q \to TT^*Q$ is a partial vector field defined at points $(v, p) \in \Delta_Q \times P$, where $P = \mathbb{F}L(\Delta_Q)$; that is, X assigns a vector in T_pT^*Q to each point $(q, v, p) \in \Delta_Q \oplus P$. We write $X(q, v, p) = (q, p, \dot{q}, \dot{p})$, so that $\dot{q}$ and $\dot{p}$ are functions of (q, v, p).

Equality of base points in (3) implies that p is given by the Legendre transformation, and so one can equivalently say that X depends only on (q, v) with p determined by the Legendre transform. That is, equation (3) means that for each $(q, v) \in \Delta_Q \subset TQ$, we have

$$(X(q, v, p), \mathfrak{D}L(q, v)) \in D_{\Delta_Q}(q, p), \tag{4}$$

where $(q, p) = \mathbb{F}L(q, v)$. It follows from equations (1), (2) and (4) that

$$p = \frac{\partial L}{\partial v}, \quad \dot{q} \in \Delta(q), \quad \dot{q} = v, \quad \text{and} \quad \dot{p} - \frac{\partial L}{\partial q} \in \Delta^\circ(q). \tag{5}$$

A *solution curve* of an implicit Lagrangian system (L, Δ_Q, X) is a curve $(q(t), v(t), p(t)) \in TQ \oplus T^*Q$, $t_1 \leq t \leq t_2$, such that it is an integral curve of X in the sense that the time derivative of $(q(t), p(t)) = \mathbb{F}L(q(t), v(t))$ coincides with the value of $X(q(t), v(t), p(t))$, which is a vector in T^*Q at the point $(q(t), p(t)) = \mathbb{F}L(q(t), v(t))$.

Note that for the case $\Delta_Q = TQ$ the condition of an implicit Lagrangian system is equivalent to the Euler–Lagrange equations $\dot{p} = \partial L/\partial q$ together with the second order condition $\dot{q} = v$.

Energy Conservation. We now show that energy is conserved for any implicit Lagrangian system (L, Δ_Q, X). Define the *generalized energy* E on $TQ \oplus T^*Q$ by

$$E(q, v, p) = \langle p, v \rangle - L(q, v).$$

Let $(q(t), v(t))$, $t_1 \leq t \leq t_2$, be the solution curve of implicit Lagrangian systems together with $(q(t), p(t)) = \mathbb{F}L(q(t), v(t))$; thus,

$$\begin{aligned}\frac{d}{dt} E(q, v, p) &= \langle \dot{p}, v \rangle + \langle p, \dot{v} \rangle - \frac{\partial L}{\partial q} \dot{q} - \frac{\partial L}{\partial v} \dot{v} \\ &= \left\langle \dot{p} - \frac{\partial L}{\partial q}, v \right\rangle\end{aligned}$$

which vanishes since $\dot{q} = v \in \Delta(q)$, $p = \partial L/\partial v$ and $\dot{p} - \partial L/\partial q \in \Delta^\circ(q)$.

Remark. Using the generalized energy E on $TQ \oplus T^*Q$, the condition for an implicit Lagrangian system (L, Δ_Q, X), namely, $(X, \mathfrak{D}L) \in D_{\Delta_Q}$, can be restated as $(X, \mathbf{d}E|_{TT^*Q}) \in D_{\Delta_Q}$ together with the Legendre transform $P = \mathbb{F}L(\Delta_Q)$. Namely, the following relation holds, for each $(q, v) \in \Delta_Q$,

$$\left(X(q, v, p), \mathbf{d}E(q, v, p)|_{T_{(q,p)}T^*Q}\right) \in D_{\Delta_Q}(q, p),$$

together with $(q, p) = \mathbb{F}L(q, v)$. The restriction $\mathbf{d}E(q, v, p)|_{T_{(q,p)}T^*Q}$ is understood in the sense that $T_{(q,p)}T^*Q$ is naturally included in $T_{(q,v,p)}(TQ \oplus T^*Q)$.

Coordinate Representation. In coordinates, since the one-forms $\omega^1, ..., \omega^m$ span a basis of the annihilator $\Delta^\circ(q)$ at each $q \in U \subset \mathbb{R}^n$, it follows that equation (5) can be represented in terms of Lagrange multipliers μ_a, $a = 1, ..., m$ as follows:

$$\begin{aligned}\begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial L}{\partial q^i} \\ v^i \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_a \, \omega_i^a(q) \end{pmatrix}, \\ p_i &= \frac{\partial L}{\partial v^i}, \\ 0 &= \omega_i^a(q) \, v^i,\end{aligned} \tag{6}$$

where $\omega^a = \omega_i^a \, dq^i$.

Later, we shall see that L–C circuits, which are a typical degenerate Lagrangian system, can be represented by equation (6) in the context of implicit Lagrangian systems [10].

Example: Lagrangians Linear in the Velocity. Consider a system with the Lagrangian $L : TQ \to \mathbb{R}$ given by

$$L(q^i, v^i) = \langle \alpha_i(q^j),\, v^i \rangle - h(q^i),\ i, j = 1, ..., n,$$

where α is a one-form on Q and h is a function on Q. This form arises in various physical systems such as point vortices and the KdV equation (see, for instance, [5, 6]). It is obvious that the Lagrangian is degenerate.

Since there are no kinematic constraints, it follows from equation (6) that equations of motion are given by

$$\begin{aligned} \dot{q}^i &= v^i, \\ \dot{p}_i &= \frac{\partial L}{\partial q^i} = \frac{\partial \alpha_j(q)}{\partial q^i} v^j - \frac{\partial h(q)}{\partial q^i}, \\ p_i &= \frac{\partial L}{\partial v^i} = \alpha_i(q). \end{aligned}$$

4 Implicit Hamiltonian Systems

Degenerate Lagrangians. Let Q be a manifold, L be a Lagrangian on TQ and Δ_Q a given constraint distribution on Q. The *constraint momentum space* $P \subset T^*Q$ is defined to be the image of Δ_Q under the Legendre transform $\mathbb{F}L : TQ \to T^*Q$; namely, $P = \mathbb{F}L(\Delta_Q)$, which in coordinates, is represented by

$$(q^i, p_i) = \left(q^i, \frac{\partial L}{\partial v^i} \right), \quad i = 1, ..., n.$$

Now, suppose that L is degenerate; that is,

$$\det \left[\frac{\partial^2 L}{\partial v^i \partial v^j} \right] = 0; \quad i, j = 1, ..., n,$$

and also that the dimension of P_q at each $q \in Q$ is a fixed integer k $(0 \leq k < n)$, and the submanifold P can be represented by, at each $q \in Q$,

$$P_q = \left\{ p \in T_q^*Q \mid \phi_A(q, p) = 0,\ A = k+1, ..., n \right\}, \tag{7}$$

where ϕ_A, $A = k+1, ..., n$, are functions on T_q^*Q. The functions $\phi_A(q,p) = 0$ in equation (7) are called *primary constraints* when $\Delta_Q = TQ$ (see, for instance, [4]), and we shall continue to call them primary constraints even in the case of $\Delta_Q \subset TQ$. Needless to say, if the Lagrangian is regular, then there are no primary constraints.

Thus, we can choose the local coordinates $(q^i,\, p_\lambda)$, $i = 1, ..., n;\ \lambda = 1, ..., k$ for $P \subset T^*Q$ together with the *partial Legendre transform*

$$p_\lambda = \frac{\partial L}{\partial v^\lambda}, \quad \lambda = 1, \ldots, k,$$

where

$$\det\left[\frac{\partial^2 L}{\partial v^\lambda \partial v^\mu}\right] \neq 0; \quad \lambda,\, \mu = 1, \ldots, k,$$

and $v = (v^\lambda, v^A)$ are local coordinates for T_qQ and with the constraints $v \in \Delta(q)$.

Generalized Legendre Transform. Define a generalized energy E on the Pontryagin bundle $TQ \oplus T^*Q$ by

$$\begin{aligned} E(q^i, v^i, p_i) &= p_i\, v^i - L(q^i, v^i) \\ &= p_\lambda\, v^\lambda + p_A\, v^A - L(q^i, v^\lambda, v^A), \end{aligned}$$

where $p_i = (p_\lambda, p_A)$. Then, the Hamiltonian H_P on P can be defined by

$$H_P(q^i, p_\lambda) = \operatorname{stat}_{v^i} E(q^i, v^i, p_i)|P,$$

where $\operatorname{stat}_{v^i}$ is the stationarity operator (defining a critical point in the variable v). In view of the primary constraints in (7), we can define a *generalized Hamiltonian* H by

$$H(q^i, v^i, p_i) = H_P(q^i, p_\lambda) + \phi_A(q^i, p_i)\, v^A.$$

which has the property that $H|P = H_P$ (but it does depend on how we split the coordinates for p_i and v^i). In the above, v_A, $A = k+1, \ldots, n$, are local coordinates for an $(n-k)$-dimensional subspace of T_qQ, which can be regarded as Lagrange multipliers for the primary constraints $\phi_A(q^i, p_i) = 0$. The range of the index A varies according to the degeneracy of the Lagrangian, namely, $0 \leq k < n$. So, the generalized Hamiltonian H may be regarded as a function on $TQ \oplus T^*Q$.

Implicit Hamiltonian Systems. The differential of the generalized Hamiltonian $H : TQ \oplus T^*Q \to \mathbb{R}$ is in coordinates given by, for each $(q, v, p) \in TQ \oplus T^*Q$,

$$\mathbf{d}H(q, v, p) = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial v}, \frac{\partial H}{\partial p}\right),$$

where we can obtain the primary constraints by setting

$$\frac{\partial H}{\partial v} = \phi_A(q^i, p_i) = 0, \quad A = k+1, \ldots, n.$$

Meanwhile, since $\mathbf{d}H(q, v, p)$ takes its values in $T^*_{(q,v,p)}(TQ \oplus T^*Q)$, the restriction of the differential of H to $T_{(q,p)}T^*Q$ is

$$\mathbf{d}H(q, v, p)|_{T_{(q,p)}T^*Q} = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right).$$

Then, an *implicit Hamiltonian system* is a triple (H, Δ_Q, X), which satisfies the condition

$$(X(q,p), \mathbf{d}H(q,v,p)|_{T_{(q,p)}T^*Q}) \in D_{\Delta_Q}(q,p), \tag{8}$$

where $X = (q, p, \dot{q}, \dot{p})$ is a vector filed on T^*Q.

The local expression for implicit Hamiltonian systems in equation (8) is given by

$$\dot{q} = \frac{\partial H(q,v,p)}{\partial p} \in \Delta(q), \quad \dot{p} + \frac{\partial H(q,v,p)}{\partial q} \in \Delta^\circ(q) \tag{9}$$

and with the primary constraints

$$\frac{\partial H(q,v,p)}{\partial v} = \phi_A(q,p) = 0. \tag{10}$$

Coordinate Representation. In coordinates, recall the one-forms $\omega^1, ..., \omega^m$ span a basis of the annihilator $\Delta^\circ(q)$ at each $q \in U \in \mathbb{R}^n$, and it follows from equations (9) and (10) that

$$\begin{aligned}
\begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial H(q,v,p)}{\partial q^i} \\ \dfrac{\partial H(q,v,p)}{\partial p^i} \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_a\, \omega_i^a(q) \end{pmatrix}, \\
0 &= \omega_i^a(q)\, \frac{\partial H(q,v,p)}{\partial p^i}, \\
0 &= \phi_A(q^i, p_i), \quad A = k+1, ..., n,
\end{aligned} \tag{11}$$

where $\omega^a = \omega_i^a\, dq^i$ and we employed the Lagrange multipliers μ_a, $a = 1, ..., m$.

Example of a Lagrangian Linear in the Velocity. Again let us consider the example the Lagrangian $L : TQ \to \mathbb{R}$, which is given by

$$L(q^i, v^i) = \langle \alpha_i(q^j),\, v^i \rangle - h(q^i),\ i,j = 1, ..., n.$$

By a direct computation, we obtain the primary constraints as

$$\phi_i(q^j, p_j) = p_i - \frac{\partial L}{\partial v^i} = p_i - \alpha_i(q^j) = 0,$$

so that the submanifold P is the graph of α in T^*Q. Define a generalized energy E on $TQ \oplus T^*Q$ by

$$\begin{aligned}
E(q^i, v^i, p_i) &= p_i\, v^i - L(q^i, v^i) \\
&= (p_i - \alpha_i(q^j))\, v^i + h(q^i)
\end{aligned}$$

and the Hamiltonian H_P on P can be defined by

$$H_P(q^i, p_i) = \mathrm{stat}_{v^i}\, E(q^i, v^i, p_i)|P = h(q^i),$$

where $p_i = \alpha_i(q^j)$. Hence, the generalized Hamiltonian H on $TQ \oplus T^*Q$ is given by

$$\begin{aligned} H(q^i, v^i, p_i) &= H_P(q^i, p_i) + \phi_i(q^i, p_i)\, v^i \\ &= h(q^i) + (p_i - \alpha_i(q^j))\, v^i, \end{aligned}$$

where we note $H|P = H_P$. Therefore, the equations of motion are given, in the context of implicit Hamiltonian systems, by

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} = v^i, \\ \dot{p}^i &= -\frac{\partial H}{\partial q^i} = \frac{\partial \alpha_j(q)}{\partial q^i}\, v^j - \frac{\partial h(q)}{\partial q^i}, \\ \frac{\partial H}{\partial v^i} &= \phi_i(q^j, p_j) = p_i - \alpha_i(q^j) = 0. \end{aligned}$$

5 Examples of L–C Circuits

As an Implicit Lagrangian System. Consider the illustrative example of an L–C circuit shown in Fig. 1, which was also investigated in [8]. In the L–C circuit, the configuration space W is a 4-dimensional vector space, that is, $W = \mathbb{R}^4$. Then, we have $TW\,(\cong W \times W)$ and $T^*W\,(\cong W \times W^*)$. Let $q = (q_L, q_{C_1}, q_{C_2}, q_{C_3}) \in W$ denote charges and $f = (f_L, f_{C_1}, f_{C_2}, f_{C_3}) \in T_qW$ currents associated with the L–C circuit.

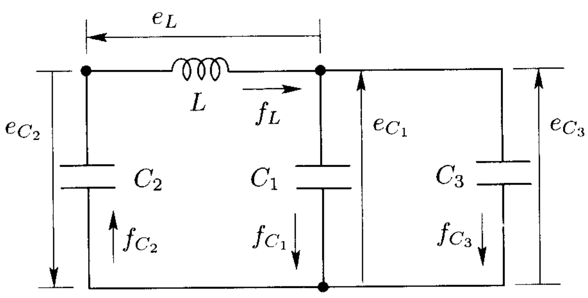

Fig. 1. L-C Circuit

The set of currents satisfying the KCL (Kirchhoff current law) constraints forms a constraint subspace $\Delta \subset TW$, which we shall call the *constraint KCL space* that is defined, for each $q \in W$, by

$$\Delta(q) = \{f \in T_qW \mid \langle \omega^a, f \rangle = 0,\ a = 1, 2\},$$

where $f = (f_1, f_2, f_3, f_4) = (f_L, f_{C_1}, f_{C_2}, f_{C_3})$ and ω^a denote 2-independent covectors (or one-forms) represented, in coordinates, by

$$\omega^a = \omega^a_k \, dq^k, \quad a = 1, 2; \; k = 1, ..., 4,$$

where $q = (q^1, q^2, q^3, q^4) = (q_L, q_{C_1}, q_{C_2}, q_{C_3})$. In this example, the coefficients ω^a_k are given in matrix representation by

$$\omega^a_k = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}.$$

Consistent with the general theory, the induced distribution Δ_{T^*W} on T^*W is defined by the KCL constraint distribution $\Delta \subset TW$ by

$$\Delta_{T^*W} = (T\pi_W)^{-1}(\Delta) \subset TT^*W,$$

where $\pi_W : T^*W \to W$ is the canonical projection and $T\pi_W : TT^*W \to TW$. Recall that the constraint set $\Delta \subset TW$ is represented as the simultaneous kernel of a number of constraint one-forms; that is, the annihilator of $\Delta(q)$, which is denoted by $\Delta^\circ(q)$, is spanned by such one-forms, that we write as $\omega^1, \omega^2, \ldots, \omega^m$. Now writing the projection map $\pi_W : T^*W \to W$ locally as $(q, p) \mapsto q$, its tangent map is locally given by $T\pi_W : (q, p, \dot{q}, \dot{p}) \mapsto (q, \dot{q})$. Then, we can locally represent Δ_{T^*W} as

$$\Delta_{T^*W} \cong \left\{ v_{(q,p)} = (q, p, \dot{q}, \dot{p}) \mid q \in U, \dot{q} \in \Delta(q) \right\}.$$

Let points in T^*T^*W be locally denoted by $\alpha_{(q,p)} = (q, p, \alpha, w)$, where α is a covector and w is a vector, and the annihilator of Δ_{T^*W} is

$$\Delta^\circ_{T^*W} \cong \left\{ \alpha_{(q,p)} = (q, p, \alpha, w) \mid q \in U, \; \alpha \in \Delta^\circ(q) \text{ and } w = 0 \right\}.$$

Recall also from equation (1) that the Dirac structure D_Δ on T^*W induced from the KCL constraint distribution Δ is locally given, for each $(q, p) \in T^*W$, by

$$D_\Delta(q, p) = \{ ((q, p, \dot{q}, \dot{p}), (q, p, \alpha, w)) \mid \dot{q} \in \Delta(q), \; w = \dot{q}, \; \alpha + \dot{p} \in \Delta^\circ(q) \}.$$

Let $T : TW \to \mathbb{R}$ be the magnetic energy of the L-C circuit, which is defined by the inductance L such that

$$T_q(f) = \frac{1}{2} L \, (f_L)^2,$$

and let $V : W \to \mathbb{R}$ be the electric potential energy of the L-C circuit, which is defined by capacitors C_1, C_2, and C_3 such that

$$V(q) = \frac{1}{2} \frac{(q_{C_1})^2}{C_1} + \frac{1}{2} \frac{(q_{C_2})^2}{C_2} + \frac{1}{2} \frac{(q_{C_3})^2}{C_3}.$$

Then, we can define the Lagrangian of the L–C circuit $\mathcal{L} : TW \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}(q, f) &= T_q(f) - V(q) \\ &= \frac{1}{2} L \, (f_L)^2 - \frac{1}{2} \frac{(q_{C_1})^2}{C_1} - \frac{1}{2} \frac{(q_{C_2})^2}{C_2} - \frac{1}{2} \frac{(q_{C_3})^2}{C_3}. \end{aligned}$$

It is obvious that the Lagrangian $\mathcal{L} : TW \to \mathbb{R}$ of the L-C circuit is degenerate, since

$$\det\left[\frac{\partial^2 \mathcal{L}}{\partial f^i \partial f^j}\right] = 0; \quad i, j = 1, ..., 4.$$

The *constraint flux linkage subspace* is defined by the Legendre transform:

$$P = \mathbb{F}\mathcal{L}(\Delta) \subset T^*W.$$

In coordinates, $(q, p) = \mathbb{F}\mathcal{L}(q, f) \in T^*W$, and it follows

$$(p_L, p_{C_1}, p_{C_2}, p_{C_3}) = \left(\frac{\partial \mathcal{L}}{\partial f_L}, \frac{\partial \mathcal{L}}{\partial f_{C_1}}, \frac{\partial \mathcal{L}}{\partial f_{C_2}}, \frac{\partial \mathcal{L}}{\partial f_{C_3}}\right),$$

from which we obtain

$$p_L = L\, f_L$$

and with the constraints

$$p_{C_1} = 0, \ p_{C_2} = 0, \ p_{C_3} = 0,$$

which correspond to primary constraints in the sense of Dirac. Needless to say, the primary constraints form the constraint flux linkage subspace $P \subset T^*W$, which immediately reads

$$(q, p) = (q_L, q_{C_1}, q_{C_2}, q_{C_3}, p_L, 0, 0, 0) \in P.$$

Let $X : TW \oplus T^*W \to TT^*W$ be a partial vector field on T^*W, defined at each point in P, with components denoted by

$$X(q, f, p) = (\dot{q}_L, \dot{q}_{C_1}, \dot{q}_{C_2}, \dot{q}_{C_3}, \dot{p}_L, 0, 0, 0)\,.$$

Since the differential of the Lagrangian $\mathbf{d}\mathcal{L}(q, f) = (\partial \mathcal{L}/\partial q, \partial \mathcal{L}/\partial f)$ is given by

$$\mathbf{d}\mathcal{L}(q, f) = \left(0, -\frac{q_{C_1}}{C_1}, -\frac{q_{C_2}}{C_2}, -\frac{q_{C_3}}{C_3}, L f_L, 0, 0. 0\right),$$

the Dirac differential of the Lagrangian $\mathfrak{D}\mathcal{L}(q, f) = (-\partial \mathcal{L}/\partial q, f)$ is given by

$$\mathfrak{D}\mathcal{L}\,(q, f) = \left(0, \frac{q_{C_1}}{C_1}, \frac{q_{C_2}}{C_2}, \frac{q_{C_3}}{C_3}, f_L, f_{C_1}, f_{C_2}, f_{C_3}\right)$$

together with $p = \partial \mathcal{L}/\partial f$.

Thus, the L-C circuit can be represented in the context of implicit Lagrangian systems $(\mathcal{L}, \Delta, X)$ by requiring that, for each $(q, f) \in \Delta \subset TW$,

$$(X(q, f, p), \mathfrak{D}\mathcal{L}(q, f)) \in D_\Delta(q, p)$$

holds and with the Legendre transform $(q, p) = \mathbb{F}\mathcal{L}(q, f)$. Therefore, the implicit Lagrangian system for this L-C circuit may be locally described by

$$\begin{pmatrix} \dot{q}_L \\ \dot{q}_{C_1} \\ \dot{q}_{C_2} \\ \dot{q}_{C_3} \\ \hline \dot{p}_L \\ 0 \\ 0 \\ 0 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} 0 \\ \frac{q_{C_1}}{C_1} \\ \frac{q_{C_2}}{C_2} \\ \frac{q_{C_3}}{C_3} \\ \hline f_L \\ f_{C_1} \\ f_{C_2} \\ f_{C_3} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

together with the Legendre transformation

$$p_L = L\, f_L.$$

The above equations of motion are supplemented by the KCL constraints

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} f_L \\ f_{C_1} \\ f_{C_2} \\ f_{C_3} \end{pmatrix}.$$

Finally, we can obtain the implicit Lagrangian system for this L-C circuit as

$$\begin{aligned} &\dot{q}_L = f_L,\ \dot{q}_{C_1} = f_{C_1},\ \dot{q}_{C_2} = f_{C_2},\ \dot{q}_{C_3} = f_{C_3}, \\ &\dot{p}_L = -\mu_1, \\ &\mu_2 = -\frac{q_{C_1}}{C_1},\ \mu_1 = -\mu_2 + \frac{q_{C_2}}{C_2},\ \mu_2 = -\frac{q_{C_3}}{C_3}, \\ &p_L = L\, f_L, \\ &f_L = f_{C_2},\ f_{C_1} = f_{C_2} - f_{C_3}. \end{aligned}$$

Representation as an Implicit Hamiltonian System. Next, let us illustrate this example of an L-C circuit in the context of implicit Hamiltonian systems via the generalized Legendre transformation.

First, define the generalized energy E on $TW \oplus T^*W$ by

$$\begin{aligned} E(q^i, f^i, p_i) &= p_i\, f^i - \mathcal{L}(q^i, f^i) \\ &= p_L\, f_L + p_{C_1} f_{C_1} + p_{C_2} f_{C_2} + p_{C_3} f_{C_3} \\ &\quad - \frac{1}{2} L\, (f_L)^2 + \frac{1}{2}\frac{(q_{C_1})^2}{C_1} + \frac{1}{2}\frac{(q_{C_2})^2}{C_2} + \frac{1}{2}\frac{(q_{C_3})^2}{C_3}. \end{aligned}$$

In the above, $(p_1, p_2, p_3, p_4) = (p_L, p_{C_1}, p_{C_2}, p_{C_3})$. Therefore, we can define the constrained Hamiltonian H_P on $P \subset T^*W$ by

$$\begin{aligned} H_P(q^i, p_\lambda) &= \operatorname{stat}_{f^i} E(q^i, f^i, p_i)|P \\ &= \frac{1}{2} L^{-1} (p_L)^2 + \frac{1}{2}\frac{(q_{C_1})^2}{C_1} + \frac{1}{2}\frac{(q_{C_2})^2}{C_2} + \frac{1}{2}\frac{(q_{C_3})^2}{C_3}, \end{aligned}$$

where $\lambda = 1$, that is, $p_1 = p_L$ and we employed the inverse partial Legendre transformation

$$f_L = L^{-1} p_L.$$

Since the primary constraints are given by

$$\phi_2 = p_{C_1} = 0, \; \phi_3 = p_{C_2} = 0, \; \phi_4 = p_{C_3} = 0,$$

we can define a generalized Hamiltonian H on $TW \oplus T^*W$ by

$$\begin{aligned} H(q^i, f^i, p_i) &= H_P(q^i, p_\lambda) + \phi_A(q^i, p_i)\, f^A \\ &= \frac{1}{2} L^{-1} (p_L)^2 + \frac{1}{2}\frac{(q_{C_1})^2}{C_1} + \frac{1}{2}\frac{(q_{C_2})^2}{C_2} + \frac{1}{2}\frac{(q_{C_3})^2}{C_3} \\ &\qquad + p_{C_1} f_{C_1} + p_{C_2} f_{C_2} + p_{C_3} f_{C_3}. \end{aligned}$$

The differential of H is given by

$$\mathbf{d}H = \left(q^i, f^i, p_i, \frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial f^i}, \frac{\partial H}{\partial p_i} \right).$$

Considering the primary constraints, we can set

$$\frac{\partial H}{\partial f^A} = \phi_A(q^i, p_i) = p_A = 0, \qquad A = 2, 3, 4.$$

restriction $\mathbf{d}H(q, f, p) : T_{(q,f,p)}(TW \oplus T^*W) \to \mathbb{R}$ to $T_{(q,p)}T^*W$ is

$$\mathbf{d}H(q, f, p)|_{T_{(q,p)}T^*W} = \left(\frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial p_i} \right),$$

which gives

$$\mathbf{d}H(q, f, p)|_{T_{(q,p)}T^*W} = \left(0, \frac{q_{C_1}}{C_1}, \frac{q_{C_2}}{C_2}, \frac{q_{C_3}}{C_3}, \; L^{-1} p_L, f_{C_1}, f_{C_2}, f_{C_3} \right).$$

Hence, the L-C circuit can be represented as an implicit Hamiltonian system (H, Δ, X) that satisfies, for each $(q, p) \in T^*W$,

$$(X(q, p), \mathbf{d}H(q, f, p)|_{T_{(q,p)}T^*W}) \in D_\Delta(q, p)$$

together with the primary constraints

$$\frac{\partial H}{\partial f} = 0.$$

Recall that the vector filed X on T^*W is given in coordinates by

$$X(q, p) = (\dot{q}_L, \dot{q}_{C_1}, \dot{q}_{C_2}, \dot{q}_{C_3}, \; \dot{p}_L, 0, 0, 0).$$

Then, it follows from equation (11) that the implicit Hamiltonian system for the L-C circuit can be represented in coordinates as

$$\begin{pmatrix} \dot{q}_L \\ \dot{q}_{C_1} \\ \dot{q}_{C_2} \\ \dot{q}_{C_3} \\ \hline \dot{p}_L \\ 0 \\ 0 \\ 0 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} 0 \\ \frac{q_{C_1}}{C_1} \\ \frac{q_{C_2}}{C_2} \\ \frac{q_{C_3}}{C_3} \\ \hline L^{-1} p_L \\ f_{C_1} \\ f_{C_2} \\ f_{C_3} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},$$

where the primary constraints

$$p_{C_2} = p_{C_3} = p_4 = 0$$

have been incorporated. The above equations of motion are accompanied with the KCL constraints

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} L^{-1} p_L \\ f_{C_1} \\ f_{C_2} \\ f_{C_3} \end{pmatrix}.$$

Finally, the implicit Hamiltonian system for the L-C circuit can be locally given as follows:

$$\begin{aligned} &\dot{q}_L = L^{-1} p_L,\ \dot{q}_{C_1} = f_{C_1},\ \dot{q}_{C_2} = f_{C_2},\ \dot{q}_{C_3} = f_{C_3}, \\ &\dot{p}_L = -\mu_1, \\ &\mu_2 = -\frac{q_{C_1}}{C_1},\ \mu_1 = -\mu_2 + \frac{q_{C_2}}{C_2},\ \mu_2 = -\frac{q_{C_3}}{C_3}, \\ &L^{-1} p_L = f_{C_2},\ f_{C_1} = f_{C_2} - f_{C_3}. \end{aligned}$$

It seems that this Hamiltonian view of this electric circuit is consistent with that presented in [8] and [2]. Note that the present approach derives the Hamiltonian structure in a systematic way from a degenerate Lagrangian, whereas the direct Hamiltonian approach requires some ingenuity to derive and its applicability to all cases is not clear.

6 Conclusions

The paper started by reviewing how a Dirac structure on a cotangent bundle is induced from a constraint distribution. In this context, implicit Lagrangian systems can be introduced in association with this induced Dirac structure, which is available for degenerate Lagrangians. It was shown how an implicit Hamiltonian system can be defined by a generalized Legendre transformation, starting with a generalized Hamiltonian on the Pontryagin bundle of a configuration manifold by incorporating the primary constraints that are present due to the possible degeneracy of the Lagrangian. The techniques were illustrated via some examples of degenerate Lagrangians with constraints, namely for L–C circuits and point vortices, as well as for nonholonomic systems, where the Lagrangian is typically nondegenerate, but constraints are present.

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References

1. Blankenstein, G. and T. S. Ratiu [2004], Singular reduction of implicit Hamiltonian systems, *Rep. Math. Phys.* **53**, 211–260.
2. Bloch, A. M. and Crouch, P. E. [1997], Representations of Dirac structures on vector spaces and nonlinear L-C circuits. Differential Geometry and Control (Boulder, CO, 1997), 103–117, *Proc. Sympos. Pure Math.* **64**. Amer. Math. Soc., Providence, RI.
3. Courant, T. and A. Weinstein [1988], Beyond Poisson structures. In *Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986)*, volume 27 of *Travaux en Cours*, pages 39–49. Hermann, Paris.
4. Dirac, P. A. M. [1950], Generalized Hamiltonian dynamics, *Canadian J. Math.* **2**, 129–148.
5. Marsden, J. E. and T. S. Ratiu [1999], *Introduction to Mechanics and Symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, second edition.
6. Rowley, C. W. and J. E. Marsden [2002], Variational integrators for degenerate Lagrangians, with application to point vortices. *Proceedings of the 41st IEEE Conference on Decision and Control, December 2002.*
7. Tulczyjew, W. M. [1977], The Legendre transformation, *Ann. Inst. H. Poincaré*, Sect. A, **27**(1), 101–114.
8. van der Schaft, A. J. [1998], Implicit Hamiltonian systems with symmetry, *Rep. Math. Phys.* **41**, 203–221.
9. van der Schaft, A. J. and B. M. Maschke [1995], The Hamiltonian formulation of energy conserving physical systems with external ports, *Archiv für Elektronik und Übertragungstechnik* **49**, 362–371.
10. Yoshimura, H. and Marsden, J. E.[2006], Dirac structures in mechanics. Part I: Implicit Lagrangian systems, *Journal of Geometry and Physics* **57-1**, 133–156.
11. Yoshimura, H. and Marsden, J. E.[2006], Dirac structures in mechanics. Part II: Variational structures. *Journal of Geometry and Physics*, **57-1**, 209–250.