BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0273-0979(XX)0000-0

NEW METHODS IN CELESTIAL MECHANICS AND MISSION DESIGN

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To Henri Poincaré on the 150th anniversary of his birth.

Abstract. The title of this paper is inspired by the work of Poincaré [1890, 1892], who introduced many key dynamical systems methods during his research on celestial mechanics and especially the three body problem. Since then, many researchers have contributed to his legacy by developing and applying these methods to problems in celestial mechanics and, more recently, with the design of space missions.

This paper will give a survey of some of these exciting ideas, and we would especially like to acknowledge the work of Michael Dellnitz, Frederic Gabern, Katalin Grubits, Oliver Junge, Wang-Sang Koon, François Lekien, Martin Lo, Sina Ober-Blöbaum, Kathrin Padberg, Robert Preis, and Bianca Thiere.

One of the purposes of the *AMS Current Events* session is to discuss work of others. Even though we were involved in the research reported on here, this short paper is intended to survey many ideas due to our collaborators and others.

This survey is by no means complete and we apologize for not having time or space to do justice to many important and fundamental works. In fact, the results reported on here rely on and were inspired by important preceding work of many others in celestial mechanics, mission design and in dynamical systems. We mention just a few whose work had positive influence on what is reported here: Brian Barden, Ed Belbruno, Robert Farquhar, Gerard Gómez, George Haller, Charles Jaffé, Kathleen Howell, Linda Petzold, Josep Masdemont, Vered Rom-Kedar, Radu Serban, Carles Simó, Turgay Uzer, Steve Wiggins, and Roby Wilson. In an upcoming monograph (see Koon, Lo, Marsden, and Ross [2005]), the dynamical systems and computational approach and its application to mission design is discussed in detail.

One of the key ideas is that the competing gravitational pull between celestial bodies creates a vast array of passageways that wind around the sun, planets and moons. The boundaries of these passageways are realized geometrically as invariant manifolds attached to equilibrium points and periodic orbits in interlinked three body problems. In particular, tube-like structures form an interplanetary transport network which will facilitate the exploration of Mercury, the Moon, the asteroids, and the outer solar system, including a mission to assess the possibility of life on Jupiter's icy moons. The use of these methods in problems in molecular dynamics of interest in Chemistry is also briefly discussed.

Received by the editors January, 2005 and, in revised form, June, 2005.

1991 *Mathematics Subject Classification.* Primary 70F07, 70F15; Secondary 37J45, 70H33.

Key words and phrases. Three-body problem, mission design, transport, celestial mechanics.

Research of JEM was supported in part by a Max Planck Research Award and NSF-ITR Grant ACI-0204932. This article is based on a lecture presented January 7, 2005, at the AMS Special Session on Current Events, Joint Mathematics Meetings, Atlanta, GA..

Research of SDR was supported by NSF Postdoctoral Fellowship DMS 0402842.

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1. Astrodynamics and Dynamical Astronomy

Astrodynamics and dynamical astronomy apply the principles of mechanics, including the law of universal gravitation to the determination of the motion of objects in space. In these subjects, orbits of astronomical bodies, such as stars, planets, and asteroids are calculated, as are spacecraft trajectories, from launch through atmospheric re-entry, including all the needed orbital maneuvers.

While there are no sharp boundaries, astrodynamics has come to denote primarily the design and control of spacecraft trajectories, while dynamical astronomy is concerned with the motion of other bodies in the solar system (origin of the moon, Kuiper belt objects, etc.) and the motion of stars in the galaxy. From the dynamical systems perspective that we have adopted, it has proven quite useful to mix these subjects rather than to isolate them. There is one obvious commonality: the model used for studying either a spacecraft or, say, the motion of an asteroid is the restricted $N+1$ body problem, where N celestial bodies move under the influence of one another and the spacecraft or asteroid moves in the field of these bodies, but has a mass too small to influence their motion.

The Ephemeris and Its Approximations. In the case of motion within the solar system, the motion of the N bodies (planets, moons, etc.) can be measured and predicted to great accuracy, producing an ephemeris. An *ephemeris* is simply a listing of positions and velocities of celestial bodies as a function of time with respect to some coordinate system. An ephemeris can be considered as the solution of the N-body gravitational problem, and forms the gravitational field which determines a spacecraft or asteroid's motion.

While the final trajectory design phase of a space mission or the long term trajectory of an asteroid will involve a solution considering the most accurate ephemeris, insight can be achieved by considering simpler, approximate ephemerides (the plural of ephemeris). An example of such an ephemeris is a simplified solution of the N-body problem, where N is small, for example, the motion of the Earth and Moon under their mutual gravitation, a two-body solution. The simplest two-body solution of massive bodies which gives rise to interesting motion for a spacecraft is the circular motion of two bodies around their common center of mass. The problem of such a spacecraft's motion is then known as the *circular restricted three-body problem*, or the CR3BP.

Introduction to the Trajectory Design Problem. The set of possible spacecraft trajectories in the three-body problem can be used as building blocks for the design of spacecraft trajectories in the presence of an arbitrary number of bodies. Consider the situation shown in Figure 1.1, where we have a spacecraft, approximated as a particle, P, that is moving in the gravitational field of N massive bodies. We assume P has a small enough mass that it does not influence the motion of the

N massive bodies, which move in prescribed orbits under their mutual gravitational attraction. In the solar system, one can think of a moon, M_2, in orbit around a planet, M_1, which is in orbit around the Sun, M_0.

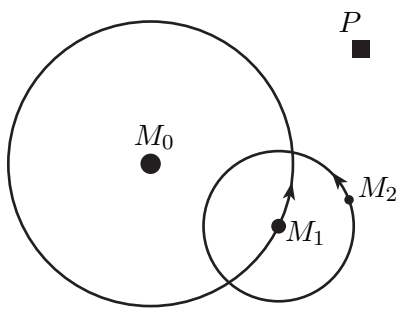


FIGURE 1.1. A spacecraft P in the gravitational field of N massive bodies which move in prescribed orbits.

One of the goals of trajectory design is to find a transfer trajectory, such as the one shown in Figure 1.2(a), which takes the spacecraft from a prescribed initial

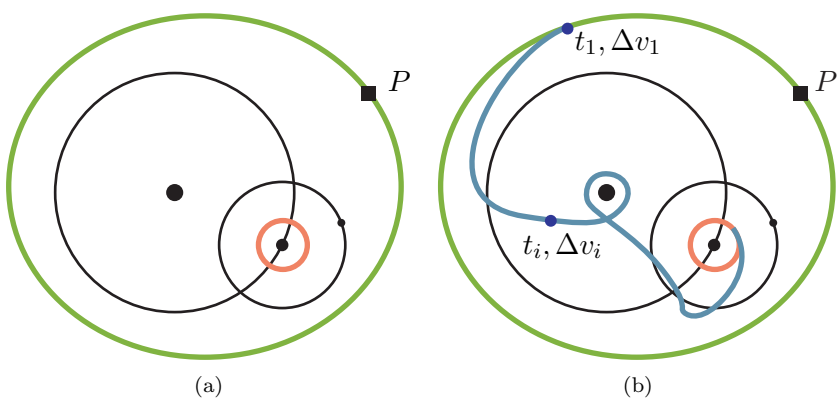


FIGURE 1.2. (a) The goal is to find a transfer trajectory which takes the spacecraft from an initial orbit to a final orbit using controls. (b) Assuming impulsive controls, i.e., several instantaneous changes in the spacecraft's velocity, with norm Δv_i at time t_i, we can effect such a transfer.

orbit to a prescribed final orbit using controls. The initial orbit may be an orbit around the Earth and the final orbit an orbit around one of the moons of Jupiter, for instance. To effect such a transfer, one can use high thrust or low thrust propulsion systems. In the low thrust case, we have a small continuous control which can operate at all times. In the high thrust case, one typically assumes that the control is discretized into several instantaneous changes in the spacecraft's velocity. These instantaneous changes have a magnitude at time t_i that is traditionally denoted Δv_i.

Under a high thrust assumption, the Δv's are proportional to the fuel consumption:

$$\Delta v = -v_e \frac{\Delta m}{m}$$

where m is the mass of the rocket and Δm is the mass of propellant ejected at an exhaust velocity v_e (see, for example, Roy [1988]). As spacecraft are limited in the amount of fuel that they can carry on-board for their journey, we often want to consider an *optimal control problem*: minimize the fuel consumed (equivalently, energy). In other words, we want to find the maneuver times t_i and sizes Δv_i to minimize

$$\sum_i \Delta v_i,$$

the total change in velocity, or "ΔV" as it is called.

It is typical in the space mission design community to use the magnitude of the required ΔV as a measure of the spacecraft fuel performance. The propellant mass is a much less stable quantity as a measure of spacecraft performance, since it is dependent on the spacecraft mass and various other parameters which change frequently as the spacecraft is being built. The ΔV measure comes from astrodynamics considerations only and is independent of the mass and type of spacecraft. Thus, for a given mission objective, one generally wants to *minimize* ΔV.

We will discuss some other optimization problems and approaches to solve them at the end of the following section.

2. The Patched Three-Body Approximation

To get a spacecraft from, say, Earth to other parts of the solar system, it is necessary to find solutions for the motion of the spacecraft under the influence of N bodies, a notoriously difficult problem. Furthermore, one needs to find solutions with a desired behavior, e.g., flying by the giant outer planets as Voyagers 1 and 2 did, while satisfying engineering constraints, e.g., low fuel consumption, short time of flight, low radiation dose, good observational geometry, etc.

Patched-Conic Approach and the Voyager Trajectory. For many purposes it is satisfactory to simplify the general trajectory problem by considering the gravitational force between the spacecraft and only one other body at a time. Even for the case of interplanetary transfer, this simplification will suffice for many calculations. That is, one may consider escape from or capture by a planet to be an interaction between the spacecraft and that particular planet alone, whereas the transfer process is considered an interaction between the spacecraft and the Sun alone. NASA's spectacular multiple flyby missions such as Voyager and Galileo are based on this Keplerian decomposition of the solar system, known as the *patched-conic approximation* (or *patched two-body approximation*), discussed in Bate, Mueller, and White [1971].

The strategy of the designers of the Voyager missions was to initially approximate the full N-body solution of the spacecraft's motion as a linkage of several two-body solutions, namely, the well known conic solutions discovered by Kepler. The spacecraft's trajectory as it coasted between two planets was considered as a heliocentric hyperbolic trajectory. The heliocentric trajectory was cleverly chosen to come close to the destination planet, in order to fly by it. When the spacecraft

came within the "sphere of influence"[1] of a planet, it was considered as a hyperbolic conic section trajectory centered on the planet. This patched-conic solution could be used as an initial guess for a numerical procedure which produced a fully integrated N-body solution.

High vs. Low Relative Velocities. For missions such as Voyager and Galileo, the speed of the spacecraft relative to the bodies is high and therefore the time during which the acceleration on the spacecraft due to two bodies is comparable is very short, and results in a minor perturbation away from a conic solution. But when one needs to deal with the unpropelled, or ballistic, capture regime of motion[2], where the relative speed is low, a three-body decomposition of the solar system is necessary.

Some Missions Cannot Be Approximated by the Patched-Conic Approach. For Voyager and Galileo, the patched-conic approach worked very well. But as space missions have become more demanding, other approaches have become necessary. For example, the Genesis, L_1 Gateway, and multi-moon orbiter trajectories discussed below resemble solutions of the restricted three- and four-body problems much more than two-body problems. In fact, methods based on a patched-conic approximation would have a very difficult time finding these complicated trajectories, as they are fundamentally non-Keplerian, restricted N-body solutions.

Taking Better Advantage of N-Body Dynamics. It is possible to satisfy mission constraints using spacecraft solutions which do not take advantage of the N-body dynamics of a system. But this may require using more fuel than is necessary[3]. Worse yet, because of the fuel restrictions on interplanetary spacecraft, some missions may not be possible if only a patched-conic approach is used. An interesting example in this category, which also served as motivation for much of our group's work, is the "rescue" of a malfunctioned Japanese space mission to the moon by Belbruno and Miller of JPL in June, 1990. The mission originally had two spacecraft, MUSES-A and MUSES-B; B was to go into orbit around the moon, with A remaining in earth orbit as a communications relay. But B failed and A did not have sufficient fuel to make the journey. However, by utilizing a trajectory concept originally discovered by Belbruno in 1986, which is more energy-efficient than the one planned for B, MUSES-A (renamed Hiten) left Earth orbit in April, 1991 and reached the moon that October. As a result, Japan became the third nation to send a spacecraft to the moon. After a series of scientific experiments, Hiten was purposely crashed into the Moon in April, 1993. See Belbruno [2004] for additional details of this fascinating story.

An ESA (European Space Agency) mission currently underway, SMART-1, which is a mission from the Earth to the Moon, slated to last until August, 2006, uses some of the same ideas as was used in the Hiten mission[4]. Its purpose is the "Testing and

[1]The sphere of influence of a planet is the radius at which the acceleration on a spacecraft due to the planet and the Sun are approximately equal (Roy [1988]).

[2]*Ballistic capture* means that no propulsion is necessary (i.e., no ΔV) to achieve a capture orbit at the destination body. In general, this "capture" is temporary.

[3]For example, Dunn [1962] proposed to use a satellite for lunar far side communications by placing it in a position where it would require approximately 1500 m/s per year for stationkeeping. A few years later, Farquhar [1966] proposed a trajectory for the same mission which used only 100 m/s per year by taking advantage of three-body dynamics.

[4]see `http://sci.esa.int/science-e/www/area/index.cfm?fareaid=10`.

proving of an ion drive and miniaturised instruments, along with investigations of lunar geochemistry and a search for ice at the south lunar pole."

A Hierarchy of Models. We want to make use of the natural dynamics in the solar system as much as possible; that is, we wish to take advantage of the phase space geometry, integrals of motion, and lanes of fast unpropelled travel. We envision generating a trajectory via a hierarchy of models. One starts with simple models which capture essential features of natural dynamics. One then uses simple model solutions as initial guess solutions in more realistic models. The approach described above does this conceptually, using the patched-conic approximation to generate the first guess solution. But there are regimes of motion where conics are simply not a good approximation to the motion of the spacecraft. There is much to be gained by starting with not two-body solutions, but three-body solutions to the spacecraft's motion.

The Patched Three-Body Approximation. Motivated by the Belbruno and Miller work, we consider a restricted four-body problem wherein a spacecraft moves under the influence of three massive bodies whose motion is prescribed, as shown schematically in Figure 1.1. For Belbruno and Miller, these four bodies were the Sun, the Earth, the Moon and the spacecraft.

To begin with, we restrict the motion of all the bodies to a common plane, so the phase space is only four-dimensional. As in the patched-conic approach, the patched three-body approach uses solutions obtained from two three-body problems as an initial guess for a numerical procedure which converges to a full four-body solution.

As an example of such a problem where there is no control, consider the four-body problem where two adjacent giant planets compete for control of the same comet (e.g., Sun-Jupiter-comet and Sun-Saturn-comet). When close to one of the planets, the comet's motion is dominated by the corresponding planet's 3-body dynamics. Between the two planets, the comet's motion is mostly heliocentric and Keplerian, but is precariously poised between two competing three-body dynamics, leading to complicated transfer dynamics between the two adjacent planets.

When we consider a spacecraft with control instead of a comet, we can intelligently exploit the transfer dynamics to construct low energy trajectories with prescribed behaviors, such as transfers between adjacent moons in the Jovian and Saturnian systems (Lo and Ross [1998]). For example, by approximating a spacecraft's motion in the $N+1$ body gravitational field of Jupiter and N of its planet-sized moons by several segments of purely three body motion—involving Jupiter, the ith moon, and the spacecraft—we can design a trajectory for the spacecraft which follows a prescribed itinerary in visiting the N moons. In an earlier study of a transfer from Ganymede to Europa, we found our fuel consumption for impulsive burns, as measured by the total norm of velocity displacements, ΔV, to be less than half the Hohmann transfer value (Koon, Lo, Marsden, and Ross [1999]). We found this to be the case for the following example of a multi-moon orbiter tour that is shown schematically in Figure 2.1: starting beyond Ganymede's orbit, the spacecraft is ballistically captured by Ganymede, orbits it once, escapes in the direction of Europa, and ends in a ballistic capture at Europa.

One advantage of this multi-moon orbiter approach as compared with the Voyager-type flybys is the "leap-frogging" strategy. In this new approach to mission design, the spacecraft can orbit a moon for a desired number of circuits, escape the moon,

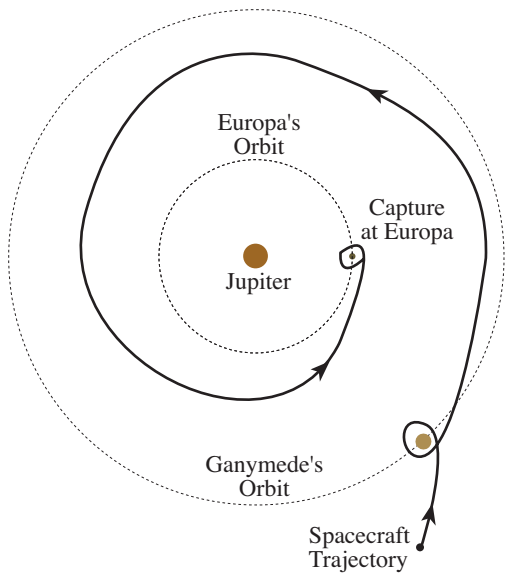


FIGURE 2.1. Leap-frogging mission concept: a multi-moon orbiter tour of Jupiter's moons Ganymede and Europa.

and then perform a transfer ΔV to become ballistically captured by a nearby moon for some number of orbits about that moon, etc. Instead of brief flybys lasting only seconds, a scientific spacecraft can orbit several different moons for any desired duration. Furthermore, the total ΔV necessary is *much less than that necessary using purely two-body motion segments*. One can also systematically construct low energy transfers from the Earth to the Moon using the Sun's perturbation, and from lunar libration point orbits to Earth libration point orbits.

Three-Body Dynamics. To patch three-body solutions (the spacecraft's motion in the presence of two bodies), one needs a good handle on what those solutions are. Studying the CR3BP solutions in detail is an interesting topic in its own right. This is a topic that goes back to the basic work of Poincaré in the late 1800s and provided the context in which he developed modern dynamical systems theory and the notion of chaos.

In the CR3BP, we have two primaries that move in circles; the smaller third body moves in the gravitational field of the primaries (without affecting them). We typically view the motion in a rotating frame so that the primaries appear stationary. It is important to consider both the planar and the spatial problems, but we shall focus on the planar problem for the moment.

One may derive the equations of motion using a little elementary mechanics as follows. Let the masses of the two primaries be denoted m_1 and m_2 and set $\mu = m_2/(m_1 + m_2)$. We can normalize the distance between the primaries to be unity and then in the rotating frame, normalized to rotate with unit angular velocity, the two bodies may be located on the x-axis at the points $(-\mu, 0)$ and $(1 - \mu, 0)$. Let the position of the third body be denoted (x, y) in the rotating frame. The kinetic energy of this third body (whose mass we take to be unity) with respect to an inertial frame but written in a frame rotating with unit angular

velocity is the usual $\frac{1}{2}mv^2$ expression:

$$K(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\left[(\dot{x} - y)^2 + (\dot{y} + x)^2\right]$$

Let r_1 be the distance from the third body to the first primary and let r_2 be its distance to the second primary. Thus,

$$r_1 = \sqrt{(x + \mu)^2 + y^2},$$
$$r_2 = \sqrt{(x - 1 + \mu)^2 + y^2)}.$$

Then the gravitational potential energy of the third body is, again in normalized units,

$$V(x, y) = -\frac{1 - \mu}{r_1} - \frac{\mu}{r_2}.$$

The Lagrangian of the third body is its kinetic minus potential energies, namely

$$L(x, y, \dot{x}, \dot{y}) = K(x, y, \dot{x}, \dot{y}) - V(x, y);$$

Now one gets the equations of motion simply by writing down the corresponding Euler-Lagrange equations:

$$\ddot{x} - 2\dot{y} = -\frac{\partial \overline{V}}{\partial x}, \qquad \ddot{y} + 2\dot{x} = -\frac{\partial \overline{V}}{\partial y} \tag{2.1}$$

where the *effective potential* is

$$\overline{V} = V - \frac{x^2 + y^2}{2}$$

Being Euler–Lagrange equations, there is a conserved energy that one computes via the Legendre transformation to be

$$E = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \overline{V}(x, y).$$

Equilibria. These occur when the the third body moves in a circular orbit with the same frequency as the primaries, so that it is stationary in the rotating frame. We find these points by finding the equilibrium points, in the standard sense of ode's, of the equations (2.1). It is clear that this task is equivalent to finding the critical points of the effective potential, an analysis that is found in every book on celestial mechanics. The result is that there are five such points. There are three collinear points on the x-axis that were discovered by Euler around 1750 and are denoted L_1, L_2, L_3 and there are two equilateral points discovered by Lagrange around 1760 and are denoted L_4, L_5. They are indicated in Figure 2.2.

Equations (2.1) may be interpreted as those of a particle moving in an effective potential plus a magnetic field. Its graph is shown in Figure 2.3. This figure also shows the region one gets by imposing conservation of energy and the inequality which simply states that the kinetic energy is positive. Thus, at a given energy level E, the third body can only move in the region given by the inequality $E - \overline{V} \geq 0$; this is called the *Hill's region* and is obtained by intersecting the graph of the effective potential with a horizontal plane. An example is shown as the white region in the right hand side of Figure 2.3 for the Sun-Jupiter-third body system. In this figure, one can see three *realms*, namely the *Sun realm*, the *Jupiter realm* and the *exterior realm* that are connected by the *neck regions*, the left hand neck containing L_1 and the right hand neck containing L_2. For other values of the energy, one or more of

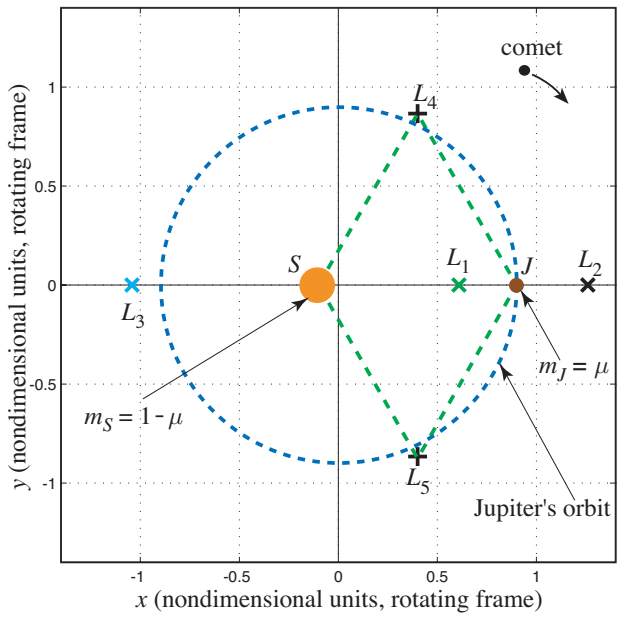


FIGURE 2.2. Equilibrium points for the three body problem.

these realms may be prohibited due to conservation of energy; that is, the necks may close off.

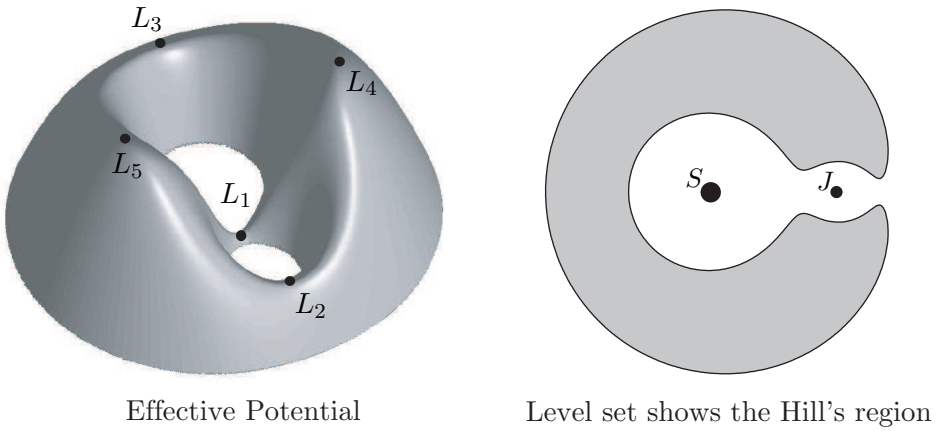


FIGURE 2.3. The graph of the effective potential in the 3-body problem. Its critical points are the equilibria.

Of special interest are the two points L_1 and L_2 closest to the secondary body, which a linearized analysis shows are center-saddle points. The famous Liapunov theorem says that there is a family of periodic orbits surrounding each of these points; one can think of this as meaning that one can "go into orbit about these

points". These planar periodic orbits are called *Liapunov orbits*, while their counterparts in the 3D problem are called *halo and Lissajous orbits* (which, by the way involves an interesting bifurcation analysis).

Tubes. In the three body problem, a key role is played by the invariant manifolds of these periodic orbits, which we call the *Conley–McGehee tubes*. Also key is a network of homoclinic and heteroclinic orbits connecting these periodic orbits, also discovered in a preliminary way in work of Conley and McGehee which was extended and thoroughly investigated in Koon, Lo, Marsden, and Ross [2000]. Some of the reasons that these tubes are important can be seen in the context of specific space missions described below.

In fact, the invariant manifold structures of L_1 and L_2 provide the framework for understanding and categorizing the motions of spacecraft as well as, for example, comets that undergo resonance hopping. Moreover, the stable and unstable invariant manifold tubes associated to periodic orbits around L_1 and L_2 are the phase space conduits transporting material between different realms in a single three body system as well as between primary bodies for separate three-body systems. These tubes can be used to construct new spacecraft trajectories as we will indicate below. It is remarkable that the connecting orbits as well as the associated Conley–McGehee tubes are critical for understanding transport in the solar system as well as in molecular systems. It is quite interesting that some of the same techniques used in the celestial context can also be used in the molecular context, and conversely, techniques from chemistry can be used in celestial problems, as was done in Jaffé, Ross, Lo, Marsden, Farrelly, and Uzer [2002].

Figure 2.4 shows some tubes (projected from phase space to configuration space) associated with periodic orbits about L_1, L_2 for the Earth-Moon system. As this figure indicates, it is the tubes that control the capture and escape properties as well as transit and non-transit orbits.

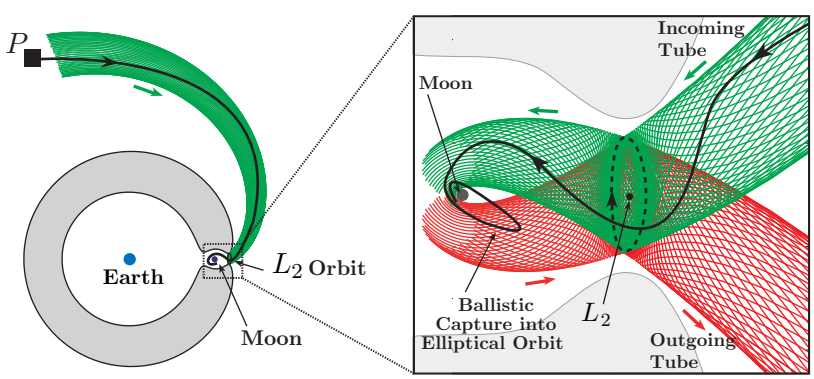


FIGURE 2.4. Tube leading to ballistic capture around the Moon (seen in rotating frame).

Tubes in Molecular Systems. In the molecular and atomic contexts, tubes control, for instance, the scattering of electrons off Rydberg atoms, as in Figure 2.5. The Conley–McGehee tubes in this case are associated with the three degree of freedom problem counterpart of the Liapunov orbit, the set of bound orbits known which are a part of the center manifold of the equilibrium point.

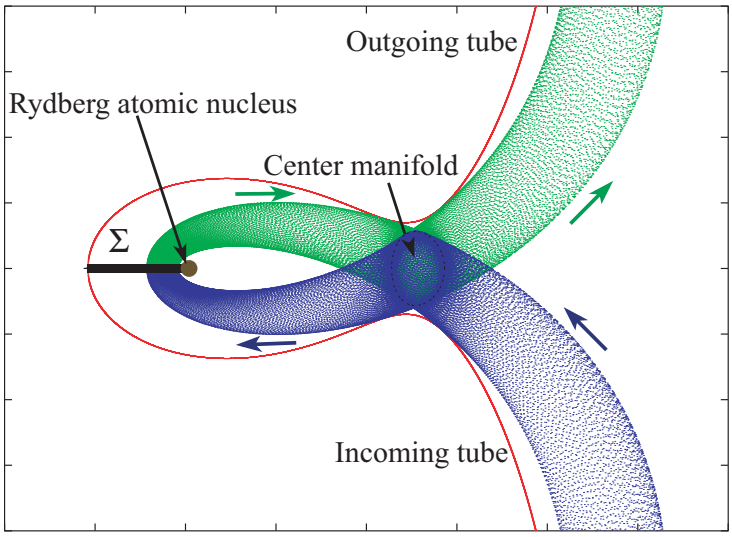


FIGURE 2.5. Tubes leading an electron toward and away from an ionized Rydberg atom in the presence of crossed electric and magnetic fields (seen in a frame rotating with the cyclotron frequency). Planar projections of the tubes are shown up to their first intersection with a Poincaré section, Σ. Figure taken from Gabern, Koon, Marsden and Ross [2005].

Another view of tubes in the Rydberg atom problem is shown in Figure 2.6. One of the interesting things about this example is that it is one for which the classical *Transition State Theory* developed in Chemistry, needs some modifications. The basic reason is because of the complex geometry of how the tubes interact with the chaotic sea, as shown in this figure.

Building on work of Jaffé, Uzer, Koon, Lo, Marsden, Ross, Wiggins and others, the papers Gabern, Koon, Marsden and Ross [2005] and Dellnitz, Grubits, Marsden, Padberg, and Thiere [2005] develop the mathematical and computational foundations of reaction rate theory that overcome some of the classical difficulties mentioned in the preceding paragraph. There are several things that are interesting about this; first of all, it is done in a way that is synergistic with the methods that are useful in dynamical astronomy. Of course it is the underlying mathematics that provides the bridge. Another interesting thing is that these computations can be done effectively for the full three dimensional system; the figures above are a little misleading as they seem to rely on two dimensional Poincaré sections, which are not available for the three degree of freedom system. Nonetheless, the computations are still feasible (for the first time, actually). Current research is directed to extending these methods to more complex system and, eventually, to conformations of biomolecules. For additional information on the celestial-molecular connection, see Porter and Cvitanović [2005].

Some Specific Missions. For the complex space missions planned for the near future, greater demands are placed on the trajectory design. In many instances,

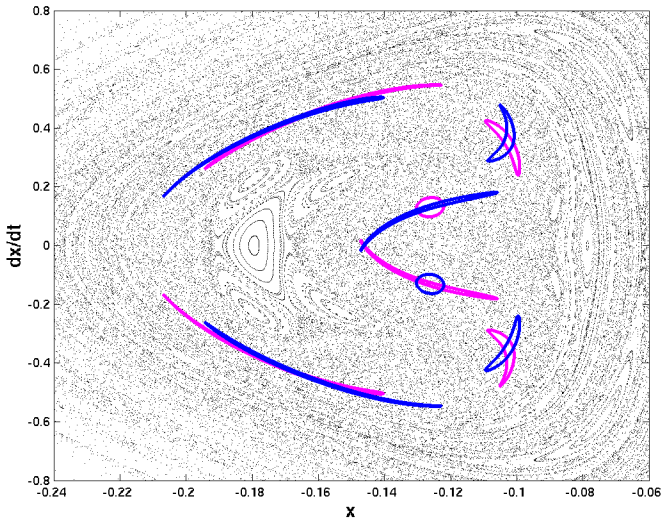


FIGURE 2.6. Tubes can intersect the chaotic sea multiple times as shown in this figure, taken from Dellnitz, Grubits, Marsden, Padberg, and Thiere [2005].

standard trajectories and classical methods such as the patched two-body approximation are inadequate to support the new mission concepts. Without appropriate and economical trajectories, these missions cannot be achieved. For nearly half a century, space mission planners have depended on trajectory concepts and tools developed in the 1950s and 1960s, based largely on a two-body decomposition of the solar system, the patched conics approach. While that approach remains very valuable for some missions, new trajectory paradigms must be developed to meet today's challenges.

A detailed understanding of the three-body problem, and in particular the dynamics associated with libration points, is absolutely necessary to continue the exploration and development of space.

Figure 2.7 shows, in "metro map format" some connections between hubs in Earth's neighborhood and beyond. NASA desires to develop a robust and flexible capability to visit several potential destinations. As shown in the figure, NASA has recognized that libration points L_1 and L_2 in the Sun-Earth and Earth-Moon system are important hubs and/or destinations. The fortuitous arrangement of low energy passageways in near-Earth space implies that lunar L_1 and L_2 orbits are connected to orbits around Earth's L_1 or L_2 via low energy pathways[5]. Therefore, a Lunar Gateway Station at the lunar L_1 would be a natural transportation hub to get humanity beyond low-Earth orbit, a stepping stone to the moon, Earth's neighborhood, Mars, the asteroids, and beyond. We will discuss the Lunar L_1 Gateway Station further below.

[5]We will sometimes refer to the Sun-Earth L_1 and L_2 as the Earth's L_1 and L_2, since they are much closer to the Earth than the Sun. Similarly, we will occasionally refer to the Earth-Moon L_1 and L_2 as the lunar or the Moon's L_1 and L_2.

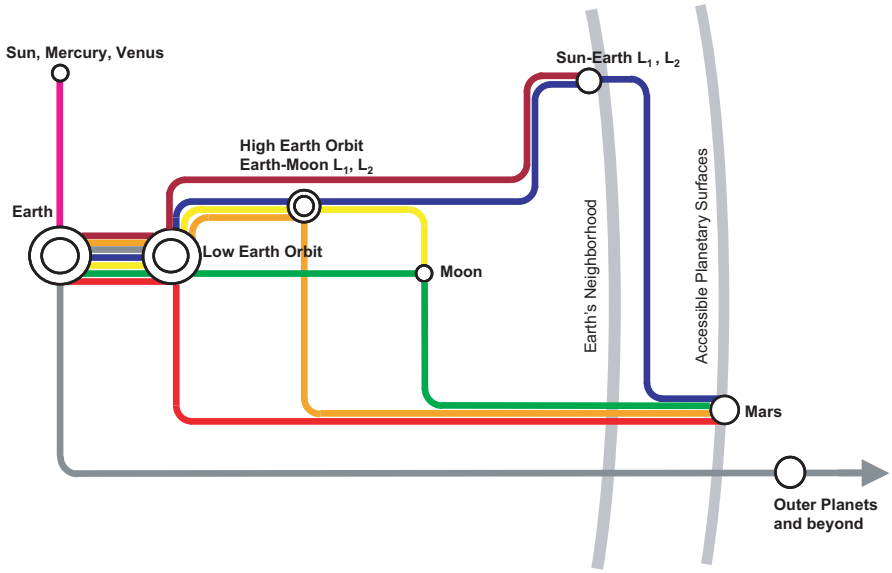


FIGURE 2.7. A metro map representation showing hubs connected by low energy passageways in the near-Earth neighborhood and beyond (source: Gary L. Martin, NASA Space Architect).

Because of its unobstructed view of the sun, the Sun-Earth L_1 is a good place to put instruments for doing solar science. NASA's Genesis Discovery Mission has been there, the first space mission designed completely using invariant manifolds and other tools from dynamical systems theory (Howell, Barden, and Lo [1997]). The Solar and Heliospheric Observatory (SOHO)[6], a joint project of the European Space Agency and NASA, and NASAs WIND and the Advanced Composition Explorer (ACE) are also there.

Genesis Discovery Mission. Launched in August 2001, the Genesis Discovery Mission spacecraft swept up specks of the sun—individual atoms of the solar wind—on five collector arrays the size of bicycle tires and in an ion concentrator. The goal was to collect solar wind samples and return them safely to the Earth for study into the origins of the solar system. Genesis returned its solar wind cargo to Earth via a sample-return capsule which returned to Earth in September 2004 (see Lo, Williams, Bollman, Han, Hahn, Bell, Hirst, Corwin, Hong, Howell, Barden, and Wilson [2001])[7]. The sample was the only extraterrestrial material brought back to Earth from deep space since the last of the Apollo landings in 1972, and the first to be collected from beyond the moon's orbit.

A reason Genesis was feasible as a mission is that it was designed using low energy passageways. Figure 2.8 shows a three-dimensional view of the Genesis trajectory (kindly supplied by Roby Wilson). The spacecraft was launched to a

[6]SOHO is a spacecraft mission designed to study the internal structure of the Sun, its extensive outer atmosphere and the origin of the solar wind, the stream of highly ionized gas that blows continuously outward through the solar system. It is a joint project of ESA and NASA. See `http://soho.estec.esa.nl` for more information.

[7]See `http://genesismission.jpl.nasa.gov/` for further information.

halo orbit in the vicinity of the Sun-Earth L_1 and uses a "heteroclinic-like return" in the three-body dynamics to return to Earth[8].

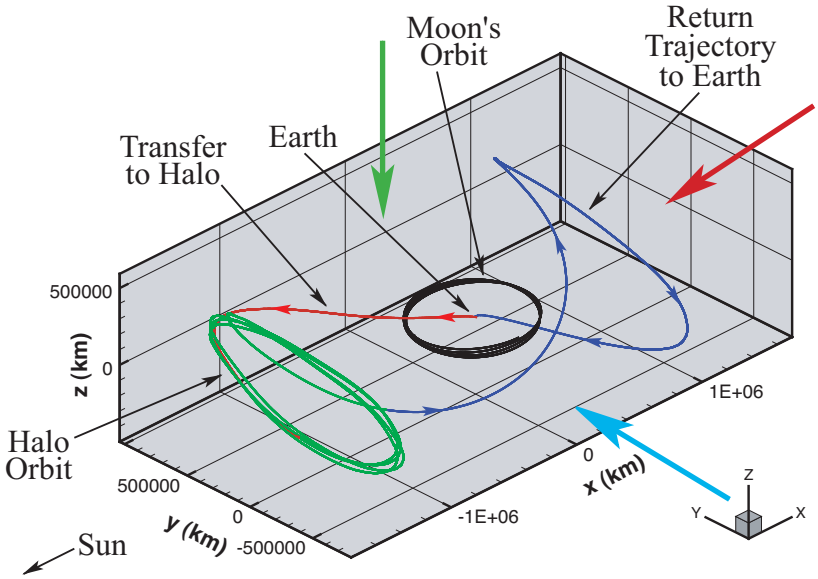


FIGURE 2.8. The Genesis Discovery Mission trajectory. The three arrows correspond to the three projections shown in Figure 2.9. Figure courtesy of Roby Wilson, Jet Propulsion Laboratory

As noted above, L_1 is the unstable equilibrium point between the Sun and the Earth at roughly 1.5 million km from the Earth in the direction of the Sun. Genesis took a low energy path to its halo orbit, stayed there collecting samples for about 2 years, and returned home on another low energy path.

Figure 2.9 shows three orthographic projections of the Genesis trajectory. These figures, plotted in a rotating frame, show the key parts of the trajectory: the transfer to the halo, the halo orbit itself, and the return to Earth. The rotating frame is defined by fixing the x-axis along the Sun-Earth line, the z-axis in the direction normal to the ecliptic, and with the y-axis completing a right-handed coordinate system. The y-amplitude of the Genesis orbit, which extends from the x-axis to the maximum y-value of the orbit, is about 780,000 km (see Figures 2.8 and 2.9). Note that this is bigger than the radius of the orbit of the Moon, which is about 380,000 km.

As Figures 2.8 and 2.9 show, the trajectory travels between neighborhoods of L_1 and L_2; L_2 is roughly 1.5 million km on the opposite side of the Earth from the Sun. In dynamical systems theory, this is closely related to the existence of a *heteroclinic connection* between the L_1 and L_2 regions.

[8]The orbit is called a "halo orbit" because, as seen from Earth, the flight path follows a halo around the sun. Such orbits were originally named for lunar halo orbits by Farquhar [1968]. By the way, setting a spacecraft exactly to the L_1 point is not a good idea, as the spacecraft's radio signals would be lost in the Suns glare.

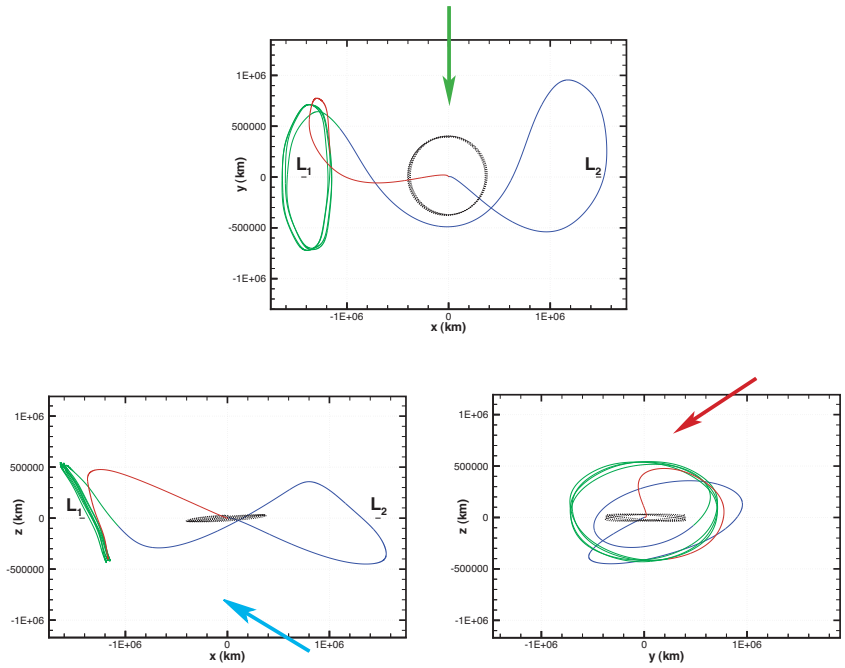


FIGURE 2.9. The xy, xz, and yz projections of the three dimensional Genesis trajectory shown in the preceding figure.

The deeper dynamical significance of the heteroclinic connection for the planar three-body problem is that it allows a classification and a construction of orbits using symbolic dynamics, as was shown in Koon, Lo, Marsden, and Ross [2000], and similar phenomena are seen when the third degree of freedom is included, as discussed in Gómez, Koon, Lo, Marsden, Masdemont, and Ross [2004].

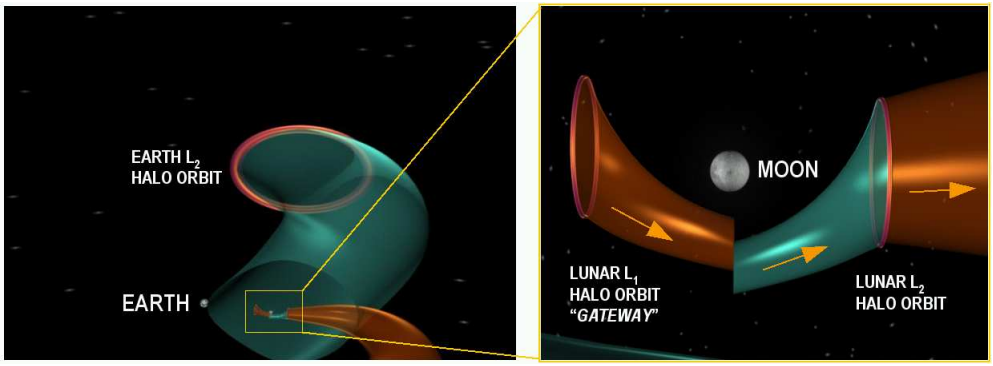
One of the attractive and interesting features of the Genesis trajectory design is that the three year mission, from launch all the way back to Earth return, requires no deterministic maneuver whatsoever and automatically injects into the halo orbit.

It is difficult to use traditional classical algorithms[9] to find a near-optimal solution like that of Genesis, so the design of such a low energy trajectory is facilitated by using dynamical systems methods. This is achieved by using the stable and unstable manifolds as guides in determining the end-to-end trajectory. That Genesis performs its huge exotic trajectory using a deterministic ΔV of *zero* (i.e., no fuel) has created a great deal of interest in both the astronautical and mathematical communities.

Lunar L_1 Gateway Station. The work on Genesis has inspired deeper exploration of the dynamics in Earth's neighborhood (see Lo and Ross [2001]). NASA desires to develop a robust and flexible capability to visit several potential destinations, as suggested by the metro map, Figure 2.7. A Lunar Gateway Station in the vicinity of the lunar L_1 libration point (between the Earth and the Moon) was

[9]See, for example, Farquhar and Dunham [1981], Farquhar, Muhonen, Newman, and Heuberger [1980], and Farquhar, Muhonen, and Richardson [1977].

proposed as a way station for transfers into the solar system and into the Earth-Sun halo orbits. This is enabled by an historical accident: the energy levels of the Sun-Earth L_1 and L_2 points differ from those of the Earth-Moon system by only 50 m/s (as measured by maneuver velocity). The significance of this coincidence to the development of space cannot be overstated. For example, this implies that the lunar L_1 halo orbits are connected to halo orbits around Earth's L_1 and L_2 via low energy pathways, as illustrated in Figure 2.10.



(a) (b)

FIGURE 2.10. (a) The fortuitous arrangement of low energy passageways in near-Earth space implies that lunar L_1 and L_2 halo orbits are connected to halo orbits around Earth's L_1 or L_2 via low energy pathways. Many of NASA's future space telescopes located around the Earth's L_1 or L_2 points may be built in a lunar L_1 orbit and conveyed to the final destination with minimal fuel requirements. (b) Shown in this close-up are two halo orbits at the lunar L_1 and L_2, respectively, and the set of invariant manifolds that provide the low energy departures from the lunar L_1 orbit.

Many of NASA's future space observatories located around the Earth's L_1 or L_2 may be built in a lunar L_1 orbit and conveyed to the final destination with minimal propulsion requirements. When the spacecraft or instruments require servicing, they may be returned from Earth libration orbits to the lunar L_1 orbit where human servicing may be performed, which was shown to be of vital importance for keeping the Hubble Space Telescope operable. Since a lunar L_1 orbit may be reached from Earth in only three days, the infrastructure and complexity of long-term space travel is greatly mitigated. The same orbit could reach any point on the surface of the Moon within hours, making it a perfect location for the return of humans to the Moon. A lunar L_1 orbit is also an excellent point of departure and arrival for interplanetary flights to Mars, the asteroids, and the outer solar system. Several lunar and Earth encounters may be added to further reduce the launch cost and open up the launch period. A lunar L_1 is therefore a versatile hub for a space transportation system.

Multi-Moon Orbiters. Using low energy passageways is in no way limited to the inner solar system. For example, consider a spacecraft in the gravity field of Jupiter and its planet-sized moons. A possible new class of missions to the outer

planet moon systems has been proposed (Koon, Lo, Marsden, and Ross [1999]; Ross, Koon, Lo, and Marsden [2003]). These are missions in which a single spacecraft orbits several moons of Jupiter (or any of the outer planets), allowing long duration observations. Using this *multi-moon orbiter* approach, a single scientific spacecraft orbits several moons of Jupiter (or any of the outer planets) for any desired duration, allowing long duration observations instead of flybys lasting only seconds. For example, a multi-moon orbiter could orbit each of the galilean moons—Callisto, Ganymede, Europa, and Io—one after the other, using a technologically feasible amount of fuel. This approach should work well with existing techniques, enhancing trajectory design capabilities for missions such as a mission to assess the possibility of life on Jupiter's icy moons.

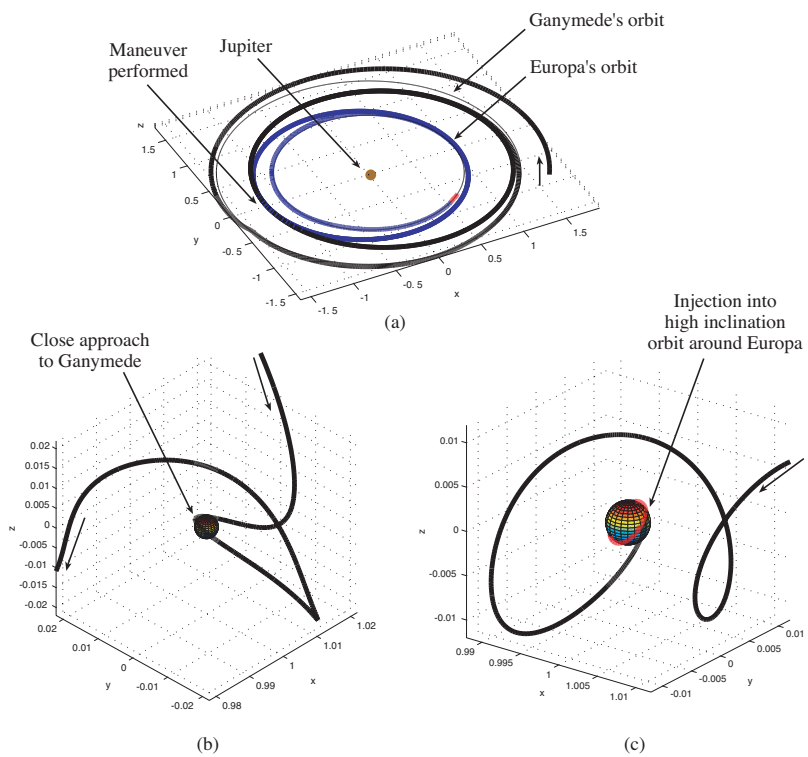


FIGURE 2.11. A multi-moon orbiter space mission concept for the Jovian moons. (a) We show a spacecraft trajectory coming into the Jupiter system and transferring from Ganymede to Europa using a single impulsive maneuver, shown in a Jupiter-centered inertial frame. (b) The spacecraft performs one loop around Ganymede, using no propulsion at all, as shown here in the Jupiter-Ganymede rotating frame. (c) The spacecraft arrives in Europa's vicinity at the end of its journey and performs a final propulsion maneuver to get into a high inclination circular orbit around Europa, as shown here in the Jupiter-Europa rotating frame.

Figure 2.11 shows a low energy transfer trajectory from an initial Jovian insertion trajectory to Ganymede. After one orbit around Ganymede including a close approach, the spacecraft heads onward to Europa, ending in a high inclination orbit around the icy moon.

The fuel requirements for such a tour can be made even smaller if resonant phenomena, as discussed in the next section, are incorporated. By utilizing resonant gravity assists with Jupiter's moons, in addition to ballistic capture and escape orbits leading toward or away from temporary capture orbits about a moon, a tour can be constructed using very little fuel, such as the one shown in Figure 2.12.

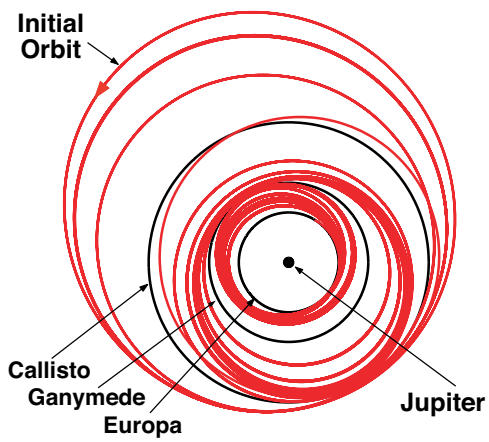


FIGURE 2.12. The trajectory shown is a simulation of a restricted 5-body problem and requires a vanishingly small ΔV. The solution likely shadows a zero ΔV solution, i.e., a purely natural solution which may bring an asteroid interloper into intimate contact with the Jupiter system. The multi-moon orbiter is a general concept that is applicable for any multi-moon system.

Optimization. As we have seen, one of the reasons that making use of the subtleties of the three body problem to design missions is that they give more fuel efficient trajectories. In other words, they are heading in the direction of *optimal* trajectories.

One study in this direction was the work Serban, Koon, Lo, Marsden, Petzold, Ross, and Wilson [2002] which used the optimization software COOPT (Control and Optimization), developed in the group of Linda Petzold, to optimize the fuel needed to get the Genesis spacecraft back on track towards its intended halo (or Lissajous) orbit in case of a launch error. That paper has a brief description of this software and how it works, but basically it is a multiple shooting method that is combined with other DAE (Differential Algebraic Equation) software and SQP (Sequential Quadratic Programming) methods.

Although many of the trajectories outlined above are already very efficient, they also need to be optimized. For example, when one is transferring from one three-body system to another, what is the optimal time to do the transfer burn? What

if one has a low thrust spacecraft? Such questions can be dealt with using optimization software such as COOPT, although each situation can have its own peculiarities.

Another approach to optimization, due to Junge, Marsden, and Ober-Blöbaum [2005], is called DMOC (Discrete Mechanics and Optimal Control) and is based on the use of discrete mechanics techniques together with SQP methods. Discrete mechanics methods are based on a discrete version of the variational principles of mechanics, namely Hamilton's principle for mechanical systems and the Lagrange-d'Alembert principle for mechanical systems subject to external forces.

Discrete mechanics methods have been available for quite some time, going back to discrete optimal control methods in the 1960s. These techniques are closely related to variational integration methods. These methods include symplectic integration algorithms, but also include systems with forcing and dissipation. As such, numerical evidence shows that they have a remarkable respect for the energy budget of a system; far better than any standard numerical error analysis would justify. A development, survey and history of such methods can be found in Marsden and West [2001]; Lew, Marsden, Ortiz, and West [2004].

Roughly speaking, the basic principle of mechanics for a system with configuration variables q (coordinates in a configuration manifold Q) and a Lagrangian (usually the kinetic minus potential energies) $L(q, \dot{q})$ and with external forcing (controls) $F(q, \dot{q})$, namely the Lagrange-d'Alembert principle,

$$\delta \int_a^b L(q(t), \dot{q}(t))\, dt + \int_a^b F \cdot \delta q\, dt = 0$$

is replace by its discrete counterpart. This is done by replacing curves $q(t)$ in the configuration manifold Q by discrete sets of points $(q_1, q_2, \ldots, q_N)$ in Q, replacing the Lagrangian by a discrete Lagrangian L_d that is a function of pairs (q_i, q_{i+1}) and replacing the integrals by an appropriate discrete sum. Asking this discrete sum to be stationary under variations of these points (but with the first and last held fixed) leads to the *discrete Lagrange-d'Alembert equations* which is a set of algebraic equations. The cost function in the optimization problem, which is normally an integral of the velocity phase space variables $(q(t), \dot{q}(t))$ along with the controls, is likewise replaced by its discrete counterpart.

One now seeks to optimize the discrete cost function subject to the discrete Lagrange-d'Alembert equations as constraints (along with any other constraints in the problem, such as actuator limits, collision constraints, etc.). This procedure discretizes the problem and as such it can be directly handed off to SQP routines.

This relatively straightforward procedure already indicates that DMOC has the considerable advantage of *simplicity* as well as respecting the geometry of Lagrangian and Hamiltonian mechanics (such as the symplectic structure, conservation laws, having good energy behavior, etc.).

One of several examples that DMOC has been applied to is of interest in finding Earth-like planets in distant solar systems. This system involves the deployment of an array of interferometers that may be used to detect the chemical composition in the atmosphere of the distant planet. The NASA version of this is called TPF—Terrestrial Planet Finder while ESA's version is called DARWIN. See Figure 2.13.

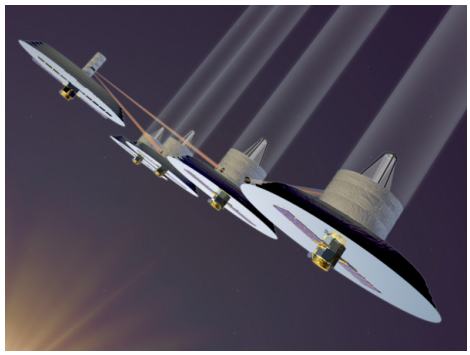


FIGURE 2.13. A system of spacecraft can act as a long baseline set of interferometers capable of detecting Earth-like planets in distant solar systems.

In this example[10], a group of six spacecraft in the vicinity of a Halo-orbit are required to adopt a planar hexagonal formation with center on the Halo orbit and to point in a given direction. The optimization problem is to have the spacecraft get to the final formation from a given initial configuration in such a way that a measure of the fuel consumed is minimized (one could also use a weighted sum of fuel consumed and the time required). Here DMOC gives a solution as indicated in Figure 2.14; the attitude dynamics of each spacecraft, as well as collision avoidance is taken into account.

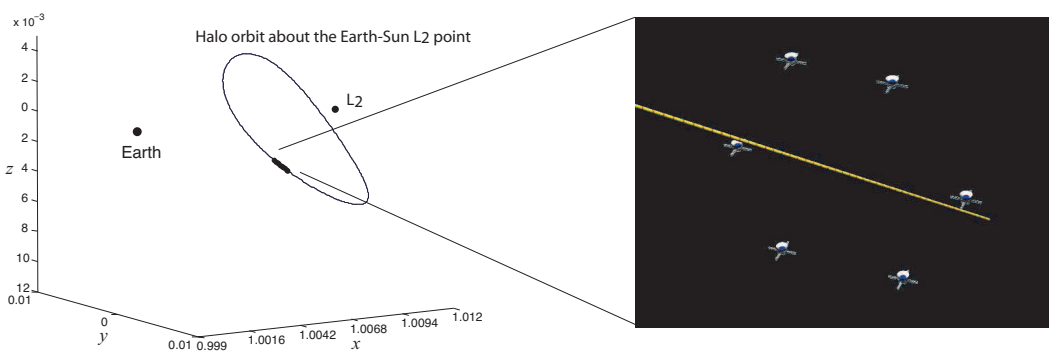


FIGURE 2.14. Using DMOC to deploy a group of six spacecraft that are in a halo orbit around L_2 to optimally assume a hexagonal pattern and point in a given direction.

Because of the good respect for the energy budget of a system, as well as its computational simplicity, one can expect that DMOC will be useful for low thrust missions as well as in formation studies.

[10]We are grateful to Oliver Junge and Sina Ober-Blöbaum for providing this example

3. Transport in the Solar System and Beyond

As we have indicated, there are many phenomena in the solar system that involve interesting transport processes. Examples include the transport of Mars rocks to Earth (the rocks could be thrown into Mars orbit by a meteor impact, for instance) and the transport of asteroids and comets from outside of Jupiter's orbit to inside of Jupiter's orbit through the two Jupiter necks shown in Figure 2.3. Several comets, such as Oterma did just that (this is described in Koon, Lo, Marsden, and Ross [2000]).

For such processes, one can ask "what is the transport rate?" More specifically, we might ask "how does one compute the percentage of a random distribution on an appropriate energy shell after 1000 years that will go from outside of Jupiter's orbit (the exterior realm) to inside (the Sun, or interior, realm)?" Similarly, "what is the probability of transport of Kuiper belt objects from outside of Neptune's orbit to inside"?

To study such questions, we need a few more tools from theoretical and computational dynamical systems.

Poincaré Sections. For the planar 3-body problem, the energy surfaces are three dimensional. Thus, using a Poincaré surface of section at fixed energy E represents the system as a 2-dimensional area preserving map. For example, in the Sun-Jupiter-third body system, we might choose a section in the interior realm as shown in Figure 3.1.

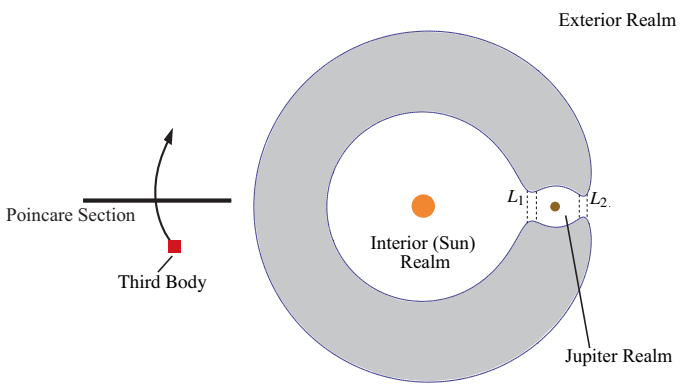


Figure 3.1. A Poincaré surface of section in the exterior realm of the Sun-Jupiter-third body system.

Such a procedure then produces a standard Poincaré map picture, as shown in Figure 3.2.

As we have indicated, and it is also important for transport in both the celestial as well as the molecular context, is that these different Poincaré sections are linked by the Conley–McGehee tubes.

MANGEN. To carry out the needed computations, software is of course required. While there are lots of packages available, we shall concentrate on two of them. First of all, MANGEN (*Manifold Generation*) computes, amongst many other things, invariant manifolds and transport rates between different resonant regions using dynamical systems methods, such as lobe dynamics (see Rom-Kedar and Wiggins

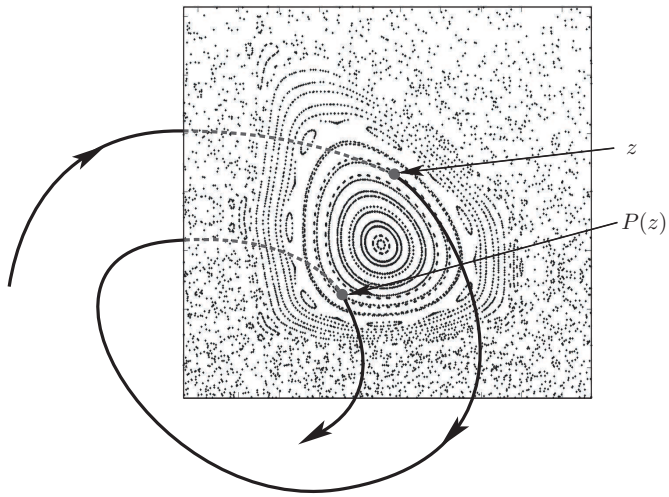


FIGURE 3.2. A Poincaré map produced by intersecting orbits with a Poincaré section.

[1990]; Meiss [1992]; Wiggins [1992]; Rom-Kedar [1999]). While this software was originally developed for the study of fluid systems (see Lekien [2003]), it has proved to be useful for astrodynamics as well as in molecular systems! Mathematics of course provides the common structures in the three areas that enable this.

A sample computation of invariant manifolds in a Poincaré section using MANGEN in the astrodynamics area is shown in Figure 3.3.

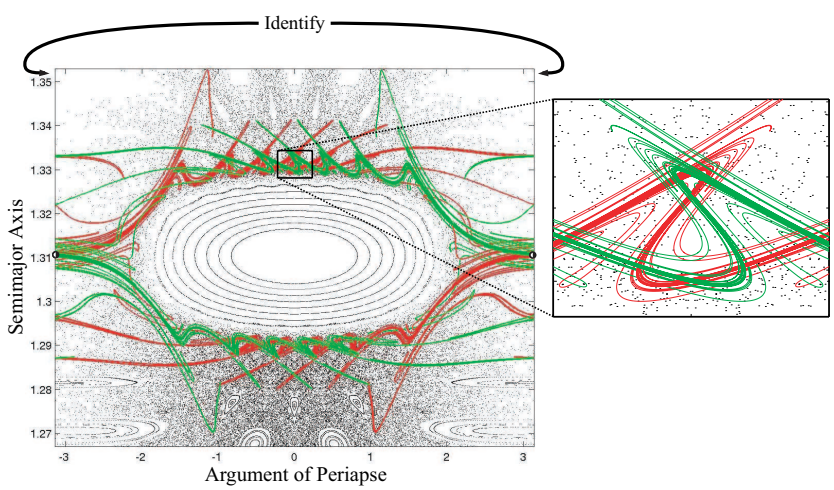


FIGURE 3.3. Unstable and stable manifolds comprising a homoclinic tangle that bounds a resonant region.

One of the interesting questions in dynamical astronomy is to ask about the transport between various regions. Some resonant regions, such at the one shown in Figure 3.4 for a region outside the orbit of Jupiter, have "leaky" boundaries due

to homoclinic tangles. Using lobe dynamics, MANGEN can compute the transport between such regions and neighboring regions.

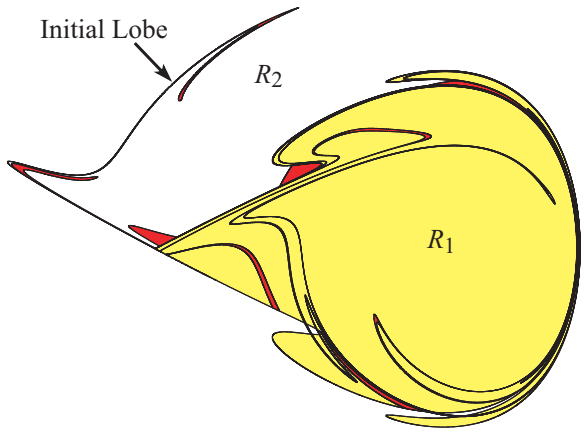


FIGURE 3.4. MANGEN can compute the transport rate between these two resonant regions R_1 and R_2 in the Sun-Jupiter-third body system. One lobe beginning in R_2 (upper left) and several of its images under the Poincaré map are shown. The axes in this Poincaré section are $(x, \dot{x})$.

GAIO. A second piece of software that is very useful is GAIO (Global Analysis of Invariant Objects); see Dellnitz, Froyland, and Junge [2001] and Dellnitz and Junge [2002]. It uses a *set-oriented* methodology, taking a global point of view to compute sets of dynamical interest rather than focusing on individual trajectories. The idea is to cover the region of phase space with boxes and to refine using box subdivisions. A given dynamical system is discretized using a map and then an associated *graph* is constructed. The nodes are the box centers and edges connect those nodes that are dynamically related. This set up allows one to make use of techniques from graph theory, such as graph partitioning.

One of the key concepts in this area is the notion of an *almost invariant set* (AIS), which corresponds to a set containing relatively long-lived dynamical trajectories. An example of an AIS are the above resonance regions. This notion is also important in, for example, biomolecules, where it corresponds to molecular conformations.

There are two related ways to compute almost invariant sets. The first is to use graph partitioning software such as PARTY (Monien, Preis and Diekmann [2000]). The idea is to find efficient ways to cut the graph so that the traffic flow (that is, the transport rate) across the cut is minimized. A second method is the use of eigenfunctions of the associated Perron–Frobenius operator (the induced map on measures). We refer to Dellnitz and Junge [2002] and to Dellnitz, Junge, Koon, Lekien, Lo, Marsden, Padberg, Preis, Ross, and Thiere [2005] for a survey of these methods and for further references. In either case, the computation of transport rates between two AISs is naturally computed within these set oriented methods.

Figure 3.5 shows the same resonance region as above, but computed using GAIO.

In some circumstances, such as shown in Figure 3.6, GAIO and MANGEN can work together to produce more efficient adaptive algorithms.

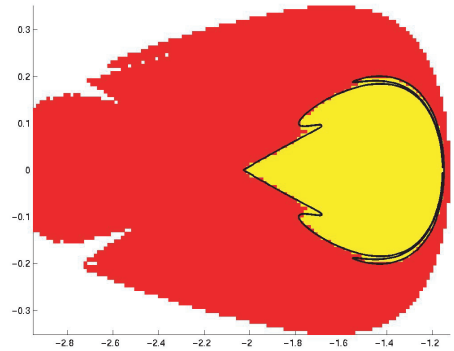


FIGURE 3.5. Resonance region for the three body Sun Jupiter system computed using GAIO.

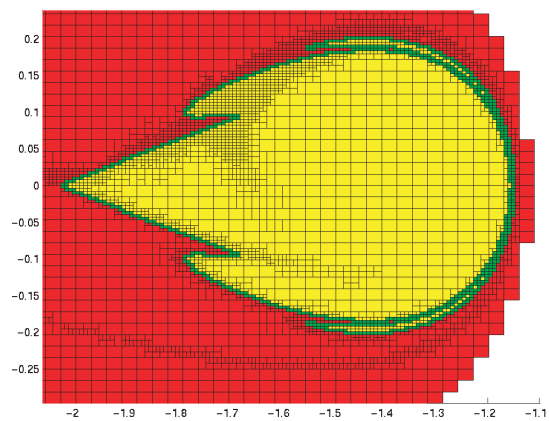


FIGURE 3.6. GAIO and MANGEN working together.

Using these techniques, one gets very concrete answers for transport rates. For example, it is shown in Dellnitz, Junge, Koon, Lekien, Lo, Marsden, Padberg, Preis, Ross, and Thiere [2005] that there is a 28% probability that a randomly chosen particle will go from region R_1 to R_2 in 1,817 Earth years.

Mars Crossing Asteroids. Figure 3.7 shows the data from the Hilda group of asteroids and comets that lie between Jupiter and Mars in a belt relatively close to Jupiter.

Figure 3.8 shows a Poincaré map for a cut chosen at an energy level appropriate to the Hilda group and cutting across it. The coordinates are x along the cut and $\dot{x}$ the corresponding velocity. Also shown is the Mars crosser line; i.e., the coordinates $(x, \dot{x})$ of points whose Keplerian orbit with that initial condition will just graze the

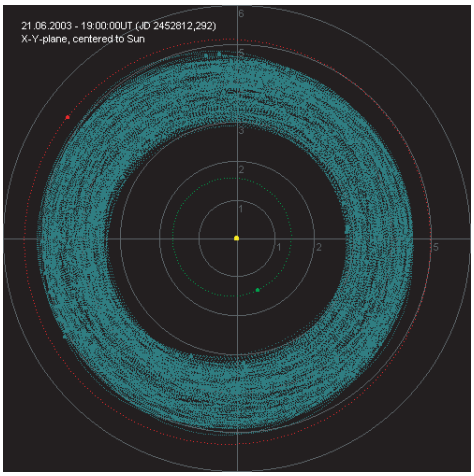


FIGURE 3.7. Orbits of asteroids in the Hilda group. Jupiter is the large dot on the outer dotted circle and Mars is the large dot on the inner dotted circle

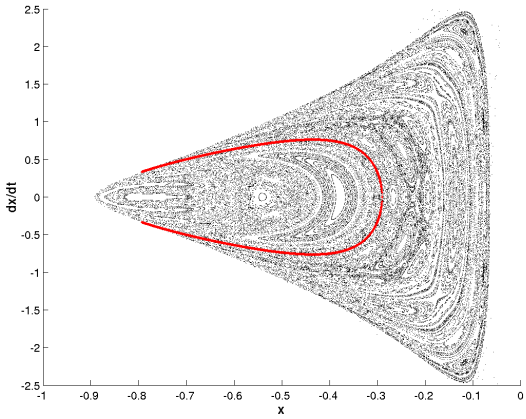


FIGURE 3.8. A Poincaré map corresponding to a section cutting across the Hilda group.

orbit of Mars. Hildas become Mars-crossing asteroids by going from the left of this curve, where they usually reside, to the right of it.

A subset of the Hilda group are the quasi-Hildas, objects known to receive strong perturbations from Jupiter (Kresák [1979]; Ross [2003]). As shown in Dellnitz, Junge, Lo, Marsden, Padberg, Preis, Ross, and Thiere [2005a], GAIO can locate the quasi-Hilda region as one of the AIS in this three body problem. Drawing in the set corresponding to the Mars crosser line, as shown in Figure 3.9, we can then ask "what is the probability that, in a certain period of time, an object in the quasi-Hilda region becomes a Mars-crossing asteroid?" GAIO gives rather concrete answers. It shows that the probability for a typical particle to leave the quasi-Hilda

region is around 6% after 200 iterates of the map, which corresponds to a transit time between 2000 and 6000 Earth years, depending on the location of the particle within the quasi-Hilda region.

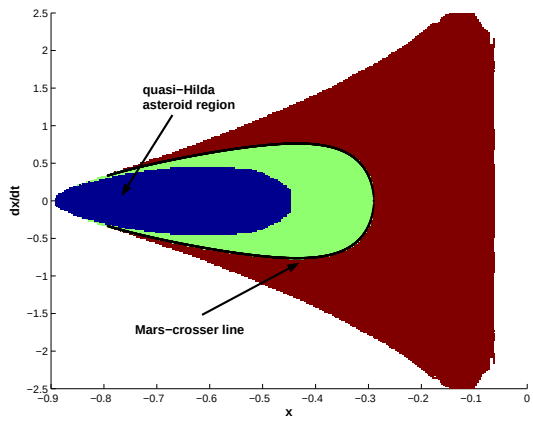


FIGURE 3.9. The quasi Hilda set (an AIS) and the Mars crosser set.

But there is much more to the story, as shown in Figure 3.10, taken from Dellnitz, Junge, Marsden, Padberg, Preis, Ross, and Thiere [2005b].

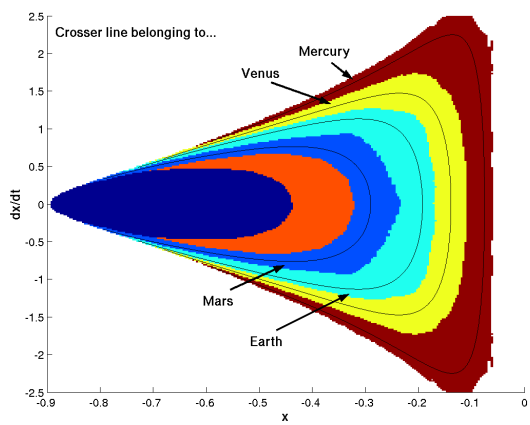


FIGURE 3.10. Crosser lines of all the inner planets superimposed on almost invariant sets for the Sun-Jupiter-third body system. Figure taken from Dellnitz, Junge, Marsden, Padberg, Preis, Ross, and Thiere [2005b]

This figure shows all of the inner planet crosser curves; notice how they lie in the middle of various of the Sun-Jupiter-third body almost invariant sets (as computed by GAIO)! The missing curve in the region between Mars and Jupiter is where the main asteroid belt lies. Since almost invariant sets, by their nature, are places

where material in the Sun-Jupiter-third body system lingers for a long time, one can speculate that material should accumulate in these locations (assuming of course that a primordial Jupiter was formed first). A possible scenario is that material was streaming from a cloud in the primordial Jupiter realm, leaking out of Jupiter's Lagrange bottlenecks, and some of this material migrated under encounters with Jupiter to the terrestrial planet region.

The almost invariant sets are computed just with the Sun-Jupiter-third body system, which in principle is independent of the knowledge of the location of the other planets. The figure clearly shows that these are not independent at all and it suggests that the Jupiter system in fact drove the formation of the whole solar system. This computation is consistent with the Titius-Bode Law for the spacing of the planets (but does not "explain it") and could be repeated for any other planetary system. Of course additional analysis and simulation are needed to make this definitive and this is planned for future research. For example, preliminary computations show that this result is not affected if Saturn is added to the Sun-Jupiter system. It is worth noting that important questions could be addressed using the almost invariant sets, beyond simply the spacing of the planets. For instance, what would this Jupiter-driven formation scenario imply, if anything, about the relative masses of the terrestrial planets? Furthermore, is there any dynamical connection between the terrestrial planets and the rocky, planet-sized moons of Jupiter?

The above result highlights one example of the many miracles hidden in the solar system and the relations between the planets! One might even be hopeful that such ideas will be useful in the search for Earth-like planets in other solar systems.

Tubes on Galactic Scales. Tubes are known to govern structure and motion even over galactic scales (Fukushige and Heggie [2000]; Ross [2005]). The huge tails emanating out of some star clusters in orbit about our galaxy are due to stars slipping into tubes connecting the star cluster with the space outside. This is shown in the left panel of Figure 3.11, from a large N-body simulation of Combes, Leon, and Meylan [1999] where N is on the order of a million. It is believed that this escape process can eventually lead to the 'evaporation' of some star clusters over tens of billions of years.

The estimation of this evaporation time scale is possible using a very simple model, that is in principle similar to the three-body model. Shown schematically in the right panel of Figure 3.11, the star cluster is modelled as a smooth potential (due to the cluster stars) plus the steady tidal field of the galaxy. Stars which are above the energy of the Lagrange points escape via Conley–McGehee tubes. From the tubes the evaporation time scale, measured as the time for the cluster to drop from N to $N/2$ stars, can be determined in a similar way to the transport calculations described above.

Tidal tails are not restricted to star clusters alone. Figure 3.12 shows the disrupted spiral galaxy Arp 188, the Tadpole Galaxy. One possible scenario is that a more compact intruder galaxy crossed in front of the Tadpole. During the close encounter, tidal forces drew out the Tadpole's stars, gas, and dust forming the spectacular tail, which is 280 thousand light-years long.

Acknowledgements. We acknowledge the help and advice of our collaborators mentioned in the abstract as well as the assistance of Wendy McKay with the typesetting.

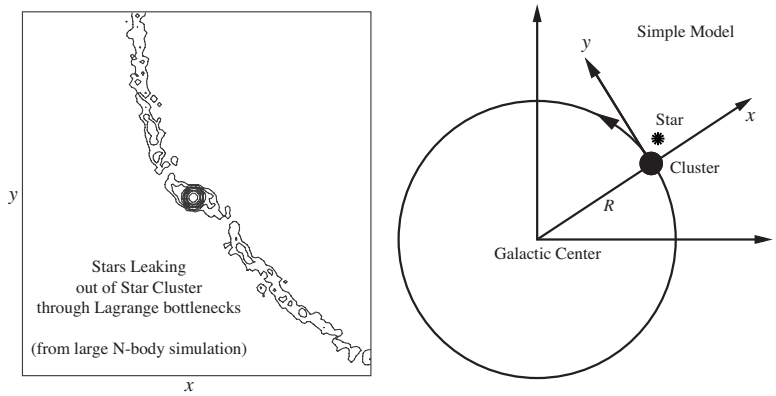


FIGURE 3.11. Large N-body simulations of star cluster 'evaporation' reveal characteristic tube features (left) out of which stars escape. The time scale of evaporation can be estimated in a simple model (right) using the tube structures associated with Lagrange points on either side of the cluster, similar to those described in the three-body problem.

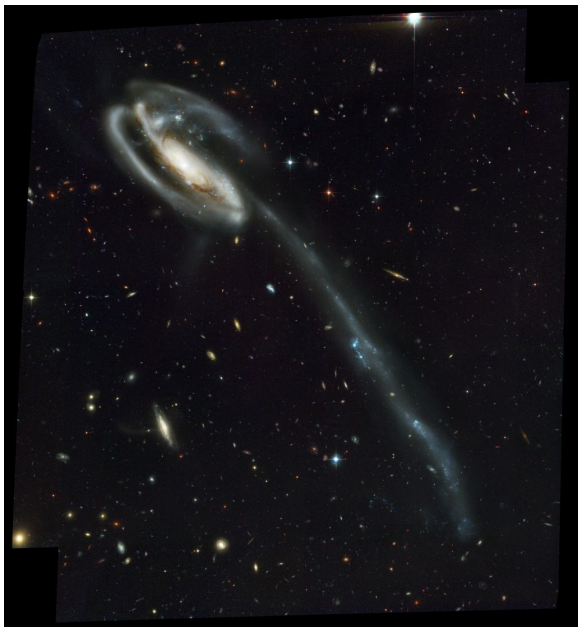


FIGURE 3.12. The Tadpole Galaxy's tidal tail. Credit: ACS Science & Engineering Team, NASA.

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