Proceedings of the 17th International Symposium on Mathematical
Theory of Networks and Systems, Kyoto, Japan, July 24-28, 2006

Dirac Structures and Implicit Lagrangian Systems in Electric Networks

Hiroaki Yoshimura and Jerrold E. Marsden

***Abstract*— In this paper, we apply Dirac structures and the associated theory of implicit Lagrangian systems to electric networks. We show how a Dirac structure on the flux linkage phase space can be induced from a KCL (Kirchhoff Current Law) constraint distribution on a configuration charge space in analogy with mechanics. In this context, a notion of implicit port–controlled Lagrangian systems is developed. As a specific illustrative example, it is demonstrated that a one–dimensional L-C transmission line can be formulated in the context of implicit port–controlled Lagrangian systems, where the transmission line may be regarded as an interconnected system of a chain of constituent primitive modules, each of which is given by an L-C circuit with external ports.**

***Keywords*— Implicit Lagrangian systems, Dirac structures, L-C transmission line**

I. INTRODUCTION

The design of devices such as L-C transmission lines for extremely wideband signal shaping has been highly exquisite and hence sophisticated mathematical modeling may be required for further developments; see, for instance, [1]. In recent years, a notion of implicit Hamiltonian systems was developed by [15], [4] and [2]. In this context, interconnections of electric circuits, such as conservative L-C circuits, were expressed using a Dirac structure and then incorporated into the implicit Hamiltonian formalism. In particular, the notion of an implicit port–controlled Hamiltonian system was developed for electric circuits with external ports, which are crucial in control design and analysis. One can argue that conservative electric circuits such as L-C circuits, are treated in a more fundamental way from the Lagrangian viewpoint, although generally one must deal with degenerate Lagrangians. Until recently, L-C circuits have not been treated in the context of degenerate Lagrangian systems (see [7]). For regular Lagrangian systems with control inputs, a notion of controlled Lagrangian systems was developed by [5].

Recently, the notion of an implicit Lagrangian system, that is, a Lagrangian analogue of implicit Hamiltonian systems, has been developed by [18], where nonholonomic mechanical systems and degenerate Lagrangian systems such as L-C circuits can be systematically formulated. In this context, use is made of the Dirac structure on the cotangent bundle, which is induced from a constraint distribution. L-C circuits were shown to be expressible by means of Pontryagin's maximum principle by [12] and an idea of implicit Lagrangian equations was developed by [13], both of which are different from our notion of implicit Lagrangian systems in the sense that they did not only regard electric circuits as degenerate Lagrangian systems; however, they did utilize a Dirac structure on a subbundle of the tangent bundle of a configuration manifold, which is consistent with Weinstein's construction of Lie algebroids (see [17]). In another context for Lagrangian systems, L-C circuits were shown to fit into a Birkhoffian formalism by [9].

In this paper, we investigate a Dirac structure induced from KCL constraints in electric networks and develop the idea of implicit port–controlled Lagrangian systems, which is a Lagrangian analogue of implicit port–controlled Hamiltonian systems. First, we give a brief review of how a geometric setting of an electric circuit can be constructed by analogy with mechanics, where the configuration space of an electric circuit may be expressed as a charge space; using this space, electric circuits can then be regarded as a degenerate Lagrangian system. Second, we show that an induced Dirac structure on the cotangent bundle of a configuration space can be defined using a given KCL constraint and also that an electric circuit with external ports can be formulated in the context of implicit port–controlled Lagrangian systems by using the Lagrange–d'Alembert–Pontryagin principle. As an example of degenerate Lagrangian systems in electric networks, we demonstrate that dynamics of a one–dimensional L-C transmission line can be formulated in the context of implicit Lagrangian systems, where the transmission line can be modeled as an interconnected system of constituent primitive modules, each of which is an L-C circuit with external ports.

II. DIRAC STRUCTURES IN ELECTRIC CIRCUITS

A. Geometric Setting for Electric Circuits

We first consider a geometric setting for electric circuits by analogy with mechanics. The configuration space W (isomorphic to an open subset of Euclidean n-space, $\mathbb{R}^n$) for an electric circuit having n elements will be taken to be charge space, where for a point $q \in W$, q^i denotes the charge associated with the i-th element (branch) of the circuit. The tangent bundle TW, the analog of the velocity phase space in mechanics, may be regarded as the current phase space, whose local coordinates are denoted by (q^i, f^i). On the other hand, the cotangent bundle T^*W,

Hiroaki Yoshimura is with the Department of Mechanical Engineering, Waseda University, Ohkubo, Shinjuku, Tokyo 169-8555, Japan yoshimura@waseda.jp

Jerrold E. Marsden is with the Department of Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125, USA marsden@cds.caltech.edu

the analog of the momentum phase space in mechanics, can be considered as the flux linkage phase space, whose local coordinates are given by (q^i, p_i). Thus, $f \in T_qW$ indicates the current, while $p \in T_q^*W$ denotes the flux linkage.

B. Dirac Structures Induced From KCL Constraints

For a given electric circuit graph (often called a topology), Kirchhoff's current law (KCL) may be written in terms of a collection of one-forms ω^a as

$$\langle \omega^a, f \rangle = 0, \quad a = 1, ..., m < n.$$

In this condition, $f = (f^1, ..., f^n) \in T_qW$ indicates the current associated with branches, and the one-forms ω^a are given by

$$\omega^a = \omega^a_k \, dq^k, \quad a = 1, ..., m; \quad k = 1, ..., n,$$

where the coefficients ω^a_k are ± 1 and 0, as determined by the given circuit. The set of all branch currents $f = (f^1, ..., f^n)$ that satisfy the KCL forms an $(n-m)$-dimensional subspace $\Delta(q)$ of T_qW defined by

$$\Delta(q) = \{ f \in T_qW \mid \langle \omega^a, f \rangle = 0, \quad a = 1, ..., m \},$$

which we call the ***KCL constraint space***. Let $\Delta^\circ(q) \subset T_q^*W$ be the annihilator of $\Delta(q)$, and suppose that it is spanned by m one-forms $\omega^1, ..., \omega^m$.

Let $T\pi_W : TT^*W \to TW$ be the tangent map of the cotangent bundle projection $\pi_W : T^*W \to W$. Define the distribution Δ_{T^*W} on T^*W by lifting the KCL constraint distribution $\Delta \subset TE$:

$$\Delta_{T^*W} := (T\pi_W)^{-1}(\Delta) \subset TT^*W.$$

The annihilator of Δ_{T^*W}, for each $z = (q, p) \in T^*W$, is of course given by

$$\Delta^\circ_{T^*W}(z) = \{ \alpha_z \in T_z^*T^*W \mid \langle \alpha_z, \ w_z \rangle = 0, \quad \text{for all } w_z \in \Delta_{T^*W}(z) \}.$$

Let Ω be the canonical symplectic structure on T^*W and $\Omega^\flat : TT^*W \to T^*T^*W$ be its associated bundle map. Then, a Dirac structure D_Δ on T^*W, which is induced from the KCL constraint distribution $\Delta \subset TW$, can be defined, for each $z = (q, p) \in T^*W$, by

$$D_\Delta(z) = \{ (v_z, \alpha_z) \in T_zT^*W \times T_z^*T^*W \mid v_z \in \Delta_{T^*W}(z), \ \text{and} \ \alpha_z - \Omega^\flat(z) \, v_z \in \Delta^\circ_{T^*W}(z) \}.$$

C. Local Expressions

Recall that the projection $\pi_W : T^*W \to W$ is locally given by $(q, p) \mapsto q$ and its tangent map is $T\pi_W : (q, p, \dot{q}, \dot{p}) \mapsto (q, \dot{q})$. Then, we can represent Δ_{T^*W} in coordinates as

$$\Delta_{T^*W} \cong \left\{ v_{(q,p)} = (q, p, \dot{q}, \dot{p}) \mid q \in W, \dot{q} \in \Delta(q) \right\}.$$

Let points in T^*T^*W be locally denoted by $\alpha_{(q,p)} = (q, p, \alpha, w)$, where α is a covector and w is a vector, and the annihilator of Δ_{T^*W} is locally, given by

$$\Delta^\circ_{T^*W} \cong \{ \alpha_{(q,p)} = (q, p, \alpha, w) \mid q \in W, \, \alpha \in \Delta^\circ(q) \text{ and } w = 0 \}.$$

It follows from the local expression of the canonical symplectic structure that, for $v_z = (q, p, \dot{q}, \dot{p})$,

$$\Omega^\flat(z) \cdot v_z = (q, p, -\dot{p}, \dot{q})$$

and also that $\alpha_z - \Omega^\flat(z) \cdot v_z \in \Delta^\circ_{T^*W}$ reads

$$(q, p, \alpha + \dot{p}, w - \dot{q}) \in \Delta^\circ_{T^*W};$$

that is, $\alpha + \dot{p} \in \Delta^\circ(q)$, and $w - \dot{q} = 0$.

Thus, the induced Dirac structure is given in local representation by

$$D_{\Delta_W}(z) = \{ ((q, p, \dot{q}, \dot{p}), (q, p, \alpha, w)) \mid \dot{q} \in \Delta(q), \ w = \dot{q}, \text{ and } \alpha + \dot{p} \in \Delta^\circ(q) \}. \quad (1)$$

III. Implicit Controlled Lagrangian Systems

A. Dirac Differential of a Lagrangian

Let $\mathcal{L} : TW \to \mathbb{R}$ be a Lagrangian (possibly degenerate). The differential of $\mathcal{L}$ is the map

$$\mathbf{d}\mathcal{L} : TW \to T^*TW,$$

which is locally given by

$$\mathbf{d}\mathcal{L} = \left(q, f, \frac{\partial \mathcal{L}}{\partial q}, \frac{\partial \mathcal{L}}{\partial f} \right).$$

As in [18], the Dirac differential of the Lagrangian is the map

$$\mathfrak{D}\mathcal{L} : TW \to T^*T^*W$$

defined by

$$\mathfrak{D}\mathcal{L} = \gamma_W \circ \mathbf{d}\mathcal{L},$$

where $\gamma_W = \Omega^\flat \circ (\kappa_W)^{-1} : T^*TW \to T^*T^*W$, which is locally given by

$$(q, \delta q, \delta p, p) \mapsto (q, p, -\delta p, \delta q).$$

In the above, the map $\kappa_W : TT^*W \to T^*TW$; $(q, p, \delta q, \delta p) \mapsto (q, \delta q, \delta p, p)$ is a symplectomorphism preserving the symplectic structure on TT^*W originally developed by [14]. The Dirac differential of L is locally given by

$$\mathfrak{D}\mathcal{L} = \left(q, \frac{\partial \mathcal{L}}{\partial f}, -\frac{\partial \mathcal{L}}{\partial q}, f \right), \quad (2)$$

where p is given by the Legendre transform

$$(q, p) = \left(q, \frac{\partial \mathcal{L}}{\partial f} \right).$$

B. External Voltage Fields

In the analysis of electric networks, we often meet circuits in which some ports are connected to external elements or other systems. We show how such electric circuits with external ports can be treated in the context of implicit Lagrangian systems. Here we take the case when the external ports are connected to external sources of voltages.

Let $e : TW \to T^*W$ be an external voltage field. Recall that an external force field induces a horizontal one–form in mechanics (see [10]), and using the analogy

$$\text{Force} \leftrightarrow \text{Voltage},$$

an external voltage field $e : TW \to T^*W$ induces a horizontal one-form $\widetilde{e}$ on T^*W by, for each $z \in T^*W$,

$$\widetilde{e} \cdot w_z = \langle e(f), T_z\pi_W(w_z)\rangle,$$

where $f \in T_qW$, $w_z \in T_zT^*W$, and $\widetilde{e} = \pi_W^* e$. In coordinates, writing $e : TW \to T^*W$ as

$$(q, f) \mapsto (q, e(f)),$$

the horizontal one–form $\widetilde{e} = \pi_W^* e$ may be written as

$$\widetilde{e} = (q, p, e(f), 0). \tag{3}$$

C. Implicit Port–Controlled Lagrangian Systems

Let X be a vector field on T^*W. The condition for implicit Lagrangian systems $(\mathcal{L}, e, \Delta, X)$ for electric circuits with external ports is given by, for each $(q, p) \in T^*W$,

$$(X(q, p), \mathfrak{D}\mathcal{L}(q, f) - \pi_W^* e(q, f)) \in D_\Delta(q, p) \tag{4}$$

together with $(q, p) = \mathbb{F}\mathcal{L}(q, f)$. Thus, the curve $(q(t), f(t), p(t))$, $t_1 \leq t \leq t_2$ in $TW \oplus T^*W$ that satisfies the condition (4) is a solution curve of the implicit Lagrangian system $(\mathcal{L}, e, \Delta, X)$.

The class of implicit Lagrangian system $(\mathcal{L}, e, \Delta, X)$ can be understood as a Lagrangian analogue of implicit port–controlled Hamiltonian systems, and so we shall call $(\mathcal{L}, e, \Delta, X)$ an ***implicit port–controlled Lagrangian system***.

Let us now develop the local expression for implicit port–controlled Lagrangian systems.

Writing a vector filed X on T^*W in coordinates as

$$X = (q, p, \dot{q}, \dot{p}), \tag{5}$$

if follows from equations (1)–(5) that

$$\begin{aligned} &\dot{q} = f \in \Delta(q), \\ &\dot{p} - \frac{\partial \mathcal{L}}{\partial q} - e(q, f) \in \Delta^\circ(q), \\ &p = \frac{\partial \mathcal{L}}{\partial f}. \end{aligned} \tag{6}$$

Using the one-forms $\omega^1, ..., \omega^m$ that span a basis of the annihilator $\Delta^\circ(q)$ at each $q \in W \subset \mathbb{R}^n$, it follows that equation (6) can be represented, in local coordinates, by

$$\begin{aligned} \begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial \mathcal{L}}{\partial q^i} \\ f^i \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ \mu_a\, \omega_i^a(q) + e_i(q, f) \end{pmatrix}, \\ p_i &= \frac{\partial \mathcal{L}}{\partial f^i}, \\ 0 &= \omega_i^a(q)\, f^i, \end{aligned} \tag{7}$$

where μ_a, $a = 1, ..., m$ are the Lagrange multipliers.

IV. The Variational Structure

We next exhibit the variational structure of implicit controlled Lagrangian systems by using the Lagrange–d'Alembert–Pontryagin principle.

Recall that a system $q : [t_1, t_2] \to W$ is said to be constrained if $\dot{q}(t) \in \Delta(q(t))$ for all t, $t_1 \leq t \leq t_2$, and the motion $q(t)$ of electric circuits with external ports is constrained to the KCL subspace $\Delta(q(t)) \subset T_{q(t)}W$. Then, the Lagrange–d'Alembert–Pontryagin principle for a curve $(q(t), f(t), p(t))$, $t_1 \leq t \leq t_2$ in $TW \oplus T^*W$ is represented by

$$\begin{aligned} \delta \int_{t_1}^{t_2} &\{\mathcal{L}(q(t), f(t)) + p(t) \cdot (\dot{q}(t) - f(t))\}\ dt \\ &+ \int_{t_1}^{t_2} e(q(t), f(t)) \cdot \delta q(t)\ dt = 0 \end{aligned}$$

for the chosen variation $\delta q(t) \in \Delta(q(t))$ and with the constraint $\dot{q}(t) \in \Delta(q(t))$. Keeping the endpoints of $q(t)$ fixed, we have

$$\begin{aligned} &\delta \int_{t_1}^{t_2} \{\mathcal{L}(q, f) + p \cdot (\dot{q} - f)\}\ dt \\ &= \int_{t_1}^{t_2} \left\{ \left(\frac{\partial \mathcal{L}}{\partial q} - \dot{p} \right) \delta q + \left(\frac{\partial \mathcal{L}}{\partial f} - p \right) \delta f + (\dot{q} - f)\delta p \right\} dt. \end{aligned}$$

Hence, the Lagrange–d'Alembert–Pontryagin principle is denoted by

$$\begin{aligned} \int_{t_!}^{t_2} &\left\{ \left(\frac{\partial \mathcal{L}}{\partial q} - \dot{p} \right) \delta q + \left(\frac{\partial \mathcal{L}}{\partial f} - p \right) \delta f + (\dot{q} - f)\delta p \right\}\ dt \\ &+ \int_{t_1}^{t_2} e(q, f)\, \delta q\, dt = 0 \end{aligned}$$

for the chosen variation $\delta q(t) \in \Delta(q(t))$, for all $\delta f(t)$ and $\delta p(t)$, and with $\dot{q}(t) \in \Delta(q(t))$.

The Lagrange–d'Alembert–Pontryagin principle gives equation (6) for an L-C circuit with external ports, which are, needless to say, also represented in local coordinates by equation (7).

In the following, we shall demonstrate that the above construction of implicit Lagrangian systems for electric circuits with external ports can be applied to an illustrative example of one–dimensional transmission lines, which is a typical degenerate Lagrangian system.

V. One–dimensional L-C Transmission Lines

A. One–dimensional L-C Transmission Line

Let us consider an electric network of one–dimensional lossless transmission lines, which comprises of inductors and capacitors illustrated in Fig.1. It is known that the one–dimensional lossless transmission line has been designed for extremely wideband signal shaping; see [1].

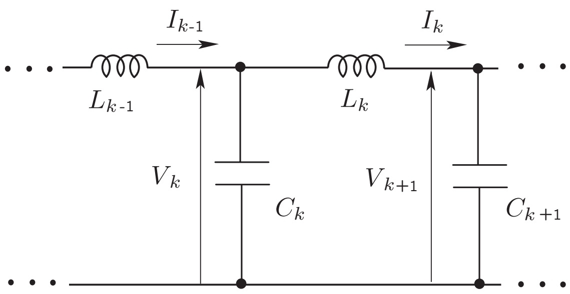

Fig. 1. One Dimensional L-C Transmission Line

B. The Primitive Module of L-C Circuits

For the analysis of infinite one–dimensional L-C transmission lines, focusing on the chain structure, we make use of a model of primitive modules $Z_k = \{L_k,\, C_k\}$, each of which is an L-C circuit with external ports as shown in Fig.2. Hence, the one–dimensional L-C transmission line may be regarded as an infinite chain of Z_k, where Z_{k-1} may be the $(k-1)$st adjacent module and Z_{k+1} the $(k+1)$st adjacent module. The configuration space W_k

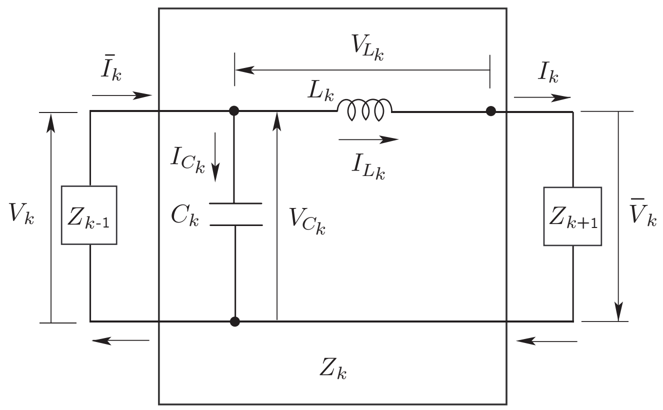

Fig. 2. The Primitive Module of L-C Circuits

for the L-C circuit of the k-th module is given by an open subset W_k of a 4–dimensional vector space $\mathbb{R}^4$. Let $x_k = (\bar{q}_k, q_{C_k}, q_{L_k}, q_k) \in W_k$ and $f_k = (\bar{I}_k, I_{C_k}, I_{L_k}, I_k) \in T_{x_k}W_k$.

By using the idea of interconnecting primitive modules, we can decentralize the dynamics of the original infinite one–dimensional transmission line into the dynamics of the k-th module Z_k, together with the interconnections with adjacent modules, which are given by

$$\text{voltage: } \bar{V}_k \cong -V_{k+1} \text{ and current: } \bar{I}_k \cong I_{k-1}.$$

Then, we shall formulate dynamics of an L-C circuit with external ports of the k-th module.

C. Kirchhoff's Current Laws

Recall that KCL constraints for the current $f_k = (\bar{I}_k, I_{C_k}, I_{L_k}, I_k) \in T_{x_k}W_k$ are given by a constraint distribution $\Delta_k \subset TW_k$ such that, for each $x_k \in W_k$,

$$\Delta_k(x_k) = \{f_k \in T_{x_k}W_k \mid \langle \omega_k^a, f_k \rangle = 0,\ a = 1, 2\}.$$

In the above, ω_k^a denote covectors (or one-forms) represented, in coordinates, by

$$\omega_k^a = \sum_{i=1}^{4} \omega_{ki}^a \, dx_k^i, \quad a = 1, 2,$$

where we set $x_k = (x_k^1, x_k^2, x_k^3, x_k^4) = (\bar{q}_k, q_{C_k}, q_{L_k}, q_k)$. In the circuit of Fig. 2, the coefficients ω_{ki}^a are given in matrix representation by

$$\omega_{ki}^a = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Hence, the KCL constraints are given in coordinates by

$$\begin{aligned} -\bar{I}_k + I_{C_k} + I_{L_k} &= 0, \\ I_{L_k} - I_k &= 0. \end{aligned} \tag{8}$$

D. Degenerate Lagrangian and Primary Constraints

Let $\mathcal{T}_k : TW_k \to \mathbb{R}$ be the magnetic energy of the k-th module of the L-C transmission line, which is defined in terms of the inductance L_k as

$$\mathcal{T}_k(x_k, f_k) = \frac{1}{2} L_k \, (I_{L_k})^2,$$

where we set $f_k = (f_k^1, f_k^2, f_k^3, f_k^4) = (\bar{I}_k, I_{C_k}, I_{L_k}, I_k)$. Let $\mathcal{V}_k : W_k \to \mathbb{R}$ be the electric potential energy, which is defined by capacitors C_k as

$$\mathcal{V}_k(x_k) = \frac{1}{2} \frac{(q_{C_k})^2}{C_k}.$$

Then, we can define the Lagrangian of the k-th module, that is, $\mathcal{L}_k : TW_k \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}_k(x_k, f_k) &= \mathcal{T}_k(x_k, f_k) - \mathcal{V}_k(x_k) \\ &= \frac{1}{2} L_k \, (I_{L_k})^2 - \frac{1}{2} \frac{(q_{C_k})^2}{C_k}. \end{aligned} \tag{9}$$

It is obvious that the k-th Lagrangian $\mathcal{L}_k : TW_k \to \mathbb{R}$ given in equation (9) is degenerate since

$$\det \left[\frac{\partial^2 \mathcal{L}_k}{\partial f_k^i \partial f_k^j} \right] = 0.$$

Meanwhile, the ***constraint flux linkage subspace*** is defined by

$$P_k = \mathbb{F}\mathcal{L}_k(\Delta_k) \subset T^*W_k,$$

where $\mathbb{F}\mathcal{L}_k : TW_k \to T^*W_k$ denotes the Legendre transform. In coordinates,

$$(x_k, \lambda_k) = \mathbb{F}\mathcal{L}_k(x_k, f_k) \in T^*W_k,$$

which is expressed by

$$\begin{aligned} &(\bar{q}_k, q_{C_k}, q_{L_k}, q_k, \bar{p}_k, p_{C_k}, p_{L_k}, p_k) \\ &= \left(\bar{q}_k, q_{C_k}, q_{L_k}, q_k, \frac{\partial \mathcal{L}}{\partial \bar{I}_k}, \frac{\partial \mathcal{L}}{\partial I_{C_k}}, \frac{\partial \mathcal{L}}{\partial I_{L_k}}, \frac{\partial \mathcal{L}}{\partial I_k} \right), \end{aligned}$$

where the current $f_k = (\bar{I}_k, I_{C_k}, I_{L_k}, I_k)$ satisfies the KCL constraints in equation (8). By direct computation, we get

$$p_{L_k} = L_k \, I_{L_k}, \ \ \bar{p}_k = p_{C_k} = p_k = 0.$$

The constraints for the flux linkages, namely,

$$\bar{p}_k = p_{C_k} = p_k = 0$$

correspond to ***primary constraints*** in the sense of Dirac, which form the constraint flux linkage subspace $P_k \subset T^*W_k$, and it immediately reads

$$(x_k, \lambda_k) = (\bar{q}_k, q_{C_k}, q_{L_k}, q_k, 0, 0, p_{L_k}, 0) \in P_k \subset T^*W_k.$$

E. External Voltage Fields

L-C circuits with external two-ports are connected to the $(k-1)$st and the $(k+1)$st adjacent modules, where the voltages V_k and $\bar{V}_k$ associated to the external two-ports may be regarded as an external voltage field, given by the fiber-preserving map

$$e_k : TW_k \to T^*W_k$$

that is expressed locally by

$$e_k = (\bar{q}_k, q_{C_k}, q_{L_k}, q_k; V_k, 0, 0, \bar{V}_k).$$

Since $e_k : TW_k \to T^*W_k$ induces a horizontal one–form on T^*W_k such that, for each $z_k = (x_k, \lambda_k) \in T^*W_k$,

$$\widetilde{e}_k \cdot w_{z_k} = \langle e_k(f_k), T_{z_k}\pi_{W_k}(w_{z_k})\rangle,$$

where $f_k \in T_{x_k}W_k$ and $w_{z_k} \in T_{z_k}T^*W_k$, one has

$$\widetilde{e_k} = \pi^*_{W_k} e_k,$$

which is given in coordinates as

$$\begin{aligned}&\widetilde{e}(\bar{q}_k, q_{C_k}, q_{L_k}, q_k, 0, 0, p_{L_k}, 0)\\ &= (V_k, 0, 0, \bar{V}_k, 0, 0, 0, 0).\end{aligned} \tag{10}$$

F. Implicit Port–Controlled Lagrangian Systems

A vector field X_k on T^*W_k, defined at each point in P_k, is expressed in coordinates by

$$\begin{aligned}&X_k(\bar{q}_k, q_{C_k}, q_{L_k}, q_k, 0, 0, p_{L_k}, 0)\\ &= (\dot{\bar{q}}_k, \dot{q}_{C_k}, \dot{q}_{L_k}, \dot{q}_k,\ 0, 0, \dot{p}_{L_k}, 0).\end{aligned} \tag{11}$$

The differential of the Lagrangian $\mathcal{L}_k$ is locally given by

$$\mathbf{d}\mathcal{L}_k(x_k, f_k) = \left(\frac{\partial \mathcal{L}_k}{\partial x_k}, \frac{\partial \mathcal{L}_k}{\partial f_k}\right)$$

and it follows that

$$\begin{aligned}&\mathbf{d}\mathcal{L}(\bar{q}_k, q_{C_k}, q_{L_k}, q_k,\ \bar{I}_k, I_{C_k}, I_{L_k}, I_{L_k})\\ &= \left(0, 0, L_k I_{L_k}, 0,\ 0, -\frac{q_{C_k}}{C_k}, 0, 0\right),\end{aligned}$$

while the Dirac differential of $\mathcal{L}_k$ is given by

$$\mathfrak{D}\mathcal{L}_k(x_k, f_k) = \left(-\frac{\partial \mathcal{L}}{\partial x_k}, f_k\right),$$

which reads

$$\begin{aligned}&\mathfrak{D}\mathcal{L}(\bar{q}_k, q_{C_k}, q_{L_k}, q_k,\ \bar{I}_k, I_{C_k}, I_{L_k}, I_{L_k})\\ &= \left(0, -\frac{q_{C_k}}{C_k}, 0, 0,\ \bar{I}_k, I_{C_k}, I_{L_k}, I_k\right).\end{aligned} \tag{12}$$

The Dirac structure D_{Δ_k} on T^*W induced from the KCL constraint distribution Δ_k can be expressed by, for each $z_k = (x_k, \lambda_k) \in T^*W_k$,

$$\begin{aligned}&D_{\Delta_k}(z_k) = \{((x_k, \lambda_k, \dot{x}_k, \dot{\lambda}_k), (x_k, \lambda_k, \alpha_k, w_k)) \mid\\ &\dot{x}_k \in \Delta_k(x_k),\ w_k = \dot{x}_k, \text{ and } \alpha_k + \dot{\lambda}_k \in \Delta_k^\circ(x_k)\}.\end{aligned} \tag{13}$$

Then, the condition of implicit port–controlled Lagrangian systems is given by

$$(X(x_k, \lambda_k), \mathfrak{D}\mathcal{L}(x_k, f_k) - \pi^*_W e(x_k, f_k)) \in D_\Delta(x_k, \lambda_k),$$

which holds for each $(x_k, f_k) \in \Delta \subset TW$ and with $(x_k, \lambda_k) = \mathbb{F}\mathcal{L}(x_k, f_k)$.

It follows from equations (10)–(13) that dynamics of the L-C transmission line can be expressed, in coordinates, in the context of implicit port–controlled Lagrangian systems $(\mathcal{L}, e, \Delta, X)$ as

$$\begin{aligned}&\dot{\bar{q}}_k = \bar{I}_k,\ \dot{q}_{C_k} = I_{C_k},\ \dot{q}_{L_k} = I_{L_k},\ \dot{q}_k = I_k,\\ &\mu_1 = -V_k,\ \mu_1 = -\frac{q_{C_k}}{C_k},\ \mu_2 = \bar{V}_k,\\ &\dot{p}_{L_k} = -\mu_1 + \mu_2\end{aligned}$$

together with the Legendre transform $p_k = L_k\, I_{L_k}$, and with the KCL constraints

$$\bar{I}_k = I_{C_k} + I_{L_k},\ I_{L_k} = I_k.$$

VI. The One–Dimensional Transmission Line as an Interconnected System

A. Interconnection between the Adjacent Modules

So far, we have derived equations of motion for the k-th primitive module of the one–dimensional transmission line. Then, in order to formulate dynamics of the original one–dimensional transmission line, we consider the following condition of the interconnection between the adjacent modules Z_k and Z_{k+1}:

$$\bar{V}_k = -V_{k+1},\quad \bar{I}_{k+1} = I_k,$$

which satisfy the so–called ***power invariance*** or ***Tellegen's theorem*** (see, for instance, [6]):

$$\langle \bar{V}_k, I_k\rangle + \langle V_{k+1}, \bar{I}_{k+1}\rangle = 0.$$

It is known that Tellegen's theorem can be incorporated into a Dirac structure (see [18]).

B. Interconnected Systems of L-C Circuits

By considering the interconnection between the adjacent modules, we can obtain dynamics of the one–dimensional L-C transmission line by the following implicit differential–algebraic equations:

$$\begin{aligned}&\dot{q}_{k-1} = I_{k-1},\ \dot{q}_{C_k} = I_{C_k},\ \dot{q}_{L_k} = I_{L_k},\ \dot{q}_k = I_k,\\ &\mu_1 = -V_k,\ \mu_1 = -\frac{q_{C_k}}{C_k},\ \mu_2 = -V_{k+1},\\ &\dot{p}_{L_k} = -\mu_1 + \mu_2,\\ &p_{L_k} = L_k\, I_{L_k},\\ &I_{k-1} = I_{C_k} + I_{L_k},\ I_{L_k} = I_k.\end{aligned}$$

Eliminating unnecessary multipliers, it follows that dynamics of the interconnected system can be given by

$$\begin{aligned} \dot{q}_{C_k} &= I_{k-1} - I_k, \\ \dot{p}_{L_k} &= V_k - V_{k+1}, \end{aligned} \tag{14}$$

where $I_k = L_k^{-1} p_{L_k}$ denotes the current of the k-th inductor and $V_k = C_k^{-1} q_{C_k}$ the voltage of the k-th capacitor.

By eliminating q_{C_k} and p_{L_k} in equation (14), we obtain the dynamics of the one–dimensional L-C transmission line:

$$\begin{aligned} C_k \frac{dV_k(t)}{dt} &= I_{k-1}(t) - I_k(t), \\ L_k \frac{dI_k(t)}{dt} &= V_k(t) - V_{k+1}(t), \end{aligned} \tag{15}$$

which are the well–known form (see, for instance, [1]).

Meanwhile, by eliminating V_k and I_k in equation (14), we can also obtain the form

$$\begin{aligned} \dot{q}_{C_k} &= L_{k-1}^{-1} p_{L_{k-1}} - L_k^{-1} p_{L_k}, \\ \dot{p}_{L_k} &= C_k^{-1} q_{C_k} - C_{k+1}^{-1} q_{C_{k+1}}, \end{aligned}$$

which are a Hamiltonian form that is equivalent with equation (15).

VII. CONCLUSIONS

In this paper, we have studied electric networks, focusing upon a one–dimensional L-C transmission line, regarded as a degenerate Lagrangian system, in the context of Dirac structures and implicit Lagrangian systems.

We first showed how a Dirac structure can be induced from KCL constraints and also showed how the dynamics of electric circuits with external ports can be formulated in the context of implicit Lagrangian systems and the newly developed notion of implicit port–controlled Lagrangian systems.

Second, we illustrated the theory of implicit port–controlled Lagrangian systems using the example of a one–dimensional L-C transmission line. We took the point of view of a model of an L-C circuit with external ports that is a constituent primitive module of the transmission line and illustrated how the primitive modules can be interconnected using implicit port–controlled Lagrangian systems. The one–dimensional transmission line is thus represented as an interconnected system of the primitive modules and we demonstrated that dynamics of the interconnected L-C circuits with external ports can reconstruct the original dynamics of the one–dimensional transmission line in the context of the implicit port–controlled Lagrangian systems.

VIII. ACKNOWLEDGMENTS

The authors are grateful to Ehsan Afshari, who kindly gave several useful remarks. The research is partially supported by JSPS Grant 16560216 and in part by NSF-ITR Grant ACI-0204932.

REFERENCES

[1] Afshari, E., Bhat, H. S., Hajimiri, A., and Marsden, L. E.[2006], Extremely wideband signal shaping using one- and two-dimensional nonuniform nonlinear transmission lines, *Journal of Applied Physics*, **99**, 05401–1–16.

[2] Blankenstein, G. [2000], *Implicit Hamiltonian Systems: Symmetry and Interconnection*, Ph.D. Dissertation. University of Twente.

[3] Bloch, A. M. [2003], *Nonholonomic Mechanics and Control*, volume 24 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York. With the collaboration of J. Baillieul, P. Crouch and J. Marsden, and with scientific input from P. S. Krishnaprasad, R. M. Murray and D. Zenkov.

[4] Bloch, A. M. and Crouch, P. E. [1997], Representations of Dirac structures on vector spaces and nonlinear L-C circuits. Differential Geometry and Control (Boulder, CO, 1997), 103–117, *Proc. Sympos. Pure Math.* **64**. Amer. Math. Soc., Providence, RI.

[5] Chang, D., A. M. Bloch, N. Leonard, J. E. Marsden, and C. Woolsey [2002], The equivalence of controlled Lagrangian and controlled Hamiltonian systems, *Control and the Calculus of Variations (special issue dedicated to J.L. Lions)* **8**, 393–422.

[6] Chua, L. O., Desoer, C. A. and Kuh, D. A. [1987], *Linear and Nonlinear Circuits*. McGraw-Hill Inc.

[7] Chua, L. O. and J. D. McPherson [1974], Explicit topological formulation of Lagrangian and Hamiltonian equations for nonlinear networks, *IEEE Transaction on Circuit and Systems*, CAS-**21**, 277–286.

[8] Courant, T. J. [1990], Dirac manifolds, *Trans. Amer. Math. Soc.* **319**, 631–661.

[9] Ionescu, D. and J. Scheurle [2005], Birkhoffian formulation of the dynamics of LC circuits. Preprint. 1–31.

[10] Marsden, J. E. and T. S. Ratiu [1999], *Introduction to Mechanics and Symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, second edition.

[11] Maschke, B. M., van der Schaft, A. J. and P. C. Breedveld [1995], An intrinsic Hamiltonian formulation of the dynamics of L-C circuits, *IEEE Trans. Circuits and Systems.* **42**, (2) 73–82.

[12] Moreau, L. and D. Aeyels, A variational principle for nonlinear LC circuits with arbitrary interconnection structure, *5th IFAC Symposium of Nonlinear Control Systems (NOLCOS '01)*, Saint-Petersburg, July 2001.

[13] Moreau, L. and A. J. van der Schaft, Implicit Lagrangian equations and the mathematical modeling of physical systems, *Proc. 41st IEEE Conf. Decision and Control*, Las Vegas, Nevada, December 2002.

[14] Tulczyjew, W. M. [1977], The Legendre transformation, *Ann. Inst. H. Poincaré*, Sect. A, **27**(1), 101–114.

[15] van der Schaft, A. J. and B. M. Maschke [1995], The Hamiltonian formulation of energy conserving physical systems with external ports, *Archiv für Elektronik und Übertragungstechnik* **49**, 362–371.

[16] van der Schaft, A. J. [1998], Implicit Hamiltonian systems with symmetry, *Rep. Math. Phys.* **41**, 203–221.

[17] Weinstein, A. [1996], Lagrangian mechanics and groupoids, *Fields Institute Communications*. **7**, 207–231.

[18] Yoshimura, H. and Marsden, J. E. [2006], Dirac structures in Lagrangian mechanics. Part I: Implicit Lagrangian systems, Part II: Variational structures. *Journal of Geometry and Physics*, published online, April, 2006.