

# Dirac Structures in Lagrangian Mechanics

## *Part II: Variational Structures*

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### Abstract

Part I of this paper introduced the notion of implicit Lagrangian systems and their geometric structure was explored in the context of Dirac structures. In this part, we develop the variational structure of implicit Lagrangian systems. Specifically, we show that the implicit Euler-Lagrange equations can be formulated using an extended variational principle of Hamilton called the Hamilton-Pontryagin principle. This variational formulation incorporates, in a natural way, the generalized Legendre transformation, which enables one to treat degenerate Lagrangian systems. The definition of this generalized Legendre transformation makes use of natural maps between iterated tangent and cotangent spaces. Then, we develop an extension of the classical Lagrange-d'Alembert principle called the Lagrange-d'Alembert-Pontryagin principle for implicit Lagrangian systems with constraints and external forces. A particularly interesting case is that of nonholonomic mechanical systems that can have both constraints and external forces. In addition, we define a constrained Dirac structure on the constraint momentum space, namely the image of the Legendre transformation (which, in the degenerate case, need not equal the whole cotangent bundle). We construct an implicit constrained Lagrangian system associated with this constrained Dirac structure by making use of an Ehresmann connection. Two examples, namely a vertical rolling disk on a plane and an L-C circuit are given to illustrate the results.

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## 1 Introduction

In the present Part II of this paper, we continue to develop the framework of implicit Lagrangian systems and their geometry in the context of Dirac structures, which was begun in Part I. This part focuses on the variational structure of implicit Lagrangian systems. An algebraic theory of Dirac structures associated with formal variational calculus is contained in the work of Dorfman [1987, 1993] in the Hamiltonian framework of integrable evolution equations. However, it has not been clear how Dirac structures are interrelated with implicit mechanical systems, whether Lagrangian or Hamiltonian, in the context of variational principles. In other words, there has been a gap between Dirac structures and variational principles in mechanics. In conjunction with Dirac’s theory of constraints, we remark that Dirac started off with Hamilton’s principle with the aim of investigating degenerate Lagrangian systems, although in the end, he did not focus on the Lagrangian formulation but rather developed the notion of a constrained Poisson structure, or the “Dirac bracket” (see, for instance, Dirac [1964]; Marsden and Ratiu [1986]).

Needless to say, in mechanics, Hamilton’s principle is employed to formulate the Euler-Lagrange equations on the Lagrangian side, while Hamilton’s phase space principle may be used to derive Hamilton’s equations on the Hamiltonian side. As is well known, both formalisms, in the case that a given Lagrangian is hyperregular, are equivalent via the Legendre transformation (see Abraham and Marsden [1978]; Marsden and Ratiu [1999]). In the case that a Lagrangian is degenerate, we need a specific treatment to deal with constraints due to the degeneracy, as in Dirac’s theory of constraints. For such degenerate Lagrangian systems, Weinstein [38] noted that a Dirac structure on a Lie algebroid may be induced from a Poisson structure on the dual bundle of the Lie algebroid. He also considered the case in which a Lie algebroid is given by the tangent bundle. L–C circuits have not been treated so far in the context of degenerated Lagrangian systems (see, for instance, Chua and McPherson [1974]). A variational principle for L–C circuits was developed using Pontryagin’s maximum principle by Moreau and Aeyels [2001] and an idea of implicit Lagrangian systems was developed by Moreau and van der Schaft [2002]. Both of these are different from our notion of implicit Lagrangian systems in the sense that they utilized a Dirac structure on a subbundle of the tangent bundle of a configuration manifold, consistent with Weinstein’s idea [38]. Furthermore, mechanical systems with nonholonomic constraints have been widely studied (see, for instance, Vershik and Faddeev [1981]; Bates and Sniatycki [1993]), specifically, from the viewpoint of symmetry and reduction by Bloch, Krishnaprasad, Marsden, and Murray [1996], where the Lagrange-d’Alembert principle played an essential role in formulating the equations of motion and, in addition, the system viewed in terms of a constrained Lagrangian was formulated using an Ehresmann connection. On the Hamiltonian side, constrained Hamiltonian systems were developed from the viewpoint of Poisson structures by van der Schaft and Maschke [1994] and then, a notion of implicit Hamiltonian systems was developed in the context of Dirac structures by van der Schaft and Maschke [1995] and van der Schaft [1998] (see also Blankenstein [2000]). Nonconservative systems with external forces that appeared in servomechanisms were also illustrated in the context

of the constrained Hamiltonian systems by Marle [1998]. The equivalence of the Lagrangian and Hamiltonian formalisms for nonholonomic mechanical systems was demonstrated by Koon and Marsden [1997, 1998] together with their intrinsic expressions. As for the details on nonholonomic mechanics and control, refer also to Bloch [2003]; Cendra, Marsden, and Ratiu [2001].

As in Part I, an implicit Lagrangian system, whose Lagrangian may be degenerate, can be defined by using a Dirac structure on  $T^*Q$  that is induced from a constraint distribution  $\Delta_Q \subset TQ$ . To carry this out, Part I utilized natural symplectomorphisms between the spaces  $TT^*Q$ ,  $T^*TQ$ , and  $T^*T^*Q$ . We also developed the Dirac differential of a Lagrangian, which amongst other things, incorporated the Legendre transformation into the context of induced Dirac structures. This procedure is consistent with the idea of a generalized Legendre transformation, which was originally developed by Tulczyjew [1974] (see also, for instance Cendra, Holm, Hoyle and Marsden [1998]). In the present Part II, we establish some basic links between variational principles on the one hand and Dirac structures and implicit Lagrangian systems, including the generalized Legendre transform, on the other.

Another important issue that is relevant with the present paper is Pontryagin's maximum principle in optimal control developed by Pontryagin, Boltyanskiĭ, Gamkrelidze and Mishchenko [1962]. It goes without saying that Pontryagin's maximum principle is the machinery that gives necessary conditions for solutions of optimal control problems; we remark that a coordinate-free version of the maximum principle was given by Sussmann [1998].

**The Hamilton-Pontryagin principle.** One of the main goals in this Part II is to provide a link between variational structures, induced Dirac structures, and implicit Lagrangian systems. To do this, we shall develop an extended variational principle called the Hamilton-Pontryagin principle.

The variational principle of Hamilton for classical holonomic mechanical systems is given by the stationary condition of the action functional for a Lagrangian  $L$  such that

$$\delta \int_{t_1}^{t_2} L(q, v) dt = 0,$$

which is subject to the second-order condition  $\dot{q} = v$  and with the endpoints of  $q(t)$  fixed. Regarding the second-order condition  $\dot{q} = v$  as a kinematic constraint, we introduce the momentum variable  $p$  as a Lagrange multiplier for this constraint, and then we rewrite Hamilton's principle as

$$\delta \int_{t_1}^{t_2} \{L(q, v) + p \cdot (\dot{q} - v)\} dt = 0. \quad (1.1)$$

We will call this form of the basic variational principle of holonomic mechanics the **Hamilton-Pontryagin principle** because of its close relation with the classical Pontryagin principle<sup>1</sup>. This principle is also closely related to, what in elasticity theory is called the Hu-Washizu principle (see, for instance, Marsden and Hughes [1983] and Washizu [1982]) that is so important in the discontinuous Galerkin method. In the form (1.1), this principle seems to be due to Livens [1919], and it also appears in Pars [1965], Section 26.2.

The Hamilton-Pontryagin principle in equation (1.1) may also be stated in the following equivalent form, by introducing an extended, or generalized, energy  $E(q, v, p) = p \cdot v - L(q, v)$ , so that

$$\delta \int_{t_1}^{t_2} \{p \cdot \dot{q} - E(q, v, p)\} dt = 0.$$

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<sup>1</sup>We thank Matt West, Tom Hughes, and David Gao for helpful comments on this principle.

Note that the Hamilton-Pontryagin variational principle gives us the second-order condition, the Legendre transformation, and the Euler-Lagrange equations:

$$\dot{q} = v, \quad p = \frac{\partial L}{\partial v}, \quad \dot{p} = \frac{\partial L}{\partial q}. \quad (1.2)$$

While this is a special case of the system that we wish to develop, it does provide a point of view for generalizing this procedure to the case of implicit Lagrangian systems. Note that this equation (1.2) includes the case of degenerate Lagrangians, and that these equations may be implicit in the sense that the Euler-Lagrange equations could be “hidden”. Of course the degenerate case of this problem and its transformation to the Hamiltonian side were the subject of the important work of Dirac [1950]. For these reasons, we refer to equations (1.2) as the *implicit Euler-Lagrange equations*.

Consistent with the above discussion of the Hamilton-Pontryagin principle, for the general case in which a constraint distribution  $\Delta_Q \subset TQ$  is given, we will show how to formulate an implicit Lagrangian system  $(L, \Delta_Q, X)$  associated with a Lagrangian  $L$  and a partial vector field  $X : TQ \oplus T^*Q \rightarrow TT^*Q$  in terms of an extended Lagrange-d’Alembert principle that will be called the Lagrange-d’Alembert-Pontryagin principle.

Furthermore, we shall develop a constrained Dirac structure on the constraint momentum space  $P = \mathbb{F}L(\Delta_Q) \subset T^*Q$ , as well as the implicit constrained Lagrangian system associated with a constrained Lagrangian  $L_c = L|_{\Delta_Q}$  in the variational context.

**Outline of the Paper.** Part II of the paper is constructed as follows. In §2, we give a brief review of the generalized Legendre transformation, which makes use of iterated tangent and cotangent bundles. In §3, the Hamilton-Pontryagin principle and the associated implicit Euler-Lagrange equations are developed in detail along with the geometry of iterated tangent and cotangent spaces. Extending this analysis to the case of nonholonomic constrained systems, it is shown that more general implicit Lagrangian systems can be intrinsically formulated in terms of the Lagrange-d’Alembert-Pontryagin principle. Furthermore, we also elucidate implicit Hamiltonian systems in the variational context for the case of regular Lagrangians. In §4, we demonstrate that nonholonomic systems with external force fields can be incorporated into the framework of implicit Lagrangian systems by employing the Lagrange-d’Alembert-Pontryagin principle with external forces. In §5, we develop the constrained Dirac structure  $D_P$  that is induced on the constraint momentum space  $P$  by utilizing an Ehresmann connection; and we construct an implicit constrained Lagrangian system  $(L_c, \Delta_Q, X_P)$  associated with the constrained Lagrangian  $L_c$  and the constrained partial vector field  $X_P$  on  $P$  in the variational context. In §6, the two examples, namely a vertical rolling disk on a plane and an L-C circuit are illustrated in the context of implicit constrained Lagrangian systems. In §7, concluding remarks and future directions are given.

### Summary of Main Results.

- The Hamilton-Pontryagin principle for holonomic but possibly degenerate Lagrangians, and its relation to implicit Lagrangian systems and Dirac structures on the cotangent bundle are developed.
- Using the geometry of iterated tangent and cotangent bundles and the Pontryagin bundle, we develop the intrinsic form of implicit Lagrangian systems in the variational context. Intrinsic implicit Hamiltonian systems are also developed.
- Nonholonomic systems with external force fields that appear in controlled mechanical systems such as robots are developed in the context of implicit Lagrangian systems

and we show that the equations of motion of such systems can be formulated in terms of the Lagrange-d'Alembert-Pontryagin principle.

- We construct a constrained Dirac structure on the constraint momentum space and show how an implicit constrained Lagrangian system can be formulated using an Ehresmann connection in the variational context.
- Two examples are presented. A vertical rolling disk illustrates implicit nonholonomic constrained Lagrangian systems. An L-C circuit is presented as a typical example of a constrained system with a degenerate Lagrangian.

## 2 The Generalized Legendre Transform

As illustrated in Part I, the spaces  $TT^*Q$ ,  $T^*TQ$ ,  $T^*T^*Q$  are interrelated with each other by two symplectomorphisms  $\kappa_Q : TT^*Q \rightarrow T^*TQ$  and  $\Omega^b : TT^*Q \rightarrow T^*T^*Q$ , which play essential roles in the construction of the generalized Legendre transformation originally developed by Tulczyjew [1974]. In this section, we shall review the generalized Legendre transformation before going into the variational framework of implicit Lagrangian systems. As to the details and required mathematical ingredients for the generalized Legendre transform, refer, for instance, to Cendra, Holm, Hoyle and Marsden [1998]; Weinstein [1977, 1971]; Abraham and Marsden [1978] and Tulczyjew and Urbański [1999]. The link between tangent Dirac structures and the spaces  $TT^*Q$  and  $T^*TQ$  was investigated by Courant [1990b].

**Symplectic Structure on  $TT^*Q$ .** Let  $Q$  be a manifold,  $TQ$  the tangent bundle and  $T^*Q$  the cotangent bundle of  $Q$ . Let  $q$ ,  $(q, \delta q)$  and  $(q, p)$  be local coordinates for  $Q$ ,  $TQ$  and  $T^*Q$  respectively. Let  $(q, \delta q, \delta p, p)$ ,  $(q, p, \delta q, \delta p)$  and  $(q, p, -\delta p, \delta q)$  be local coordinates for  $T^*TQ$ ,  $TT^*Q$  and  $T^*T^*Q$  respectively. Let  $\pi_Q : T^*Q \rightarrow Q$ ;  $(q, p) \mapsto q$  be the cotangent projection and  $T\pi_Q : TT^*Q \rightarrow TQ$ ;  $(q, p, \delta q, \delta p) \mapsto (q, \delta q)$ , be the tangent map of  $\pi_Q$ . Furthermore,  $\pi_{TQ} : T^*TQ \rightarrow TQ$ ;  $(q, \delta q, \delta p, p) \mapsto (q, \delta q)$  and  $\tau_{T^*Q} : TT^*Q \rightarrow T^*Q$ ;  $(q, p, \delta q, \delta p) \mapsto (q, p)$ . In Part I, we showed that there is a natural diffeomorphism

$$\kappa_Q : TT^*Q \rightarrow T^*TQ; \quad (q, p, \delta q, \delta p) \mapsto (q, \delta q, \delta p, p)$$

that is determined by how it intertwines the two sets of maps:

$$\pi_{TQ} \circ \kappa_Q = T\pi_Q \quad \text{and} \quad \pi^1 \circ \kappa_Q = \tau_{T^*Q}.$$

In the above, we recall the projection  $\pi^1 : T^*TQ \rightarrow T^*Q$ ;  $(q, \delta q, \delta p, p) \mapsto (q, p)$  is defined, for  $\alpha_{v_q} \in T_{v_q}^*TQ$  and  $u_q \in T_qQ$ , such that  $\langle \pi^1(\alpha_{v_q}), u_q \rangle = \langle \alpha_{v_q}, \text{ver}(u_q, v_q) \rangle$ , where  $\text{ver}(u_q, v_q)$  is the vertical lift of  $u_q$  along  $v_q$ .

On the other hand, the map  $\Omega^b : TT^*Q \rightarrow T^*T^*Q$ ;  $(q, p, \delta q, \delta p) \mapsto (q, p, -\delta p, \delta q)$  is the natural diffeomorphism associated with the canonical symplectic structure  $\Omega$  on  $T^*Q$ . Recall that the manifold  $TT^*Q$  is the symplectic manifold with a particular symplectic form  $\Omega_{TT^*Q}$  that can be defined by the two distinct but intrinsic one-forms:

$$\begin{aligned} \lambda &= (\kappa_Q)^* \Theta_{T^*TQ} = \delta p dq + p d\delta q, \\ \chi &= (\Omega^b)^* \Theta_{T^*T^*Q} = -\delta p dq + \delta q dp, \end{aligned}$$

where  $\Theta_{T^*TQ}$  is the canonical one-form on  $T^*TQ$  and  $\Theta_{T^*T^*Q}$  is the canonical one-form on  $T^*T^*Q$ . Recall also that the symplectic form  $\Omega_{TT^*Q}$  is defined by

$$\Omega_{TT^*Q} = -\mathbf{d}\lambda = \mathbf{d}\chi = dq \wedge d\delta p + d\delta q \wedge dp.$$

Let us see how the Lagrangian and Hamiltonian may be interrelated with each other throughout the symplectic structure  $\Omega_{TT^*Q}$ .

**Lagrangian Constraints.** Let  $L$  be a Lagrangian on  $\Delta_Q \subset TQ$ . The symplectic manifold  $(TT^*Q, \Omega_{TT^*Q} = -\mathbf{d}\lambda)$  is defined by the quadruple  $(TT^*Q, TQ, T\pi_Q, \lambda)$  and the set

$$N = \{x \in TT^*Q \mid T\pi_Q(x) \in \Delta_Q, \lambda_x(w) = \langle \mathbf{d}L(T\pi_Q(x)), T_x T\pi_Q(w) \rangle \\ \text{for all } w \in T_x(TT^*Q) \text{ such that } T_x T\pi_Q(w) \in T_{T\pi_Q(x)}\Delta_Q\} \quad (2.1)$$

is a Lagrangian submanifold of  $(TT^*Q, \Omega_{TT^*Q} = -\mathbf{d}\lambda)$  with  $\frac{1}{2}\dim TT^*Q$ , where the submanifold

$$\Delta_Q = T\pi_Q(N) \subset TQ$$

is the constraint distribution on  $Q$  called a *Lagrangian constraint*. Hence, the Lagrangian  $L$  is a generating function of  $N$ , since  $N \subset TT^*Q$  is the graph of  $(\kappa_Q)^{-1}(\mathbf{d}L)$ .

**Hamiltonian Constraints.** Let  $H$  be a Hamiltonian on  $P \subset T^*Q$  and the symplectic manifold  $(TT^*Q, \Omega_{TT^*Q} = \mathbf{d}\chi)$  is defined by the quadruple  $(TT^*Q, T^*Q, \tau_{T^*Q}, \chi)$ . The set

$$N = \{x \in TT^*Q \mid \tau_{T^*Q}(x) \in P, \chi_x(w) = \langle \mathbf{d}H(\tau_{T^*Q}(x)), T_x \tau_{T^*Q}(w) \rangle \\ \text{for all } w \in T_x(TT^*Q) \text{ such that } T_x \tau_{T^*Q}(w) \in T_{\tau_{T^*Q}(x)}P\} \quad (2.2)$$

is a Lagrangian submanifold of  $(TT^*Q, \Omega_{TT^*Q} = \mathbf{d}\chi)$  with  $\frac{1}{2}\dim TT^*Q$ , where the submanifold

$$P = \tau_{T^*Q}(N) \subset T^*Q$$

is the *constraint momentum space* called a *Hamiltonian constraint*. Similarly, the Hamiltonian  $H$  is a generating function of  $N$ , because  $N \subset TT^*Q$  is the graph of  $(\Omega^b)^{-1}(\mathbf{d}H)$ .

**Symplectomorphism and the Momentum Function.** Consider the identity map, which we can regard as a symplectomorphism  $\varphi : (P_1 = TT^*Q, \Omega_{P_1} = -\mathbf{d}\lambda) \rightarrow (P_2 = TT^*Q, \Omega_{P_2} = \mathbf{d}\chi)$ , and it follows

$$\varphi^* \Omega_{P_2} = \Omega_{P_1},$$

since  $P_1 = P_2$ , and, as we have seen in Part I,  $\Omega_{P_2} = \Omega_{P_1}$ . The graph of the symplectomorphism  $\varphi$  is a submanifold of  $P_1 \times P_2$ , which is denoted by

$$\Gamma(\varphi) \subset P_1 \times P_2.$$

Let  $i_\varphi : \Gamma(\varphi) \rightarrow P_1 \times P_2$  be the inclusion and let  $\pi_i : P_1 \times P_2 \rightarrow P_i$  be the canonical projection. Define

$$\omega = \pi_1^* \Omega_{P_1} - \pi_2^* \Omega_{P_2} \\ = \pi_1^*(-\mathbf{d}\lambda) - \pi_2^*\mathbf{d}\chi.$$

Since  $\varphi$  is symplectic, it follows that

$$i_\varphi^* \omega = (\pi_1|_{\Gamma(\varphi)})^*(\Omega_{P_1} - \varphi^* \Omega_{P_2}) \\ = (\pi_1|_{\Gamma(\varphi)})^*(-\mathbf{d}\lambda - \varphi^* \mathbf{d}\chi) \\ = 0.$$

In the above,  $\pi_1 \circ i_\varphi$  is the projection restricted to  $\Gamma(\varphi)$  and  $\pi_2 \circ i_\varphi = \varphi \circ \pi_1$  on  $\Gamma(\varphi)$ . So, we can write  $\omega = -\mathbf{d}\theta$  where  $\theta = \lambda \oplus \chi = \pi_1^* \lambda + \pi_2^* \chi$ . Clearly,  $\Gamma(\varphi)$  is a maximally isotropic submanifold with half of the dimension of  $P_1 \times P_2 = TT^*Q \times TT^*Q$ .

Letting  $\Psi : TT^*Q \rightarrow TT^*Q \times TT^*Q$  be the diagonal map, we have the one-form  $\Psi^*\theta$  on  $TT^*Q$ , which is represented, by using local coordinates  $(q, \delta q)$ ,  $(q, p)$  and  $(q, p, \delta q, \delta p)$  for  $TQ$ ,  $T^*Q$  and  $TT^*Q$ , as

$$\begin{aligned} \Psi^*\theta &= \Psi^*(\lambda \oplus \chi) = \lambda + \varphi^*\chi \\ &= (\delta p dq + p d\delta q) + (-\delta p dq + \delta q dp) \\ &= p d\delta q + \delta q dp \\ &= \mathbf{d}(p \cdot \delta q) \\ &= \mathbf{d}(G \circ \rho_{TT^*Q}), \end{aligned}$$

where we recall from Part I that the map  $\rho_{TT^*Q} : TT^*Q \rightarrow TQ \oplus T^*Q$  is given in coordinates by  $(q, p, \delta q, \delta p) \mapsto (q, \delta q, p)$ . We shall also need the function  $G$  defined on the Pontryagin bundle  $TQ \oplus T^*Q$  that simply pairs an element of  $T_qQ$  with that of  $T_q^*Q$ ; it is given in local coordinates by

$$G(q, \delta q, p) = p \cdot \delta q,$$

which we shall call the *momentum function*.

**The Generalized Legendre Transform.** There are two different ways of realizing the submanifold  $N$  in  $TT^*Q$ , as shown in equations (2.1) and (2.2), as graphs of one-forms on  $TQ$  and  $T^*Q$  and the passage between them implies the Legendre transformation. This procedure is called the *generalized Legendre transform*, which enables us to treat the case in which a given Lagrangian is degenerate.

The generalized Legendre transformation is the procedure to obtain a submanifold  $\mathcal{K}$  of the Pontryagin bundle  $TQ \oplus T^*Q$  from the submanifold  $N$  of  $TT^*Q$  associated with  $(TT^*Q, TQ, T\pi_Q, \lambda)$  and with a Lagrangian  $L$  (possibly degenerate) on  $\Delta_Q \subset TQ$  as in equation (2.1). This can be understood by the passage of the identity symplectomorphism

$$\varphi : (TT^*Q, \Omega_{TT^*Q} = -\mathbf{d}\lambda) \rightarrow (TT^*Q, \Omega_{TT^*Q} = \mathbf{d}\chi).$$

Let  $T\pi_Q \times \tau_{T^*Q} : TT^*Q \times TT^*Q \rightarrow TQ \times T^*Q$  and define the map

$$(T\pi_Q \times \tau_{T^*Q}) \circ \Psi : TT^*Q \rightarrow TQ \times T^*Q.$$

We can define the submanifold  $\mathcal{K}$  by the image of  $N$  by the map  $(T\pi_Q \times \tau_{T^*Q}) \circ \Psi$  such that

$$\mathcal{K} = (T\pi_Q \times \tau_{T^*Q}) \circ \Psi(N) \subset TQ \times T^*Q,$$

which is to be the graph of the Legendre transform  $\mathbb{F}L : TQ \rightarrow T^*Q$  with respect to a constraint distribution  $\Delta_Q = T\pi_Q(N) \subset TQ$ . Define the *generalized energy*  $E$  on  $TQ \oplus T^*Q$ , using local coordinates  $(q, v)$ ,  $(q, p)$  and  $(q, v, p)$  for  $TQ$ ,  $T^*Q$  and  $TQ \oplus T^*Q$  and  $\text{pr}_{TQ} : TQ \oplus T^*Q \rightarrow TQ$ , by

$$\begin{aligned} E(q, v, p) &= G(q, v, p) - L(\text{pr}_{TQ}(q, v, p)) \\ &= p \cdot v - L(q, v). \end{aligned}$$

In fact, the submanifold  $\mathcal{K}$  may be given by

$$\mathcal{K} = \left\{ (q, v, p) \in TQ \oplus T^*Q \mid (q, v) \in \Delta_Q \subset TQ \text{ is a stationary point of } E(q, v, p) \text{ for each } (q, p) \in T^*Q \right\},$$

which is eventually represented by

$$\mathcal{K} = \left\{ (q, v, p) \in TQ \oplus T^*Q \mid (q, v) \in \Delta_Q, p = \frac{\partial L}{\partial v} \right\}.$$

### 3 The Variational Framework

In this section, we illustrate the variational framework of implicit Lagrangian systems. First, we show *implicit Euler-Lagrange equations* can be formulated by using an extended variational principle of Hamilton called the *Hamilton-Pontryagin principle*, which we develop by inspiring from Pontryagin's maximum principle (see [Pontryagin, Boltyanskiĭ, Gamkrelidze and Mishchenko \[1962\]](#)). This variational principle naturally includes the generalized Legendre transformation. Second, we investigate an implicit Lagrangian system by developing an extended Lagrange-d'Alembert principle called the *Lagrange-d'Alembert-Pontryagin principle* together with its intrinsic expression. Then, we show the variational link between an implicit Lagrangian system and the induced Dirac structure on  $T^*Q$ . Third, in the case where a given Lagrangian  $L$  is hyperregular, a Hamiltonian  $H$  is well defined by the usual Legendre transformation. Then, we also show how an implicit Hamiltonian system that is defined by an induced Dirac structure on  $T^*Q$  can be naturally associated with an extension of Hamilton's phase space principle that we call the *Hamilton-d'Alembert principle in phase space*.

**Variational Principle of Hamilton.** Before going into an extended variational principle of Hamilton, we shall review the variational principle of Hamilton.

Let  $L$  be a Lagrangian on  $TQ$  and  $q(t)$ ,  $t_1 \leq t \leq t_2$  be a curve in the manifold  $Q$ . Define the path space from  $q_1$  to  $q_2$  by

$$C(q_1, q_2, [t_1, t_2]) = \{q : [t_1, t_2] \rightarrow Q \mid q(t_1) = q_1, q(t_2) = q_2\}$$

and the map called the action functional  $\mathfrak{S} : C(q_1, q_2, [t_1, t_2]) \rightarrow \mathbb{R}$  by

$$\mathfrak{S}(q(t)) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t)) dt.$$

The variation of the action functional  $\mathfrak{S}(q(t))$  at  $q(t)$  in direction of  $\delta q(t)$  is

$$\begin{aligned} \mathbf{d}\mathfrak{S}(q(t)) \cdot \delta q(t) &= \delta \int_{t_1}^{t_2} L(q(t), \dot{q}(t)) dt \\ &= \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt \\ &= \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \cdot \delta q \Big|_{t_1}^{t_2}. \end{aligned}$$

In the above,  $\dot{q}$  denotes  $dq/dt$  and we employ  $\delta \dot{q} = d(\delta q)/dt$ . When  $q(t)$  is a critical point of the action functional  $\mathfrak{S}$ , that is,  $\mathbf{d}\mathfrak{S}(q(t)) \cdot \delta q(t) = 0$  for all  $\delta q(t) \in T_{q(t)}C(q_1, q_2, [t_1, t_2])$ , the curve  $q(t)$  satisfies, keeping the endpoints fixed, the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

**The Hamilton-Pontryagin Principle.** We shall illustrate implicit Euler-Lagrange equations can be formulated by an extended variational principle of Hamilton, which we call the *Hamilton-Pontryagin principle*, that incorporates the second-order condition  $v = \dot{q}$  into the action functional for a Lagrangian as a kinematical constraint. This variational principle naturally includes the generalized Legendre transform.



**Proposition 3.1.** *Let  $q$ ,  $(q, v)$  and  $(q, p)$  be local coordinates respectively for  $Q$ ,  $TQ$  and  $T^*Q$ . Let  $(q, v, p)$  be local coordinates for the Pontryagin bundle  $TQ \oplus T^*Q$ . Let  $L : TQ \rightarrow \mathbb{R}$  be a Lagrangian (possibly degenerate). Consider the action functional defined by*

$$\int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) (\dot{q}(t) - v(t))\} dt = \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - E(q(t), v(t), p(t))\} dt.$$

In the above, as previously illustrated,  $E$  is the generalized energy on  $TQ \oplus T^*Q$  given by

$$\begin{aligned} E(q, v, p) &= G(q, v, p) - L(q, v) \\ &= p \cdot v - L(q, v), \end{aligned}$$

where  $G(q, v, p) = p \cdot v$  is the momentum function. Keeping the endpoints of  $q(t)$  fixed whereas the endpoints of  $v(t)$  and  $p(t)$  are allowed to be free, the stationary condition for the action functional implies

$$\dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}, \quad p = \frac{\partial L}{\partial v}, \quad (3.1)$$

which we shall call the **implicit Euler-Lagrange equations**.

**Proof.** The variation of the action functional is given by

$$\begin{aligned} & \delta \int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) (\dot{q}(t) - v(t))\} dt \\ &= \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - E(q(t), v(t), p(t))\} dt \\ &= \int_{t_1}^{t_2} \{\delta p \dot{q} + p \delta \dot{q} - \delta E(q, v, p)\} dt \\ &= \int_{t_1}^{t_2} \left( -\dot{p} \delta q + \dot{q} \delta p - \frac{\partial E}{\partial q} \delta q - \frac{\partial E}{\partial v} \delta v - \frac{\partial E}{\partial p} \delta p \right) dt + p \delta q \Big|_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2} \left\{ \left( \dot{q} - \frac{\partial E}{\partial p} \right) \delta p + \left( -\dot{p} - \frac{\partial E}{\partial q} \right) \delta q - \frac{\partial E}{\partial v} \delta v \right\} dt + p \delta q \Big|_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2} \left\{ (\dot{q} - v) \delta p + \left( -\dot{p} + \frac{\partial L}{\partial q} \right) \delta q + \left( -p + \frac{\partial L}{\partial v} \right) \delta v \right\} dt + p \delta q \Big|_{t_1}^{t_2}, \end{aligned} \quad (3.2)$$

where  $(\delta q, \delta v, \delta p) \in T_{(q,v,p)}(TQ \oplus T^*Q)$ . Keeping the endpoints of  $q(t)$  fixed, that is,  $q(t_1) = q_1$  and  $q(t_2) = q_2$ , the stationary condition for the action functional for all  $(\delta q, \delta v, \delta p)$  provides equation (3.1). ■

Notice that the Hamilton-Pontryagin principle naturally includes the Legendre transform and also that equation (3.1) is nothing but the local expression of an implicit Lagrangian system  $(L, \Delta_Q, X)$  for the case in which  $\Delta_Q = TQ$ , as shown in Part I.

**The Intrinsic Form of Implicit Euler-Lagrange Equations.** We shall develop the intrinsic form of the implicit Euler-Lagrange equations in the context of the Hamilton-Pontryagin principle.

Consider the following maps

$$\begin{aligned}
 \rho_{TT^*Q} &: TT^*Q \rightarrow TQ \oplus T^*Q \\
 \text{pr}_{TQ} &: TQ \oplus T^*Q \rightarrow TQ \\
 \text{pr}_{T^*Q} &: TQ \oplus T^*Q \rightarrow T^*Q \\
 \tau_{TQ \oplus T^*Q} &: T(TQ \oplus T^*Q) \rightarrow TQ \oplus T^*Q.
 \end{aligned}$$

Recall that the map  $\rho_{TT^*Q} : TT^*Q \rightarrow TQ \oplus T^*Q$  is given in coordinates by  $(q, p, \delta q, \delta p) \mapsto (q, \delta q, p)$ , while the other maps are self explanatory. Define the path space of curves  $x(t) = (q(t), v(t), p(t))$ ,  $t_1 \leq t \leq t_2$  in  $TQ \oplus T^*Q$  by

$$\begin{aligned}
 \tilde{C}(q_1, q_2, [t_1, t_2]) &= \{(q, v, p) : [t_1, t_2] \rightarrow TQ \oplus T^*Q \mid \\
 &\quad \text{pr}_Q(q(t_1), v(t_1), p(t_1)) = q_1, \text{pr}_Q(q(t_2), v(t_2), p(t_2)) = q_2\},
 \end{aligned}$$

where  $\text{pr}_Q : TQ \oplus T^*Q \rightarrow Q$ .

The action functional on  $\tilde{C}(q_1, q_2, [t_1, t_2])$  of curves  $x(t) = (q(t), v(t), p(t))$ ,  $t_1 \leq t \leq t_2$  in  $TQ \oplus T^*Q$  is represented by

$$\begin{aligned}
 &\int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) \cdot (\dot{q}(t) - v(t))\} dt \\
 &= \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - E(q(t), v(t), p(t))\} dt \\
 &= \int_{t_1}^{t_2} \{G(\rho_{TT^*Q} \circ T\text{pr}_{T^*Q}(x(t), \dot{x}(t))) - E(\tau_{TQ \oplus T^*Q}(x(t), \dot{x}(t)))\} dt, \quad (3.3)
 \end{aligned}$$

where  $\dot{x}(t)$  denotes the time derivative of  $x(t)$ ,  $T\text{pr}_{T^*Q} : T(TQ \oplus T^*Q) \rightarrow TT^*Q$  is the tangent map of  $\text{pr}_{T^*Q}$  and then  $\rho_{TT^*Q} \circ T\text{pr}_{T^*Q} : T(TQ \oplus T^*Q) \rightarrow TQ \oplus T^*Q$ .

Let us call equation (3.3) the **Hamilton-Pontryagin integral**.

**Tangent Bundle of the Pontryagin Bundle.** Let  $\tau_Q : TQ \rightarrow Q$  be the tangent projection and  $\pi_Q : T^*Q \rightarrow Q$  be the cotangent projection. The tangent space of  $TQ \oplus T^*Q$  at a point  $(q, v, p)$ , that is,  $T_{(q, v, p)}(TQ \oplus T^*Q)$  is the subset of  $T_{v_q}TQ \oplus T_{p_q}T^*Q$  consisting of vectors that project to the same point of  $TQ$ ; that is,

$$T\tau_Q(q, v, \dot{q}, \dot{v}) = T\pi_Q(q, p, \dot{q}, \dot{p}),$$

where the tangent map  $T\tau_Q : TTQ \rightarrow TQ$  is given in coordinates by  $(q, v, \dot{q}, \dot{v}) \mapsto (q, \dot{q})$  and the tangent map  $T\pi_Q : TT^*Q \rightarrow TQ$  is given in coordinates by  $(q, p, \dot{q}, \dot{p}) \mapsto (q, \dot{q})$ . Thus, tangent vectors of the Pontryagin bundle in coordinates have a base point  $(q, v, p)$  and a vector part  $(\dot{q}, \dot{v}, \dot{p})$ . That is, the  $\dot{q}$  piece for the two tangent vectors of  $TQ$  and  $T^*Q$  agree.

There are two different maps from  $T(TQ \oplus T^*Q)$  to  $TQ \oplus T^*Q$ ; namely,

$$\begin{aligned}
 \tau_{TQ \oplus T^*Q} &: T(TQ \oplus T^*Q) \rightarrow TQ \oplus T^*Q; & (q, v, p, \dot{q}, \dot{v}, \dot{p}) &\mapsto (q, v, p), \\
 \rho_{TT^*Q} \circ T\text{pr}_{T^*Q} &: T(TQ \oplus T^*Q) \rightarrow TQ \oplus T^*Q; & (q, v, p, \dot{q}, \dot{v}, \dot{p}) &\mapsto (q, \dot{q}, p),
 \end{aligned}$$

where we recall that the map  $\rho_{TT^*Q} : TT^*Q \rightarrow TQ \oplus T^*Q$  is given by  $(q, p, \dot{q}, \dot{p}) \mapsto (q, \dot{q}, p)$ . The map  $T\text{pr}_{T^*Q}$  is the tangent of the projection map  $\text{pr}_{T^*Q} : TQ \oplus T^*Q \rightarrow T^*Q$  and is given by  $(q, v, p, \dot{q}, \dot{v}, \dot{p}) \mapsto (q, p, \dot{q}, \dot{p})$ .

**Proposition 3.2.** *Let  $\Omega^b : TT^*Q \rightarrow T^*T^*Q$  be the bundle map associated with the canonical symplectic structure  $\Omega$  on  $T^*Q$  and  $\Theta_{T^*Q}$  be the canonical one-form on  $T^*Q$ . Let  $\chi = (\Omega^b)^* \Theta_{T^*T^*Q}$  be the induced one-form on  $TT^*Q$ , where  $\Theta_{T^*T^*Q}$  is the canonical one-form on  $T^*T^*Q$ . Then, the variation of the Hamilton-Pontryagin integral in equation (3.3) is represented by*

$$\begin{aligned}
 & \delta \int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) \cdot (\dot{q}(t) - v(t))\} dt \\
 &= \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - E(q(t), v(t), p(t))\} dt \\
 &= \delta \int_{t_1}^{t_2} \left\{ G(\rho_{TT^*Q} \circ T\text{pr}_{T^*Q}(x, \dot{x})) - E(\tau_{TQ \oplus T^*Q}(x, \dot{x})) \right\} dt \\
 &= \int_{t_1}^{t_2} \left\{ \Theta_{T^*T^*Q} \left( \Omega^b \cdot T\text{pr}_{T^*Q}(x, \dot{x}) \right) \cdot T\Omega^b(T_{(x, \dot{x})}(T\text{pr}_{T^*Q})(w)) \right. \\
 &\quad \left. - \mathbf{d}E(\tau_{TQ \oplus T^*Q}(x, \dot{x})) \cdot T_{(x, \dot{x})}\tau_{TQ \oplus T^*Q}(w) \right\} dt \\
 &\quad + \Theta_{T^*Q}(\text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q})(x, \dot{x})) \cdot (T\tau_{T^*Q} \circ T(T\text{pr}_{T^*Q}))(w) \Big|_{t_1}^{t_2} \\
 &= \int_{t_1}^{t_2} \left\{ (T\text{pr}_{T^*Q})^* \chi(x, \dot{x}) - (\tau_{TQ \oplus T^*Q})^* \mathbf{d}E(x, \dot{x}) \right\} \cdot w dt \\
 &\quad + \Theta_{T^*Q}(\text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q})(x, \dot{x})) \cdot (T\tau_{T^*Q} \circ T(T\text{pr}_{T^*Q}))(w) \Big|_{t_1}^{t_2}.
 \end{aligned} \tag{3.4}$$

In the above,  $(x, \dot{x}) \in T(TQ \oplus T^*Q)$ ,  $w \in T_{(x, \dot{x})}T(TQ \oplus T^*Q)$ , and

$$T(T\text{pr}_{T^*Q}) : TT(TQ \oplus T^*Q) \rightarrow T(TT^*Q),$$

where  $T_{(x, \dot{x})}(T\text{pr}_{T^*Q})(w) \in T_{T\text{pr}_{T^*Q}(x, \dot{x})}(TT^*Q)$ .

**Proof.** Let us check, by using local coordinates, that equation (3.4) is the intrinsic representation of equation (3.2). Let  $(q, v)$ ,  $(q, p)$  and  $(q, v, p)$  be local coordinates for  $TQ$ ,  $T^*Q$  and  $TQ \oplus T^*Q$ .

First, by using the two different maps  $\tau_{TQ \oplus T^*Q} : T(TQ \oplus T^*Q) \rightarrow TQ \oplus T^*Q$  and  $\rho_{TT^*Q} \circ T\text{pr}_{T^*Q} : T(TQ \oplus T^*Q) \rightarrow TQ \oplus T^*Q$ , for each  $x = (q, v, p) \in TQ \oplus T^*Q$ , one obtains

$$\begin{aligned}
 \rho_{TT^*Q} \circ T\text{pr}_{T^*Q}(x, \dot{x}) &= \rho_{TT^*Q} \circ T\text{pr}_{T^*Q}(q, v, p, \dot{q}, \dot{v}, \dot{p}) \\
 &= \rho_{TT^*Q} \circ (q, p, \dot{q}, \dot{p}) \\
 &= (q, \dot{q}, p) \in TQ \oplus T^*Q,
 \end{aligned}$$

while

$$\begin{aligned}
 \tau_{TQ \oplus T^*Q}(x, \dot{x}) &= \tau_{TQ \oplus T^*Q}(q, v, p, \dot{q}, \dot{v}, \dot{p}) \\
 &= (q, v, p) \in TQ \oplus T^*Q,
 \end{aligned}$$

where  $(x, \dot{x}) = (q, v, p, \dot{q}, \dot{v}, \dot{p})$ . Since the momentum function  $G$  on  $TQ \oplus T^*Q$  is locally given by

$$\begin{aligned}
 G(\rho_{TT^*Q} \circ T\text{pr}_{T^*Q}(x, \dot{x})) &= G(q, \dot{q}, p) \\
 &= p \cdot \dot{q},
 \end{aligned}$$

it reads, using local coordinates,

$$\begin{aligned}
 \delta \int_{t_1}^{t_2} G(\rho_{TT^*Q} \circ T\text{pr}_{T^*Q}(x, \dot{x})) dt &= \delta \int_{t_1}^{t_2} G(q, \dot{q}, p) dt \\
 &= \delta \int_{t_1}^{t_2} (p \cdot \dot{q}) dt \\
 &= \int_{t_1}^{t_2} (p \delta \dot{q} + \delta p \dot{q}) dt \\
 &= \int_{t_1}^{t_2} (-\dot{p} \delta q + \delta p \dot{q}) dt + p \delta q \Big|_{t_1}^{t_2}. \quad (3.5)
 \end{aligned}$$

Second, let us check the terms concerning with the canonical one-forms  $\Theta_{T^*T^*Q}$  in equation (3.4) by using local coordinates. From the map  $\Omega^b \circ T\text{pr}_{T^*Q} : T(TQ \oplus T^*Q) \rightarrow T^*T^*Q$ , we can easily see that

$$\begin{aligned}
 \Theta_{T^*T^*Q}(\Omega^b \cdot T\text{pr}_{T^*Q}(x, \dot{x})) &= \Theta_{T^*T^*Q}(q, p, -\dot{p}, \dot{q}) \\
 &= -\dot{p} dq + \dot{q} dp,
 \end{aligned}$$

where  $T\text{pr}_{T^*Q}(x, \dot{x}) = (q, p, \dot{q}, \dot{p})$  and then  $\Omega^b \circ T\text{pr}_{T^*Q}(x, \dot{x}) = (q, p, -\dot{p}, \dot{q})$ . Furthermore, we can write  $w \in T_{(x, \dot{x})}T(TQ \oplus T^*Q)$  in coordinates as

$$w = (q, v, p, \dot{q}, \dot{v}, \dot{p}, \delta q, \delta v, \delta p, \delta \dot{q}, \delta \dot{v}, \delta \dot{p}),$$

and it follows that

$$\begin{aligned}
 T_{(x, \dot{x})}(T\text{pr}_{T^*Q})(w) &= T_{(x, \dot{x})}(T\text{pr}_{T^*Q})(q, v, p, \dot{q}, \dot{v}, \dot{p}, \delta q, \delta v, \delta p, \delta \dot{q}, \delta \dot{v}, \delta \dot{p}) \\
 &= (q, p, \dot{q}, \dot{p}, \delta q, \delta p, \delta \dot{q}, \delta \dot{p})
 \end{aligned}$$

and then

$$\begin{aligned}
 (T\Omega^b \circ T_{(x, \dot{x})}(T\text{pr}_{T^*Q}))(w) &= T\Omega^b(q, p, \dot{q}, \dot{p}, \delta q, \delta p, \delta \dot{q}, \delta \dot{p}) \\
 &= (q, p, -\dot{p}, \dot{q}, \delta q, \delta p, -\delta \dot{p}, \delta \dot{q}),
 \end{aligned}$$

where  $T(T\text{pr}_{T^*Q}) : TT(TQ \oplus T^*Q) \rightarrow T(TT^*Q)$  and  $T\Omega^b : T(TT^*Q) \rightarrow T(T^*T^*Q)$ . Therefore, noting  $\chi = (\Omega^b)^* \Theta_{T^*T^*Q}$ , we have

$$\begin{aligned}
 \Theta_{T^*T^*Q}(\Omega^b \circ T\text{pr}_{T^*Q}(x, \dot{x})) \cdot (T\Omega^b \circ T_{(x, \dot{x})}(T\text{pr}_{T^*Q}))(w) &= (T\text{pr}_{T^*Q})^* \chi(x, \dot{x}) \cdot w \\
 &= -\dot{p} \delta q + \dot{q} \delta p. \quad (3.6)
 \end{aligned}$$

As to the term associated with the canonical one-form  $\Theta_{T^*Q}$  in equation (3.4), by employing the map  $\text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q}) : T(TQ \oplus T^*Q) \rightarrow T^*Q$ , we can obtain

$$\begin{aligned}
 \text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q})(x, \dot{x}) &= \text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q})(q, v, p, \dot{q}, \dot{v}, \dot{p}) \\
 &= \text{pr}_{T^*Q}(q, v, p) \\
 &= (q, p),
 \end{aligned}$$

and then

$$\begin{aligned}
 \Theta_{T^*Q}(\text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q})(x, \dot{x})) &= \Theta_{T^*Q}(q, p) \\
 &= p dq.
 \end{aligned}$$

Recall  $\tau_{T^*Q} : TT^*Q \rightarrow T^*Q$  and  $T\tau_{T^*Q} : T(TT^*Q) \rightarrow TT^*Q$ , and it follows that

$$\Theta_{T^*Q}(\text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q})(x, \dot{x})) \cdot (T\tau_{T^*Q} \circ T(\text{pr}_{T^*Q})(w)) = p \delta q, \quad (3.7)$$

where the map  $T\tau_{T^*Q} \circ T(\text{pr}_{T^*Q}) : TT(TQ \oplus T^*Q) \rightarrow TT^*Q$  is locally indicated by

$$\begin{aligned} T\tau_{T^*Q} \circ T(\text{pr}_{T^*Q})(w) &= T\tau_{T^*Q} \circ T(\text{pr}_{T^*Q})(q, v, p, \dot{q}, \dot{v}, \dot{p}, \delta q, \delta v, \delta p, \delta \dot{q}, \delta \dot{v}, \delta \dot{p}) \\ &= T\tau_{T^*Q}(q, p, \dot{q}, \dot{p}, \delta q, \delta p, \delta \dot{q}, \delta \dot{p}) \\ &= (q, p, \delta q, \delta p). \end{aligned}$$

From equations (3.5)–(3.7), we can easily check the following relation holds:

$$\begin{aligned} &\delta \int_{t_1}^{t_2} G(\rho_{TT^*Q} \circ T\text{pr}_{T^*Q}(x, \dot{x})) dt \\ &= \int_{t_1}^{t_2} \Theta_{T^*T^*Q}(\Omega^b \cdot T\text{pr}_{T^*Q}(x, \dot{x})) \cdot (T\Omega^b \circ T_{(x, \dot{x})}(T\text{pr}_{T^*Q})(w)) dt \\ &\quad + \Theta_{T^*Q}(\text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q})(x, \dot{x})) \cdot (T\tau_{T^*Q} \circ T(\text{pr}_{T^*Q})(w)) \Big|_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2} (T\text{pr}_{T^*Q})^* \chi(x, \dot{x}) \cdot w dt \\ &\quad + \Theta_{T^*Q}(\text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q})(x, \dot{x})) \cdot (T\tau_{T^*Q} \circ T(\text{pr}_{T^*Q})(w)) \Big|_{t_1}^{t_2}. \quad (3.8) \end{aligned}$$

Third, let us check the terms relating to  $E$  on  $TQ \oplus T^*Q$  in equation (3.4). Since the local representation of  $E$  is given by

$$\begin{aligned} E(\tau_{TQ \oplus T^*Q}(x, \dot{x})) &= E(q, v, p) \\ &= p \cdot v - L(q, v), \end{aligned}$$

one can directly compute the differential of  $E$  in local coordinates such that

$$\begin{aligned} \mathbf{d}E(\tau_{TQ \oplus T^*Q}(x, \dot{x})) &= \mathbf{d}E(q, v, p) \\ &= \frac{\partial E}{\partial q} dq + \frac{\partial E}{\partial v} dv + \frac{\partial E}{\partial p} dp \\ &= -\frac{\partial L}{\partial q} dq + \left( p - \frac{\partial L}{\partial v} \right) dv + v dp. \end{aligned}$$

Recall the map  $T\tau_{TQ \oplus T^*Q} : TT(TQ \oplus T^*Q) \rightarrow T(TQ \oplus T^*Q)$  is locally represented by

$$\begin{aligned} T_{(x, \dot{x})} \tau_{TQ \oplus T^*Q}(w) &= T_{(x, \dot{x})} \tau_{TQ \oplus T^*Q}(q, v, p, \dot{q}, \dot{v}, \dot{p}, \delta q, \delta v, \delta p, \delta \dot{q}, \delta \dot{v}, \delta \dot{p}) \\ &= (q, v, p, \delta q, \delta v, \delta p), \end{aligned}$$

and hence it follows

$$\begin{aligned} \mathbf{d}E(\tau_{TQ \oplus T^*Q}(x, \dot{x})) \cdot T_{(x, \dot{x})} \tau_{TQ \oplus T^*Q}(w) &= (\tau_{TQ \oplus T^*Q})^* \mathbf{d}E(x, \dot{x}) \cdot w \\ &= \frac{\partial E}{\partial q} \delta q + \frac{\partial E}{\partial v} \delta v + \frac{\partial E}{\partial p} \delta p \\ &= -\frac{\partial L}{\partial q} \delta q + \left( p - \frac{\partial L}{\partial v} \right) \delta v + v \delta p. \quad (3.9) \end{aligned}$$

Thus, by equations (3.8) and (3.9), it reads that equation (3.4) is the intrinsic representation of equation (3.2).  $\blacksquare$

**Proposition 3.3.** *A curve  $x(t) = (q(t), v(t), p(t))$ ,  $t_1 \leq t \leq t_2$  in  $TQ \oplus T^*Q$  joining  $\text{pr}_Q(x(t_1)) = q_1$  and  $\text{pr}_Q(x(t_2)) = q_2$  satisfies*

$$(\text{Pr}_{T^*Q})^* \chi(x(t), \dot{x}(t)) = (\tau_{TQ \oplus T^*Q})^* \mathbf{d}E(x(t), \dot{x}(t)), \quad (3.10)$$

*if and only if  $x(t)$  is a stationary point of the Hamilton-Pontryagin integral in equation (3.3).*

**Proof.** The stationary condition of the Hamilton-Pontryagin integral is given by

$$\begin{aligned} & \delta \int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) \cdot (\dot{q}(t) - v(t))\} dt \\ &= \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - E(q(t), v(t), p(t))\} dt \\ &= \delta \int_{t_1}^{t_2} \left\{ G(\rho_{TT^*Q} \circ T\text{pr}_{T^*Q}(x, \dot{x})) - E(\tau_{TQ \oplus T^*Q}(x, \dot{x})) \right\} dt \\ &= \int_{t_1}^{t_2} \left\{ (T\text{pr}_{T^*Q})^* \chi(x, \dot{x}) - (\tau_{TQ \oplus T^*Q})^* \mathbf{d}E(x, \dot{x}) \right\} \cdot w dt \\ & \quad + \Theta_{T^*Q}(\text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q})(x, \dot{x})) \cdot (T\tau_{T^*Q} \circ T(T\text{pr}_{T^*Q})(w)) \Big|_{t_1}^{t_2} \\ &= 0, \end{aligned}$$

which satisfies for all  $w \in T_{(x, \dot{x})}T(TQ \oplus T^*Q)$ . Since the endpoints of  $q(t)$  are fixed, one has

$$\begin{aligned} \Theta_{T^*Q}(\text{pr}_{T^*Q} \circ (\tau_{TQ \oplus T^*Q})(x, \dot{x})) \cdot (T\tau_{T^*Q} \circ T(T\text{pr}_{T^*Q})(w)) \Big|_{t_1}^{t_2} &= p \delta q \Big|_{t_1}^{t_2} \\ &= 0. \end{aligned}$$

Thus, we obtain equation (3.10). ■

Needless to say, equation (3.10) is the intrinsic expression of equation (3.1), and so we shall call equation (3.10) the ***intrinsic implicit Euler-Lagrange equations***, which are equivalent with an implicit Lagrangian system  $(L, \Delta_Q, X)$  for the case in which  $\Delta_Q = TQ$ .

**The Lagrange-d'Alembert-Pontryagin Principle.** Next, we shall investigate an implicit Lagrangian system for the case in which a regular constraint distribution is given. To do this, we introduce an extended Lagrange-d'Alembert principle called the Lagrange-d'Alembert-Pontryagin principle.

Let  $L$  be a Lagrangian on  $TQ$  and  $\Delta_Q \subset TQ$  be a constraint distribution on  $Q$ . Define a generalized energy  $E(q, v, p) = p \cdot v - L(q, v)$  on  $TQ \oplus T^*Q$ . Keeping the endpoints of  $q(t)$  fixed, the ***Lagrange-d'Alembert-Pontryagin principle*** is expressed by

$$\begin{aligned} & \delta \int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) \cdot (\dot{q}(t) - v(t))\} dt \\ &= \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - E(q(t), v(t), p(t))\} dt \\ &= \int_{t_1}^{t_2} \left\{ \left( \frac{\partial L}{\partial q} - \dot{p} \right) \delta q + \left( \frac{\partial L}{\partial v} - p \right) \delta v + (\dot{q} - v) \delta p \right\} dt \\ &= 0, \end{aligned} \quad (3.11)$$

where we choose a variation  $(\delta q(t), \delta v(t), \delta p(t))$  of the curve  $(q(t), v(t), p(t))$ ,  $t_1 \leq t \leq t_2$  in  $TQ \oplus T^*Q$  such that  $\delta q(t) \in \Delta_Q(q(t))$  and with the constraint  $v(t) \in \Delta_Q(q(t))$ . Hence, the Lagrange-d'Alembert-Pontryagin principle is represented by

$$\int_{t_1}^{t_2} \left\{ \left( \frac{\partial L}{\partial q} - \dot{p} \right) \delta q + \left( \frac{\partial L}{\partial v} - p \right) \delta v + (\dot{q} - v) \delta p \right\} dt = 0,$$

which is equivalent to the equation

$$\left( \frac{\partial L}{\partial q} - \dot{p} \right) \delta q + \left( \frac{\partial L}{\partial v} - p \right) \delta v + (\dot{q} - v) \delta p = 0 \quad (3.12)$$

for all variations  $\delta q(t) \in \Delta_Q(q(t))$ , for arbitrary  $\delta p(t)$  and  $\delta v(t)$ , and with the constraint  $v(t) \in \Delta_Q(q(t))$ .

**Proposition 3.4.** *Let a distribution  $\Delta_Q$  be locally denoted by  $\Delta(q) \subset \mathbb{R}^n$  at each  $q \in U \subset \mathbb{R}^n$ . The Lagrange-d'Alembert-Pontryagin principle for a curve  $(q(t), v(t), p(t))$  provides equations of motion, in coordinates, such that*

$$\dot{q} = v \in \Delta(q), \quad \dot{p} - \frac{\partial L}{\partial q} \in \Delta^\circ(q), \quad p = \frac{\partial L}{\partial v}. \quad (3.13)$$

**Proof.** From equation (3.12), we obtain the second-order condition  $\dot{q} = v$ , the Legendre transform  $p = \partial L / \partial v$ , the equations of motion  $\dot{p} - \partial L / \partial q \in \Delta^\circ$  and with the constraints  $v \in \Delta(q)$ . Thus, we obtain equation (3.13). ■

Notice that, as shown in Part I, equation (3.13) is the local expression of an implicit Lagrangian system.

**Constraint Distributions.** We shall define constraint distributions for the intrinsic expression of the Lagrange-d'Alembert-Pontryagin principle.

Consider a regular constraint distribution  $\Delta_Q$  on  $Q$ . Let  $\text{pr}_Q : TQ \oplus T^*Q \rightarrow Q$  and  $T\text{pr}_Q : T(TQ \oplus T^*Q) \rightarrow TQ$ . Let  $\tau_{TQ \oplus T^*Q} : T(TQ \oplus T^*Q) \rightarrow TQ \oplus T^*Q$  and  $T\tau_{TQ \oplus T^*Q} : TT(TQ \oplus T^*Q) \rightarrow T(TQ \oplus T^*Q)$ .

Define the submanifold  $\mathcal{K} \subset TQ \oplus T^*Q$  by

$$\mathcal{K} = \left\{ (q, v, p) \in TQ \oplus T^*Q \mid (q, v) \in \Delta_Q, \quad p = \frac{\partial L}{\partial v} \right\}, \quad (3.14)$$

and also define the distribution  $\mathcal{B}$  on  $TQ \oplus T^*Q$  by

$$\mathcal{B} = (T\text{pr}_Q)^{-1}(\Delta_Q) \subset T(TQ \oplus T^*Q).$$

Let  $\mathcal{C}$  be the restriction of  $\mathcal{B}$  to  $\mathcal{K}$ ; that is,

$$\mathcal{C} = \mathcal{B} \cap T\mathcal{K} \subset T(TQ \oplus T^*Q).$$

In the above, we assume that  $\mathcal{C}$  is a regular distribution on  $\mathcal{K}$ . Furthermore, note that  $T\text{pr}_Q \circ T\tau_{TQ \oplus T^*Q} : TT(TQ \oplus T^*Q) \rightarrow TQ$  and define the distribution  $\mathcal{F}$  on  $T(TQ \oplus T^*Q)$  by

$$\mathcal{F} = (T\text{pr}_Q \circ T\tau_{TQ \oplus T^*Q})^{-1}(\Delta_Q) \subset TT(TQ \oplus T^*Q).$$

Let  $\mathcal{G}$  be defined by the restriction of  $\mathcal{F}$  to  $\mathcal{C}$  as

$$\mathcal{G} = \mathcal{F} \cap T\mathcal{C} \subset TT(TQ \oplus T^*Q), \quad (3.15)$$

where we assume  $\mathcal{G}$  is a regular distribution on  $\mathcal{C}$ .

**The Intrinsic Lagrange-d'Alembert-Pontryagin Equations.** We can intrinsically represent the Lagrange-d'Alembert-Pontryagin principle for a curve  $x(t) = (q(t), v(t), p(t))$ ,  $t_1 \leq t \leq t_2$  in  $TQ \oplus T^*Q$ , with the endpoints of  $q(t)$  fixed, by

$$\begin{aligned}
 & \delta \int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) \cdot (\dot{q}(t) - v(t))\} dt \\
 &= \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - E(q(t), v(t), p(t))\} dt \\
 &= \delta \int_{t_1}^{t_2} \left\{ G(\rho_{TT^*Q} \circ T\text{pr}_{T^*Q}(x, \dot{x})) - E(\tau_{TQ \oplus T^*Q}(x, \dot{x})) \right\} dt \\
 &= \int_{t_1}^{t_2} \left\{ (T\text{pr}_{T^*Q})^* \chi(x, \dot{x}) - (\tau_{TQ \oplus T^*Q})^* \mathbf{d}E(x, \dot{x}) \right\} \cdot w dt \\
 &= 0,
 \end{aligned} \tag{3.16}$$

which holds for all  $w = (q, v, p, \dot{q}, \dot{v}, \dot{p}, \delta q, \delta v, \delta p, \delta \dot{q}, \delta \dot{v}, \delta \dot{p}) \in \mathcal{G}(x, \dot{x}) \subset T_{(x, \dot{x})}T(TQ \oplus T^*Q)$ .

**Proposition 3.5.** *The Lagrange-d'Alembert-Pontryagin principle is equivalent to the equation*

$$(T\text{pr}_{T^*Q})^* \chi(x(t), \dot{x}(t)) \cdot w(t) = (\tau_{TQ \oplus T^*Q})^* \mathbf{d}E(x(t), \dot{x}(t)) \cdot w(t) \tag{3.17}$$

for all  $w \in \mathcal{G}(x(t), \dot{x}(t))$ .

**Proof.** Note that equation (3.16) is the intrinsic expression of equation (3.11). From equation (3.16), it is obvious that we obtain equation (3.17), which is the intrinsic expression of equation (3.13). ■

Notice that we have derived (3.17) from the variational viewpoint, and that the result is consistent with the geometry of the generalized Legendre transform that we studied in §2.

**The Intrinsic Implicit Lagrangian Systems.** Let  $X : TQ \oplus T^*Q \rightarrow TT^*Q$  be a **partial vector field** on  $T^*Q$ ; that is, a map that assigns to a point  $x = (v_q, p_q) \in TQ \oplus T^*Q$ , a vector in  $T_{p_q}T^*Q$ ; we write  $X$  in local representation as

$$X(x) = (q, p, \dot{q}, \dot{p}),$$

where  $\dot{q}$  and  $\dot{p}$  are functions of  $x = (q, v, p)$ .

Define the **vertical lift** of the partial vector field  $X(x) = (q, p, \dot{q}, \dot{p})$  to be the vector field  $\tilde{X}$  on  $TQ \oplus T^*Q$  whose value at  $x = (q, v, p)$  is given (in local representation) by

$$\tilde{X}(x) = (q, v, p, \dot{q}, 0, \dot{p}).$$

By an **integral curve** of  $X$  we mean a curve  $(q(t), v(t), p(t)) \in TQ \oplus T^*Q$  such that for each  $t$ , we have

$$X(x) = (q, p, \dot{q}, \dot{p}),$$

where  $(q, p) = \text{pr}_{T^*Q}(x)$ . Notice that the following algebraic relation holds:

$$X(x) = T_x \text{pr}_{T^*Q}(\tilde{X}(x)).$$



**Proposition 3.6.** *Let  $X : TQ \oplus T^*Q \rightarrow TT^*Q$  be a partial vector field and  $\tilde{X}$  be its vertical lift. If an integral curve  $x(t) = (q(t), v(t), p(t)) \in TQ \oplus T^*Q$  of  $X$  satisfies*

$$(T\text{pr}_{T^*Q})^*\chi(\tilde{X}(x(t))) \cdot w(t) = (\tau_{TQ \oplus T^*Q})^*\mathbf{d}E(\tilde{X}(x(t))) \cdot w(t) \quad (3.18)$$

for all variations  $w(t) \in \mathcal{G}(\tilde{X}(x(t))) \subset T_{\tilde{X}(x(t))}T(TQ \oplus T^*Q)$ , then, it satisfies the Lagrange-d'Alembert-Pontryagin equation in (3.13).

**Proof.** Since  $(q(t), v(t), p(t))$  is an integral curve of the partial vector field  $X : TQ \oplus T^*Q \rightarrow TT^*Q$ , we have

$$X(x) = (q, p, \dot{q}, \dot{p}).$$

By construction, the vertical lift of  $X$  at each  $x = (q, v, p)$ , takes the value

$$\tilde{X}(x) = (q, v, p, \dot{q}, 0, \dot{p}).$$

On the left-hand side of equation (3.18), recall that  $\chi$  is the one-form on  $TT^*Q$  induced from the canonical one-form  $\Theta_{T^*T^*Q}$  on  $T^*T^*Q$  as  $\chi = (\Omega^\flat)^*\Theta_{T^*T^*Q}$ , which is given, in coordinates  $(q, p, -\delta p, \delta q)$ , for  $T^*T^*Q$  by

$$\chi = -\delta p dq + \delta q dp,$$

and

$$(T\text{pr}_{T^*Q})^*\chi = -\delta p dq + \delta q dp$$

is the pulled-back one-form on  $T(TQ \oplus T^*Q)$  by the tangent map  $T\text{pr}_{T^*Q} : T(TQ \oplus T^*Q) \rightarrow TT^*Q$  of the projection  $\text{pr}_{T^*Q} : TQ \oplus T^*Q \rightarrow T^*Q$ . Then, we have

$$(T\text{pr}_{T^*Q})^*\chi(\tilde{X}(x)) = -\dot{p} dq + \dot{q} dp.$$

Further, the variation  $w(t) \in \mathcal{G}(\tilde{X}(x(t)))$  may be locally denoted by

$$w = (q, v, p, \dot{q}, 0, \dot{p}, \delta q, \delta v, \delta p, \delta \dot{q}, 0, \delta \dot{p}),$$

and hence it follows that the left-hand side of equation (3.18) is locally denoted by

$$(T\text{pr}_{T^*Q})^*\chi(\tilde{X}(x)) \cdot w = -\dot{p} \delta q + \dot{q} \delta p. \quad (3.19)$$

On the other hand, as to the right-hand side of equation (3.18), we have

$$\begin{aligned} (\tau_{TQ \oplus T^*Q})^*\mathbf{d}E(\tilde{X}(x)) &= \frac{\partial E}{\partial q} dq + \frac{\partial E}{\partial v} dv + \frac{\partial E}{\partial p} dp \\ &= \left(-\frac{\partial L}{\partial q}\right) dq + \left(p - \frac{\partial L}{\partial v}\right) dv + v dp. \end{aligned}$$

Hence, the right-hand side of equation (3.18) is given, in coordinates, by

$$(\tau_{TQ \oplus T^*Q})^*\mathbf{d}E(\tilde{X}(x)) \cdot w = \left(-\frac{\partial L}{\partial q}\right) \delta q + \left(p - \frac{\partial L}{\partial v}\right) \delta v + v \delta p. \quad (3.20)$$

From equations (3.19) and (3.20), it follows

$$\left(\frac{\partial L}{\partial q} - \dot{p}\right) \delta q + \left(\frac{\partial L}{\partial v} - p\right) \delta v + (\dot{q} - v) \delta p = 0$$

for all variations  $\delta q \in \Delta_Q(q)$  and for all  $\delta p$  and  $\delta v$ , together with  $v \in \Delta_Q(q)$ .

Thus, we obtain the local expression of an implicit Lagrangian system that is given by

$$\dot{q} = v \in \Delta_Q(q), \quad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^\circ(q), \quad p = \frac{\partial L}{\partial v},$$

which is the Lagrange-d'Alembert-Pontryagin equation (3.13). ■

**Remarks.** Let us rewrite equation (3.18) using the partial vector field  $X : TQ \oplus T^*Q \rightarrow TT^*Q$ . Associated with the partial vector field  $X(q, v, p) = (q, p, \dot{q}, \dot{p})$  and its vertical lift  $\tilde{X}$ , recall that the algebraic equation

$$X(x) = T_x \text{pr}_{T^*Q}(\tilde{X}(x))$$

holds for each  $x = (q, v, p)$  and recall also that

$$\tau_{TQ \oplus T^*Q}(\tilde{X}(x)) = x.$$

Then, equation (3.18) can be equivalently restated as

$$\chi(X(x)) \cdot T_{\tilde{X}(x)}(T \text{pr}_{T^*Q})(w) = \mathbf{d}E(x) \cdot T_{\tilde{X}(x)}(\tau_{TQ \oplus T^*Q})(w). \quad (3.21)$$

In coordinates, equation (3.21) may be given by

$$-\dot{p} \delta q + \dot{q} \delta p = \left( -\frac{\partial L}{\partial q} \right) \delta q + \left( p - \frac{\partial L}{\partial v} \right) \delta v + v \delta p \quad (3.22)$$

for all variations  $\delta q \in \Delta_Q(q)$ , for all  $\delta p$  and  $\delta v$ , and with  $v \in \Delta_Q(q)$ .

In fact, we can check this directly by computations. Since

$$\chi(X(x)) = -\dot{p} dq + \dot{q} dp$$

and

$$\begin{aligned} T_{\tilde{X}(x)}(T \text{pr}_{T^*Q})(w) &= T_{(q, v, p, \dot{q}, 0, \dot{p})}(T \text{pr}_{T^*Q})(\delta q, \delta v, \delta p, \delta \dot{q}, 0, \delta \dot{p}) \\ &= (q, p, \dot{q}, \dot{p}, \delta q, \delta p, \delta \dot{q}, \delta \dot{p}), \end{aligned}$$

the left-hand side of equation (3.21) is given by

$$\chi(X(x)) \cdot T_{\tilde{X}(x)}(T \text{pr}_{T^*Q})(w) = -\dot{p} \delta q + \dot{q} \delta p,$$

while we have

$$\mathbf{d}E(x) = \left( -\frac{\partial L}{\partial q} \right) dq + \left( p - \frac{\partial L}{\partial v} \right) dv + v dp$$

and

$$\begin{aligned} T_{\tilde{X}(x)}(\tau_{TQ \oplus T^*Q})(w) &= T_{(q, v, p, \dot{q}, 0, \dot{p})}(\tau_{TQ \oplus T^*Q})(q, v, p, \dot{q}, 0, \dot{p}, \delta q, \delta v, \delta p, \delta \dot{q}, 0, \delta \dot{p}) \\ &= (q, v, p, \delta q, \delta v, \delta p). \end{aligned}$$

Then, the right-hand side of equation (3.21) is locally denoted by

$$\mathbf{d}E(x) \cdot T_{\tilde{X}(x)}(\tau_{TQ \oplus T^*Q})(w) = \left( -\frac{\partial L}{\partial q} \right) \delta q + \left( p - \frac{\partial L}{\partial v} \right) \delta v + v \delta p.$$

Hence, we obtain equation (3.22), from which it follows that

$$\left( \frac{\partial L}{\partial q} - \dot{p} \right) \delta q + \left( \frac{\partial L}{\partial v} - p \right) \delta v + (\dot{q} - v) \delta p = 0$$

holds for all variations  $\delta q \in \Delta_Q(q)$ , for all  $\delta p$  and  $\delta v$ , and with  $v \in \Delta_Q(q)$ . Thus, we obtain equation (3.13).

Let us call equation (3.18), or equivalently, equation (3.21) the *intrinsic implicit Lagrangian system*.

**Variational Link with Dirac Structures.** Next, we shall see the variational link of an implicit Lagrangian systems  $(L, \Delta_Q, X)$  with the induced Dirac structure  $D_{\Delta_Q}$  on  $T^*Q$ , where  $X$  is the partial vector field defined.

Recall from Part I that  $D_{\Delta_Q}$  is defined, for each  $z \in T^*Q$ ,

$$\begin{aligned} D_{\Delta_Q}(z) &= \{(u_z, \alpha_z) \in T_z T^*Q \times T_z^* T^*Q \mid u_z \in \Delta_{T^*Q}(z), \text{ and} \\ &\quad \alpha_z(w_z) = \Omega_{\Delta_Q}(u_z, w_z) \text{ for all } w_z \in \Delta_{T^*Q}(z)\}, \end{aligned} \quad (3.23)$$

where  $\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q)$  and  $\Omega_{\Delta_Q}$  is the restriction of the canonical symplectic form  $\Omega$  on  $T^*Q$  to  $\Delta_{T^*Q}$ .

As to the variational link of implicit Lagrangian systems, we have the following proposition.

**Proposition 3.7.** *If a curve  $x(t) = (q(t), v(t), p(t))$  is a solution curve of the intrinsic implicit Lagrangian system associated with equation (3.18), then, it satisfies, for each  $v(t) \in \Delta_Q(q(t))$ ,*

$$(X(q(t), v(t), p(t)), \mathfrak{D}L(q(t), v(t))) \in D_{\Delta_Q}(q(t), p(t)), \quad (3.24)$$

where  $p_{q(t)}(t) = \mathbb{F}L(q(t), v(t))$  and  $(q(t), v(t), p(t))$  is an integral curve of the partial vector field  $X$ .

**Proof.** It is logically obvious that the above proposition holds; however, we shall prove this by direct computations.

Let us rewrite the left-hand side of equation (3.18). Recall the one-form  $\chi$  on  $TT^*Q$  is defined by  $\chi = (\Omega^b)^* \Theta_{T^*T^*Q}$ , and we have

$$\begin{aligned} (T\text{pr}_{T^*Q})^* \chi(\tilde{X}(x)) \cdot w &= \chi(T\text{pr}_{T^*Q}(\tilde{X}(x))) \cdot T_{\tilde{X}(x)} T\text{pr}_{T^*Q}(w) \\ &= \Theta_{T^*T^*Q}(\Omega^b \circ T\text{pr}_{T^*Q}(\tilde{X}(x))) \cdot T\Omega^b(T_{\tilde{X}(x)}(T\text{pr}_{T^*Q}(w))). \end{aligned} \quad (3.25)$$

Recall also that the canonical one-form on  $T^*T^*Q$  is defined by

$$\Theta_{T^*T^*Q}(\alpha) \cdot V = \langle \alpha, T\pi_{T^*Q}(V) \rangle,$$

where  $\alpha \in T^*T^*Q$ ,  $V \in T_\alpha(T^*T^*Q)$ ,  $\pi_{T^*Q} : T^*T^*Q \rightarrow T^*Q$  is the canonical projection and  $T\pi_{T^*Q} : TT^*T^*Q \rightarrow TT^*Q$ . Using local coordinates  $x = (q, v, p)$ , equation (3.25) reads that

$$\begin{aligned} &\Theta_{T^*T^*Q}(\Omega^b \circ T\text{pr}_{T^*Q}(\tilde{X}(x))) \cdot (T\Omega^b \circ T_{\tilde{X}(x)}(T\text{pr}_{T^*Q}(w))) \\ &= \Omega^b \circ T\text{pr}_{T^*Q}(\tilde{X}(x)) \cdot T\pi_{T^*Q}(T\Omega^b \circ T_{\tilde{X}(x)}(T\text{pr}_{T^*Q}(w))) \\ &= -\dot{p} \delta q + \dot{q} \delta p. \end{aligned} \quad (3.26)$$

In the above, the vertical lift of  $X$  is given, using local coordinates, by  $\tilde{X}(x) = (q, v, p, \dot{q}, 0, \dot{p})$  and hence  $\Omega^b \circ T\text{pr}_{T^*Q}(\tilde{X}(x)) = \Omega^b(q, p, \dot{q}, \dot{p}) = (q, p, -\dot{p}, \dot{q})$ , and further, noting that  $w = (q, v, p, \dot{q}, 0, \dot{p}, \delta q, \delta v, \delta p, \delta \dot{q}, 0, \delta \dot{p})$ ,

$$\begin{aligned} T\pi_{T^*Q}(T\Omega^b \circ T_{\tilde{X}(x)}(T\text{pr}_{T^*Q}(w))) &= T\pi_{T^*Q}(q, p, -\dot{p}, \dot{q}, \delta q, \delta p, -\delta \dot{p}, \delta \dot{q}) \\ &= (q, p, \delta q, \delta p). \end{aligned}$$

On the other hand, the right-hand side of equation (3.18) is locally expressed by

$$\begin{aligned}
 (\tau_{TQ \oplus T^*Q})^* \mathbf{d}E(\tilde{X}(x)) \cdot w &= \mathbf{d}E \left( \tau_{TQ \oplus T^*Q}(\tilde{X}(x)) \right) \cdot T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w) \\
 &= \mathbf{d}E(x) \cdot T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w) \\
 &= \left( \frac{\partial E}{\partial q} dq + \frac{\partial E}{\partial v} dv + \frac{\partial E}{\partial p} dp \right) \cdot \left( \delta q \frac{\partial}{\partial q} + \delta v \frac{\partial}{\partial v} + \delta p \frac{\partial}{\partial p} \right) \\
 &= \left( -\frac{\partial L}{\partial q} \right) \delta q + \left( p - \frac{\partial L}{\partial v} \right) \delta v + v \delta p
 \end{aligned}$$

where  $T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w) = (q, v, p, \delta q, \delta v, \delta p) \in \mathcal{C}(x)$  and  $\tau_{TQ \oplus T^*Q}(\tilde{X}(x)) = x$ . For each point  $x = (q, v, p) \in \mathcal{K}$ , one has

$$\begin{aligned}
 \frac{\partial E(x)}{\partial v} &= p - \frac{\partial L}{\partial v} \\
 &= 0.
 \end{aligned}$$

Hence, using the projection  $\text{pr}_{T^*Q} : TQ \oplus T^*Q \rightarrow T^*Q$  and its tangent map  $T\text{pr}_{T^*Q} : T(TQ \oplus T^*Q) \rightarrow T(T^*Q)$ , it follows that

$$\begin{aligned}
 (\tau_{TQ \oplus T^*Q})^* \mathbf{d}E(\tilde{X}(x)) \cdot w &= \mathbf{d}E(x) |_{T_{\text{pr}_{T^*Q}(x)} T^*Q} \cdot T\text{pr}_{T^*Q}(T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w)) \\
 &= \left( \frac{\partial E}{\partial q} dq + \frac{\partial E}{\partial p} dp \right) \cdot \left( \delta q \frac{\partial}{\partial q} + \delta p \frac{\partial}{\partial p} \right) \\
 &= \left( -\frac{\partial L}{\partial q} \right) \delta q + v \delta p. \tag{3.27}
 \end{aligned}$$

In the above, the restriction of  $\mathbf{d}E(x) : T_x(TQ \oplus T^*Q) \rightarrow \mathbb{R}$  to  $T_{\text{pr}_{T^*Q}(x)} T^*Q$  is given by

$$\mathbf{d}E(x) |_{T_{\text{pr}_{T^*Q}(x)} T^*Q} = \left( -\frac{\partial L}{\partial q} \right) dq + v dp,$$

where  $\text{pr}_{T^*Q}(x) = (q, p = \partial L / \partial v)$ .

Recall that the Dirac differential of a Lagrangian, that is,  $\mathfrak{D}L : TT^*Q \rightarrow T^*T^*Q$  is defined by using the diffeomorphism  $\gamma_Q = \Omega^b \circ (\kappa_Q)^{-1} : T^*TQ \rightarrow T^*T^*Q$ , as shown in Part I, and is given by

$$\begin{aligned}
 \mathfrak{D}L &= \gamma_Q \circ \mathbf{d}L \\
 &= \left( q, \frac{\partial L}{\partial v}, -\frac{\partial L}{\partial q}, v \right).
 \end{aligned}$$

Recall also that  $\text{pr}_{TQ} : TQ \oplus T^*Q \rightarrow TQ$  is given in coordinates by  $\text{pr}_{TQ}(x) = (q, v)$ , and it reads

$$\mathbf{d}E(x) |_{T_{\text{pr}_{T^*Q}(x)} T^*Q} = \mathfrak{D}L(\text{pr}_{TQ}(x)) \tag{3.28}$$

From equations (3.18) and (3.25)–(3.28), we have

$$\begin{aligned}
 &\Omega^b(T\text{pr}_{T^*Q}(\tilde{X}(x))) \cdot T\pi_{T^*Q}(T\Omega^b \circ T_{\tilde{X}(x)}(T\text{pr}_{T^*Q})(w)) \\
 &= \mathbf{d}E(x) |_{T_{\text{pr}_{T^*Q}(x)} T^*Q} \cdot T\text{pr}_{T^*Q}(T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w)) \\
 &= \mathfrak{D}L(\text{pr}_{TQ}(x)) \cdot T\text{pr}_{T^*Q}(T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w)). \tag{3.29}
 \end{aligned}$$

Notice that there exists the identity

$$T\pi_{T^*Q}(T\Omega^\flat \circ T(\text{Pr}_{T^*Q})) = T\text{pr}_{T^*Q} \circ T\tau_{TQ \oplus T^*Q},$$

and we set

$$\begin{aligned} u &= T\pi_{T^*Q}(T\Omega^\flat \circ T_{\tilde{X}(x)}(\text{Pr}_{T^*Q})(w)) \\ &= T\text{pr}_{T^*Q}(T_{\tilde{X}(x)}\tau_{TQ \oplus T^*Q}(w)) \\ &= (q, p, \delta q, \delta p). \end{aligned} \quad (3.30)$$

From equations (3.29) and (3.30), it follows that equation (3.18) can be restated as

$$\Omega^\flat(X(q(t), v(t), p(t))) \cdot u(t) = \mathfrak{D}L(q(t), v(t)) \cdot u(t)$$

for all  $u(t) \in \Delta_{T^*Q}(p_{q(t)}(t))$ , where  $\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q$ . In other words, a curve  $(q(t), v(t)) \in \Delta_Q(q(t))$  satisfies

$$\Omega_{\Delta_Q}(X(q(t), v(t), p(t)), u(t)) = \mathfrak{D}L(q(t), v(t)) \cdot u(t), \quad (3.31)$$

for all  $u(t) \in \Delta_{T^*Q}(p_{q(t)}(t))$ , where  $p_{q(t)}(t) = \mathbb{F}L(q(t), v(t))$  and  $(q(t), v(t), p(t))$  is the integral curve in  $P = \mathbb{F}L(\Delta_Q)$  of the partial vector field  $X : TQ \oplus T^*Q \rightarrow TT^*Q$ .

As shown in equation (3.23), the set of  $\Delta_{T^*Q}$  and the skew-symmetric bilinear form  $\Omega_{\Delta_Q}$  defines an *induced Dirac structure*  $D_{\Delta_Q}$  on  $T^*Q$ . Then, needless to say, equation (3.31) represents the condition of an implicit Lagrangian system  $(L, \Delta_Q, X)$  associated with the induced Dirac structure  $D_{\Delta_Q}$  on  $T^*Q$ ; that is,

$$(X, \mathfrak{D}L) \in D_{\Delta_Q}$$

together with the Legendre transform  $P = \mathbb{F}L(\Delta_Q)$ . ■

**Remarks.** It follows from equation (3.28), equation (3.24) can be also denoted by, for each  $(q, v) \in \Delta_Q$ ,

$$(X(q, v, p), \mathbf{d}E(q, v, p)|_{T_{(q,p)}T^*Q}) \in D_{\Delta_Q}(q, p), \quad (3.32)$$

where  $(q, p) = \mathbb{F}L(q, v)$  and the restriction  $\mathbf{d}E(q, v, p)|_{T_{(q,p)}T^*Q}$  is understood in the sense that  $TT^*Q$  is naturally included in  $T(TQ \oplus T^*Q)$ . Hence, the condition for an implicit Lagrangian system  $(L, \Delta_Q, X)$ , namely,  $(X, \mathfrak{D}L) \in D_{\Delta_Q}$ , can be restated as

$$(X, \mathbf{d}E|_{T_P T^*Q}) \in D_{\Delta_Q}$$

together with the Legendre transform  $P = \mathbb{F}L(\Delta_Q)$ .

We can summarize the results obtained so far in the following theorem.

**Theorem 3.8.** *Let  $L$  be a Lagrangian on  $TQ$  (possibly degenerate) and  $\Delta_Q$  be a constraint distribution on  $Q$ . Let  $X : TQ \oplus T^*Q \rightarrow TT^*Q$  be a partial vector field on  $T^*Q$ , which assigns to each point  $x = (q, v, p) \in TQ \oplus T^*Q$  a vector  $X(x) = (q, p, \dot{q}, \dot{p})$  in  $T_{p_q}T^*Q$ , where  $\dot{q}$  and  $\dot{p}$  are functions of  $x = (q, v, p)$ . Denote by  $x(t) = (q(t), v(t), p(t))$ ,  $t_1 \leq t \leq t_2$ , a curve in  $TQ \oplus T^*Q$ . The following statements are equivalent:*

- (a)  $x(t)$  satisfies the Lagrange-d'Alembert-Pontryagin principle in equation (3.16);
- (b)  $x(t)$  is a solution curve of the intrinsic implicit Lagrangian system, which is denoted by equation (3.18);
- (c)  $x(t)$  satisfies the condition of an implicit Lagrangian system  $(L, \Delta_Q, X)$ , which is given in equation (3.24).

**Hamilton's Phase Space Principle.** If a given Lagrangian is hyperregular, then, a hyperregular Hamiltonian is well defined on the cotangent bundle via the Legendre transform. Hence, we can also develop an *implicit Hamiltonian system* for the hyperregular case in terms of the induced Dirac structure on the cotangent bundle. Before going into the construction of implicit Hamiltonian systems, we first show how *intrinsic Hamilton's equations* can be developed in the context of Hamilton's phase space principle.

Let  $L$  be a hyperregular Lagrangian on  $TQ$ . Define the energy  $E$  on  $TQ$ , by employing local coordinates  $(q, v)$  for  $TQ$ , such that

$$E(q, v) = \frac{\partial L}{\partial v} \cdot v - L(q, v).$$

Since the Legendre transform  $\mathbb{F}L : TQ \rightarrow T^*Q$  is diffeomorphism, a hyperregular Hamiltonian  $H$  can be defined on  $T^*Q$  such that

$$H = E \circ (\mathbb{F}L)^{-1}.$$

Then, define the path space of curves  $z(t) = (q(t), p(t))$ ,  $t_1 \leq t \leq t_2$  in  $T^*Q$  as

$$S(q_1, q_2, [t_1, t_2]) = \{z = (q, p) : [t_1, t_2] \rightarrow T^*Q \mid \pi_Q(z(t_1)) = q_1, \pi_Q(z(t_2)) = q_2\},$$

where  $\pi_Q : T^*Q \rightarrow Q$  and also define the action functional on the path space  $S(q_1, q_2, [t_1, t_2])$  of curves  $z(t) = (q(t), p(t))$ ,  $t_1 \leq t \leq t_2$  by

$$\int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - H(q(t), p(t))\} dt, \quad (3.33)$$

which is called the *Poincaré-Cartan integral*.

**Proposition 3.9.** *Keeping the endpoints  $q(t_1)$  and  $q(t_2)$  of  $q(t)$  fixed whereas  $p(t_1)$  and  $p(t_2)$  of  $p(t)$  are allowed to be free, the stationary condition for the Poincaré-Cartan integral in equation (3.33) gives Hamilton's equations*

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \quad (3.34)$$

**Proof.** The variation of the Poincaré-Cartan integral is locally represented by

$$\begin{aligned} & \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - H(q(t), p(t))\} dt \\ &= \int_{t_1}^{t_2} \left( \delta p \dot{q} + p \delta \dot{q} - \frac{\partial H}{\partial q} \delta q - \frac{\partial H}{\partial p} \delta p \right) dt \\ &= \int_{t_1}^{t_2} \left\{ \left( \dot{q} - \frac{\partial H}{\partial p} \right) \delta p + \left( -\dot{p} - \frac{\partial H}{\partial q} \right) \delta q \right\} dt + p \delta q \Big|_{t_1}^{t_2}. \end{aligned} \quad (3.35)$$

Keeping the endpoints  $q(t_1)$  and  $q(t_2)$  of  $q(t)$  fixed, the stationary condition for the Poincaré-Cartan integral gives Hamilton's equations in equation (3.34). ■

**The Intrinsic Form of Hamiltonian's Equations** Let us demonstrate the intrinsic expression for Hamilton's phase space principle in the following.

**Proposition 3.10.** *Let  $\Theta_{T^*Q}$  be the canonical one-form on  $T^*Q$  and  $\Omega^b : TT^*Q \rightarrow T^*T^*Q$  be the bundle map associated with the canonical symplectic form  $\Omega$ . Let  $\Theta_{T^*T^*Q}$  be the canonical one-form on  $T^*T^*Q$ . Let  $\tau_{T^*Q} : TT^*Q \rightarrow T^*Q$  be the tangent projection and  $\rho_{TT^*Q} : TT^*Q \rightarrow TQ \oplus T^*Q$  be the projection. Denote by  $H$ , a hyperregular Hamiltonian on  $T^*Q$ . Let  $G(q, v, p) = p \cdot v$  be the momentum function on  $TQ \oplus T^*Q$ , where  $(q, v, p) \in TQ \oplus T^*Q$ . The variation of the Poincaré-Cartan integral in equation (3.33) is represented by*

$$\begin{aligned}
 & \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - H(q(t), p(t))\} dt \\
 &= \delta \int_{t_1}^{t_2} \{G(\rho_{TT^*Q}(z, \dot{z})) - H(\tau_{T^*Q}(z, \dot{z}))\} dt \\
 &= \int_{t_1}^{t_2} \left\{ \Theta_{T^*T^*Q}(\Omega_z^b(\dot{z})) \cdot T\Omega^b(w) - \mathbf{d}H(\tau_{T^*Q}(z, \dot{z})) \cdot T\tau_{T^*Q}(w) \right\} dt \\
 & \quad + \Theta_{T^*Q}(\tau_{T^*Q}(z, \dot{z})) \cdot T\tau_{T^*Q}(w) \Big|_{t_1}^{t_2} \\
 &= \int_{t_1}^{t_2} \left\{ \chi(z, \dot{z}) - (\tau_{T^*Q})^* \mathbf{d}H(z, \dot{z}) \right\} \cdot w dt + \Theta_{T^*Q}(\tau_{T^*Q}(z, \dot{z})) \cdot T\tau_{T^*Q}(w) \Big|_{t_1}^{t_2},
 \end{aligned} \tag{3.36}$$

where  $z = (q, p) \in T^*Q$ ,  $\dot{z} = dz/dt \in T_z T^*Q$  and  $w \in T_{(z, \dot{z})}(TT^*Q)$ .

**Proof.** Let us check that equation (3.36) is the intrinsic representation of equation (3.35). Recall that the one-form  $\chi$  on  $TT^*Q$  is defined by the canonical one-form  $\Theta_{T^*T^*Q}$  on  $T^*T^*Q$  such that

$$\chi = (\Omega^b)^* \Theta_{T^*T^*Q}.$$

Since  $(z, \dot{z}) \in TT^*Q$  is locally denoted by  $(q, p, \dot{q}, \dot{p})$ , we have

$$\chi(z, \dot{z}) = -\dot{p} dq + \dot{q} dp \in T_{(z, \dot{z})}^*(TT^*Q).$$

Then, noting  $T\Omega^b : T(TT^*Q) \rightarrow T(T^*T^*Q)$ , the following relation holds:

$$\begin{aligned}
 \Theta_{T^*T^*Q}(\Omega_z^b(\dot{z})) \cdot T\Omega^b(w) &= \chi(z, \dot{z}) \cdot w \\
 &= -\dot{p} \delta q + \dot{q} \delta p
 \end{aligned}$$

for all  $\dot{z} \in T_z T^*Q$  and  $w = (q, p, \dot{q}, \dot{p}, \delta q, \delta p, \delta \dot{q}, \delta \dot{p}) \in T_{(z, \dot{z})}(TT^*Q)$ .

On the other hand, recall that the canonical one-form  $\Theta_{T^*T^*Q}$  is defined such that

$$\Theta_{T^*T^*Q}(\Omega_z^b(\dot{z})) \cdot T\Omega^b(w) = \Omega_z^b(\dot{z}) \cdot T\pi_{T^*Q}(T\Omega^b(w)), \tag{3.37}$$

where  $\pi_{T^*Q} : T^*T^*Q \rightarrow T^*Q$  is the cotangent projection,  $\Omega_z^b(\dot{z}) = (q, p, -\dot{p}, \dot{q}) \in T^*T^*Q$  and  $T\pi_{T^*Q}(T\Omega^b(w)) = (q, p, \delta q, \delta p) \in TT^*Q$ .

The differential of the Hamiltonian  $\mathbf{d}H : T^*Q \rightarrow T^*T^*Q$  is locally denoted by

$$\mathbf{d}H = \left( q, p, \frac{\partial H}{\partial q}, \frac{\partial H}{\partial p} \right).$$

By the projection  $T\tau_{T^*Q} : T(TT^*Q) \rightarrow TT^*Q$ , we have

$$\begin{aligned}
 T\tau_{T^*Q}(w) &= T\tau_{T^*Q}(q, p, \dot{q}, \dot{p}, \delta q, \delta p, \delta \dot{q}, \delta \dot{p}) \\
 &= (q, p, \delta q, \delta p).
 \end{aligned}$$

Hence, it follows that, for all  $z \in T^*Q$  and for all  $w \in T_{(z,\dot{z})}TT^*Q$ ,

$$\begin{aligned} \mathbf{d}H(\tau_{T^*Q}(z, \dot{z})) \cdot T\tau_{T^*Q}(w) &= (\tau_{T^*Q})^* \mathbf{d}H(z, \dot{z}) \cdot w \\ &= \frac{\partial H}{\partial q} \delta q + \frac{\partial H}{\partial p} \delta p. \end{aligned} \quad (3.38)$$

Furthermore, we have the local expression

$$\Theta_{T^*Q}(\tau_{T^*Q}(z, \dot{z})) \cdot T\tau_{T^*Q}(w) = p \delta q. \quad (3.39)$$

From equations (3.37) to (3.39), it immediately reads that equation (3.36) is the intrinsic representation of equation (3.35).  $\blacksquare$

**Remarks.** Recall that  $\theta = \lambda \oplus \chi$  is the one-form on  $TT^*Q \times TT^*Q$  and also that  $\Psi^*\theta = \mathbf{d}(G \circ \rho_{TT^*Q})$  using the diagonal map  $\Psi : TT^*Q \rightarrow TT^*Q \times TT^*Q$ . Then, the following relation is satisfied, for each  $z \in T^*Q$ ,

$$\begin{aligned} \delta \int_{t_1}^{t_2} G(\rho_{TT^*Q}(z, \dot{z})) dt &= \int_{t_1}^{t_2} \mathbf{d}G(\rho_{TT^*Q}(z, \dot{z})) \cdot T\rho_{TT^*Q}(w) dt \\ &= \int_{t_1}^{t_2} \Psi^*\theta(z, \dot{z}) \cdot w dt \\ &= \int_{t_1}^{t_2} \{ \Theta_{T^*T^*Q}(\Omega_z^b(\dot{z})) \cdot T\Omega^b(w) + \Theta_{T^*TQ}(\kappa_Q(z, \dot{z})) \cdot T\kappa_Q(w) \} dt \\ &= \int_{t_1}^{t_2} \Theta_{T^*T^*Q}(\Omega_z^b(\dot{z})) \cdot T\Omega^b(w) dt \\ &\quad + \frac{d}{dt} \int_{t_1}^{t_2} \Theta_{T^*Q}(\tau_{T^*Q}(z, \dot{z})) \cdot T\tau_{T^*Q}(w) dt \\ &= \int_{t_1}^{t_2} \Theta_{T^*T^*Q}(\Omega_z^b(\dot{z})) \cdot T\Omega^b(w) dt + \Theta_{T^*Q}(\tau_{T^*Q}(z, \dot{z})) \cdot T\tau_{T^*Q}(w) \Big|_{t_1}^{t_2}. \end{aligned}$$

Using local coordinates  $z = (q, p) \in T^*Q$ ,  $\dot{z} = (\dot{q}, \dot{p}) \in T_zT^*Q$ , and  $w = (\delta q, \delta p, \delta \dot{q}, \delta \dot{p}) \in T_{(z,\dot{z})}(TT^*Q)$ , it is easy to check the above relation:

$$\begin{aligned} \delta \int_{t_1}^{t_2} G(q, \dot{q}, p) dt &= \delta \int_{t_1}^{t_2} (p \cdot \dot{q}) dt \\ &= \int_{t_1}^{t_2} (p \delta \dot{q} + \dot{q} \delta p) dt \\ &= \int_{t_1}^{t_2} \{ (-\dot{p} \delta q + \dot{q} \delta p) + (\dot{p} \delta q + p \delta \dot{q}) \} dt \\ &= \int_{t_1}^{t_2} (-\dot{p} \delta q + \dot{q} \delta p) dt + \frac{d}{dt} \int_{t_1}^{t_2} p \delta q dt \\ &= \int_{t_1}^{t_2} (-\dot{p} \delta q + \dot{q} \delta p) dt + p \delta q \Big|_{t_1}^{t_2}. \end{aligned}$$

**Proposition 3.11.** *Keeping the endpoints  $\pi_Q(z(t_1)) = q(t_1)$  and  $\pi_Q(z(t_2)) = q(t_2)$  of the curve  $\pi_Q(z(t)) = q(t)$  fixed, the stationary condition for the Poincaré-Cartan integral in equation (3.36) provides **intrinsic Hamilton's equations** such that, for each  $z = (q, p) \in T^*Q$ ,*

$$\chi(z, \dot{z}) = (\tau_{T^*Q})^* \mathbf{d}H(z, \dot{z}). \quad (3.40)$$



**Proof.** The stationary condition of the Poincaré-Cartan integral is given by

$$\begin{aligned} & \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - H(q(t), p(t))\} dt \\ &= \delta \int_{t_1}^{t_2} \{G(\rho_{TT^*Q}(z, \dot{z})) - H(\tau_{T^*Q}(z, \dot{z}))\} dt \\ &= \int_{t_1}^{t_2} \{\chi(z, \dot{z}) - (\tau_{T^*Q})^* \mathbf{d}H(z, \dot{z})\} \cdot w dt + \Theta_{T^*Q}(\tau_{T^*Q}(z, \dot{z})) \cdot T\tau_{T^*Q}(w) \Big|_{t_1}^{t_2} \end{aligned}$$

for all  $w \in T_{(z, \dot{z})}(TT^*Q)$ . Keeping the endpoints of  $q(t)$  fixed, we can obtain the intrinsic Hamilton's equations in equation (3.40). ■

**Implicit Hamiltonian Systems.** Let us illustrate an *implicit Hamiltonian system* for the case in which a hyperregular Hamiltonian is given on the cotangent bundle and with a constraint distribution on a configuration manifold.

**Definition 3.12.** Let  $H$  be a hyperregular Hamiltonian on  $T^*Q$  and  $\Delta_Q \subset TQ$  be a constraint distribution on  $Q$ . Let  $X$  be a vector field on  $T^*Q$  and  $\Omega$  be the canonical symplectic form on  $T^*Q$ . Let  $D_{\Delta_Q}$  be the induced Dirac structure on  $T^*Q$  defined by equation (3.23).

Then, an **implicit Hamiltonian system** is the triple  $(H, \Delta_Q, X)$  that satisfies, for each point  $z \in T^*Q$ ,

$$(X(z), \mathbf{d}H(z)) \in D_{\Delta_Q}(z),$$

that is

$$(X, \mathbf{d}H) \in D_{\Delta_Q}.$$

**Definition 3.13.** A **solution curve** of an implicit Hamiltonian system  $(H, \Delta_Q, X)$  is a curve  $(q(t), p(t))$ ,  $t_1 \leq t \leq t_2$ , in  $T^*Q$  such that the curve  $(q(t), p(t))$  is an integral curve of  $X$ .

**Proposition 3.14.** Using local coordinates  $(q, p)$  for  $T^*Q$ , it follows from the condition  $(X, \mathbf{d}H) \in D_{\Delta_Q}$  that the local expression for an implicit Hamiltonian system is given by

$$\dot{q} = \frac{\partial H}{\partial p} \in \Delta(q), \quad \dot{p} + \frac{\partial H}{\partial q} \in \Delta^\circ(q), \quad (3.41)$$

where the distribution  $\Delta_Q$  is locally denoted by  $\Delta(q) \subset \mathbb{R}^n$  at each  $q \in U \subset \mathbb{R}^n$ .

**Proof.** Recall the local expression for the canonical symplectic form is given by

$$\Omega((q, p, u_1, \alpha_1), (q, p, u_2, \alpha_2)) = \langle \alpha_2, u_1 \rangle - \langle \alpha_1, u_2 \rangle,$$

and the condition for an implicit Hamiltonian system  $(H, \Delta_Q, X)$  is given by, for each  $(q, p) \in T^*Q$ ,

$$(X(q, p), \mathbf{d}H(q, p)) \in D_{\Delta_Q}(q, p).$$

Using local expressions  $X(q, p) = (\dot{q}, \dot{p})$  and  $\mathbf{d}H(q, p) = (\partial H/\partial q, \partial H/\partial p)$ , it follows that

$$\left\langle \frac{\partial H}{\partial q}, \delta q \right\rangle + \left\langle \frac{\partial H}{\partial p}, \delta p \right\rangle = \langle \delta p, \dot{q} \rangle - \langle \dot{p}, \delta q \rangle$$

for all  $\delta q \in \Delta(q)$ , for all  $\delta p$ , and with  $\dot{q} \in \Delta(q)$ . Thus, we obtain equation (3.41). ■

Notice that equation (3.41) is the local expression for an **implicit Hamiltonian system**.

**The Hamilton-d'Alembert Principle in Phase Space.** We now show how to obtain an *implicit Hamiltonian system* in the context of a generalization of Hamilton's phase space principle that we refer to as the *Hamilton-d'Alembert principle in phase space*.

The *Hamilton-d'Alembert principle in phase space* for a curve  $(q(t), p(t))$ ,  $t_1 \leq t \leq t_2$ , in  $T^*Q$  is given by

$$\delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - H(q(t), p(t))\} dt = 0, \quad (3.42)$$

and with  $\dot{q}(t) \in \Delta(q(t))$ . The variation of the left-hand side in equation (3.42) is locally given, keeping the endpoints of  $q(t)$  fixed, by

$$\begin{aligned} & \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - H(q(t), p(t))\} dt \\ &= \int_{t_1}^{t_2} \left\{ \left( \dot{q} - \frac{\partial H}{\partial p} \right) \delta p + \left( -\dot{p} - \frac{\partial H}{\partial q} \right) \delta q \right\} dt, \end{aligned} \quad (3.43)$$

where we choose a variation  $\delta q(t)$  of curves  $q(t)$  such that  $\delta q(t) \in \Delta(q(t))$ .

Note that in the case of regular Lagrangians, if one starts with the Lagrange-d'Alembert-Pontryagin principle, and optimizes first over  $v$ , then one arrives at the Hamilton-d'Alembert principle in phase space.

**Proposition 3.15.** *The Hamilton-d'Alembert principle in phase space for a curve  $(q(t), p(t))$ ,  $t_1 \leq t \leq t_2$ , in  $T^*Q$  gives the implicit Hamiltonian systems in equation (3.41).*

**Proof.** From equation (3.43), the Hamilton-d'Alembert principle in phase space is equivalent to

$$\left( \dot{q} - \frac{\partial H}{\partial p} \right) \delta p + \left( -\dot{p} - \frac{\partial H}{\partial q} \right) \delta q = 0$$

for all  $\delta q \in \Delta(q)$ , for all  $\delta p$ , and with  $\dot{q} \in \Delta(q)$ . Thus, we obtain equation (3.41).  $\blacksquare$

**Coordinate Representation.** Suppose that the dimension of  $\Delta(q)$  is  $n - m$  at each point  $q$ . Let  $\Delta^\circ(q)$  be the annihilator of  $\Delta(q)$  spanned by  $m$  one-forms  $\omega^1, \dots, \omega^m$ , and it follows that equation (3.41) can be represented, in coordinates, by employing the Lagrange multipliers  $\mu_a$ ,  $a = 1, \dots, m$  such that

$$\begin{aligned} \begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q^i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_a \omega_i^a(q) \end{pmatrix}, \\ 0 &= \omega_i^a(q) \frac{\partial H}{\partial p_i}, \end{aligned}$$

where we use the local expression  $\omega^a = \omega_i^a dq^i$ .

**Constraint Distributions.** Let  $\Delta_Q \subset TQ$  be a constraint distribution on  $Q$ . Define the distribution on  $T^*Q$  by

$$\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q,$$

where  $\pi_Q : T^*Q \rightarrow Q$  and  $T\pi_Q : TT^*Q \rightarrow TQ$ . Let  $P$  be defined by the image of  $\Delta_Q$  under the Legendre transformation  $\mathbb{F}L : TQ \rightarrow T^*Q$ , that is,  $P = \mathbb{F}L(\Delta_Q) \subset T^*Q$  and let  $\Delta_P$  be the restriction of  $\Delta_{T^*Q}$  to  $P$  such that

$$\Delta_P = \Delta_{T^*Q} \cap TP \subset TT^*Q,$$

where we assume  $\Delta_P$  is a regular distribution on  $P$ . Define the distribution on  $TT^*Q$  by

$$\mathcal{I} = (T\pi_Q \circ \tau_{TT^*Q})^{-1}(\Delta_Q) \subset T(TT^*Q),$$

where  $\tau_{TT^*Q} : T(TT^*Q) \rightarrow TT^*Q$  and hence  $T\pi_Q \circ \tau_{TT^*Q} : T(TT^*Q) \rightarrow TQ$ . Let  $\mathcal{J}$  be the restriction of  $\mathcal{I}$  to  $\Delta_P$  such that

$$\mathcal{J} = \mathcal{I} \cap T\Delta_P \subset T(TT^*Q),$$

where  $\mathcal{J}$  is assumed to be a regular distribution on  $\Delta_P$ .

**Intrinsic Implicit Hamiltonian Systems.** Let us see how the *intrinsic implicit Hamiltonian system* is related to the Hamilton-d'Alembert principle in phase space.

**Proposition 3.16.** *The Hamilton-d'Alembert principle in phase space for a curve  $z(t) = (q(t), p(t))$ ,  $t_1 \leq t \leq t_2$ , in  $T^*Q$  is intrinsically represented, keeping the endpoints of  $q(t)$  fixed, by*

$$\begin{aligned} & \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - H(q(t), p(t))\} dt \\ &= \delta \int_{t_1}^{t_2} \{G(\rho_{TT^*Q}(z, \dot{z})) - H(\tau_{T^*Q}(z, \dot{z}))\} dt \\ &= \int_{t_1}^{t_2} \{\chi(z, \dot{z}) - (\tau_{T^*Q})^* \mathbf{d}H(z, \dot{z})\} \cdot w dt \\ &= 0 \end{aligned} \tag{3.44}$$

for a chosen variation  $w = (q, p, \dot{q}, \dot{p}, \delta q, \delta p, \delta \dot{q}, \delta \dot{p}) \in \mathcal{J}(z, \dot{z}) \subset T_{(z, \dot{z})}(TT^*Q)$ .

Then, the *intrinsic Hamilton-d'Alembert principle in phase space* is equivalent to the equations

$$\chi(z, \dot{z}) \cdot w = (\tau_{T^*Q})^* \mathbf{d}H(z, \dot{z}) \cdot w \tag{3.45}$$

for all variations  $w \in \mathcal{J}(z, \dot{z})$ .

**Proof.** From equation (3.36), it is apparent that equation (3.44) is the intrinsic expression of the Hamilton-d'Alembert principle in phase space. Thus, we obtain equation (3.45). ■

**Proposition 3.17.** *Let  $z(t)$ ,  $t_1 \leq t \leq t_2$ , be the integral curve of a vector field  $X$  on  $T^*Q$ . If  $z(t)$  is a solution curve of the Hamilton-d'Alembert principle, then it satisfies*

$$\chi(X(z)) \cdot w = (\tau_{T^*Q})^* \mathbf{d}H(X(z)) \cdot w \tag{3.46}$$

for all  $w \in \mathcal{J}(X(z))$ .

**Proof.** Since  $z(t)$  is the integral curve of  $X$ , we have  $\dot{z} = X(z)$ . By substituting this into equation (3.45), we can obtain equation (3.46). ■

**Proposition 3.18.** *If a curve  $z(t) = (q(t), p(t))$ ,  $t_1 \leq t \leq t_2$  is an integral curve of the vector field  $X$  associated with equation (3.46), then, it satisfies the condition of an implicit Hamiltonian system  $(H, \Delta_Q, X)$ , namely,*

$$(X(z(t)), \mathbf{d}H(z(t))) \in D_{\Delta_Q}(z(t)), \tag{3.47}$$

where  $D_{\Delta_Q}(z)$  is the induced Dirac structure on  $T^*Q$ .

**Proof.** It is logically obvious that the above proposition holds; however, we shall prove this by direct computations in coordinates.

Recall that the one-form  $\chi$  is defined, by using the map  $\Omega^b : TT^*Q \rightarrow T^*T^*Q$ , such that  $\chi = (\Omega^b)^*\Theta_{T^*T^*Q}$ , and the left-hand side of equation (3.46) can be restated as, for each  $z = (q, p) \in T^*Q$ ,

$$\begin{aligned}\chi(X(z)) \cdot w &= (\Omega^b)^*\Theta_{T^*T^*Q}(X(z)) \cdot w \\ &= \Theta_{T^*T^*Q}(\Omega^b(X(z))) \cdot T\Omega^b(w) \\ &= -\dot{p} \delta q + \dot{q} \delta p\end{aligned}\tag{3.48}$$

for all  $w = (q, p, \dot{q}, \dot{p}, \delta q, \delta p, \delta \dot{q}, \delta \dot{p}) \in \mathcal{J}(X(z))$ , where  $X(z) = (q, p, \dot{q}, \dot{p})$ ,  $\Omega^b(X(z)) = (q, p, -\dot{p}, \dot{q})$  and  $T\Omega^b(w) = (q, p, -\dot{p}, \dot{q}, \delta q, \delta p, -\delta \dot{p}, \delta \dot{q})$ . Recall also that the canonical one-form  $\Theta_{T^*T^*Q}$  on  $T^*T^*Q$  is defined by

$$\Theta_{T^*T^*Q}(\alpha) \cdot V = \langle \alpha, T\pi_{T^*Q}(V) \rangle$$

for all  $\alpha \in T^*T^*Q$  and  $V \in T_\alpha(T^*T^*Q)$ . So, it follows

$$\begin{aligned}\Theta_{T^*T^*Q}(\Omega^b(X(z))) \cdot T\Omega^b(w) &= \Omega^b(X(z)) \cdot T\pi_{T^*Q}(T\Omega^b(w)) \\ &= -\dot{p} \delta q + \dot{q} \delta p,\end{aligned}\tag{3.49}$$

where one can easily check

$$\begin{aligned}T\pi_{T^*Q}(T\Omega^b(w)) &= (T\pi_{T^*Q} \circ T\Omega^b)(q, p, \dot{q}, \dot{p}, \delta q, \delta p, \delta \dot{q}, \delta \dot{p}) \\ &= T\pi_{T^*Q}(q, p, -\dot{p}, \dot{q}, \delta q, \delta p, -\delta \dot{p}, \delta \dot{q}) \\ &= (q, p, \delta q, \delta p).\end{aligned}$$

From equations (3.48) and (3.49), it reads

$$\chi(X(z)) \cdot w = \Omega^b(X(z)) \cdot T\pi_{T^*Q}(T\Omega^b(w)).\tag{3.50}$$

On the other hand, the right-hand side of equation (3.46) is locally expressed by

$$\begin{aligned}(\tau_{T^*Q})^*\mathbf{d}H(X(z)) \cdot w &= \mathbf{d}H(\tau_{T^*Q}(X(z))) \cdot T_{X(z)}(\tau_{T^*Q})(w) \\ &= \frac{\partial H}{\partial q} \delta q + \frac{\partial H}{\partial p} \delta p.\end{aligned}\tag{3.51}$$

Noting the identity

$$T(\tau_{T^*Q}) = T\pi_{T^*Q} \circ T\Omega^b,$$

we can set

$$\begin{aligned}u &= T_{X(z)}(\tau_{T^*Q})(w) \\ &= T\pi_{T^*Q} \circ T\Omega^b(w) \\ &= (q, p, \delta q, \delta p).\end{aligned}\tag{3.52}$$

From equations (3.50) to (3.52), an integral curve  $z(t) = (q(t), p(t))$ ,  $t_1 \leq t \leq t_2$ , of the vector field  $X$  on  $T^*Q$  satisfies

$$\Omega^b(X(z(t))) \cdot u(t) = \mathbf{d}H(z(t)) \cdot u(t)$$

for all  $u(t) = (\delta q(t), \delta p(t)) \in \Delta_{T^*Q}(z(t))$ . This equation indicates the condition for an implicit Hamiltonian system  $(H, \Delta_Q, X)$  associated with the induced Dirac structure  $D_{\Delta_Q}$  on  $T^*Q$ , namely,

$$(X, \mathbf{d}H) \in D_{\Delta_Q}.$$

■

Thus, we shall call equation (3.46) the *intrinsic implicit Hamiltonian system*.

We can summarize the obtained results in the following theorem:

**Theorem 3.19.** *Let  $H$  be a hyperregular Hamiltonian on  $T^*Q$  and  $\Delta_Q$  be a given distribution on  $Q$ . Denote by  $X$  a vector field on  $T^*Q$ . Let  $z(t) = (q(t), p(t))$ ,  $t_1 \leq t \leq t_2$ , be a curve in  $T^*Q$ . The following statements are equivalent:*

- (a)  $z(t)$  satisfies the Hamilton-d'Alembert principle in phase space in equation (3.44);
- (b)  $z(t)$  is a solution curve of the intrinsic implicit Hamiltonian system, which is represented by equation (3.46);
- (c)  $z(t)$  satisfies the condition of an implicit Hamiltonian system  $(H, \Delta_Q, X)$ , which is given by equation (3.47).

## 4 Nonholonomic Systems with External Forces

In this section, we demonstrate that nonholonomic mechanical systems with external forces can be naturally incorporated into the context of implicit Lagrangian systems by employing the Lagrange-d'Alembert-Pontryagin principle. Needless to say, nonholonomic mechanical systems have been widely investigated from the viewpoint of geometric mechanics in conjunction with the analysis of stability and control problems (see, for instance, Zenkov, Bloch, and Marsden [1998]; Bloch [2003]). In particular, what is developed in this section will be quite useful in the analysis of interconnected systems, and are also relevant to controlled Lagrangian systems (see Bloch, Leonard, and Marsden [2000]; Bloch, Chang, Leonard, and Marsden [2001]).

**External Force Fields.** Consider a mechanical system with an external force field and let  $Q$  be a configuration manifold. Let  $\pi_Q : T^*Q \rightarrow Q$  be the cotangent projection. Recall that an *external force field*  $F : TQ \rightarrow T^*Q$  is a fiber-preserving map over the identity, which induces the horizontal one-form  $F'$  on  $T^*Q$  as

$$F'(z) \cdot u = \langle F(v), T_z \pi_Q(u) \rangle,$$

where  $z \in T_q^*Q$ ,  $v \in T_qQ$  and  $u \in T_z T^*Q$ . Further, using the projection  $\text{pr}_{T^*Q} : TQ \oplus T^*Q \rightarrow T^*Q$ , the horizontal one-form  $F'$  on  $T^*Q$  can be lifted as the horizontal one-form  $\tilde{F}$  on  $TQ \oplus T^*Q$  such that, for  $x \in TQ \oplus T^*Q$ ,

$$\tilde{F}(x) \cdot W = F'(\text{pr}_{T^*Q}(x)) \cdot T_x \text{pr}_{T^*Q}(W),$$

where  $W \in T_x(TQ \oplus T^*Q)$ .

**The Lagrange-d'Alembert-Pontryagin Principle.** Consider kinematic constraints that are given by a constraint distribution  $\Delta_Q$  on  $Q$ , which is locally represented by  $\Delta(q) \subset \mathbb{R}^n$  at each  $q \in U \subset \mathbb{R}^n$ . We assume the dimension of  $\Delta(q)$  is  $n - m$  at each point  $q$  and let  $\Delta^\circ(q)$  be the annihilator of  $\Delta(q)$  spanned by  $m$  one-forms  $\omega^1, \dots, \omega^m$ .

Recall that the motion of the mechanical system  $c : [t_1, t_2] \rightarrow Q$  is said to be constrained if  $\dot{c}(t) \in \Delta_Q(c(t))$  for all  $t$ ,  $t_1 \leq t \leq t_2$ . Further, the distribution  $\Delta_Q$  is not involutive in general; that is,  $[X(q), Y(q)] \notin \Delta(q)$  for any two vector fields  $X, Y$  on  $Q$  with values in  $\Delta_Q$ .

Let  $L$  be a (possibly degenerate) Lagrangian on  $TQ$  and let  $F : TQ \rightarrow T^*Q$  be an external force field. The Lagrange-d'Alembert-Pontryagin principle for a curve  $(q(t), v(t), p(t))$ ,  $t_1 \leq$

$t \leq t_2$ , in  $TQ \oplus T^*Q$  is represented by

$$\begin{aligned}
 & \delta \int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) \cdot (\dot{q}(t) - v(t))\} dt + \int_{t_1}^{t_2} F(q(t), v(t)) \cdot \delta q(t) dt \\
 &= \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - E(q(t), v(t), p(t))\} dt + \int_{t_1}^{t_2} F(q(t), v(t)) \cdot \delta q(t) dt \\
 &= 0
 \end{aligned} \tag{4.1}$$

for a given variation  $\delta q(t) \in \Delta(q(t))$  and with the constraint  $v(t) \in \Delta(q(t))$ . Keeping the endpoints of  $q(t)$  fixed, we have

$$\begin{aligned}
 & \delta \int_{t_1}^{t_2} \{L(q, v) + p \cdot (\dot{q} - v)\} dt \\
 &= \delta \int_{t_1}^{t_2} \{p \cdot \dot{q} - E(q, v, p)\} dt \\
 &= \int_{t_1}^{t_2} \left\{ \left( \frac{\partial L}{\partial q} - \dot{p} \right) \delta q + \left( \frac{\partial L}{\partial v} - p \right) \delta v + (\dot{q} - v) \delta p \right\} dt.
 \end{aligned}$$

Hence, the Lagrange-d'Alembert-Pontryagin principle is represented by

$$\int_{t_1}^{t_2} \left\{ \left( \frac{\partial L}{\partial q} - \dot{p} \right) \delta q + \left( \frac{\partial L}{\partial v} - p \right) \delta v + (\dot{q} - v) \delta p \right\} dt + \int_{t_1}^{t_2} F(q, v) \delta q dt = 0 \tag{4.2}$$

for a chosen variation  $\delta q(t) \in \Delta(q(t))$ , for all  $\delta v(t)$  and  $\delta p(t)$ , and with  $v(t) \in \Delta(q(t))$ .

**Proposition 4.1.** *The Lagrange-d'Alembert-Pontryagin principle gives the local expression of equations of motion for nonholonomic mechanical systems with external forces such that*

$$\dot{q} = v \in \Delta(q), \quad \dot{p} - \frac{\partial L}{\partial q} - F(q, v) \in \Delta^\circ(q), \quad p = \frac{\partial L}{\partial v}. \tag{4.3}$$

**Proof.** From equation (4.2), it reads

$$\left( \frac{\partial L}{\partial q} - \dot{p} + F(q, v) \right) \delta q + \left( \frac{\partial L}{\partial v} - p \right) \delta v + (\dot{q} - v) \delta p = 0,$$

which is satisfied for a given variation  $\delta q(t) \in \Delta(q(t))$ , for all  $\delta v(t)$  and  $\delta p(t)$ , and with the constraint  $v(t) \in \Delta(q(t))$ . Thus, we obtain equation (4.3).  $\blacksquare$

**Coordinate Representation.** Recall the one-forms  $\omega^1, \dots, \omega^m$  span a basis of the annihilator  $\Delta^\circ(q)$  at each  $q \in U \subset \mathbb{R}^n$ , and it follows that equation (4.3) can be represented, in coordinates, by employing the Lagrange multipliers  $\mu_a$ ,  $a = 1, \dots, m$ , as follows:

$$\begin{aligned}
 \begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial L}{\partial q^i} - F_i(q, v) \\ v^i \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_a \omega_i^a(q) \end{pmatrix}, \\
 p_i &= \frac{\partial L}{\partial v^i}, \\
 0 &= \omega_i^a(q) v^i,
 \end{aligned}$$

where we employ the local expression  $\omega^a = \omega_i^a dq^i$ .

**Intrinsic Formulation.** Denote by  $\Delta_Q \subset TQ$  a constraint distribution. Let  $\mathcal{K} \subset TQ \oplus T^*Q$  be the submanifold defined in equation (3.14) and  $\mathcal{G} \subset TT(TQ \oplus T^*Q)$  be the regular distribution defined by equation (3.15). Let  $F : TQ \rightarrow T^*Q$  be an external force field and  $\tilde{F}$  on  $TQ \oplus T^*Q$  be the horizontal one-form. The Lagrange-d'Alembert-Pontryagin principle for a curve  $x(t) = (q(t), v(t), p(t))$  in  $TQ \oplus T^*Q$  is intrinsically represented by

$$\begin{aligned}
 & \delta \int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) \cdot (\dot{q}(t) - v(t))\} dt + \int_{t_1}^{t_2} F(q(t), v(t)) \cdot \delta q(t) dt \\
 &= \delta \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - E(q(t), v(t), p(t))\} dt + \int_{t_1}^{t_2} F(q(t), v(t)) \cdot \delta q(t) dt \\
 &= \delta \int_{t_1}^{t_2} \left\{ G(\rho_{TT^*Q} \circ T\text{pr}_{T^*Q}(x(t), \dot{x}(t))) - E(\tau_{TQ \oplus T^*Q}(x(t), \dot{x}(t))) \right\} dt \\
 & \quad + \int_{t_1}^{t_2} \tilde{F}(\tau_{TQ \oplus T^*Q}(x(t), \dot{x}(t))) \cdot T\tau_{TQ \oplus T^*Q}(w(t)) dt \tag{4.4} \\
 &= \int_{t_1}^{t_2} \left\{ (T\text{pr}_{T^*Q})^* \chi(x(t), \dot{x}(t)) - (\tau_{TQ \oplus T^*Q})^* \mathbf{d}E(x(t), \dot{x}(t)) \right\} \cdot w(t) dt \\
 & \quad + \int_{t_1}^{t_2} (\tau_{TQ \oplus T^*Q})^* \tilde{F}(x(t), \dot{x}(t)) \cdot w(t) dt \\
 &= 0,
 \end{aligned}$$

which is satisfied for all variations  $w = (q, v, p, \dot{q}, \dot{v}, \dot{p}, \delta q, \delta v, \delta p, \delta \dot{q}, \delta \dot{v}, \delta \dot{p}) \in \mathcal{G}(x, \dot{x}) \subset T_{(x, \dot{x})}T(TQ \oplus T^*Q)$  and with the endpoints of  $q(t)$  fixed. Using the projections  $\text{pr}_Q : TQ \oplus T^*Q \rightarrow Q$  and  $\tau_{TQ \oplus T^*Q} : T(TQ \oplus T^*Q) \rightarrow TQ \oplus T^*Q$  with their tangent maps, it follows that  $(T\text{pr}_Q \circ T\tau_{TQ \oplus T^*Q})(w) = (q, \delta q) \in \Delta_Q$ .

**Proposition 4.2.** *The Lagrange-d'Alembert-Pontryagin principle in equation (4.4) for a curve  $x(t)$  in  $TQ \oplus T^*Q$  induces the **intrinsic Lagrange-d'Alembert-Pontryagin equation with external forces** as*

$$\begin{aligned}
 & (T\text{pr}_{T^*Q})^* \chi(x(t), \dot{x}(t)) \cdot w(t) \\
 &= (\tau_{TQ \oplus T^*Q})^* \left( \mathbf{d}E(x(t), \dot{x}(t)) - \tilde{F}(x(t), \dot{x}(t)) \right) \cdot w(t) \tag{4.5}
 \end{aligned}$$

for all  $w(t) \in \mathcal{G}(x(t), \dot{x}(t))$ .

**Proof.** Recall that the variation of the Hamilton-Pontryagin integral is given in equation (3.4) and recall also that the external force field  $F : TQ \rightarrow T^*Q$  induces the horizontal one-form  $\tilde{F}$  on  $TQ \oplus T^*Q$ , and it is apparent that equation (4.4) is the intrinsic expression for equation (4.1). Thus, we can obtain equation (4.5).  $\blacksquare$

**Proposition 4.3.** *Let  $X : TQ \oplus T^*Q \rightarrow TT^*Q$  be a partial vector field on  $T^*Q$ , and let  $\tilde{X}$  be the vertical lift of  $X$ . If a curve  $x(t) = (q(t), v(t), p(t))$ ,  $t_1 \leq t \leq t_2$  in  $TQ \oplus T^*Q$  satisfies*

$$(T\text{pr}_{T^*Q})^* \chi(\tilde{X}(x(t))) \cdot w(t) = (\tau_{TQ \oplus T^*Q})^* \left( \mathbf{d}E(\tilde{X}(x(t))) - \tilde{F}(\tilde{X}(x(t))) \right) \cdot w(t),$$

or equivalently,

$$\begin{aligned}
 & \chi(X(x(t))) \cdot T_{\tilde{X}(x(t))}(T\text{pr}_{T^*Q})(w(t)) \\
 &= \left( \mathbf{d}E(x(t)) - \tilde{F}(x(t)) \right) \cdot T_{\tilde{X}(x(t))}(\tau_{TQ \oplus T^*Q})(w(t)), \tag{4.6}
 \end{aligned}$$

for all variations  $w(t) \in \mathcal{G}(\tilde{X}(x(t)))$  such that  $(q(t), v(t), p(t))$  is an integral curve of  $X$ , then, it satisfies the intrinsic Lagrange-d'Alembert-Pontryagin equation with external forces in equation (4.5).

**Proof.** We shall illustrate that equation (4.6) induces the Lagrange-d'Alembert-Pontryagin equations with external forces in equation (4.5) by direct computations using local coordinates.

Recall that the vertical lift of  $X(q, v, p) = (q, p, \dot{q}, \dot{p})$  is represented, in local representation, by  $\tilde{X}(q, v, p) = (q, v, p, \dot{q}, 0, \dot{p})$ , and also that the horizontal one-form  $\tilde{F}$  on  $TQ \oplus T^*Q$  is defined by lifting the horizontal one-form  $F'$  on  $T^*Q$  such that, for  $x = (q, v, p) \in TQ \oplus T^*Q$ ,

$$\begin{aligned} (\tau_{TQ \oplus T^*Q})^* \tilde{F}(\tilde{X}(x)) \cdot w &= \tilde{F}(\tau_{TQ \oplus T^*Q}(\tilde{X}(x)) \cdot T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w)) \\ &= F'(\text{pr}_{T^*Q}(x)) \cdot T_x \text{pr}_{T^*Q}(T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w)) \end{aligned}$$

for all  $w = (q, v, p, \dot{q}, 0, \dot{p}, \delta q, \delta v, \delta p, \delta \dot{q}, 0, \delta \dot{p}) \in \mathcal{G}(\tilde{X}(x))$ . In the above, the horizontal one-form  $F'$  is induced from the external force field  $F : TQ \rightarrow T^*Q$ , and it follows

$$\begin{aligned} F'(\text{pr}_{T^*Q}(x)) \cdot T_x \text{pr}_{T^*Q}(T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w)) \\ &= \left\langle F(\text{pr}_{TQ}(x)), T_{\text{pr}_{T^*Q}(x)} \pi_Q(T_x \text{pr}_{T^*Q}(T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w))) \right\rangle \\ &= \langle F(q, v), T_{(q,p)} \pi_Q(q, p, \delta q, \delta p) \rangle \\ &= F(q, v) \delta q, \end{aligned}$$

where  $(q, p) = \text{pr}_{T^*Q}(x)$ ,  $(q, v) = \text{pr}_{TQ}(x)$  and  $(q, p, \delta q, \delta p) = T_x \text{pr}_{T^*Q}(T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w))$ . Therefore, one has

$$\begin{aligned} (\tau_{TQ \oplus T^*Q})^* \tilde{F}(\tilde{X}(x)) \cdot w &= \tilde{F}(\tau_{TQ \oplus T^*Q}(\tilde{X}(x)) \cdot T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w)) \\ &= F(q, v) \delta q \end{aligned}$$

and hence, in view of the proof of Proposition 3.6, the coordinate expression of equation (4.6) may be given by

$$-\dot{p} \delta q + \dot{q} \delta p = \left( -\frac{\partial L}{\partial q} - F(q, v) \right) \delta q + \left( p - \frac{\partial L}{\partial v} \right) \delta v + v \delta p$$

Then, it follows that

$$\left( \frac{\partial L}{\partial q} - \dot{p} + F(q, v) \right) \delta q + \left( \frac{\partial L}{\partial v} - p \right) \delta v + (\dot{q} - v) \delta p = 0$$

is satisfied for a given variation  $\delta q(t) \in \Delta(q(t))$ , for all  $\delta v(t)$  and  $\delta p(t)$ , and with the constraint  $v(t) \in \Delta(q(t))$ .

Thus, we can obtain equation (4.3), which is the local expression of the intrinsic Lagrange-d'Alembert-Pontryagin equations with external forces in equation (4.5).  $\blacksquare$

**Proposition 4.4.** *If a curve  $x(t) = (q(t), v(t), p(t))$  is solution curve of equation (4.6), then, it satisfies the condition of an implicit Lagrangian system for nonholonomic systems with external forces  $(L, F, \Delta_Q, X)$  such that, for each  $(q(t), v(t)) \in \Delta_Q(q(t))$ ,*

$$(X(q(t), v(t), p(t)), \mathfrak{D}L(q(t), v(t)) - \pi_Q^* F(q(t), v(t))) \in D_{\Delta_Q}(q(t), p(t)), \quad (4.7)$$

where  $(q(t), p(t)) = \mathbb{F}L(q(t), v(t))$  and  $(q(t), v(t), p(t))$  is an integral curve of the partial vector field  $X : TQ \oplus T^*Q \rightarrow TT^*Q$  and  $\mathfrak{D}L : TQ \rightarrow T^*T^*Q$  is the Dirac differential of  $L$ ,  $\pi_Q : T^*Q \rightarrow Q$  and  $D_{\Delta_Q}$  is the induced Dirac structure defined in equation (3.23).



**Proof.** In view of Proposition 3.7, it suffices to check the terms of the external force field in equations (4.6) and (4.7). As previously shown, the horizontal one-form  $\tilde{F}$  on  $TQ \oplus T^*Q$  is induced from the external force field  $F : TQ \rightarrow T^*Q$  such that, for each  $x \in TQ \oplus T^*Q$ ,

$$\begin{aligned} (\tau_{TQ \oplus T^*Q})^* \tilde{F}(\tilde{X}(x)) \cdot w &= \tilde{F}(\tau_{TQ \oplus T^*Q}(\tilde{X}(x)) \cdot T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w)) \\ &= \left\langle F(\text{pr}_{TQ}(x)), T_{\text{pr}_{T^*Q}(x)} \pi_Q (T_x \text{pr}_{T^*Q} (T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w))) \right\rangle \\ &= \pi_Q^* F(v_q) \cdot u \end{aligned}$$

for all  $w \in \mathcal{G}(\tilde{X}(x))$ , where  $v_q = \text{pr}_{TQ}(x) \in \Delta_Q(q)$  and  $u = T_x \text{pr}_{T^*Q} (T_{\tilde{X}(x)} \tau_{TQ \oplus T^*Q}(w)) \in \Delta_{T^*Q}(\text{pr}_{T^*Q}(x))$ .

In combination with the proof of Proposition 3.7, it follows that a curve  $(q(t), v(t))$ ,  $t_1 \leq t \leq t_2$ , in  $\Delta_Q$  satisfies

$$\Omega_{\Delta_Q}(X(q(t), v(t), p(t)), u(t)) = (\mathfrak{D}L(q(t), v(t)) - \pi_Q^* F(q(t), v(t))) \cdot u(t),$$

where  $(q(t), p(t)) = \mathbb{F}L(q(t), v(t))$ ,  $u(t) = (q(t), p(t), \delta q(t), \delta p(t))$ , and  $\Omega_{\Delta_Q}$  is the restriction of the canonical symplectic form  $\Omega$  to  $\Delta_{T^*Q}$ . This can be restated by the condition of an implicit Lagrangian system  $(L, F, \Delta_Q, X)$ ; that is, for each  $v(t) \in \Delta_Q(q(t))$ ,

$$(X(q(t), v(t), p(t)), \mathfrak{D}L(q(t), v(t)) - \pi_Q^* F(q(t), v(t))) \in D_{\Delta_Q}(q(t), p(t)),$$

where  $p(t) = \mathbb{F}L(q(t), v(t))$  and  $(q(t), v(t), p(t))$  is an integral curve of  $X$ . Thus, we can check that a solution curve  $x(t) = (q(t), v(t), p(t))$  of equation (4.6) satisfies the condition of an implicit Lagrangian system for nonholonomic systems with external forces  $(L, F, \Delta_Q, X)$ . ■

Thus, it follows from Proposition 4.4 that equation (4.6) is the intrinsic expression of an implicit Lagrangian system for nonholonomic systems with external forces, which we shall call an ***intrinsic implicit Lagrangian system for nonholonomic systems with external forces***.

We can summarize the results obtained so far in the following theorem.

**Theorem 4.5.** *Consider a Lagrangian  $L$  on  $TQ$  (possibly degenerate) and with a given distribution  $\Delta_Q$  on  $Q$ . Let  $F : TQ \rightarrow T^*Q$  be an external force field. Let  $X : TQ \oplus T^*Q \rightarrow TT^*Q$  be a partial vector field on  $T^*Q$  and let  $x(t) = (q(t), v(t), p(t))$ ,  $t_1 \leq t \leq t_2$  be a curve in  $TQ \oplus T^*Q$ . The following statements are equivalent:*

- (a)  $x(t)$  satisfies the Lagrange-d'Alembert-Pontryagin principle in equation (4.5);
- (b)  $x(t)$  is a solution curve of the intrinsic implicit Lagrangian system, which is given by equation (4.6);
- (c)  $x(t)$  satisfies the condition of an implicit Lagrangian system for nonholonomic systems with external forces  $(L, F, \Delta_Q, X)$ , which is given in equation (4.7).

## 5 Implicit Constrained Lagrangian Systems

In this section, we investigate a *constrained Dirac structure*  $D_P$  on the constraint momentum space  $P = \mathbb{F}L(\Delta_Q) \subset T^*Q$  by using an Ehresmann connection and we also develop an *implicit constrained Lagrangian system* associated with  $D_P$ .

**Ehresmann Connections.** We briefly review an Ehresmann connection associated with nonholonomic mechanical systems; for details, refer to [Koon and Marsden \[1998\]](#) and [Bloch \[2003\]](#).

Assume that there is a bundle structure with a projection  $\pi : Q \rightarrow \mathcal{R}$  for our space  $Q$ ; that is, there exists another manifold  $\mathcal{R}$  called the base. We call the kernel of  $T_q\pi$  at any point  $q \in Q$  the *vertical space* denoted by  $\mathcal{V}_q$ .

Recall that an *Ehresmann connection*  $A$  is a vertical vector-valued one-form on  $Q$ , which satisfies

1.  $A_q : T_qQ \rightarrow \mathcal{V}_q$  is a linear map at each point  $p \in Q$ ,
2.  $A$  is a projection :  $A(v_q) = v_q$ , for all  $v_q \in \mathcal{V}_q$ .

Thus, we can split the tangent space at  $q$  such that  $T_qQ = \mathcal{H}_q \oplus \mathcal{V}_q$ , where  $\mathcal{H}_q = \text{Ker } A_q$  is the horizontal space at  $q$ .

Let  $\Delta_Q \subset TQ$  be a constraint distribution, which is locally given by

$$\Delta_Q(q) = \{v_q \in T_qQ \mid \langle \omega^a, v_q \rangle = 0, a = 1, \dots, m\},$$

where  $\omega^a$  are  $m$  independent one-forms that form the basis for the annihilator  $\Delta_Q^\circ \subset T^*Q$ . Let us choose an Ehresmann connection  $A$  in such a way that  $\mathcal{H}_q = \Delta_Q(q)$ . In other words, we assume that the connection is chosen such that the constraints are written as  $A \cdot v_q = 0$ .

Using the bundle coordinates  $q = (r, s) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ , the coordinate representation of  $\pi$  is just projection onto the factor  $r$ , and the connection  $A$  can be locally expressed by a vector-valued differential form  $\omega^a$  as

$$A = \omega^a \frac{\partial}{\partial s^a}, \quad \omega^a(q) = ds^a + A_\alpha^a(r, s) dr^\alpha, \quad a = 1, \dots, m; \quad \alpha = 1, \dots, n - m.$$

Let

$$v_q = u^\alpha \frac{\partial}{\partial r^\alpha} + w^a \frac{\partial}{\partial s^a}$$

be an element of  $T_qQ$ . Then,

$$\omega^a(v_q) = w^a + A_\alpha^a u^\alpha$$

and

$$A(v_q) = (w^a + A_\alpha^a u^\alpha) \frac{\partial}{\partial s^a}.$$

**Horizontal Lift.** Given an Ehresmann connection  $A$ , a point  $q \in Q$  and a vector  $v_r \in T_r\mathcal{R}$  tangent to the base at a point  $r = \pi(q) \in \mathcal{R}$ , we can define the *horizontal lift* of  $v_r$  to be the unique vector  $v_r^h$  in  $\mathcal{H}_q$  that projects to  $v_r$  under  $T_q\pi$ . If we have a vector  $X_q \in T_qQ$ , we shall write its vertical part as

$$\text{ver } X_q = A(q) \cdot X_q,$$

and we shall also write its horizontal part as

$$\text{hor } X_q = X_q - A(q) \cdot X_q.$$

In coordinates, the vertical projection is the map  $(u^\alpha, w^a) \mapsto (0, w^a + A_\alpha^a u^\alpha)$ , while the horizontal projection is the map  $(u^\alpha, w^a) \mapsto (u^\alpha, -A_\alpha^a u^\alpha)$ .

**The Lagrange-d'Alembert-Pontryagin Principle.** Let  $L$  be a (possibly degenerate) Lagrangian on  $TQ$ . Define a generalized energy  $E$  by  $E(q, v, p) = p \cdot v - L(q, v)$  using local coordinates  $(q, v, p)$  for  $TQ \oplus T^*Q$ . Recall that the Lagrange-d'Alembert-Pontryagin principle is given by

$$\begin{aligned} & \delta \int_{t_1}^{t_2} \{L(q, v) + p \cdot (\dot{q} - v)\} dt \\ &= \delta \int_{t_1}^{t_2} \{p \cdot \dot{q} - E(q, v, p)\} dt \\ &= \int_{t_1}^{t_2} \left\{ \left( \frac{\partial L}{\partial q^i} - \dot{p}_i \right) \delta q^i + \left( \frac{\partial L}{\partial v^i} - p_i \right) \delta v^i + (\dot{q}^i - v^i) \delta p_i \right\} dt \\ &= 0 \end{aligned}$$

for chosen variations  $\delta q^i(t) \in \Delta_Q(q(t))$ , with the endpoints of  $q(t)$  fixed and with the constraint  $\omega^a(q) \cdot v_q = 0$ . Hence, the Lagrange-d'Alembert-Pontryagin principle is equivalent to the equation

$$\left( \frac{\partial L}{\partial q^i} - \dot{p}_i \right) \delta q^i + \left( \frac{\partial L}{\partial v^i} - p_i \right) \delta v^i + (\dot{q}^i - v^i) \delta p_i = 0,$$

for all  $\delta q^i(t) \in \Delta_Q(q(t))$  that satisfy, in coordinates  $q^i = (r^\alpha, s^a)$ ,

$$\delta s^a + A_\alpha^a \delta r^\alpha = 0,$$

where the distribution  $\Delta_Q$  is denoted, in coordinates, by

$$\Delta_Q = \text{span} \left\{ \frac{\partial}{\partial r^\alpha} - A_\alpha^a \frac{\partial}{\partial s^a} \right\}.$$

Since the kinematic constraints are given by  $\omega^a(q) \cdot v_q = w^a + A_\alpha^a u^\alpha = 0$ , the Lagrange-d'Alembert-Pontryagin principle may be restated by

$$\begin{aligned} & \left\{ -A_\alpha^a \left( \frac{\partial L}{\partial s^a} - \dot{p}_a \right) + \left( \frac{\partial L}{\partial r^\alpha} - \dot{p}_\alpha \right) \right\} \delta r^\alpha + \left( \frac{\partial L}{\partial u^\alpha} - p_\alpha \right) \delta u^\alpha \\ & \quad + \left( \frac{\partial L}{\partial w^a} - p_a \right) \delta w^a + (\dot{r}^\alpha - u^\alpha) \delta p_\alpha + (\dot{s}^a - w^a) \delta p_a = 0 \end{aligned}$$

for all  $\delta r^\alpha$  and for all  $\delta v^i = (\delta u^\alpha, \delta w^a)$  and  $\delta p_i = (\delta p_\alpha, \delta p_a)$ . Then, equations of motion are given, in coordinates, by

$$\begin{aligned} \dot{r}^\alpha &= u^\alpha, \\ \dot{s}^a &= w^a, \\ \dot{p}_\alpha - \frac{\partial L}{\partial r^\alpha} &= A_\alpha^a \left( \dot{p}_a - \frac{\partial L}{\partial s^a} \right), \end{aligned} \tag{5.1}$$

and with the Legendre transformation

$$p_\alpha = \frac{\partial L}{\partial u^\alpha}, \quad p_a = \frac{\partial L}{\partial w^a}. \tag{5.2}$$

Note that equations of motion in equations (5.1) and the Legendre transformation in equation (5.2) are to be combined with the kinematic constraints

$$w^a = -A_\alpha^a u^\alpha. \tag{5.3}$$

**The Constrained Lagrangian.** Define the constrained Lagrangian  $L_c$  on  $\Delta_Q \subset TQ$  by  $L_c(q, v) = L(q, \text{hor } v)$  for  $(q, v) \in TQ$ . By substituting equation (5.3) into the Lagrangian  $L$  on  $TQ$ , we can obtain, in local coordinates,

$$L_c(r^\alpha, s^a, u^\alpha) = L(r^\alpha, s^a, u^\alpha, -A_\alpha^a(r, s)u^\alpha)$$

and define

$$\begin{aligned} \tilde{p}_\alpha &= \frac{\partial L_c}{\partial u^\alpha} \\ &= \frac{\partial L}{\partial u^\alpha} - A_\alpha^a \frac{\partial L}{\partial w^a}, \end{aligned} \quad (5.4)$$

which is equivalent with

$$\tilde{p}_\alpha = p_\alpha - A_\alpha^a p_a$$

in view of equation (5.2). By computations, it follows that

$$\begin{aligned} \frac{\partial L_c}{\partial r^\alpha} &= \frac{\partial L}{\partial r^\alpha} - \frac{\partial L}{\partial w^a} \left( \frac{\partial A_\beta^a}{\partial r^\alpha} u^\beta \right) \\ &= \frac{\partial L}{\partial r^\alpha} - p_a \left( \frac{\partial A_\beta^a}{\partial r^\alpha} u^\beta \right), \\ \frac{\partial L_c}{\partial s^a} &= \frac{\partial L}{\partial s^a} - \frac{\partial L}{\partial w^a} \left( \frac{\partial A_\beta^a}{\partial s^a} u^\beta \right) \\ &= \frac{\partial L}{\partial s^a} - p_a \left( \frac{\partial A_\beta^a}{\partial s^a} u^\beta \right). \end{aligned} \quad (5.5)$$

Substituting equations (5.4) and (5.5) into equation (5.1) together with the kinematic constraints in equation (5.3), we can eventually obtain the *implicit Lagrange-d'Alembert equations of motion* for the constrained Lagrangian  $L_c$  as

$$\begin{aligned} \dot{r}^\alpha &= u^\alpha, \\ \dot{s}^a &= -A_\beta^a u^\beta, \\ \dot{\tilde{p}}_\alpha &= \frac{\partial L_c}{\partial r^\alpha} - A_\alpha^b \frac{\partial L_c}{\partial s^b} - p_b K_{\alpha\beta}^b u^\beta, \\ \tilde{p}_\alpha &= \frac{\partial L_c}{\partial u^\alpha}, \end{aligned} \quad (5.6)$$

where

$$K_{\alpha\beta}^b = \frac{\partial A_\alpha^b}{\partial r^\beta} - \frac{\partial A_\beta^b}{\partial r^\alpha} + A_\alpha^a \frac{\partial A_\beta^b}{\partial s^a} - A_\beta^a \frac{\partial A_\alpha^b}{\partial s^a}$$

is the *curvature* of the Ehresmann connection  $A$ .

**Remarks.** The curvature of the Ehresmann connection  $A$  is the vertical valued two-form on  $Q$  defined by its action on two vector fields  $Y$  and  $Z$  on  $Q$  such that

$$K(Y, Z) = -A([\text{hor}Y, \text{hor}Z]),$$

where the bracket on the right-hand side is the Jacobi-Lie bracket of vector fields.

Recall the identity for the exterior derivative  $\mathbf{d}\alpha$  of a one-form  $\alpha$  on a manifold  $Q$  acting on two vector fields  $Y, Z$  is

$$(\mathbf{d}\alpha)[Y, Z] = Y[(\alpha(Z))] - Z[\alpha(Y)] - \alpha([Y, Z]).$$

This identity indicates that one can evaluate, in coordinates, the curvature by writing the connection as a one-form  $\omega^b$  by computing its exterior derivative (component by component) and restricting the result to horizontal vectors, that is, to the constraint distribution. Then, one has

$$K(Y, Z) = d\omega^b(\text{hor}Y, \text{hor}Z) \frac{\partial}{\partial s^b},$$

where the local expression for the curvature is denoted by

$$K^b(Y, Z) = K_{\alpha\beta}^b Y^\alpha Z^\beta.$$

**Proposition 5.1.** *Let  $\mathbf{d}\omega^b$  be the exterior derivative of  $\omega^b$ . The **Lagrange-d'Alembert-Pontryagin principle** for the constrained Lagrangian  $L_c$  may be written as*

$$\delta \left\{ L_c(r^\alpha, s^a, u^\alpha) + \tilde{p}_\alpha (\dot{r}^\alpha - u^\alpha) \right\} = p_b \mathbf{d}\omega^b(v, \delta r), \quad (5.7)$$

which provides the equations of motion in equation (5.6).

**Proof.** The left-hand side of equation (5.7) is

$$\begin{aligned} \delta \left\{ L_c(r^\alpha, s^a, u^\alpha) + \tilde{p}_\alpha (\dot{r}^\alpha - u^\alpha) \right\} &= \left\{ \frac{\partial L_c}{\partial r^\alpha} - A_\alpha^a \frac{\partial L_c}{\partial s^a} - \dot{\tilde{p}}_\alpha \right\} \delta r^\alpha \\ &\quad + \left( \frac{\partial L_c}{\partial u^\alpha} - \tilde{p}_\alpha \right) \delta u^\alpha + (\dot{r}^\alpha - u^\alpha) \delta \tilde{p}_\alpha, \end{aligned}$$

while the direct computation using properties of differential forms shows that

$$\mathbf{d}\omega^b(v, \cdot) = K_{\alpha\beta}^b u^\beta dr^\alpha.$$

It follows that

$$\begin{aligned} \left\{ \frac{\partial L_c}{\partial r^\alpha} - A_\alpha^a \frac{\partial L_c}{\partial s^a} - \dot{\tilde{p}}_\alpha \right\} \delta r^\alpha + \left( \frac{\partial L_c}{\partial u^\alpha} - \tilde{p}_\alpha \right) \delta u^\alpha + (\dot{r}^\alpha - u^\alpha) \delta \tilde{p}_\alpha \\ = (p_b K_{\alpha\beta}^b u^\beta) \delta r^\alpha \end{aligned}$$

for all  $\delta r^\alpha$ ,  $\delta u^\alpha$  and  $\delta \tilde{p}_\alpha$ . Combining with the kinematic constraints, we obtain equation (5.6).  $\blacksquare$

**Remarks.** The constrained energy  $E_c$  may be defined by

$$E_c(r^\alpha, s^a, u^\alpha, \tilde{p}_\alpha) = \tilde{p}_\alpha u^\alpha - L_c(r^\alpha, s^a, u^\alpha)$$

and then, the Lagrange-d'Alembert-Pontryagin principle in equation (5.7) can be restated as the equivalent form

$$\delta \left\{ \tilde{p}_\alpha \dot{r}^\alpha - E_c(r^\alpha, s^a, u^\alpha, \tilde{p}_\alpha) \right\} = p_b \mathbf{d}\omega^b(v, \delta r).$$

**Restriction of a Dirac Structure.** Before going into the construction of a constrained Dirac structure, we briefly discuss the restriction of a Dirac structure on a manifold to its submanifold.

Let  $M$  be a manifold and let  $N$  be a submanifold of  $M$ . Recall the definition of a Dirac structure on a manifold  $M$ ; that is, a subbundle  $D_M \subset TM \oplus T^*M$  is called a Dirac structure if for every fiber  $D_M(x) \subset T_x M \times T_x^* M$ ,  $x \in M$ , one has  $D_M(x) = D_M^\perp(x)$ , where

$$D_M^\perp(x) = \{(v_x, \alpha_x) \in T_x M \times T_x^* M \mid \langle \bar{\alpha}_x, v_x \rangle + \langle \alpha_x, \bar{v}_x \rangle = 0, \text{ for all } (\bar{v}_x, \bar{\alpha}_x) \in D_M(x)\}.$$

Now, let  $D_M$  be a Dirac structure on  $M$ . Define a map  $\sigma : TN \times T^*M \rightarrow TN \times T^*N$  as follows: For each  $y \in N$ , let  $\sigma(y) : T_yN \times T_y^*M \rightarrow T_yN \times T_y^*N$  be defined by

$$\sigma(y)(v_y, \alpha_y) = (v_y, \alpha_y|_{T_yN}),$$

where  $v_y \in T_yN$ ,  $\alpha_y \in T_y^*M$ , and  $\alpha_y|_{T_yN}$  denotes the restriction of the covector  $\alpha_y$  to the subspace  $T_yN$ . This map  $\sigma$  is a vector bundle projection.

Assume that  $D_M(y) \cap (T_yN \times T_y^*M)$  has constant dimension for each  $y \in N$ , namely, it is a vector subbundle of  $TN \times T^*M$ . Define the subbundle  $D_N \subset TN \oplus T^*N$  by, for each  $y \in N$ ,

$$D_N(y) = \sigma(y) (D_M(y) \cap (T_yN \times T_y^*M)).$$

**Proposition 5.2.** *The subbundle  $D_N \subset TN \oplus T^*N$  is a Dirac structure on  $N$ .*

**Proof.** Let us check  $D_N(y) = D_N^\perp(y)$  for each  $y \in N$ . It is obvious that  $D_N(y) \subset D_N^\perp(y)$  for  $y \in N$ . Then, let us check  $D_N^\perp(y) \subset D_N(y)$  for  $y \in N$ . Suppose that  $(w_y, \beta_y) \in D_N^\perp(y) \subset T_yN \times T_y^*N$ . So, we have  $\langle \alpha_y, w_y \rangle + \langle \beta_y, v_y \rangle = 0$  for all  $(v_y, \alpha_y) \in D_N(y)$ . Then, there exists  $\alpha'_y \in T_y^*M$  such that  $(v_y, \alpha_y) = \sigma(y)(v_y, \alpha'_y) \in D_N(y)$ , where  $(v_y, \alpha'_y) \in D_M(y)$  and  $\alpha'_y|_{T_yN} = \alpha_y$ . Therefore, we have  $\langle \alpha_y, w_y \rangle + \langle \beta_y, v_y \rangle = \langle \alpha'_y, w_y \rangle + \langle \beta'_y, v_y \rangle = 0$  for all  $(v_y, \alpha'_y) \in D_M(y)$ , where  $\beta'_y$  is an arbitrary extension of  $\beta_y$  to  $T_yM$  and  $v_y \in T_yN$ . Hence, one obtains

$$\begin{aligned} (w_y, \beta'_y) \in \left( D_M(y) \cap (T_yN \times T_y^*M) \right)^\perp &= D_M^\perp(y) + (T_yN \times T_y^*M)^\perp \\ &= D_M(y) + (\{0\} \times T_yN^\circ). \end{aligned}$$

Then, there exists  $\gamma'_y \in T_yN^\circ \subset T_y^*M$  such that  $(w_y, \beta'_y + \gamma'_y) \in D_M(y)$ . Noting  $\sigma(y)(w_y, \beta'_y + \gamma'_y) = (w_y, (\beta'_y + \gamma'_y)|_{T_yN}) = (w_y, \beta_y) \in D_N(y)$ , it follows  $D_N^\perp(y) \subset D_N(y)$  for  $y \in N$ . ■

We call the Dirac structure  $D_N$  the *restriction of a Dirac structure*  $D_M$  to  $N$ .

**Remarks.** Proposition 5.2 was originally developed by Courant [1990a]; we follow the exposition in Blankenstein and Ratiu [2004].

**Proposition 5.3.** *Let  $D_N$  be constructed as in Proposition 5.2 and let  $\iota : N \rightarrow M$  denote the inclusion map. Then,  $(w, \beta)$  is a local section of  $D_N$  if and only if there exists a local section  $(v, \alpha)$  of  $D_M$  such that  $T\iota \circ w = v \circ \iota$  and  $\iota^* \alpha = \beta$ . In other words, a Dirac structure  $D_N$  on  $N$  is represented by, for each  $y \in N$ ,*

$$\begin{aligned} D_N(y) = \{ (w_y, \beta_y) \in T_yN \times T_y^*N \mid &\text{there is a } (v_{\iota(y)}, \alpha_{\iota(y)}) \in D_M(\iota(y)) \\ &\text{such that } T\iota \circ w = v \circ \iota \text{ and } \iota^* \alpha = \beta \}. \end{aligned}$$

**Proof.** As demonstrated in Part I, given a distribution  $\Delta_M$  on  $M$  and a two-form  $\Omega$  on  $M$ , there exists a Dirac structure  $D_M \subset TM \oplus T^*M$  on a manifold  $M$ , whose fiber is defined by, for each  $x \in M$ ,

$$\begin{aligned} D_M(x) = \{ (v_x, \alpha_x) \in T_xM \times T_x^*M \mid &v_x \in \Delta_M(x) \text{ and} \\ &\alpha_x(v'_x) = \Omega_{\Delta_M}(x)(v_x, v'_x) \text{ for all } v'_x \in \Delta_M(x) \}, \end{aligned}$$

where  $\Omega_{\Delta_M} = \Omega|_{\Delta_M \times \Delta_M}$ .

Define a distribution  $\Delta_N$  on  $N$  by restricting  $\Delta_M$  to  $N$  such that

$$\Delta_N = TN \cap \Delta_M,$$

where we assume  $\Delta_N$  to be a regular distribution. By construction, we can define the restricted Dirac structure  $D_N$  on  $N$  such that, for each  $y \in N$ ,

$$\begin{aligned} D_N(y) &= \sigma(y) \left( D_M(\iota(y)) \cap (T_y N \times T_{\iota(y)}^* M) \right) \\ &= \{(w_y, \beta_y) \in T_y N \times T_y^* N \mid w_y \in \Delta_N(y) \text{ and} \\ &\quad \beta_y(w'_y) = \Omega_{\Delta_N}(y)(w_y, w'_y) \text{ for all } w'_y \in \Delta_N(y)\}, \end{aligned}$$

where  $\beta = \iota^* \alpha$ ,  $\Omega_{\Delta_N} = \iota^* \Omega_{\Delta_M}$ ,  $T\iota \cdot w = v \circ \iota$  and  $T\iota \cdot w' = v' \circ \iota$ . ■

**Constrained Dirac Structures.** Let  $L$  be a Lagrangian on  $TQ$  and  $\Delta_Q \subset TQ$  a constraint distribution. Let  $\pi_Q : T^*Q \rightarrow Q$  be the cotangent projection. Recall that an induced Dirac structure  $D_{\Delta_Q}$  on  $T^*Q$  is defined such that, for each  $z \in T^*Q$ ,

$$\begin{aligned} D_{\Delta_Q}(z) &= \{(v_z, \alpha_z) \in T_z T^*Q \times T_z^* T^*Q \mid v_z \in \Delta_{T^*Q}(z) \text{ and} \\ &\quad \alpha_z(v'_z) = \Omega_{\Delta_Q}(z)(v_z, v'_z) \text{ for all } v'_z \in \Delta_{T^*Q}(z)\}, \end{aligned}$$

where  $\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q)$  and  $\Omega_{\Delta_Q}$  is the restriction of the canonical symplectic structure  $\Omega$  on  $T^*Q$  to  $\Delta_{T^*Q}$ .

Our initial goal is to define a Dirac structure on the constraint momentum space  $P = \mathbb{F}L(\Delta_Q) \subset T^*Q$ . To do this purpose, we define a regular distribution on  $P$  by restricting  $\Delta_{T^*Q}$  to  $P$  such that

$$\Delta_P = TP \cap \Delta_{T^*Q}.$$

Letting  $\iota : P \rightarrow T^*Q$  be the inclusion, we can define a **constrained Dirac structure** on  $P = \mathbb{F}L(\Delta_Q) \subset T^*Q$  by the subbundle  $D_P \subset TP \oplus T^*P$ , whose fiber is given such that, for each  $y \in P$ ,

$$\begin{aligned} D_P(y) &= \{(w_y, \beta_y) \in T_y P \times T_y^* P \mid w_y \in \Delta_P(y) \text{ and} \\ &\quad \beta_y(w'_y) = \Omega_{\Delta_P}(y)(w_y, w'_y) \text{ for all } w'_y \in \Delta_P(y)\}, \end{aligned}$$

where  $\beta = \iota^* \alpha$ ,  $\Omega_{\Delta_P} = \iota^* \Omega_{\Delta_Q}$ ,  $T\iota \cdot w = v \circ \iota$  and  $T\iota \cdot w' = v' \circ \iota$ . It goes without saying that  $D_P$  is the restriction of the induced Dirac structure  $D_{\Delta_Q}$ .

It is obvious that the constrained Dirac structure  $D_P$  may be restated as follows: for each  $y \in P$ ,

$$\begin{aligned} D_P(y) &= \{(v_y, \alpha_y) \in T_y P \times T_y^* P \mid v_y \in \Delta_P(y) \text{ and} \\ &\quad \alpha_y - \Omega_P^\flat(y) \cdot v_y \in \Delta_P^\circ(y)\}, \end{aligned}$$

where  $\Delta_P^\circ$  is the annihilator of  $\Delta_P$  and  $\Omega_P^\flat : TP \rightarrow T^*P$  is the bundle map associated with the skew-symmetric bilinear form  $\Omega_P = \Omega|_{TP \times TP}$ .

We can also construct the constrained Dirac structure by employing the canonical Poisson structure. Recall that the canonical Poisson tensor  $B : T^*T^*Q \times T^*T^*Q \rightarrow \mathbb{R}$  is defined by, for any smooth function  $F, G$  on  $T^*Q$ ,

$$\begin{aligned} B(\mathbf{d}F, \mathbf{d}G) &= \Omega(X_F, X_G) \\ &= \langle \mathbf{d}F, B^\sharp \mathbf{d}G \rangle \\ &= \{F, G\}, \end{aligned}$$

where  $X_F$  and  $X_G$  are vector fields on  $T^*Q$ ,  $B^\sharp : T^*T^*Q \rightarrow TT^*Q$  is the associated bundle map, and  $\{, \}$  is the Poisson bracket. Further,  $B_P^\sharp : T^*P \rightarrow TP$  is the associated bundle map

of the contravariant antisymmetric two-tensor  $B_P = B|_{T^*P \times T^*P}$ . Let  $\Delta_P^* = \Omega_P^b(\Delta_P) \subset T^*P$  be the codistribution and  $(\Delta_P^*)^\circ$  be its annihilator, and then the constrained Dirac structure on  $P$  is defined by, for each  $y \in P$ ,

$$D_P(y) = \{(v_y, \alpha_y) \in T_y P \times T_y^* P \mid \alpha_y \in \Delta_P^*(y) \text{ and } v_y - B_P^\sharp(y) \cdot \alpha_y \in (\Delta_P^*)^\circ(y)\}. \quad (5.8)$$

**A Local Representation Using an Ehresmann Connection.** Let us construct a local representation of a constrained Dirac structure using an Ehresmann connection to represent the set  $D_P$  given in (5.8). Needless to say,  $T^*Q$  is naturally equipped with the canonical Poisson bracket  $\{, \}$  or the Poisson structure  $B : T^*T^*Q \times T^*T^*Q \rightarrow \mathbb{R}$  such that, for each  $(q, p) \in T^*Q$ ,

$$\begin{aligned} \{F, G\}(q, p) &= B(q, p)(\mathbf{d}F(q, p), \mathbf{d}G(q, p)) \\ &= \langle \mathbf{d}F(q, p), B^\sharp(q, p) \mathbf{d}G(q, p) \rangle. \end{aligned}$$

In the above,  $F, G$  are smooth functions on  $T^*Q$  and the canonical Poisson bracket is represented, in local coordinates  $(q^i, p_i)$  for  $T^*Q$ , by

$$\{F, G\}(q, p) := \left( \frac{\partial F^T}{\partial q^i}, \frac{\partial F^T}{\partial p_i} \right) B^\sharp(q, p) \begin{pmatrix} \frac{\partial G}{\partial q^i} \\ \frac{\partial G}{\partial p_i} \end{pmatrix},$$

where

$$B^\sharp(q, p) = \begin{pmatrix} \{q^i, q^j\} & \{q^i, p_j\} \\ \{p_i, q^j\} & \{p_i, p_j\} \end{pmatrix} = \begin{pmatrix} 0 & \delta_j^i \\ -\delta_i^j & 0 \end{pmatrix}.$$

As previously illustrated, we choose an Ehresmann connection  $A$  such that  $\mathcal{H}_q = \Delta_Q(q)$ , where the constraint distribution  $\Delta_Q$  is spanned by a set of  $m$  independent one-forms, which is given, in local coordinates  $q^i = (r^\alpha, s^a)$  for  $Q$ , by

$$\omega^\alpha = ds^a + A_\alpha^a(r, s) dr^\alpha.$$

Define the new coordinates  $(q^i, \tilde{p}_i) = (r^\alpha, s^a, \tilde{p}_\alpha, \tilde{p}_a)$  for  $T^*Q$ , as in [van der Schaft and Maschke \[1994\]](#) and [Koon and Marsden \[1998\]](#), by

$$\tilde{p}_\alpha = p_\alpha - A_\alpha^a p_a,$$

with some choice of complementary coordinates  $\tilde{p}_a$ . We then employ the induced coordinates  $(q^i, \tilde{p}_\alpha) = (r^\alpha, s^a, \tilde{p}_\alpha)$  for  $P$ .

In this context, the bundle map  $B_P^\sharp : T^*P \rightarrow TP$  associated with  $B_P = B|_{T^*P \times T^*P}$  can be constructed, using local coordinates  $(q^i, \tilde{p}_\alpha)$  for  $P$ , by computing  $\{q^i, q^j\}$ ,  $\{q^i, \tilde{p}_\alpha\}$ ,  $\{\tilde{p}_\alpha, \tilde{p}_\beta\}$ ; one finds that

$$\{q^i, q^j\} = 0, \{r^\beta, \tilde{p}_\alpha\} = \delta_\alpha^\beta, \{s^b, \tilde{p}_\alpha\} = -A_\alpha^b, \{\tilde{p}_\alpha, \tilde{p}_\beta\} = -K_{\alpha\beta}^b p_b.$$

Hence, it follows that

$$B_P^\sharp(q; \tilde{p}) = \begin{pmatrix} 0 & 0 & \delta_\beta^\alpha \\ 0 & 0 & -A_\beta^a \\ -\delta_\alpha^\beta & (A_\alpha^b)^T & -p_b K_{\alpha\beta}^b \end{pmatrix}, \quad (5.9)$$



where  $\delta_\beta^\alpha$  is Kronecker's delta and  $B_P^\sharp(q, \tilde{p})$  is the  $(2n - m) \times (2n - m)$  truncated matrix representation of the bundle map  $B_P^\sharp : T^*P \rightarrow TP$ , for each  $(q^i, \tilde{p}_\alpha) \in P$ . This bundle map defines the bracket  $\{, \}_P$  on the constrained submanifold  $P$  such that

$$\{F_P, G_P\}_P(q, \tilde{p}) := \left( \frac{\partial F_P^T}{\partial q^i}, \frac{\partial F_P^T}{\partial \tilde{p}_\alpha} \right) B_P^\sharp(q, \tilde{p}) \begin{pmatrix} \frac{\partial G_P}{\partial q^i} \\ \frac{\partial G_P}{\partial \tilde{p}_\alpha} \end{pmatrix} \quad (5.10)$$

for smooth functions  $F_P, G_P$  on  $P$ .

Thus, we can construct the constrained Dirac structure  $D_P$  in the form of (5.8).

**Remarks.** In equation (5.9), notice that the curvature  $K_{\alpha\beta}^b$  measures the failure of the constraint distribution to be an integrable bundle and then, after restricting all terms to  $P$ , the term  $-p_b K_{\alpha\beta}^b$  should be understood as  $-(p_b)_P K_{\alpha\beta}^b$ . However, we write it as  $-p_b K_{\alpha\beta}^b$  for simplicity. Needless to say,  $p_b = \partial L / \partial w^b$  holds and the Poisson bracket  $\{, \}_P$  in equation (5.10) *does not satisfy the Jacobi identity* when the distribution is nonholonomic.

**The Constrained Partial Vector Field.** Let  $\Delta_Q \subset TQ$  be a constraint distribution and  $P = \mathbb{F}L(\Delta_Q) \subset T^*Q$ . Let  $V$  be the vector subbundle of  $T_P(T^*Q)$  defined, for each  $p \in T_q^*Q$ , by

$$V_{(q,p)} = \{ \text{vert}(\eta, p) \mid \eta \in \Delta_Q^\circ(q) \}.$$

In the above,  $T_P(T^*Q)$  is the restriction of the tangent bundle of  $T(T^*Q)$  to the constraint momentum space  $P$  and  $\text{vert}(\eta, p)$  is the vertical lift of  $\eta \in T_q^*Q$  with respect to  $p \in T_q^*Q$ , which is described in coordinates as

$$\text{vert}(\eta, p) = (q, p, 0, \eta).$$

Marle [1998] shows that the vector subbundle  $T_P(T^*Q)$  is a direct sum of the vector subbundles  $TP$  and  $V$ :

$$T_P(T^*Q) = TP \oplus V.$$

In this context, the restriction of a partial vector field  $X : TQ \oplus T^*Q \rightarrow TT^*Q$  on  $T^*Q$  to the subbundle  $\Delta_Q \oplus P \subset TQ \oplus T^*Q$ , that is,  $X|_{\Delta_Q \oplus P}$  splits into a sum

$$X|_{\Delta_Q \oplus P} = X_P + X_V, \quad (5.11)$$

where  $X_P : \Delta_Q \oplus P \rightarrow TP$  is the **constrained partial vector field** on the constraint momentum space  $P$  that is tangent to  $P$  and  $X_V$  is a smooth section of the subbundle  $V$ , whose negative is called the **constraint force field**.

In conjunction with the above construction, refer also to Bloch [2003], §5.8.

**Implicit Constrained Lagrangian Systems.** We now develop the notion of an implicit constrained Lagrangian system in the context of the constrained Dirac structure.

Let  $L : TQ \rightarrow \mathbb{R}$  be a Lagrangian and let  $\Delta_Q \subset TQ$  be a constraint distribution. The constraint momentum space  $P \subset T^*Q$  is defined as  $P = \mathbb{F}L(\Delta_Q)$ . Let  $L_c$  be the constrained Lagrangian on  $\Delta_Q$  defined by  $L_c = L|_{\Delta_Q}$ . Let  $D_P$  be the constrained Dirac structure defined by the restriction of the induced Dirac structure  $D_{\Delta_Q}$  on  $T^*Q$  to  $P$ . Denote by  $X : TQ \oplus T^*Q \rightarrow TT^*Q$  a partial vector field on  $T^*Q$ , and denote by  $X_P : \Delta_Q \oplus P \rightarrow TP$  the constrained partial vector field on  $P$  given in equation (5.11).

**Definition 5.4.** Recall that the Dirac differential of  $L$  can be regarded as a function on  $T_{\mathbb{F}L(u)}(T^*Q)$  for each  $u \in \Delta_Q(q)$  such that  $\mathfrak{D}L(u) : T_{\mathbb{F}L(u)}(T^*Q) \rightarrow \mathbb{R}$ , and let us define the **Dirac differential of the constrained Lagrangian** by, for each  $u \in \Delta_Q(q)$ ,

$$\mathfrak{D}L_c(u) = \mathfrak{D}L(u)|_{TP},$$

where  $\mathfrak{D}L_c(u) : T_{\mathbb{F}L_c(u)}P \rightarrow \mathbb{R}$ .

Then, an **implicit constrained Lagrangian system** is a triple  $(L_c, \Delta_Q, X_P)$ , which satisfies the condition, for each  $u \in \Delta_Q(q)$ ,

$$(X_P(q, u, \tilde{p}), \mathfrak{D}L_c(q, u)) \in D_P(q, \tilde{p}), \quad (5.12)$$

where  $\tilde{p}_q = \mathbb{F}L_c(q, u)$  is the partial Legendre transformation.

**Remarks.** As shown below, the coordinate expression for  $\mathfrak{D}L_c$ , in fact, depends only on derivatives of  $L_c$ .

**Coordinate Representation.** Suppose that the constraint distribution  $\Delta_Q$  is denoted by, in coordinates  $q = (r^\alpha, s^a)$  for  $Q$ , by

$$\omega^\alpha(q) = ds^a + A_\alpha^a(r, s) dr^\alpha, \quad a = 1, \dots, m; \quad \alpha = 1, \dots, n - m,$$

where  $\omega^\alpha$  indicate  $m$ -independent one-forms that consist of the basis of the annihilator of  $\Delta_Q$  and an Ehresmann connection  $A$  is chosen such that  $\mathcal{H}_q = \Delta_Q(q)$ . The constrained Lagrangian  $L_c = L|\Delta_Q$  is given, in coordinates, by

$$L_c(r^\alpha, s^a, u^\alpha) = L(r^\alpha, s^a, u^\alpha, -A_\alpha^a(r, s)u^\alpha),$$

and it follows that

$$dL_c = \left( r^\alpha, s^a, u^\alpha, \frac{\partial L_c}{\partial r^\alpha}, \frac{\partial L_c}{\partial s^a}, \frac{\partial L_c}{\partial u^\alpha} \right).$$

Hence, we obtain the Dirac differential of  $L_c$  as

$$\mathfrak{D}L_c = \left( r^\alpha, s^a, \frac{\partial L_c}{\partial u^\alpha}, -\frac{\partial L_c}{\partial r^\alpha}, -\frac{\partial L_c}{\partial s^a}, u^\alpha \right) \quad (5.13)$$

and with

$$\tilde{p}_\alpha = \frac{\partial L_c}{\partial u^\alpha}.$$

Let  $X_P : \Delta_Q \oplus P \rightarrow TP$  be the constrained partial vector field, which is denoted, by using coordinates  $(q, u, \tilde{p}) = (r^\alpha, s^a, u^\alpha, \tilde{p}_\alpha)$  for  $\Delta_Q \oplus P$ , such that

$$X_P(r^\alpha, s^a, u^\alpha, \tilde{p}_\alpha) = (\dot{r}^\alpha, \dot{s}^a, \dot{\tilde{p}}_\alpha). \quad (5.14)$$

Recall that the skew-symmetric bundle map  $B_P^\sharp : T^*P \rightarrow TP$  can be constructed, by using the Ehresmann connection associated with the constraints, as in equation (5.9), and recall also that the constrained Dirac structure on  $P$  can be defined such that, for each  $y = (q, \tilde{p}) \in P$ ,

$$D_P(y) = \{(w_y, \alpha_y) \in T_yP \times T_y^*P \mid \alpha_y \in \Delta_P^*(y), \\ w_y - B_P^\sharp(y) \alpha_y \in (\Delta_P^*)^\circ(y)\}. \quad (5.15)$$

Since an implicit constrained Lagrangian system is a triple  $(L_c, \Delta_Q, X_P)$  that satisfies the condition, for each  $u \in \Delta_Q(q)$ ,

$$(X_P(q, u, \tilde{p}), \mathfrak{D}L_c(q, u)) \in D_P(q, \tilde{p}),$$

where  $\tilde{p}_q = \mathbb{F}L_c(q, u) \in P$ , it reads from equations (5.9) and (5.13)–(5.15) that the coordinate expression of the implicit constrained Lagrangian system  $(L_c, \Delta_Q, X_P)$  may be given, using coordinates  $(q, u, \tilde{p}) = (r^\alpha, s^a, u^\alpha, \tilde{p}_\alpha)$  for  $\Delta_Q \oplus P$ , by

$$\begin{pmatrix} \dot{r}^\alpha \\ \dot{s}^a \\ \dot{\tilde{p}}_\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & \delta_\beta^\alpha \\ 0 & 0 & -A_\beta^a \\ -\delta_\alpha^\beta & (A_\alpha^b)^T & -p_b K_{\alpha\beta}^b \end{pmatrix} \begin{pmatrix} -\frac{\partial L_c}{\partial r^\beta} \\ -\frac{\partial L_c}{\partial s^b} \\ u^\beta \end{pmatrix} \quad (5.16)$$

and with the partial Legendre transform  $\tilde{p}_\alpha = \partial L_c / \partial u^\alpha$ .

Needless to say, equation (5.16) is equivalent with the implicit Lagrange-d'Alembert equation in (5.6).

**Definition 5.5.** A *solution curve* of the implicit constrained Lagrangian system  $(L_c, \Delta_Q, X_P)$  is a curve  $(q(t), u(t), \tilde{p}(t))$ ,  $t_1 \leq t \leq t_2$  such that  $\tilde{p}_{q(t)}(t) = \mathbb{F}L_c(q(t), u(t))$  and  $(q(t), u(t), \tilde{p}(t))$  is an integral curve in  $P$  of  $X_P$ , where  $u(t) \in \Delta_Q(q(t))$ .

We can summarize the results so far in the following theorem.

**Theorem 5.6.** Consider a Lagrangian  $L$  (possibly degenerate) on  $TQ$  and with a constraint distribution  $\Delta_Q$  on  $Q$ . Let  $P = \mathbb{F}L(\Delta_Q)$  be the constraint momentum space. Let  $L_c = L|_{\Delta_Q}$  be the constrained Lagrangian and  $D_P$  be the constrained Dirac structure on  $P$ . Let  $X_P : \Delta_Q \oplus P \rightarrow TP$  be the constrained partial vector field on  $P$ . Denote by  $\tilde{x}(t) = (q(t), u(t), \tilde{p}(t))$ ,  $t_1 \leq t \leq t_2$ , a curve in  $\Delta_Q \oplus P$ . Then, the following statements are equivalent:

- (a)  $\tilde{x}(t)$  satisfies the implicit Lagrange-d'Alembert equations in equation (5.6);
- (b)  $\tilde{x}(t)$  is a solution curve of the Lagrange-d'Alembert-Pontryagin principle in equation (5.7);
- (c)  $\tilde{x}(t)$  satisfies the condition of an implicit constrained Lagrangian system  $(L_c, \Delta_Q, X_P)$  in equation (5.12).

## 6 Examples

In this section, we demonstrate the implicit constrained Lagrangian system together with two examples. Namely, we illustrate the same examples of a vertical rolling disk on a plane and an L-C circuit as in Part I, for this purpose.

**Example: The Vertical Rolling Disk.** Consider a vertical rolling disk on the  $xy$ -plane. Recall the configuration space of the system is denoted by  $Q = \mathbb{R}^2 \times S^1 \times S^1$ , whose coordinates are given by  $q = (x, y, \theta, \varphi)$ , where  $x, y$  indicate the position of the contact point of the disk,  $\theta$  the rotation angle of the disk and  $\varphi$  the orientation of the disk. Recall the Lagrangian is given by

$$L(x, y, \theta, \varphi, v_x, v_y, v_\theta, v_\varphi) = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}Iv_\theta^2 + \frac{1}{2}Jv_\varphi^2.$$

In the above,  $m$  indicates the mass, and  $I$  and  $J$  the moment of inertia. Recall also the constraints are given by the constraint distribution  $\Delta_Q \subset TQ$  such that, for each  $q \in Q$ ,

$$\Delta_Q(q) = \{v_q \in T_q Q \mid \langle \omega^a(q), v_q \rangle = 0, a = 1, 2\},$$

where  $v_q = (v_x, v_y, v_\theta, v_\varphi)$  and the one-forms  $\omega^a$  are given by

$$\begin{aligned}\omega^1 &= dx - R(\cos \varphi) d\theta, \\ \omega^2 &= dy - R(\sin \varphi) d\theta,\end{aligned}$$

where  $R$  denotes radius of the disk.

Let us choose a bundle structure  $\pi : Q \rightarrow \mathcal{R}$  such that the base  $\mathcal{R}$  is to be  $S^1 \times S^1$  parameterized by  $\theta$  and  $\varphi$  together with the projection to  $\mathcal{R}$ , that is,  $(r^1, r^2, s^1, s^2) = (\theta, \varphi, x, y) \mapsto (r^1, r^2) = (\theta, \varphi)$ . Then, the Ehresmann connection can be constructed by

$$A = \omega^a \frac{\partial}{\partial s^a}, \quad \omega^a(q) = ds^a + A_\alpha^a(r, s) dr^\alpha.$$

The components of the Ehresmann connection are given by

$$A_1^1 = -R(\cos \varphi), \quad A_1^2 = -R(\sin \varphi)$$

and the remaining components are zero.

As in equation (5.9), the bundle map  $B_P^\sharp : T^*P \rightarrow TP$  is locally denoted by

$$B_P^\sharp(r, s, \tilde{p}) = \left( \begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & R(\cos \varphi) & 0 \\ 0 & 0 & 0 & 0 & R(\sin \varphi) & 0 \\ \hline -1 & 0 & -R(\cos \varphi) & -R(\sin \varphi) & 0 & p_b K_{12}^b \\ 0 & -1 & 0 & 0 & -p_b K_{21}^b & 0 \end{array} \right),$$

and the components of the curvature  $K$  is given by

$$K_{12}^1 = -K_{21}^1 = R \sin \varphi, \quad K_{12}^2 = -K_{21}^2 = -R \cos \varphi.$$

Thus, the constrained Dirac structure  $D_P$  on  $P$  can be defined as in equation (5.15).

Meanwhile, the constrained Lagrangian  $L_c(r^\alpha, s^a, u^\alpha) = L(r^\alpha, s^a, u^\alpha, -A_\alpha^a(r, s)u^\alpha)$  is given by

$$L_c(\theta, \varphi, x, y, v_\theta, v_\varphi) = \frac{1}{2} (mR^2 + I) v_\theta^2 + \frac{1}{2} J v_\varphi^2,$$

where  $(r^\alpha, s^a, u^\alpha) = (\theta, \varphi, x, y, v_\theta, v_\varphi)$ . Then, the differential of  $L_c$  is locally expressed by

$$\begin{aligned}dL_c(\theta, \varphi, x, y, v_\theta, v_\varphi) &= \left( \frac{\partial L_c}{\partial \theta}, \frac{\partial L_c}{\partial \varphi}, \frac{\partial L_c}{\partial x}, \frac{\partial L_c}{\partial y}, \frac{\partial L_c}{\partial v_\theta}, \frac{\partial L_c}{\partial v_\varphi} \right) \\ &= (0, 0, 0, 0, (mR^2 + I) v_\theta, J v_\varphi)\end{aligned}$$

and hence the Dirac differential of  $L_c$  is locally denoted by

$$\begin{aligned}\mathfrak{D}L_c(\theta, \varphi, x, y, v_\theta, v_\varphi) &= \left( -\frac{\partial L_c}{\partial \theta}, -\frac{\partial L_c}{\partial \varphi}, -\frac{\partial L_c}{\partial x}, -\frac{\partial L_c}{\partial y}, v_\theta, v_\varphi \right) \\ &= (0, 0, 0, 0, v_\theta, v_\varphi),\end{aligned}$$

and with the partial Legendre transform

$$\begin{aligned}\tilde{p}_\theta &= \frac{\partial L_c}{\partial v_\theta} = (mR^2 + I) v_\theta, \\ \tilde{p}_\varphi &= \frac{\partial L_c}{\partial v_\varphi} = Jv_\varphi.\end{aligned}$$

Since  $(r^\alpha, s^\alpha, u^\alpha, \tilde{p}_\alpha) = (\theta, \varphi, x, y, v_\theta, v_\varphi, \tilde{p}_\theta, \tilde{p}_\varphi)$  are local coordinates for  $\Delta_Q \oplus P$ , the constrained partial vector field  $X_P : \Delta_Q \oplus P \rightarrow TP$  is locally denoted by

$$X_P(\theta, \varphi, x, y, v_\theta, v_\varphi, \tilde{p}_\theta, \tilde{p}_\varphi) = (\dot{\theta}, \dot{\varphi}, \dot{x}, \dot{y}, \dot{\tilde{p}}_\theta, \dot{\tilde{p}}_\varphi).$$

Hence, we obtain the coordinate representation of the implicit constrained Lagrangian system  $(L_c, \Delta_Q, X_P)$  such that

$$\begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \\ \dot{x} \\ \dot{y} \\ \dot{\tilde{p}}_\theta \\ \dot{\tilde{p}}_\varphi \end{pmatrix} = \left( \begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & R(\cos \varphi) & 0 \\ 0 & 0 & 0 & 0 & R(\sin \varphi) & 0 \\ \hline -1 & 0 & -R(\cos \varphi) & -R(\sin \varphi) & 0 & p_b K_{12}^b \\ 0 & -1 & 0 & 0 & -p_b K_{21}^b & 0 \end{array} \right) \begin{pmatrix} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline v_\theta \\ v_\varphi \end{pmatrix},$$

where  $\tilde{p}_\theta = (mR^2 + I) v_\theta$  and  $\tilde{p}_\varphi = Jv_\varphi$ .

By computations, it follows that

$$\begin{aligned}\dot{\tilde{p}}_\theta &= p_b K_{12}^b v_\varphi \\ &= \{m v_x (R \sin \varphi) - m v_y (R \cos \varphi)\} v_\varphi \\ &= \{m (R \cos \varphi) (R \sin \varphi) v_\theta - m (R \sin \varphi) (R \cos \varphi) v_\theta\} v_\varphi \\ &= 0, \\ \dot{\tilde{p}}_\varphi &= -p_b K_{21}^b v_\theta \\ &= \{m v_x (R \sin \varphi) - m v_y (R \cos \varphi)\} v_\theta \\ &= \{m (R \cos \varphi) (R \sin \varphi) v_\theta - m (R \sin \varphi) (R \cos \varphi) v_\theta\} v_\theta \\ &= 0,\end{aligned}$$

where  $p_b = (p_x, p_y)$  are given by

$$\begin{aligned}p_x &= \frac{\partial L}{\partial v_x} = m v_x = m R(\cos \varphi) v_\theta, \\ p_y &= \frac{\partial L}{\partial v_y} = m v_y = m R(\sin \varphi) v_\theta.\end{aligned}$$

Thus, we obtain the equations of motion in the context of implicit constrained Lagrangian systems as

$$\begin{aligned}\dot{\theta} &= v_\theta, & \dot{\varphi} &= v_\varphi, & \dot{\tilde{p}}_\theta &= 0, & \dot{\tilde{p}}_\varphi &= 0, \\ \tilde{p}_\theta &= (mR^2 + I) v_\theta, & \tilde{p}_\varphi &= Jv_\varphi.\end{aligned}$$

**Example: L-C Circuits.** Let us consider the same example of an L-C circuit as illustrated in Part I in the context of implicit constrained Lagrangian systems.

Recall that the configuration space  $E = \mathbb{R}^4$ , whose element denotes the charge  $q$  and its local coordinates are given by  $(q_L, q_{C_1}, q_{C_2}, q_{C_3})$  and recall also that the KCL constraints form a constraint subspace called the constraint KCL space  $\Delta \subset TE$ , which is defined, for each  $q \in E$ , by

$$\Delta_q = \{f \in T_q E \mid \langle \omega^a, f \rangle = 0, a = 1, 2\}.$$

Note that  $f = (f_L, f_{C_1}, f_{C_2}, f_{C_3}) \in T_q E$  denotes the current vector and  $\omega^a$  indicate independent covectors (or one-forms), which form the basis for the annihilator  $\Delta_q^\circ \subset T_q^* E$  and are given, in coordinates, by

$$\begin{aligned}\omega^1 &= -dq_L + dq_{C_2}, \\ \omega^2 &= -dq_{C_1} + dq_{C_2} - dq_{C_3}.\end{aligned}$$

Therefore, the KCL constraints for currents  $f = (f_L, f_{C_1}, f_{C_2}, f_{C_3}) \in T_q E$  are represented, in coordinates, by

$$\begin{aligned}-f_L + f_{C_2} &= 0, \\ -f_{C_1} + f_{C_2} - f_{C_3} &= 0.\end{aligned}$$

Choose a bundle structure  $\pi : E \rightarrow \mathcal{R}$  such that the base  $\mathcal{R}$  is to be  $\mathbb{R}^2$  parameterized by  $(r^1, r^2) = (q_{C_2}, q_{C_3})$  together with the projection to  $\mathcal{R}$ , that is,

$$(r^1, r^2, s^1, s^2) = (q_{C_2}, q_{C_3}, q_L, q_{C_1}) \mapsto (r^1, r^2) = (q_{C_2}, q_{C_3}),$$

where we choose an Ehresmann connection in such a way that  $\mathcal{H}_q = \Delta_q$ . The connection  $A$  is described, in local coordinates  $q = (r^1, r^2, s^1, s^2) = (q_{C_2}, q_{C_3}, q_L, q_{C_1})$  for  $E = \mathbb{R}^4$ , by a vertical valued one-form  $\omega^a$  such that

$$A = \omega^a \frac{\partial}{\partial s^a}, \quad \omega^a = ds^a + A_\alpha^a dr^\alpha, \quad a = 1, 2, \quad \alpha = 1, 2,$$

where the components of  $A$  are locally represented by

$$A_\alpha^a = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Therefore, the KCL constraints may be rewritten as

$$f^a = -A_\alpha^a f^\alpha,$$

that is, in matrix representation,

$$\begin{pmatrix} f_L \\ f_{C_1} \end{pmatrix} = - \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} f_{C_2} \\ f_{C_3} \end{pmatrix}.$$

As in Part I, recall that the Lagrangian  $\mathcal{L}$  on  $TE$  is locally given by

$$\mathcal{L} = \frac{1}{2}L(f_L)^2 - \frac{1}{2}\frac{(q_{C_1})^2}{C_1} - \frac{1}{2}\frac{(q_{C_2})^2}{C_2} - \frac{1}{2}\frac{(q_{C_3})^2}{C_3}.$$

Hence, we have the equations of motion

$$\dot{p}_\alpha - \frac{\partial \mathcal{L}}{\partial r^\alpha} = A_\alpha^a \left( \dot{p}_a - \frac{\partial \mathcal{L}}{\partial s^a} \right),$$

which may be denoted, in matrix form, by

$$\begin{pmatrix} \dot{p}_{C_2} + \frac{q_{C_2}}{C_2} \\ \dot{p}_{C_3} + \frac{q_{C_3}}{C_3} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{p}_L \\ \dot{p}_{C_1} + \frac{q_{C_2}}{C_3} \end{pmatrix}.$$

The flux linkages  $p = (p_\alpha, p_a) = (p_{C_2}, p_{C_3}, p_L, p_{C_1})$  are defined by

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial f^\alpha}, \quad p_a = \frac{\partial \mathcal{L}}{\partial f^a},$$

and it reads

$$p_{C_2} = 0, \quad p_{C_3} = 0, \quad p_L = L f_L, \quad p_{C_1} = 0.$$

Note that the constraint flux linkage space is defined by

$$P = \mathbb{F}\mathcal{L}(\Delta) \subset T^*E,$$

where the distribution  $\Delta \subset TE$  is represented, in coordinates, by

$$\Delta = \text{span} \left\{ \frac{\partial}{\partial r^\alpha} - A_\alpha^a \frac{\partial}{\partial s^a} \right\}.$$

Define the new coordinates  $(r^\alpha, s^a, \tilde{p}_\alpha, \tilde{p}_a)$  for  $T^*E$  such that

$$\tilde{p}_\alpha = p_\alpha - A_\alpha^a p_a$$

with some choice of complementary coordinates  $\tilde{p}_a$ . Hence, we employ the induced coordinates  $(r^\alpha, s^a, \tilde{p}_\alpha) = (q_{C_2}, q_{C_3}, q_L, q_{C_1}, \tilde{p}_{C_2}, \tilde{p}_{C_3})$  for  $P$ .

The bundle map  $B_P^\sharp : T^*P \rightarrow TP$  associated with the constrained Poisson structure  $B_P$  on  $P$  can be constructed by computing  $\{q^i, q^j\} = 0$ ,  $\{r^\beta, \tilde{p}_\alpha\} = \delta_\alpha^\beta$ ,  $\{s^b, \tilde{p}_\alpha\} = -A_\alpha^b$ ,  $\{\tilde{p}_\alpha, \tilde{p}_\beta\} = 0$  such that

$$B_P^\sharp(q, \tilde{p}_\alpha) = \begin{pmatrix} 0 & 0 & \delta_\beta^\alpha \\ 0 & 0 & -A_\beta^a \\ -\delta_\alpha^\beta & (A_\alpha^b)^T & 0 \end{pmatrix}.$$

Since the KCL constraints are holonomic, the curvature  $K_{\alpha\beta}^d$  of the Ehresmann connection  $A$  does not appear in  $B_P^\sharp(q, \tilde{p}_\alpha)$ , and it immediately reads that the Jacobi identity holds.

As previously mentioned, the set of  $B_P^\sharp$  and  $\Delta_P^*$  defines the Dirac structure  $D_P \subset TP \oplus T^*P$  on  $P$ , whose fiber is given, for each  $y \in P$ , by

$$D_P(y) = \{(v_y, \alpha_y) \in T_yP \times T_y^*P \mid \alpha_y \in \Delta_P^*(y) \text{ and } v_y - B_P^\sharp(y) \alpha_y \in (\Delta_P^*)^\circ(y)\}.$$

The constrained Lagrangian for the L-C circuit may be constructed as

$$\mathcal{L}_c(r^\alpha, s^a, f^\alpha) = \mathcal{L}(r^\alpha, s^a, f^\alpha, -A_\alpha^a f^\alpha)$$

and we obtain

$$\mathcal{L}_c(q_{C_2}, q_{C_3}, q_L, q_{C_1}, f_{C_2}, f_{C_3}) = \frac{1}{2} L (f_{C_2})^2 - \frac{1}{2} \frac{(q_{C_1})^2}{C_1} - \frac{1}{2} \frac{(q_{C_2})^2}{C_2} - \frac{1}{2} \frac{(q_{C_3})^2}{C_3}.$$

Then, it follows from the partial Legendre transformation that

$$\tilde{p}_{C_2} = \frac{\partial \mathcal{L}_c}{\partial f_{C_2}} = L f_{C_2}, \quad \tilde{p}_{C_3} = \frac{\partial \mathcal{L}_c}{\partial f_{C_3}} = 0,$$

which exactly correspond to the equality of the base points. By using local coordinates  $(r^\alpha, s^a, f^\alpha, \tilde{p}_\alpha) = (q_{C_2}, q_{C_3}, q_L, q_{C_1}, f_{C_2}, f_{C_3}, \tilde{p}_{C_2}, 0)$  for  $\Delta \oplus P$ , the constrained partial vector field  $X_P : \Delta \oplus P \rightarrow TP$  can be denoted as

$$\begin{aligned} X_P(r^\alpha, s^a, f^\alpha, \tilde{p}_\alpha) &= (\dot{r}^\alpha, \dot{s}^a, \dot{\tilde{p}}_\alpha), \\ &= \left( \dot{q}_{C_2}, \dot{q}_{C_3}, \dot{q}_L, \dot{q}_{C_1}, \dot{\tilde{p}}_{C_2}, 0 \right), \end{aligned}$$

while the Dirac differential of  $\mathcal{L}_c$  is given by

$$\begin{aligned} \mathfrak{D}\mathcal{L}_c(r^\alpha, s^a, f^\alpha) &= \left( -\frac{\partial \mathcal{L}_c}{\partial r^\alpha}, -\frac{\partial \mathcal{L}_c}{\partial s^a}, f^\alpha \right) \\ &= \left( \frac{q_{C_2}}{C_2}, \frac{q_{C_3}}{C_3}, 0, \frac{q_{C_1}}{C_1}, f_{C_2}, f_{C_3} \right). \end{aligned}$$

Then, the L-C circuit can be expressed as an implicit constrained Lagrangian system, since the triple  $(\mathcal{L}_c, \Delta, X_P)$  satisfies the condition

$$(X_P, \mathfrak{D}\mathcal{L}_c) \in D_P,$$

which is represented, in coordinates, by using the bundle map  $B_P^\sharp(q, \tilde{p})$  as

$$\begin{pmatrix} \dot{r}^\alpha \\ \dot{s}^a \\ \dot{\tilde{p}}_\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & \delta_\beta^\alpha \\ 0 & 0 & -A_\beta^a \\ -\delta_\alpha^\beta & (A_\alpha^b)^T & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial \mathcal{L}_c}{\partial r^\beta} \\ -\frac{\partial \mathcal{L}_c}{\partial s^b} \\ f^\beta \end{pmatrix}.$$

Hence, we obtain

$$\begin{pmatrix} \dot{q}_{C_2} \\ \dot{q}_{C_3} \\ \dot{q}_L \\ \dot{q}_{C_1} \\ \dot{\tilde{p}}_{C_2} \\ 0 \end{pmatrix} = \left( \begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ \hline -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{array} \right) \begin{pmatrix} \frac{q_{C_2}}{C_2} \\ \frac{q_{C_3}}{C_3} \\ 0 \\ \frac{q_{C_1}}{C_1} \\ f_{C_2} \\ f_{C_3} \end{pmatrix},$$

where  $\tilde{p}_{C_2} = L f_{C_2}$  holds. Furthermore, we can eliminate the components associated with the current  $\dot{q}_L$  of the inductor  $L$ , and it reads

$$\begin{pmatrix} \dot{q}_{C_2} \\ \dot{q}_{C_3} \\ \dot{q}_{C_1} \\ \dot{\tilde{p}}_{C_2} \\ 0 \end{pmatrix} = \left( \begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & -1 \\ \hline -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array} \right) \begin{pmatrix} \frac{q_{C_2}}{C_2} \\ \frac{q_{C_3}}{C_3} \\ \frac{q_{C_1}}{C_1} \\ f_{C_2} \\ f_{C_3} \end{pmatrix}.$$



Thus, the reduced equations of motion of the L-C circuit are derived in the context of the implicit constrained Lagrangian system as follows:

$$\begin{aligned}\dot{q}_{C_1} &= f_{C_2} - f_{C_3}, & \dot{q}_{C_2} &= f_{C_2}, & \dot{q}_{C_3} &= f_{C_3}, \\ \dot{\tilde{p}}_{C_2} &= -\frac{q_{C_1}}{C_1} - \frac{q_{C_2}}{C_2}, & \tilde{p}_{C_2} &= L f_{C_2}, \\ 0 &= -\frac{q_{C_3}}{C_3} + \frac{q_{C_1}}{C_1}.\end{aligned}$$

**Remarks.** Note that the original Lagrangian  $\mathcal{L}(q^i, f^i)$  and the constrained Lagrangian  $\mathcal{L}_c(r^\alpha, s^\alpha, f^\alpha)$  are independent of  $q_L$ , which implies  $q_L$  might be a "secret" variable that is related to symmetries.

## 7 Conclusions

Part I showed that a constrained distribution on a manifold together with the canonical two-form induces a Dirac structure on the cotangent bundle. It was shown that some basic examples, such as KCL and KVL constraints in electric circuits, interconnections, as well as nonholonomic constraints, naturally fit into this context. Utilizing the symplectomorphisms between the iterated tangent and cotangent bundles, Part I also developed the notion of an implicit Lagrangian system  $(L, \Delta_Q, X)$  in this context.

In Part II of the paper, we established the link between variational structures and implicit Lagrangian systems in mechanics. To do this, use was made of an extension of the variational principle of Hamilton, called the Hamilton-Pontryagin principle, which, in the case  $\Delta_Q = TQ$  leads to a set of equations that naturally includes the Legendre transformation as well as the Euler-Lagrange equations themselves. For the case of a general constraint distribution  $\Delta_Q \subset TQ$ , we showed that an implicit Lagrangian system can be derived from a generalization of the Hamilton-Pontryagin principle, namely an extended Lagrange-d'Alembert principle called the Lagrange-d'Alembert-Pontryagin principle. We also proposed a generalization of Hamilton's phase space principle called the Hamilton-d'Alembert principle in phase space and used this to establish the relationship with implicit Hamiltonian systems for the case of regular Lagrangians.

In conjunction with applications to controlled interconnected systems such as robots and electromechanical systems, we demonstrated that nonholonomic mechanical systems with external forces naturally fall into the context of implicit Lagrangian systems.

Furthermore, we developed the notion of an implicit constrained Lagrangian system  $(L_c, \Delta_Q, X_P)$ , by introducing the constrained Dirac structure  $D_P$  on the constraint momentum space  $P$  and the constrained partial vector field  $X_P : \Delta_Q \oplus P \rightarrow TP$ ; the constrained Dirac structure  $D_P$  on  $P$  can be naturally defined by restricting the induced Dirac structure  $D_{\Delta_Q}$  on  $T^*Q$  to  $P$  and we have shown that  $D_P$  can be constructed by using an Ehresmann connection associated with the constraint distribution. Also, implicit constrained Lagrangian systems were shown to fit naturally into the context of the Lagrange-d'Alembert-Pontryagin principle. Finally, two examples were given, namely, a vertical rolling disk on a plane as an example of a nonholonomic mechanical system and an L-C circuit as an example of a degenerate Lagrangian system with holonomic constraints.

Some interesting topics for future work are as follows:

- Implicit Lagrangian systems with symmetry, Dirac reduction and links with, for example, [Cendra, Marsden, Pekarsky, and Ratiu \[2003\]](#). Specifically, it would be interesting to explore the Euler-Poincaré and Lie-Poisson equations from this point of view.

- The momentum equations and the reduced Lagrange-d'Alembert equations in the context of Dirac structures; see [Cendra, Marsden, and Ratiu \[2001\]](#) and references therein.
- The relationship between implicit Lagrangian systems and implicit Hamiltonian systems for degenerate Lagrangians, using a generalized Legendre transform and the theory of Dirac constraints.
- An analog of controlled Lagrangians and related stability problems for implicit Lagrangian systems; [Bloch, Leonard, and Marsden \[2000\]](#); [Bloch, Chang, Leonard, and Marsden \[2001\]](#); [Zenkov, Bloch, and Marsden \[1998\]](#).
- Discrete mechanics and variational integrators for implicit Lagrangian systems from the viewpoint of the Hamilton-Pontryagin principle.
- Applications to interconnected systems such as multibody systems, general electric circuits, and networks including sensing and communications, electromechanical systems, biochemical systems, etc.

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