DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement Volume 2005

Website: http://AIMsciences.org

pp. **297–306**

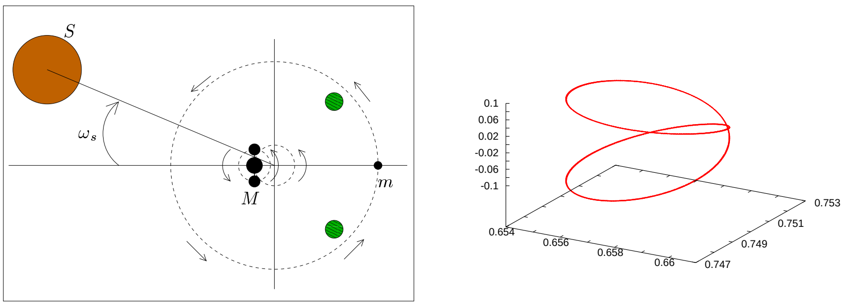
SPACECRAFT DYNAMICS NEAR A BINARY ASTEROID

F. Gabern, W.S. Koon and J.E. Marsden
Control and Dynamical Systems, California Institute of Technology
107–81, 1200 E. California Boulevard, Pasadena, CA 91125, USA

Abstract. We study a simple model for an asteroid pair, namely a planar system consisting of a rigid body and a sphere. This model is interesting because it is one of the simplest that captures the coupling between rotational and translational dynamics. By assuming that the binary is in a relative equilibria of the system, we construct a model for the motion of a spacecraft about this asteroid pair without affecting its motion (that is, we consider a restricted problem). This model can be studied as a perturbation of the standard Restricted Three Body Problem (RTBP). We use the stable zones near the triangular relative equilibrium points of the binary and a normal form of the Hamiltonian to compute stable periodic and quasi-periodic orbits for the spacecraft, which enable it to observe the binary while the binary orbits around the Sun.

1. **Introduction.** In August 1993, the Galileo probe approached 243 Ida, the second asteroid ever encountered by a spacecraft. The greatest discovery from this Galileo fly-by was that Ida has a natural satellite. This moon has been named Dactyl and is the first natural satellite of an asteroid ever found. Since then, over 50 binary asteroids have been discovered and the interest in studying asteroid pairs has grown significantly.

Thus, the study of spacecraft motion about an asteroid pair is an extremely relevant topic for future missions to asteroids. An important question is to find stable zones and orbits for the spacecraft to observe the binary as the pair orbits around the Sun. See Figure 1.



(a) (b)

Figure 1. (a) Schematic diagram showing the two stable zones for the spacecraft (gray circles) to observe the binary (M and m) as the asteroid pair orbits around the Sun (S). (b) A high inclination observation orbit for the spacecraft.

2000 *Mathematics Subject Classification.* 37N05,70F07.

Key words and phrases. Binary asteroids, normal forms, spacecraft dynamics.

Work partly supported by the California Institute of Technology President's Fund and NSF-ITR grant ACI-0204932. F.G. acknowledges the support of the Fulbright–GenCat postdoctoral program, the MCyT/FEDER Grant BFM2003-07521-C02-01 and the CIRIT grant 2001SGR–70.

In solving this problem, we draw on some basic facts of the circular Restricted Three Body Problem (RTBP) which describes the motion of a massless particle under the gravitational attraction of two point masses. As it is well known, the three-body system has two triangular fixed points (see [8]) that are linearly stable if the mass ratio between the primaries is small. In this paper, we study how these equilibria are perturbed in two important ways:

1. When one of the primaries is not a point mass any more but an extended rigid body and
2. When the effect of orbiting the Sun is also considered.

We use a simple model for the asteroid pair, a planar system of a rigid body and a sphere (known as the "sphere restriction" of the Full Two Body Problem [7]). The potential of the rigid body will be approximated by the gravitational potential of three (rigidly) connected masses, as shown in Figure 1(a). As a model for the spacecraft motion, we assume the binary to be in a relative equilibria and we also consider only the direct effect of the Sun on the spacecraft.

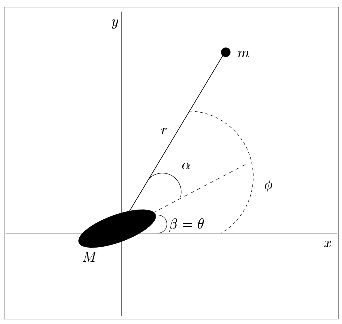
The basic tools used in the paper come from geometric mechanics and dynamical systems theory. The use of Hamiltonian reduction methods allows us to reduce the dimension of the problem and *Normal Form* techniques are central to our numerical exploration. The software (adapted from [3] and [1]) is "handcrafted" and uses an algebraic manipulator to obtain high-order expansions. These expansions cannot be achieved with a commercial-type manipulator and are crucial, for instance, to obtain relatively high inclination observation orbits for the spacecraft.

The paper is organized as follows: In §2, we derive the reduced equations for the asteroid pair via reduction theory and make a preliminary study of this reduced model. In §3, we construct the models for the spacecraft motion based on a particular solution of the asteroid pair. In §4, we study the dynamics of these models in the vicinity of the stable triangular points and use this study to find stable periodic and quasi-periodic orbits for the spacecraft to observe the binary, while the binary orbits the Sun.

2. **Reduced Model of Asteroid Pair.**

2.1. **Reduced Equations for the Binary.** Consider the mechanical system of a rigid body and a sphere in a plane, as in Figure 2.

(a)



(b)

FIGURE 2. (a) Ida and Dactyl. (b) Gravitational interaction of a rigid body and a sphere in the plane.

Reduction of the translational symmetry. Relative to a given inertial frame, the kinetic energy of the binary system is

$$K = \frac{1}{2}m\|\dot{\mathbf{r}}\|^2 + \frac{1}{2}M\|\dot{\mathbf{R}}\|^2 + \frac{1}{2}I_{zz}\dot{\theta}^2$$

where $\mathbf{r}$ and $\mathbf{R}$ are the positions of the sphere's center and the rigid body's barycenter, m and M are the masses of the sphere and the rigid body, I_{zz} is the inertia tensor of the rigid body and the angle θ is as shown in Figure 2(b).

After reducing the translational symmetry (using the fact that $m\mathbf{r} + M\mathbf{R} = 0$ at the system's center of masses), the kinetic energy can be re-written as

$$K = \frac{1}{2}\frac{mM}{m+M}\|\dot{\mathbf{q}}\|^2 + \frac{1}{2}I_{zz}\dot{\theta}^2$$

where $\mathbf{q} = \mathbf{r} - \mathbf{R}$. Furthermore, if the unit of mass is defined such that $\frac{mM}{m+M} = 1$, the unit of time such that $G(m+M) = 1$, and the unit of length as the longest axis of inertia of the rigid body, the kinetic energy can be simplified as

$$K = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 + \frac{1}{2}I_{zz}\dot{\theta}^2.$$

Notice that the configuration space Q of this reduced system is the planar Euclidean group $SE(2)$. Its Lagrangian is of the type kinetic minus potential and can be written locally as

$$L(\mathbf{q}, \theta, \dot{\mathbf{q}}, \dot{\theta}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 + \frac{1}{2}I_{zz}\dot{\theta}^2 - V(\mathbf{q}, \theta). \tag{1}$$

From the Lagrangian, one can define the momenta corresponding to the variables $(\mathbf{q}, \theta)$ via the Legendre transformation: $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}}$, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I_{zz}\dot{\theta}$. Thus, its corresponding Hamiltonian is

$$H = \frac{1}{2}\|\mathbf{p}\|^2 + \frac{1}{2I_{zz}}p_\theta^2 + V(\mathbf{q}, \theta). \tag{2}$$

This system still has an overall rotational symmetry which we would like to reduce next.

Reduction of the rotational symmetry. Let us first perform two preliminary (canonical) changes of variables that will trivialize the action of the symmetry group $G = S^1$ on the configuration space $Q = SE(2)$. The first change is the introduction of the polar coordinates

$$q_x = r\cos\phi, \qquad p_x = p_r\cos\phi - \frac{p_\phi}{r}\sin\phi,$$
$$q_y = r\sin\phi, \qquad p_y = p_r\sin\phi + \frac{p_\phi}{r}\cos\phi.$$

The second one is the use of the relative angles

$$\alpha = \phi - \theta, \qquad p_\alpha = p_\phi,$$
$$\beta = \theta, \qquad p_\beta = p_\phi + p_\theta.$$

See Figure 2(b). These changes are the first steps in rewriting the equations of the system using the *body frame* of the rigid body. After performing these changes, the Lagrangian is

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\alpha}^2 + \frac{1}{2}(r^2 + I_{zz})\dot{\beta}^2 + r^2\dot{\alpha}\dot{\beta} - V(r, \alpha)$$

where the potential does not depend on the "orientation" angle θ of the rigid body due to its invariance under rotations. Notice that the group action on the (r, α, β)

variables is trivial: $\Phi_\varphi(r,\alpha,\beta) = (r,\alpha,\beta+\varphi)$. Moreover, the Hamiltonian in these new coordinates is given by

$$H = \frac{1}{2}p_r^2 + \left(\frac{1}{2r^2} + \frac{1}{2I_{zz}}\right)p_\alpha^2 + \frac{1}{2I_{zz}}p_\beta^2 - \frac{1}{I_{zz}}p_\alpha p_\beta + V(r,\alpha) \tag{3}$$

where $p_\alpha = r^2\dot\alpha + r^2\dot\beta$ and $p_\beta = r^2\dot\alpha + (r^2+I_{zz})\dot\beta$. Notice that β is a cyclic variable of the Hamiltonian (3). Hence, its conjugate momenta p_β is conserved.

To perform the reduction on the Hamiltonian side, we apply the theory in [4] and [5]. The *momentum map* is given by

$$\mathbf{J}(r,\alpha,\beta,p_r,p_\alpha,p_\beta) = p_\beta$$

which correspond to the angular momentum of the system. The *locked inertia tensor* is $\mathbb{I}(r,\alpha,\beta) = r^2 + I_{zz}$ which is the instantaneous tensor of inertia when the relative motion of the two body is locked. The *mechanical connection* 1-form can be written as

$$\mathcal{A}(r,\alpha,\beta) = \frac{r^2}{r^2+I_{zz}}d\alpha + d\beta.$$

For a fixed $p_\beta = \gamma$, we can perform the momentum shift from $J^{-1}(\gamma)$ to $J^{-1}(0)$ as

$$\tilde p_r = p_r, \qquad \tilde p_\alpha = p_\alpha - \frac{\gamma r^2}{r^2+I_{zz}}, \qquad \tilde p_\beta = 0.$$

The reduced Hamiltonian in $J^{-1}(0)/S^1$ has only two degrees of freedom

$$H = \frac{1}{2}\tilde p_r^2 + \frac{1}{2}\left(\frac{1}{r^2} + \frac{1}{I_{zz}}\right)\tilde p_\alpha^2 + V(r,\alpha) + \frac{\gamma^2}{2(r^2+I_{zz})} \tag{4}$$

with the non-canonical reduced symplectic form given by

$$\omega_\gamma = dr\wedge d\tilde p_r + d\alpha\wedge d\tilde p_\alpha - \frac{2\gamma I_{zz} r}{(r^2+I_{zz})^2}dr\wedge d\alpha. \tag{5}$$

Finally, the reduced Hamiltonian equations can be easily derived from the Hamiltonian (4) and the symplectic form (5). It is a two degrees of freedom system which describes the motion of the sphere in the body frame of the rigid body.

Simple potential of the rigid body. For simplicity, we approximate the potential of the rigid body by the gravitational potential of three masses stuck together by two massless rods as a stand-in, or an approximate model for the potential of an ellipsoid-type body. See Figure 3.

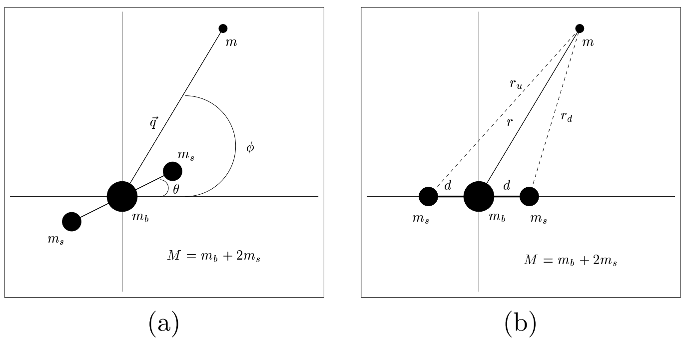


FIGURE 3. Simple model for the potential of the rigid body. (a) Unreduced system. (b) Body-frame.

Following the results in the last section, the Hamiltonian with this potential is:

$$H = \frac{1}{2}\tilde{p}_r^2 + \frac{1}{2}\left(\frac{r^2 + I_{zz}}{r^2 I_{zz}}\right)\tilde{p}_\alpha^2 + V_\gamma(r, \alpha) \tag{6}$$

where

$$V_\gamma = -\frac{1-2\mu}{r} - \mu\left(\frac{1}{r_u} + \frac{1}{r_d}\right) + \frac{\gamma^2}{2(r^2 + I_{zz})}.$$

Here,

$$\mu = \frac{m_s}{m_b + 2m_s}, \quad \nu = \frac{m}{m+M},$$

$$r_u^2 = r^2 + 2dr\cos\alpha + d^2, \text{ and } r_d^2 = r^2 - 2dr\cos\alpha + d^2.$$

See Figure 3(b). The moment of inertia of the system is $I_{zz} = \frac{\mu}{2\nu}$.

2.2. **Relative Equilibria for the Binary.** The relative equilibria of the asteroid pair in the reduced system can be obtained from the Hamiltonian (6). They satisfy the following conditions:

$$\tilde{p}_r = \tilde{p}_\alpha = 0, \qquad \frac{\partial V_\gamma}{\partial r} = \frac{\partial V_\gamma}{\partial \alpha} = 0.$$

After simple computation, we have

$$p_r = 0, \quad p_\alpha = \frac{\gamma r^2}{r^2 + I_{zz}}, \quad \mu dr \sin\alpha\left(\frac{1}{r_d^3} - \frac{1}{r_u^3}\right) = 0. \tag{7}$$

The last equation has two class of solutions:

1. *Collinear configurations*, with $\sin\alpha = 0$, $\alpha \in \{0, \pi\}$; and
2. *T-configurations*, with $r_d = r_u$, which is equivalent to $\alpha = \pm\pi/2$.

In this paper, we do not intend to study the stability of these relative equilibria. Instead, we will just offer a few observations of this topic and its dependency on the parameters of the problem (For more details, see [6]). The collinear configurations are likely to be linearly stable and the T-configurations unstable if the sphere is a "big" body and the rigid body plays the role of a small satellite (this is known for the $\nu = 1$ limit, that corresponds to the usual theory of gravity gradient stabilization). On the other extreme, if the rigid body is "big" ($\nu << 1$), the collinear configurations are unstable and the T-configurations may be linearly stable if the rotation is not too fast (i.e., small γ).

In the next section, we will use these qualitative observations to choose appropriate values of the parameters so that the T-configuration is linearly stable.

Reconstruction of relative equilibrium solution. Since we are interested in visualizing the relative equilibria in the initial configuration space, we need to reconstruct the dynamics from the reduced coordinates. In our case, it is not difficult to see that the reconstruction equations for the group variables are given by

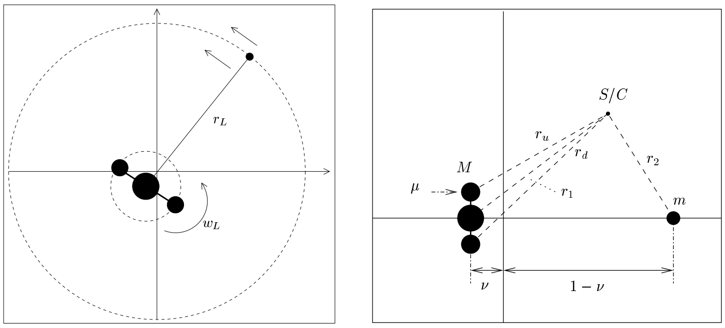
$$\dot{\theta} = \frac{p_\theta}{I_{zz}}, \qquad p_\theta = \gamma - p_\alpha. \tag{8}$$

If a solution in the reduced space $(r(t), \alpha(t), \tilde{p}_r(t), \tilde{p}_\alpha(t))$ is known, one may integrate equation (8) to obtain the evolution of the orientation angle of the rigid body $\theta(t)$.

For instance, if the reduced system is in one of the relative equilibria described above, the conjugate momenta of the orientation variable is constant, $p_\theta =$ constant, and the equation for the orientation angle (8) is trivial to integrate:

$$\dot{\theta} = \frac{p_\theta}{I_{zz}} \equiv \omega_L \implies \theta = \omega_L t + \theta_0.$$

Hence, in the unreduced system, the equilibrium points are periodic orbits of period $T_L = 2\pi/\omega_L$. For example, for one of the T-configuration points ($r \equiv r_L$, $\alpha = \pi/2$), the solution for a rigid body and a sphere is a T-shaped object which is rotating uniformly at rate ω_L. See Figure 4(a).



(a) (b)

FIGURE 4. (a) Relative equilibria for the T-configuration visualized in the unreduced reference frame. The system is rotating uniformly with angular velocity ω_L. (b) Basic T-model for the spacecraft (S/C).

3. **Models for the Motion of the Spacecraft.** In this section, we will construct models for the motion of a spacecraft near the asteroid pair. Let us assume that the binary is in a T-configuration relative equilibria, i.e., we choose the solution with $\alpha = \pi/2$ in (7), and the system is rotating uniformly with frequency ω_L, as in Figure 4(a). In the first model, we will assume that the motion of the spacecraft is affected only by the gravitational interaction of the asteroid pair. In the second model, we will add the perturbation of the Sun to the motion of the spacecraft.

3.1. **Basic T-model.** Let us suppose that $\mathbf{R}_0$ and $\{\mathbf{R}_u, \mathbf{R}_d\}$ are, respectively, the position vectors of the central and the two external masses of the rigid body in an inertial reference frame centered at the system's barycenter. Let us also call $\mathbf{R}_s$ the position of the sphere and $\mathbf{R}$ the position of the spacecraft in the same frame. In this inertial reference frame, the equations of motion for the spacecraft are

$$\ddot{\mathbf{R}} = -\frac{\partial V}{\partial \mathbf{R}}, \qquad \text{with } V = -G\left(\frac{m_b}{\|\mathbf{R}_{0p}\|} + \frac{m_s}{\|\mathbf{R}_{up}\|} + \frac{m_s}{\|\mathbf{R}_{dp}\|} + \frac{m}{\|\mathbf{R}_{sp}\|}\right).$$

Here, $M = m_b + 2m_s$, $\nu = m/(m+M)$, and $\mathbf{R}_{jp} = \mathbf{R} - \mathbf{R}_j$, for $j = 0, u, d, s$.

We now perform a rotation to fix the rigid body's longest principal axis orthogonal to the x-axis: $\mathbf{R} = R_{\theta_L}\mathbf{q}$, where $\mathbf{q} = (x, y, z)$ and R_{θ_L} is the counterclockwise rotation of angle $\theta_L = \omega_L t + \theta_0$ in the xy plane. In this *rotating frame*, the equations of motion for the spacecraft are

$$\begin{aligned} \dot{x} &= p_x + \dot{\theta}_L y, & \dot{y} &= p_y - \dot{\theta}_L x, & \dot{z} &= p_z, \\ \dot{p_x} &= \dot{\theta}_L p_y - \frac{\partial V}{\partial x}, & \dot{p_y} &= -\dot{\theta}_L p_x - \frac{\partial V}{\partial y}, & \dot{p_z} &= -\frac{\partial V}{\partial z}, \end{aligned}$$

where

$$V = -G\left(\frac{m_b}{\|\mathbf{q}_{0p}\|} + \frac{m_s}{\|\mathbf{q}_{up}\|} + \frac{m_s}{\|\mathbf{q}_{dp}\|} + \frac{m}{\|\mathbf{q}_{sp}\|}\right).$$

Here, $\dot{\theta}_L = \omega_L$, and $\mathbf{q}_{jp} = \mathbf{q} - \mathbf{q}_j$, for $j = 0, u, d, s$. Note that $\mathbf{q}_j$ are known from the T-configuration relative equilibria: $\mathbf{q}_0 = (-\nu r_L, 0, 0)$, $\mathbf{q}_u = (-\nu r_L, 1/2, 0)$, $\mathbf{q}_d = (-\nu r_L, -1/2, 0)$ and $\mathbf{q}_s = ((1-\nu) r_L, 0, 0)$. These equations are Hamiltonian (in a canonical way) with the following Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + \omega_L\left(y p_x - x p_y\right) + V(x, y, z).$$

Let us redefine non-dimensional units for this problem as follows: take the new unit of length to be the distance between the center of masses of the rigid body and the sphere, the unit of time such that $\omega_L = 1$ and the asteroid pair does a complete revolution in 2π units of time, and the unit of mass such that $GmM = 1$. Then, the Hamiltonian for the motion of the spacecraft can be written as a $O(\mu)$-perturbation of the RTBP with mass-ratio ν

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + \left(y p_x - x p_y\right) + V(x, y, z)$$

where

$$V = -\frac{(1-\nu)(1-2\mu)}{r_1} - \frac{\nu}{r_2} - \mu(1-\nu)\left(\frac{1}{r_u} + \frac{1}{r_d}\right).$$

Here, $r_1^2 = (x+\nu)^2 + y^2 + z^2$, $r_2^2 = (x-(1-\nu))^2 + y^2 + z^2$, along with $r_u^2 = (x+\nu)^2 + (y-d)^2 + z^2$, $r_d^2 = (x+\nu)^2 + (y+d)^2 + z^2$, and $d = \frac{1}{2r_L}$. See Figure 4(b).

3.2. **T-Model: Perturbation of the Sun.** We now take into consideration the direct effect of Sun's perturbation on the spacecraft. For simplicity, we assume that the uniform rotation of the binary is not affected by the Sun and that the center of masses of the binary is also rotating uniformly around the Sun with a different rate, say ω_s. See Figure 5(b). This idea is similar to the construction of the well-known Bicircular Problem (BCP) that has been used to model some restricted four-body problems in the Solar System (see [2]). In a similar way as it is derived for the BCP, we can add Sun's effect on the spacecraft in the T-model and write it as a periodic perturbation of the T-model:

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + \left(y p_x - x p_y\right) - \frac{m_s}{a_s^2}\left(x\cos\theta_s - y\sin\theta_s\right) + V_s(x, y, z, \theta_s),$$

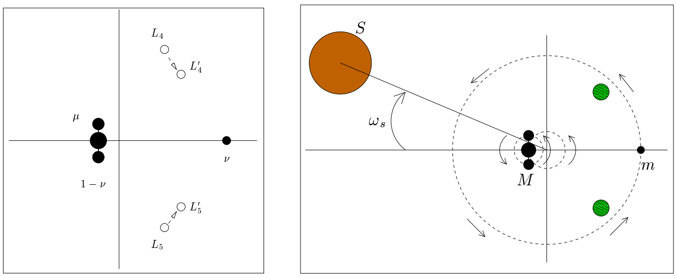
where

$$V_s = -\frac{(1-\nu)(1-2\mu)}{r_1} - \frac{\nu}{r_2} - \mu(1-\nu)\left(\frac{1}{r_u} + \frac{1}{r_d}\right) - \frac{m_s}{r_s}.$$

Here, $r_s^2 = (x + a_s\cos\theta_s)^2 + (y - a_s\sin\theta_s)^2 + z^2$.

4. **Nonlinear dynamics for the Spacecraft models.** In order to study the motion of a spacecraft near an asteroid pair, the parameters of the problem are chosen in such a way that the binary is in a linearly stable relative equilibria. The mass ratio between the sphere and the rigid body is taken as a typical value for certain type of binaries, $\nu = m/(m+M) = 0.001$, which assumes that about 5% of the rigid body's mass is on the extreme part ($\mu = 0.02$). The moment of inertia $I_{zz} = 20$ and the angular momentum of the system $\gamma = 4$ are chosen so that the T-configuration is linearly stable. With this parameters, the T-configuration solution is found at $r_L \approx 5.08$ times the largest dimension of the rigid body.

When the perturbation of the Sun is taken into account, we assume that the binary is in the main asteroid belt ($a_s \approx 3$ A.U.) and that its total mass is the one of a medium/large size asteroid (10^{17} kg). That give us the rest of the parameters for the second model in adimensional units: $a_s = 1.5 \times 10^6$ and $m_s = 10^{13}$.



(a) (b)

FIGURE 5. (a) Schematic diagram showing the triangular equilibria of the RTBP persist when one of the primaries is not a point mass but an extended rigid body. (b) Perturbation of the Sun on the T-model.

Using these parameter values in the actual computations, we proceed to make a local study of the dynamics for the spacecraft near the Lagrangian stable regions, knowing that the qualitative results will be valid for a wide range of parameters. It can be shown by using the Implicit Function Theorem that, if the perturbations are small and under some non-resonance conditions, the RTBP triangular points persist in the basic T-model and are replaced by stable periodic orbits after taking into consideration of the Sun's perturbation. See Figure 5.

In this paper, we will focus on the L_4 case, although the same results can be obtained for L_5. The new fixed point that substitutes L_4 in the T-model will be called L_4' and the periodic orbit that has the same period as the Sun's perturbation $T_s = \frac{2\pi}{w_s}$ will be named $PO(L_4')$.

4.1. **Study of the T-model at L_4' and $PO(L_4')$.** It is not difficult to compute the eigenvalues of the linear vector field at L_4' or the Floquet modes of the periodic orbit $PO(L_4')$. For the actual example, they correspond to elliptic objects and their frequencies are displayed in Table 1. Thus, the T-model system is linearly elliptic at L_4' and around $PO(L_4')$ and we can study the nonlinear dynamics around these objects by constructing a high-order Normal Form around the fixed point L_4' and around the periodic orbit $PO(L_4')$. The Normal Form expansion of the Hamiltonian at the equilibrium points provides a way to obtain all possible motions in a vicinity of these points. Let us briefly describe the main points of the procedure as follows (readers can consult [3] and [1] for details): (1) Translate the origin of coordinates to L_4' or $PO(L_4')$ (in the second case, the translation depends periodically on time). (2) Construct the quadratic Normal Form using the real Jordan form of the linearization of the vector field in the autonomous case and the Floquet Theorem in the time-periodic case. (3) Perform an expansion of the Hamiltonian in a Taylor series (Fourier–Taylor series in the time-periodic case) up to degree N. (4) Construct a high-order Normal Form with a Lie series method (see [3]).

Quadratic Normal Form. The Normal Form up to degree 2 only contains monomials of order 2. The order 0 term is irrelevant in the equations of motion and the order 1 terms are eliminated because the origin is a fixed point after the translation. So, the normal form up to the degree two is a quadratic form

$$H_2 = i\omega_1 q_1 p_1 + i\omega_2 q_2 p_2 + i\omega_3 q_3 p_3,$$

where the values for the frequencies can be found in Table 1.

	L_4'	$PO(L_4')$
ω_1	-0.10702011607983	-0.10702058242758
ω_2	0.99366842989866	0.99366615570514
ω_3	1.00058470215019	1.00058692342681

Table 1. Normal frequencies for the linear oscillators around the elliptic objects L_4' and $PO(L_4')$.

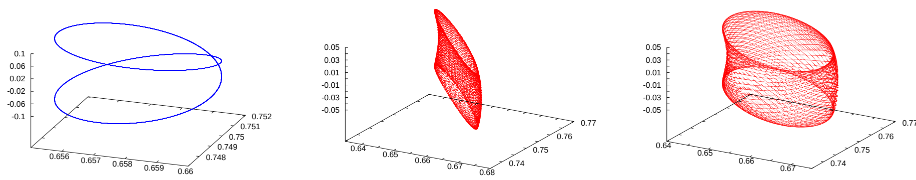


Figure 6. Examples of stable orbits near the Lagrangian zones of the asteroid pair. Left: Periodic orbit in the autonomous case. Center/Right: Two examples of 3D tori for the time-periodic case.

High-order Normal Form. To build the Normal Form of order higher than 2, Lie series method is implemented as in [3]. We use a hand-made software with an algebraic manipulator that is able to deal with the Taylor and Fourier-Taylor series appearing in the computations. To give a flavor of the method, we sketch one step of the process for the time-periodic case. For the autonomous case, just skip the dependence of time.

Let us assume that the Hamiltonian is already in normal form up to degree $r-1$:

$$H = \omega_s p_{\theta_s} + H_2^{(n)}(qp) + \sum_{j=4,\ j=\dot{2}}^{r-1} H_j^{(n)}(qp) + H_r(q,p,\theta_s) + H_{r+1}(q,p,\theta_s) + \cdots$$

where $H_r(q,p,\theta_s) = \sum_{|k|=r} h_r^k(\theta_s) q^{k^1} p^{k^2}$, $\theta_s = \omega_s t + \theta_0$ and $k = (k^1, k^2) \in \mathbb{Z}^3 \times \mathbb{Z}^3$. The extra term $\omega_s p_{\theta_s}$ has been inserted to autonomize the Hamiltonian and p_{θ_s} is the momenta conjugated to the θ_s variable.

We want to make a change of variables that removes the maximum number of terms of $H_r(q,p,\theta_s)$ and autonomizes all the monomials. This canonical change is given by the following generating function

$$G_r = G_r(q,p,\theta_s) = \sum_{\substack{|k|=r \\ k^1 \neq k^2}} \frac{-h_r^k(\theta_s)}{\langle \omega, k^2 - k^1 \rangle} q^{k^1} p^{k^2},$$

where $\langle \cdot, \cdot \rangle$ denotes the dot product. The new Hamiltonian obtained with this change of variables is

$$H' = H + \{H, G_r\} + \frac{1}{2!}\{\{H, G_r\}, G_r\} + \cdots$$

and it is of the following form

$$H' = \omega_s p_{\theta_s} + H_2^{(n)}(qp) + \sum_{j=4,\ j=\dot{2}}^{r-1} H_j^{(n)}(qp) + H_r^{(n)}(qp) + H'_{r+1}(q,p,\theta_s) + \cdots$$

After performing all this changes up to high-order, $N = 32$ in the first case and $N = 24$ in the time-periodic case, we write the final Hamiltonian in action-angle variables by defining $I_j = iq_jp_j$

$$H = \mathcal{N}(I) + \mathcal{R}(I, \varphi, \theta_s)\,, \qquad \mathcal{N}(I) = \sum_{|k|=1}^{N/2} h_k I_1^{k_1} I_2^{k_2} I_3^{k_3}.$$

Since the Normal Form Hamiltonian only depends on actions, it is trivially integrable. All motions in a (small) vicinity of L_4' and $PO(L_4')$ are periodic or quasi-periodic. They take place on invariant tori of dimensions 1, 2 and 3 (autonomous case) or 2, 3 and 4 (periodic case). See Figure 6 for some examples.

4.2. **Parking Orbits for the Spacecraft.** Some of these practically stable orbits are very interesting because we can use them to "park" the spacecraft to do observations of the binary as the pair orbits around the Sun. For instance, if observations of the asteroid pair with relatively high inclinations are required, one could select $I_1 = I_2 = 0$ to minimize the planar motion and maximize the vertical one, and choose $I_3 = 0.02$ to make the error in the Normal Form small. The result is a periodic orbit (in the autonomous case) or a 2-D invariant torus (in the periodic case) that are quite extended in the vertical direction. See Figure 7. Ongoing work is devoted to apply the ideas developed in this paper to more realistic potentials.

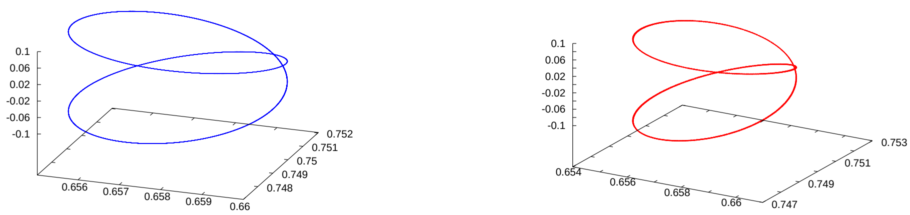


FIGURE 7. Spacecraft orbits suitable for binary observations. Left: Periodic orbit for the autonomous case. Right: 2D torus for the time-periodic case.

References

[1] F. Gabern and À. Jorba, *A restricted four-body model for the dynamics near the Lagrangian points of the Sun-Jupiter system*, Discrete Contin. Dynam. Systems - B, **1** (2001) no. 2, 143–182.
[2] G. Gómez and À. Jorba and J. Masdemont and C. Simó, "Dynamics and mission design near libration points. Vol. IV", World Scientific Monograph Series in Mathematics, 2001.
[3] À. Jorba, *A methodology for the numerical computation of normal forms, centre manifolds and first integrals of Hamiltonian systems*, Exp. Math., **8** (1999) no. 2, 155–195.
[4] J.E. Marsden, "Lectures on mechanics", volume 174 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 1992.
[5] J.E. Marsden and T.S. Ratiu, "Introduction to mechanics and symmetry", volume 17 of *Texts in Applied Mathematics*, Springer-Verlag, New York, second edition, 1999.
[6] D. Scheeres, *Stability of relative equilibria in the full two-body problem*, Annals of the New York Academy of Science, **1017** (2004), 81–94.
[7] D. J. Scheeres and J. Bellerose, *The Restricted Hill Full 4-Body Problem: Application to spacecraft motion about binary asteroids*, Dynam. Syst., **20** (2005) no. 1, 23–44.
[8] V. Szebehely. "Theory of Orbits", Academic Press, 1967.

Received September, 2004; revised May, 2005.

E-mail address: gabern@cds.caltech.edu