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DISCRETE MECHANICS AND OPTIMAL CONTROL FOR CONSTRAINED MULTIBODY DYNAMICS

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ABSTRACT

This paper formulates the dynamical equations of mechanics subject to holonomic constraints in terms of the states and controls using a constrained version of the Lagrange-d'Alembert principle. Based on a discrete version of this principle, a structure preserving time-stepping scheme is derived. It is shown that this respect for the mechanical structure (such as a reliable computation of the energy and momentum budget, without numerical dissipation) is retained when the system is reduced to its minimal dimension by the discrete null space method. Together with initial and final conditions on the configuration and conjugate momentum, the reduced time-stepping equations serve as nonlinear equality constraints for the minimisation of a given cost functional. The algorithm yields a sequence of discrete configurations together with a sequence of actuating forces, optimally guiding the system from the initial to the desired final state. The resulting discrete optimal control algorithm is shown to have excellent energy and momentum properties, which are illustrated by two specific examples, namely reorientation and repositioning of a rigid body subject to external forces and the reorientation of a rigid body with internal momentum wheels.

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INTRODUCTION

This work combines two recently developed methods, namely the discrete null space method which is suitable for the accurate, robust and efficient time integration of constrained dynamical systems (in particular for multibody dynamics) and a new approach to discrete mechanics and optimal control (DMOC) based on a discretisation of the Lagrange-d'Alembert principle.

From the variety of methods to enforce holonomic constraints in the framework of the Hamiltonian or Lagrangian formalism (see e.g. [1, 2] and for a computational approach [3]), the focus here is on two methods yielding exact constraint fulfilment, the Lagrange multiplier method and a null space method, described e.g. in [4]. Because of the relatively simple structure of the evolution equations emanating from the Lagrange multiplier method, their temporal discrete form can be derived easily using mechanical integrators as demonstrated among others in [5–7]. However, the presence of the Lagrange multipliers in the set of unknowns enlarges the number of equations to solve and causes the discrete system to be ill-conditioned for small time-steps as reported by [8, 9]. In contrast to that, the use of a specific null space method, especially in conjunction with a reparametrisation in generalised coordinates, has the advantageous property of a small dimensional system of equations. On the other hand, these evolution equations have a highly complicated structure, causing the derivation of their temporal discrete form to be expensive and therefore, in most cases, not recommended [10, 11].

A remedy for these difficulties is found in the discrete null space method introduced in [12] which proposes a reversal of the two main steps when designing a specific numerical method. First of all, the discrete form of the simple structured DAEs resulting from the use of the Lagrange multiplier method is derived using a mechanical integrator, e.g. an energy-momentum conserving integrator [5, 6] or a variational integrator leading to a symplectic-momentum conserving scheme [7]. For forced systems, both methods correctly compute the change in momentum maps. The evolution of energy is represented accurately by the first class of schemes while the latter captures these changes qualitatively. The transition to the reduced scheme and finally the nodal reparametrisation are performed in the temporal discrete setting in complete analogy to the procedure described in the continuous case according to the so-called discrete null space method. The resulting time-stepping scheme performs excellently in all relevant categories. First of all, it yields the smallest possible dimension for the system of equations, promising lower computational costs than other schemes. Secondly, it is second order accurate and inherits the conservation properties from the constrained scheme and thirdly, the condition number of the scheme is independent of the time-step. Summarising, the discrete null space method is especially suited for the accurate simulation of large dimensional systems subject to a high number of constraints. In particular the resulting equations lend themselves as dynamics constraints in an optimisation algorithm since only the exactly required number of unknowns has to be determined.

To find local solutions of nonlinear optimal control problems consisting of a given cost functional and equations describing the underlying dynamics of the system, a numerical method falling into the class of direct methods is used here. Thereby, the state and control variables are discretised directly in order to transform the optimal control problem. The resulting finite dimensional nonlinear constrained optimisation problem can be solved by standard nonlinear optimisation techniques like sequential quadratic programming [13–15]. In contrast to other methods like, e.g. shooting [16–18], multiple shooting [19–21], or collocation methods [22, 23], relying on a direct integration of the associated ordinary differential equations or on its fulfillment at certain grid points (see also [24, 25] for an overview of the current state of the art), a recently developed method DMOC (Discrete Mechanics and Optimal Control, [26]) is used here. It is based on the discretisation of the variational structure of the mechanical system directly. In the context of variational integrators [27], the discretisation of the Lagrange-d'Alembert principle leads to structure preserving time-stepping equations which serve as equality constraints for the resulting finite dimensional nonlinear optimisation problem. In [26, 28, 29] the described method was firstly applied to low orbital thrust transfers and the optimal control of formation flying satellites including an algorithm that exploits a hierarchical structure of that problem. In [30], it has been applied to a multibody system formulated in generalised coordinates.

In this work, DMOC is used to find optimal trajectories of state and control variables for systems of rigid bodies combined with joint constraints. Each rigid body is viewed as a constrained continuum, i.e. it is described in terms of redundant coordinates subject to holonomic constraints [31, 32]. Then the equations of motion assume the form of DAEs with a constant mass matrix. Their temporal discrete form can be derived and reduced according to the discrete null space method. This procedure has the advantage of circumventing the difficulties associated with rotational parameters [33, 34] and it can be generalised easily to the modelling of geometrically exact beams and shells and to multibody systems consisting of theses structures as developed in [35–37]. The reduced time-stepping equations then serve as constraints in the optimisation algorithm.

The combination of the two proposed methods involves several specific benefits. First of all, the discrete dynamics equations constraining the optimal control problem when using DMOC can be formulated easily. Using the discrete Lagrange-d'Alembert principle, they are derived as the discrete analog to the simple structured evolution equations whereby the configuration constraints are enforces using Lagrange multipliers. Secondly, the discrete null space method reduces the dynamics constraints to the smallest possible number of equations and variables which leads to lower computational cost for the optimisation algorithm. Thirdly, the benefit of exact constraint fulfilment, correct computation of the change in momentum maps and good energy behaviour is guaranteed by the optimisation algorithm. These benefits are of high importance especially for high dimensional rigid body systems combined with joint constraints.

CONSTRAINED DYNAMICS AND OPTIMAL CONTROL

Consider an n-dimensional mechanical system with the time-dependent configuration vector $q(t) \in Q$ and velocity vector $\dot{q}(t) \in T_{q(t)}Q$, where $t \in [t_0, t_N] \subset \mathbb{R}$ denotes the time. Let the configuration be constrained by the function $g(q) = 0 \in \mathbb{R}^m$ and influenced by the force field $f : W \times TQ \to T^*Q$. Due to the presence of constraints, the forces f are not independent. They can be calculated in terms of the time dependent generalised control forces $\tau(t) \in W \subseteq \mathbb{R}^{n-m}$.

Optimisation problem. The goal is to determine the optimal force field, such that the system is moved from the initial state $(q^0, \dot{q}^0)$ to the final state $(q^N, \dot{q}^N)$ while the cost functional

$$J(q,\dot{q},f) = \int_{t_0}^{t_N} B(q,\dot{q},f(\tau,q,\dot{q}))\,dt \tag{1}$$

is minimised.

Constrained Lagrange-d'Alembert principle. Contemporaneously, the motion $(q,\dot{q})$ has to be in accordance with an equation of motion which in the present case is based on a constrained version of the Lagrange-d'Alembert principle (see e.g. [38]) requiring

$$\delta \int_{t_0}^{t_N} L(q,\dot{q}) - g^T(q)\cdot\lambda\,dt + \int_{t_0}^{t_N} f(\tau,q,\dot{q})\cdot\delta q\,dt = 0 \tag{2}$$

for all variations $\delta q \in TQ$ and $\delta\lambda \in \mathbb{R}^m$ vanishing at the endpoints. The Lagrangian $L: TQ \to \mathbb{R}$ comprises the kinetic energy $\frac{1}{2}\dot{q}^T \cdot M \cdot \dot{q}$ including the consistent mass matrix $M \in \mathbb{R}^{n\times n}$ and a potential function $V: Q \to \mathbb{R}$. Furthermore, $\lambda(t) \in \mathbb{R}^m$ represents the vector of time dependent Lagrange multipliers. The constrained Lagrange-d'Alembert principle (2) leads to the differential-algebraic system of equations of motion

$$\begin{aligned} \frac{\partial L(q,\dot{q})}{\partial q} - \frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) - G^T(q)\cdot\lambda + f(\tau,q,\dot{q}) &= 0 \\ g(q) &= 0 \end{aligned} \tag{3}$$

where $G(q) = Dg(q)$ denotes the Jacobian of the constraints. The vector $G^T(q)\cdot\lambda$ represents the constraint forces that prevent the system from deviations of the constraint manifold

$$C = \{q \in Q | g(q) = 0\} \tag{4}$$

Null space method. Assuming that the constraints are independent, for every $q \in C$ the basis vectors of T_qC form an $n \times (n-m)$ matrix $P(q)$ with corresponding linear map $P(q): \mathbb{R}^{n-m} \to T_qC$. This matrix is called null space matrix, since

$$\text{range}\,(P(q)) = \text{null}\,(G(q)) = T_qQ \tag{5}$$

Thus a premultiplication of the differential equation $(3)_1$ by $P^T(q)$ eliminates the constraint forces including the Lagrange multipliers from the system. The resulting equations of motion read

$$\begin{aligned} P^T(q)\cdot\left[\frac{\partial L(q,\dot{q})}{\partial q} - \frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) + f(\tau,q,\dot{q})\right] &= 0 \\ g(q) &= 0 \end{aligned} \tag{6}$$

Reparametrisation. For many applications it is possible to find a reparametrisation of the constraint manifold $F: U \subseteq \mathbb{R}^{n-m} \to C$ in terms of independent generalised coordinates $u \in U$. Then the Jacobian $DF(u)$ of the coordinate transformation plays the role of a null space matrix. Since the constraints $(3)_2$ are fulfilled automatically by the reparametrised configuration variable $q = F(u)$, the system is reduced to $n-m$ second order differential equations. This is the minimal possible dimension for the present mechanical system which consists of precisely $n-m$ configurational degrees of freedom. Consequently, there are $n-m$ independent generalised forces $\tau \in W \subseteq \mathbb{R}^{n-m}$ acting on the degrees of freedom. These can be calculated as $\tau = \left(\frac{\partial F}{\partial u}\right)^T \cdot f$, see e.g. [39].

CONSTRAINED DISCRETE DYNAMICS AND OPTIMAL CONTROL

A variational integrator is chosen to derive the temporal discrete version of the dynamical problem at hand. In [7], a variational integrator has been employed to simulate a constrained problem, whereby Lagrange multipliers have been used to enforce the constraints. For a detailed introduction to discrete mechanics and variational integrators see [27].

Corresponding to the configuration manifold Q, the discrete phase space is defined by $Q\times Q$ which is locally isomorphic to TQ. For a constant time-step $h \in \mathbb{R}$, a path $q: [t_0, t_N] \to Q$ is replaced by a discrete path $q_d: \{t_0, t_0+h, \ldots, t_0+Nh = t_N\} \to Q$, $N \in \mathbb{N}$, where $q_n = q_d(t_0+nh)$ is viewed as an approximation to $q(t_0+nh)$. Similarly, $\lambda_n = \lambda_d(t_n)$ approximates the Lagrange multiplier at $t_n = t_0 + nh$, $\tau_n = \tau_d(t_n)$ approximates the generalised control force and the force field f is approximated by two discrete forces $f_n^-, f_n^+: W \times Q \to T^*C$.

Discrete constrained Lagrange-d'Alembert principle. According to the variational integrator in use, the action integral in (2) is approximated in a time interval $[t_n, t_{n+1}]$ using the discrete Lagrangian $L_d: Q\times Q \to \mathbb{R}$ via

$$L_d(q_n,q_{n+1}) - g_d^T(q_{n+1})\cdot\lambda_{n+1} \approx \int_{t_n}^{t_{n+1}} L(q,\dot{q}) - g^T(q)\cdot\lambda\,dt \tag{7}$$

Among various possible choices to approximate this integral, the midpoint rule is in use for the Lagrangian, i.e.

$$L_d(q_n,q_{n+1}) = hL\left(\frac{q_{n+1}+q_n}{2}, \frac{q_{n+1}-q_n}{h}\right) \tag{8}$$

and the constraints and multipliers are evaluated at the time nodes themselves

$$g_d^T(q_{n+1})\cdot\lambda_{n+1} = hg^T(q_{n+1})\cdot\lambda_{n+1} \tag{9}$$

Likewise, the virtual work is approximated in a time interval $[t_n, t_{n+1}]$ by

$$\int_{t_n}^{t_{n+1}} f(\tau, q, \dot{q}) \cdot \delta q \, dt \approx f_n^- \cdot \delta q_n + f_n^+ \cdot \delta q_{n+1} \tag{10}$$

where f_n^+, f_n^- are called the left and right discrete forces, respectively. They are specified in (16).

The discrete version of the constrained Lagrange-d'Alembert principle (2) requires the discrete path $\{q_n\}_{n=0}^N$ and multipliers $\{\lambda_n\}_{n=1}^N$ to fulfil

$$\delta \sum_{n=0}^{N-1} L_d(q_n, q_{n+1}) - g_d^T(q_{n+1}) \cdot \lambda_{n+1} + \sum_{n=0}^{N-1} f_n^- \cdot \delta q_n + f_n^+ \cdot \delta q_{n+1} = 0 \tag{11}$$

for all variations $\{\delta q_n\}_{n=0}^N$ and $\{\delta \lambda_n\}_{n=1}^N$ with $\delta q_0 = \delta q_N = 0$ and $\delta\lambda_1 = \delta\lambda_N = 0$, which is equivalent to the constrained forced discrete Euler-Lagrange equations

$$\begin{aligned} D_2 L_d(q_{n-1}, q_n) + D_1 L_d(q_n, q_{n+1}) - G_d^T(q_n) \cdot \lambda_n + f_{n-1}^+ + f_n^- &= 0 \\ g(q_{n+1}) &= 0 \end{aligned} \tag{12}$$

for $n = 1, \ldots, N-1$ where $G_d(q_n)$ denotes the Jacobian of $g_d(q_n)$. Note that the time-stepping scheme (12) has not been deduced discretising (3), but via a discrete variational principle.

Discrete null space method. The reduction of the time-stepping scheme (12) can be accomplished in analogy to the continuous case according to the discrete null space method. In order to eliminate the discrete constraint forces from the equations, a discrete null space matrix fulfilling

$$\mathrm{range}\,(P(q_n)) = \mathrm{null}\,(G(q_n)) \tag{13}$$

is employed.

Remark It is important to note, that the choice to evaluate the constraints and the Lagrange multipliers at the time nodes in (9) causes the evaluation of the constraint Jacobian in (12) at the time nodes. Therefore a discrete null space matrix with the property (13) can simply be found by evaluation of the continuous null space matrix at the time nodes. Acquaintance of the continuous null space matrix for a specific mechanical system always yields an explicit representation of the discrete null space matrix for the symplectic-momentum conserving time-stepping scheme emanating from the discrete variational principle in conjunction with the chosen approximation. This is in contrast to energy-momentum conserving time-stepping schemes based on the concept of discrete derivatives [12, 40] or on finite elements in time [5], where the discrete constraint Jacobian $\mathsf{G}(q_n, q_{n+1})$ depends on both the present and the unknown configuration.

Analogue to (6), the premultiplication of (12) by the transposed discrete null space matrix cancels the constraint forces from the system, i.e. the Lagrange multipliers are eliminated from the set of unknowns and the system's dimension is reduced to n.

$$\begin{aligned} P^T(q_n) \cdot \left[D_2 L_d(q_{n-1}, q_n) + D_1 L_d(q_n, q_{n+1}) + f_{n-1}^+ + f_n^- \right] &= 0 \\ g(q_{n+1}) &= 0 \end{aligned} \tag{14}$$

Nodal Reparametrisation. Similar to the continuous case, a reduction of the system to the minimal possible dimension can be accomplished by a local reparametrisation of the constraint manifold in the neighbourhood of the discrete configuration variable $q_n \in C$. At the time nodes, q_n is expressed in terms of the discrete generalised coordinates $u_n \in U \subseteq \mathbb{R}^{n-m}$, such that the constraints are fulfilled.

$$F : U \subseteq \mathbb{R}^{n-m} \times Q \to C \quad \text{i.e.} \quad g(q_n) = g(F(u_n, q_{n-1})) = 0 \tag{15}$$

Furthermore, the components of the discrete force vectors f_n^+ and f_n^- are also not independent. They can be calculated using the the discrete generalised forces $\tau_{n-1}, \tau_n \in W \subseteq \mathbb{R}^{n-m}$ as follows.

$$\begin{aligned} & f_{n-1}^+ = \frac{h}{2} f(\tau_{n-1}, q_n), \quad f_n^- = \frac{h}{2} f(\tau_n, q_n) \quad \in T_{q_n}^* C \\ & f_n = \frac{1}{2} \left(f(\tau_n, q_{n+1}) + f(\tau_n, q_n) \right) \\ & f_d = \{f_n\}_{n=0}^{N-1} \end{aligned} \tag{16}$$

Note that the discrete generalised control forces are assumed to be constant in each time interval, see Fig. 1. Thus f_{n-1}^+ denotes the effect of the generalised force τ_{n-1} acting in $[t_{n-1}, t_n]$ on q_n while f_n^- denotes the effect on q_n of τ_n acting in $[t_n, t_{n+1}]$.

Insertion of the nodal reparametrisations for the configuration (15) and the force (16) into the scheme redundantises $(14)_2$. The resulting scheme

$$P^T(q_n) \cdot \left[D_2 L_d(q_{n-1}, q_n) + D_1 L_d(q_n, q_{n+1}) + f_{n-1}^+ + f_n^- \right] = 0 \tag{17}$$

is equivalent to the constrained scheme (12), thus it also has the key properties of exact constraint fulfilment, symplecticity and momentum consistency, i.e. any change in the value of a momentum map reflects exactly the applied forces. When no load

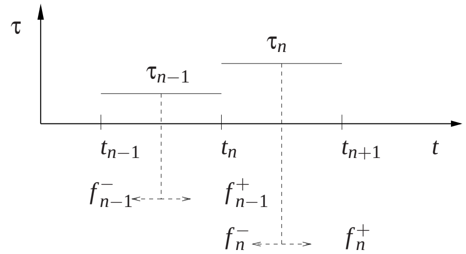

Figure 1. Relation of redundant forces at t_n to piecewise constant discrete generalised forces.

is present, momentum maps are conserved exactly. While the constrained scheme becomes increasingly ill-conditioned for decreasing time-steps, the condition number of (17) is independent of the time-step.

Boundary conditions. In the next step, the boundary conditions $q(t_0) = q^0, \dot{q}(t_0) = \dot{q}^0$ and $q(t_N) = q^N, \dot{q}(t_N) = \dot{q}^N$ have to be specified. Those on configuration level can be used as constraints for the optimisation algorithm in a straightforward way as $u_0 = u^0$ and $u_N = u^N$. However, since in the present formulation velocities are approximated in a time interval $[t_n, t_{n+1}]$ according to (8) (as opposed to an approximation at the time nodes), the velocity conditions have to be transformed to conditions on the conjugate momentum, which is defined at each and every time node using the discrete Legendre transform. The presence of forces at the time nodes has to be incorporated into that transformation leading to the so called forced discrete Legendre transforms $\mathbb{F}^{f^-} L_d : Q \times Q \to T^*Q$ and $\mathbb{F}^{f^+} L_d : Q \times Q \to T^*Q$ (see [27]) reading

$$\begin{aligned} \mathbb{F}^{f^-} L_d &: (q_{n-1}, q_n) \mapsto (q_{n-1}, p_{n-1}) \\ p_{n-1} &= -D_1 L_d(q_{n-1}, q_n) - f^-_{n-1} \\ \mathbb{F}^{f^+} L_d &: (q_{n-1}, q_n) \mapsto (q_n, p_n) \\ p_n &= D_2 L_d(q_{n-1}, q_n) + f^+_{n-1} \end{aligned} \tag{18}$$

In these transformations, constraints have not been taken into account. To do so, the discrete momenta can be projected using the transposed discrete null space matrix $P^T(q) : T^*_q Q \to T^*_u U$.

Prescribed initial and final velocities of course should be consistent with the constraints on velocity level. Using the standard continuous Legendre transform $\mathbb{F}L : TC \to T^*C$

$$\mathbb{F}L : (q, \dot{q}) \mapsto (q, p) = (q, D_2 L(q, \dot{q})) \tag{19}$$

yields momenta which are consistent with the constraints on momentum level as well. Since the algorithm enforces constraints on configuration level only, one can hardly expect the computed discrete momenta to equal those prescribed in T^*C. However, one can request their projection to $T^*_u U$ to be equal. Thus the velocity boundary conditions are transformed to the following conditions on momentum level $p(t_0) = p^0, p(t_N) = p^N$, which read in detail

$$\begin{aligned} P^T(q_0) \cdot \left[D_2 L(q_0, \dot{q}_0) + D_1 L_d(q_0, q_1) + f^-_0 \right] &= 0 \\ P^T(q_N) \cdot \left[-D_2 L(q_N, \dot{q}_N) + D_2 L_d(q_{N-1}, q_N) + f^+_{N-1} \right] &= 0 \end{aligned} \tag{20}$$

Discrete constrained optimisation problem. With the described preliminaries at hand, now the optimal control problem for the constrained discrete dynamical problem can be formulated. To begin with, an approximation

$$B_d(q_n, q_{n+1}, f_n) \approx \int_{t_n}^{t_{n+1}} B(q, \dot{q}, f(\tau, q, \dot{q}))\, dt \tag{21}$$

of the continuous cost functional (1) has to be defined. Similar to the approximations in (8) and (10) the midpoint rule is applied.

$$B_d(q_n, q_{n+1}, f_n) = hB\left(\frac{q_{n+1} + q_n}{2}, \frac{q_{n+1} - q_n}{h}, f_n \right) \tag{22}$$

with the discrete forces given in (16). This yields the discrete cost functional

$$J_d(q_d, f_d) = \sum_{n=0}^{N-1} B_d(q_n, q_{n+1}, f_n) \tag{23}$$

where the discrete configurations and forces are expressed in terms of their corresponding independent generalised quantities. Alternatively a new cost functional can be formulated directly in the generalised quantities

$$\bar{J}_d(u_d, \tau_d) = \sum_{n=0}^{N-1} \bar{B}_d(u_n, u_{n+1}, \tau_n) \tag{24}$$

depending on the desired interpretation of the optimisation problem. In any case, (23) or (24) has to be minimised with respect to u_d, τ_d subject to the constraints

$$\begin{aligned} u_0 - u^0 &= 0 \\ u_N - u^N &= 0 \\ P^T(q_0) \cdot \left[D_2 L(q_0, \dot{q}_0) + D_1 L_d(q_0, q_1) + f^-_0 \right] &= 0 \\ P^T(q_N) \cdot \left[-D_2 L(q_N, \dot{q}_N) + D_2 L_d(q_{N-1}, q_N) + f^+_{N-1} \right] &= 0 \\ P^T(q_n) \cdot \left[D_2 L_d(q_{n-1}, q_n) + D_1 L_d(q_n, q_{n+1}) + f^+_{n-1} + f^-_n \right] &= 0 \end{aligned} \tag{25}$$

for $n = 1, \ldots, N-1$.

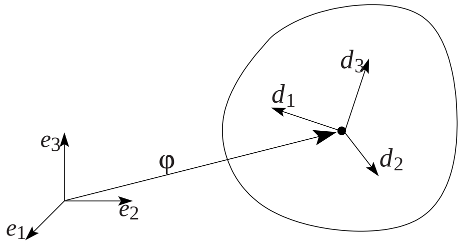

Figure 2. Configuration of a rigid body with respect to an orthonormal frame $\{e_I\}$ fixed in space.

OPTIMAL CONTROL FOR RIGID BODY DYNAMICS

The treatment of rigid bodies as structural elements relies on the kinematic assumptions illustrated in Fig. 2 (see [41]) that the placement of a material point in the body's configuration $X = X_I d_I \in \mathcal{B} \subset \mathbb{R}^3$ relative to an orthonormal basis $\{e_I\}$ fixed in space can be described as

$$x(X,t) = \varphi(t) + X_I d_I(t) \tag{26}$$

Here $X_I \in \mathbb{R}, I = 1,2,3$ represent coordinates in the body-fixed director triad $\{d_I\}$. The time-dependent configuration variable of a rigid body

$$q(t) = \begin{bmatrix} \varphi(t) \\ d_1(t) \\ d_2(t) \\ d_3(t) \end{bmatrix} \in \mathbb{R}^{12} \tag{27}$$

consists of the placement of the center of mass $\varphi \in \mathbb{R}^3$ and the directors $d_I \in \mathbb{R}^3, I = 1,2,3$ which are constrained to stay orthonormal during the motion, representing the rigidity of the body and its orientation. These orthonormality conditions pertaining to the kinematic assumptions of the underlying theory are termed internal constraints. There are $m_{int} = 6$ independent internal constraints for the rigid body with associated constraint functions

$$g_{int}(q) = \begin{bmatrix} \frac{1}{2}[d_1^T \cdot d_1 - 1] \\ \frac{1}{2}[d_2^T \cdot d_2 - 1] \\ \frac{1}{2}[d_3^T \cdot d_3 - 1] \\ d_1^T \cdot d_2 \\ d_1^T \cdot d_3 \\ d_2^T \cdot d_3 \end{bmatrix} \tag{28}$$

For simplicity, it is assumed that the axes of the body frame coincide with the principal axes of inertia of the rigid body. Then the body's Euler tensor with respect to the center of mass can be related to the inertia tensor J via

$$E = \frac{1}{2}(\mathrm{tr}J)I - J \tag{29}$$

where I denotes the 3×3 identity matrix. The principal values of the Euler tensor E_i together with the body's total mass M_φ build the rigid body's constant symmetric positive definite mass matrix

$$M = \begin{bmatrix} M_\varphi I & 0 & 0 & 0 \\ 0 & E_1 I & 0 & 0 \\ 0 & 0 & E_2 I & 0 \\ 0 & 0 & 0 & E_3 I \end{bmatrix} \tag{30}$$

where 0 denotes the 3×3 zero matrix. This description of rigid body dynamics has been expatiated in [36] where also the null space matrix

$$P_{int}(q) = \begin{bmatrix} I & 0 \\ 0 & -\widehat{d_1} \\ 0 & -\widehat{d_2} \\ 0 & -\widehat{d_3} \end{bmatrix} \tag{31}$$

corresponding to the constraints (28) has been derived. When the nodal reparametrisation of unknowns is applied, the configuration of the free rigid body is specified by six unknowns $u = (u_\varphi, \theta) \in U \subset \mathbb{R}^3 \times \mathbb{R}^3$, characterising the displacement and rotation, respectively. Accordingly, in the present case the nodal reparametrisation $F : U \to C$ introduced in (15) assumes the form

$$q_{n+1} = F(u_n) = \begin{bmatrix} \varphi_n + (u_\varphi) \\ \exp(\widehat{\theta}) \cdot (d_1)_n \\ \exp(\widehat{\theta}) \cdot (d_2)_n \\ \exp(\widehat{\theta}) \cdot (d_3)_n \end{bmatrix} \tag{32}$$

where Rodrigues' formula is used to obtain a closed form expression of the exponential map, see e.g. [38]. Translational forces $\tau_\varphi \in \mathbb{R}^3$ can directly be applied to the body's center of mass, thus $f_\varphi = \tau_\varphi$. However, torques $\tau_\theta \in \mathbb{R}^3$ have to be transformed to follower forces perpendicular to the directors according to $f_{d_I} = \frac{1}{2}\tau_\theta \times d_I$. This ensures $\tau_\theta = d_I \times f_{d_I}$. Consistency of momentum maps is guaranteed by the following discrete forces.

$$\begin{array}{lclcl} (f_\varphi)^+_{n-1} & = & (\tau_\varphi)^+_{n-1} & = & \frac{h}{2}(\tau_\varphi)_{n-1} \\ (f_\varphi)^-_n & = & (\tau_\varphi)^-_n & = & \frac{h}{2}(\tau_\varphi)_n \\ (f_{d_I})^+_{n-1} & = & -\frac{1}{2}(\widehat{d_I})_n \cdot (\tau_\theta)^+_{n-1} & = & -\frac{1}{2}(\widehat{d_I})_n \cdot \frac{h}{2}(\tau_\theta)_{n-1} \\ (f_{d_I})^-_n & = & -\frac{1}{2}(\widehat{d_I})_n \cdot (\tau_\theta)^-_n & = & -\frac{1}{2}(\widehat{d_I})_n \cdot \frac{h}{2}(\tau_\theta)_n \end{array} \tag{33}$$

Numerical example

Optimal control of a rigid sphere As a first example to demonstrate the performance of the proposed procedure,

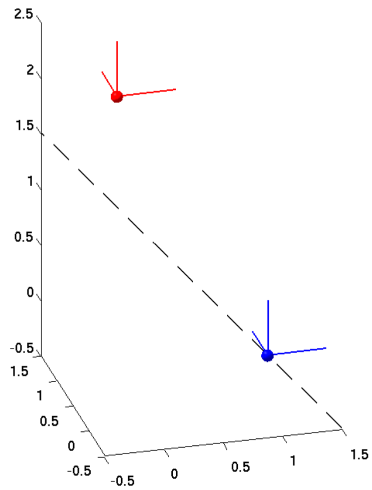

Figure 3. Rigid sphere: initial and final configuration and axis of rotation.

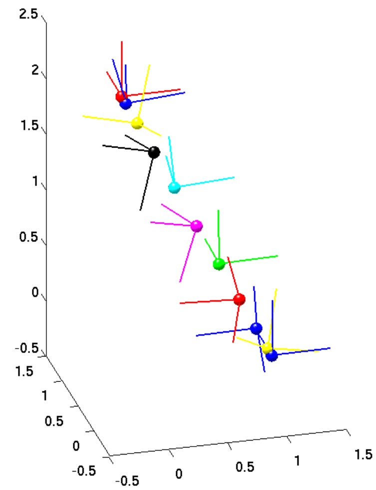

Figure 4. Rigid sphere: configuration at $t = 3nh, n = 0, \ldots, 10$ ($h = 0.1$).

the actuation of a rigid sphere in three-dimensional space is investigated. The sphere has the radius $r = 0.05$ and a density of $\rho = 27000$. In the initial position, its center of mass is located at $u_\varphi^0 = [R, 0, 0]^T$ with $R = 1$ and the directors are aligned with the axes of the inertial frame, thus $\theta^0 = [0, 0, 0]^T$. The body is forced to move to the position $u_\varphi^N = [0, R, 2R]^T$ while performing three full rotations around the axis $[-1, 1, 1]^T / \sqrt{3}$, hence $\theta^N = \frac{6\pi}{\sqrt{3}} [-1, 1, 1]^T$. The motion starts and ends at rest and takes places within $N = 30$ time-steps of size $h = 0.1$. The cost function in use is of type (24) and reads $J_d = h \sum_{n=0}^{N-1} ||\tau_n||^2$.

Figure 3 shows the initial (blue) and final (red) configuration of the sphere as well as the specified axis of rotation (dashed line). While the sphere moves in space, this axis is translated in parallel. The motion of the sphere is depicted in Fig. 4 at every third time-step. The corresponding motion of the center of mass and evolution of the directors are depicted in Fig. 5. The evolution of the generalised forces, consisting of the translational forces and the torques can be observed in Fig. 6. According to the assumptions made, the generalised forces are constant in each time interval. Figure 7 shows the evolution of the kinetic energy and the components of the angular momentum. Apparently the initial and final conditions of zero motion are met. The first diagram in Fig. 8 depicts the change of angular momentum in each time interval while the second diagram reveals its consistency in the sense that the change of angular momentum $L_{n+1} - L_n$ equals exactly the sum of the applied torques and the momentum induced by the translational forces during that time interval $\Lambda_n^+ + \Lambda_n^- = (\tau_\theta)_n^+ + (\tau_\theta)_n^- + \varphi_{n+1} \times (\tau_\varphi)_n^+ + \varphi_n \times (\tau_\varphi)_n^-$.

Optimal control of a rigid body with rotors Inspired by space telescopes like e.g. the Hubble telescope, whose change

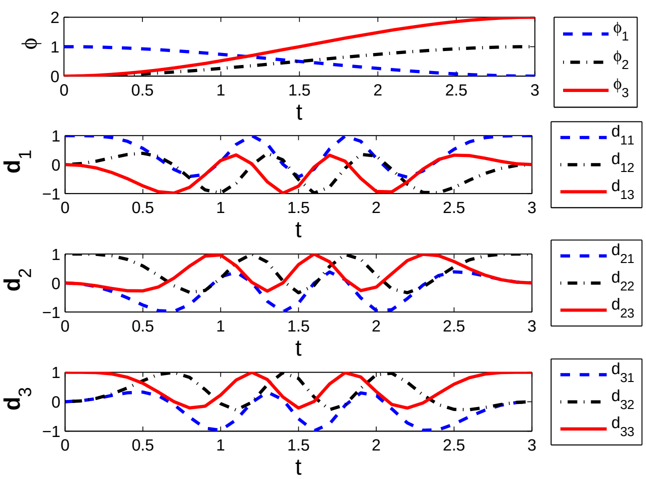

Figure 5. Rigid sphere: motion of center of mass and directors ($h = 0.1$).

in orientation is induced by external spinning rotors, a multibody system consisting of a main body to which rotors are connected by revolute joints has been analysed. The revolute joints allow each rotor to rotate relative to the main body around an axis through its center which is fixed in the main body. Therefore the torque in each revolute joint is a scalar quantity. The goal is to determine optimal torques to guide the main body into the final position $u_\theta^N = \frac{\pi}{14}[1, 2, 3]$, whereby the system starts and ends at rest. The motion takes 5 seconds and the time-step is $h = 0.1$, thus $N = 50$. As in the first example, the objective function rep-

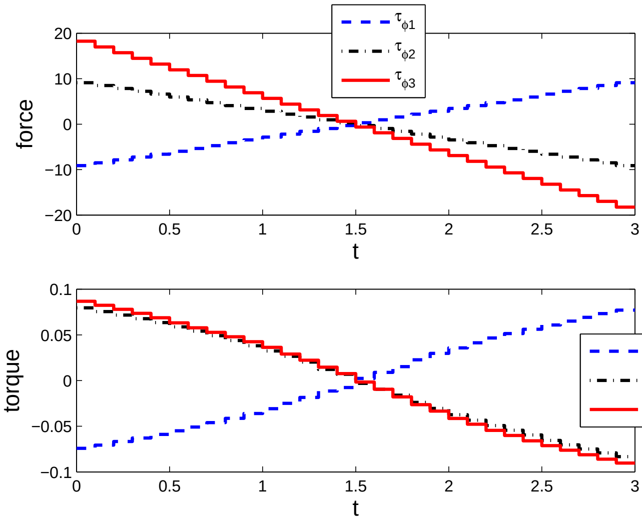

Figure 6. Rigid sphere: force and torque ($h = 0.1$).

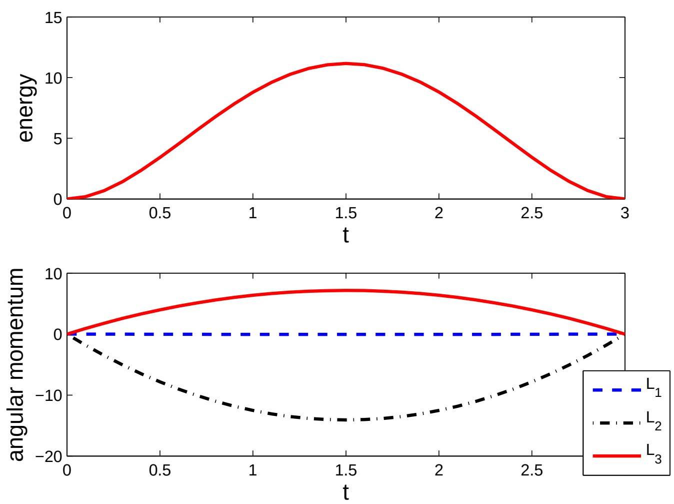

Figure 7. Rigid sphere: energy and components of angular momentum vector $L = L_i e_i$ ($h = 0.1$).

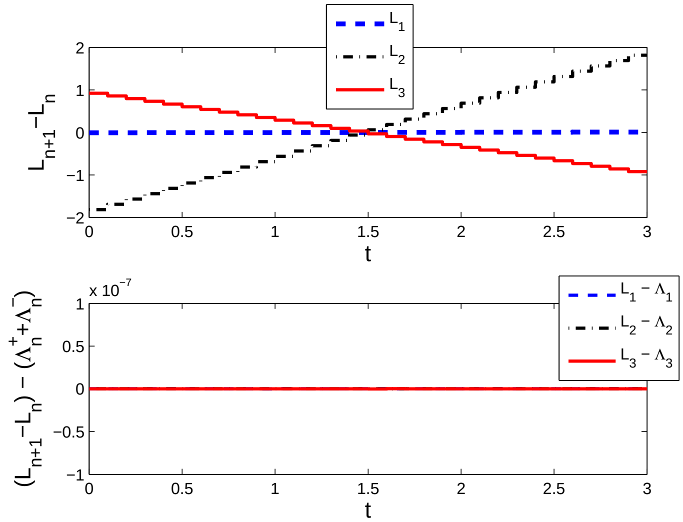

Figure 8. Rigid sphere: change and consistency of angular momentum ($h = 0.1$).

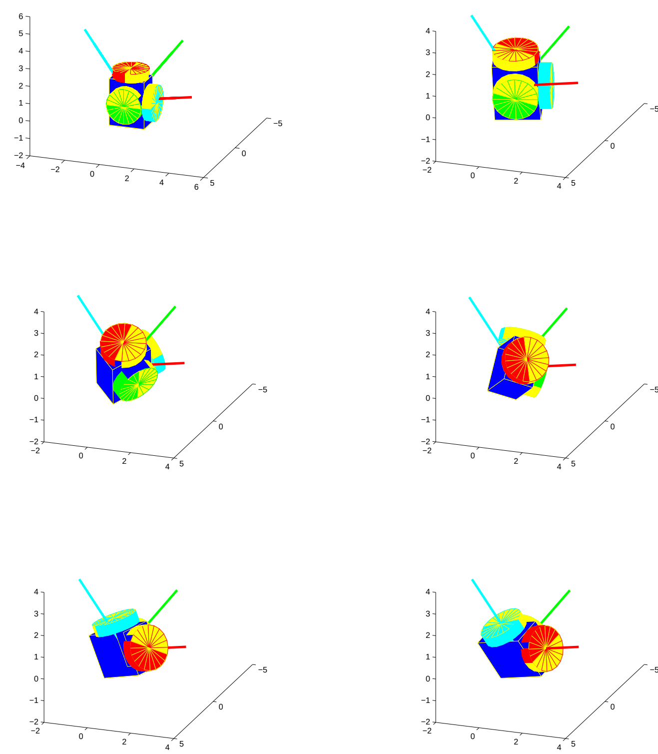

Figure 9. Rigid body with rotors: configuration at $t = 10nh, n = 0, \ldots, 5$ ($h = 0.1$).

resents the control effort which has to be minimised.

Figure 9 shows the configuration of the system at $t = 0, 1, \ldots, 5$ seconds. The static frame represents the required final orientation whereby the axes must coincide with the centers of the rotors as the motion ends (see last picture). The optimal torques which are constant in each time interval are depicted in Fig. 10. Finally Fig. 11 illustrates the evolution of the kinetic energy and a special attribute of the system under consideration. It has a geometric phase which means that the motion occurs although the total angular momentum remains zero at all times.

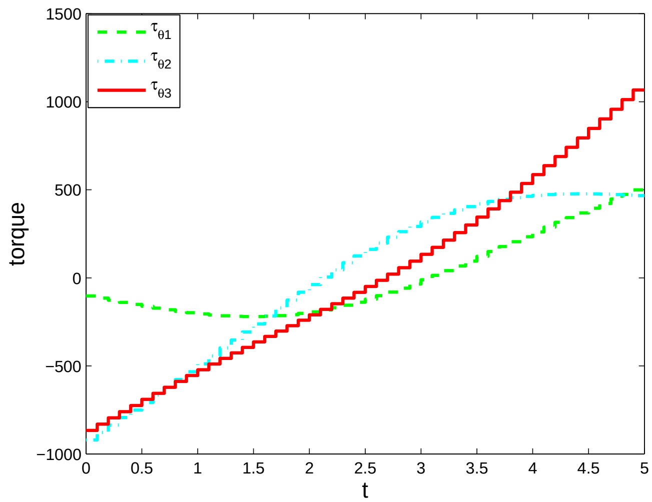

Figure 10. Rigid body with rotors: torque ($h = 0.1$).

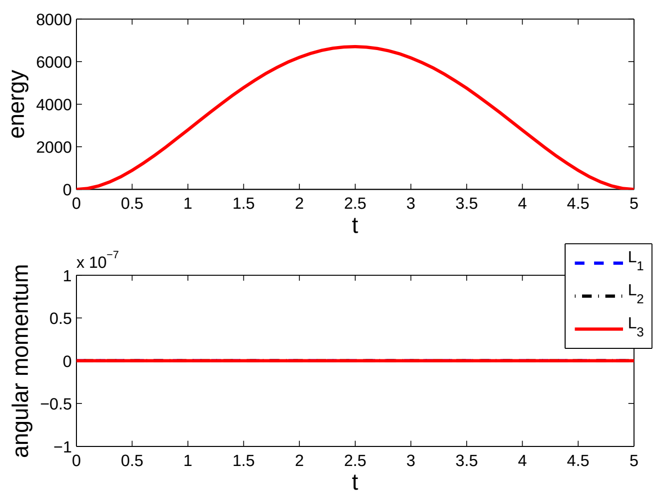

Figure 11. Rigid body with rotors: energy and components of angular momentum vector $L = L_i e_i$ ($h = 0.1$).

CONCLUSION

This paper proposes a new approach to the solution of optimal control problems for constrained dynamical systems via the combination of two recently developed methods: the discrete null space method, which is suitable for the accurate, robust and efficient time integration of such kind of systems, and the optimal control method DMOC.

DMOC is used to compute trajectories for a mechanical system that is optimally guided from an initial to a final configuration via external forces. Thereby, the given cost functional is extremised subject to the dynamics of the constrained mechanical system. Starting form the constrained Lagrange-d'Alembert principle, the discrete null space method yields reduced time-stepping equations that lend themselves as constraints for the resulting optimisation problem.

The proposed method benefits from an easy derivation of the constraint equation for the optimisation algorithm and ensures exactly constraint fulfillment and structure preserving properties of the computed solutions.

As a first example to demonstrate the performance of the proposed procedure, it has been applied to enforce a translational and rotational motion of a rigid sphere in three-dimensional space starting and ending at rest. Furthermore an example involving the actuation of a multibody system with joint constraints has been investigated. Since the system under consideration has a geometric phase, it is of great importance that the change of angular momentum according to the applied forces (which is zero for this example) is captured correctly. This property is demonstrated in the documentation of both examples.

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