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Regularization of the Amended Potential and the Bifurcation of Relative Equilibria

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Summary. In the context of simple mechanical systems with symmetry, we give a method based on blowing up the amended potential for obtaining symmetry-breaking branches of relative equilibria bifurcating from a given set of symmetric relative equilibria. The general method is illustrated with two concrete mechanical examples, the double spherical pendulum and the symmetric coupled rigid bodies.

1. Introduction

1.1. Background

The search for special orbits, such as equilibria and periodic orbits and their bifurcations, is a major theme in the theory of dynamical systems. In the presence of symmetry, it is also natural to study these issues for *relative equilibria*, that is, dynamic orbits generated by the symmetry group, which correspond to equilibrium points in the quotient space. This paper is a contribution to the study of the bifurcation of symmetry-breaking branches from relative equilibria in mechanical systems with symmetry. Our main tool will be that of blowing up the amended potential in the context of simple mechanical systems; that is, systems whose Hamiltonian is of the form kinetic plus potential energies on phase space.

The symplectic context is often used for the study of relative equilibria and their bifurcation. In this setting, one considers a (finite-dimensional) symplectic manifold (P, Ω) with a Lie group G (whose Lie algebra is denoted $\mathfrak{g}$) acting symplectically on P and with an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ and a G-invariant Hamiltonian H. A relatively simple situation occurs when the action of the symmetry group is locally free in a neighborhood of (the group orbit of) a relative equilibrium z_e, when its momentum value μ is a regular point of the coadjoint action and z_e is a nondegenerate critical point of the symplectically reduced Hamiltonian. Then, for every ν close to μ, there is a unique group orbit of relative equilibria (close to the orbit of z_e) with momentum equal to ν. (See, for example, Marsden and Weinstein [1974] and Arnold [1989, appendix 2] for this elementary consequence of the implicit function theorem).

In other words, the relative equilibrium persist to nearby momentum level sets. Patrick [1995] showed that in this situation the set of relative equilibria persisting from z_e form a smooth manifold and, under some extra generic assumptions, this set is a smooth $(\dim G + \operatorname{rank} G)$-dimensional symplectic submanifold of P. This persistence result has been extended in Lerman and Singer [1998], where the case when the isotropy of z_e has positive dimension is considered; in Patrick and Roberts [2000], where a stratification of the set of relative equilibria is induced from the lattice of momentum-generator isotropy subgroups; and in Roberts, Wulff, and Lamb [2002] (Corollary 4.3) and Wulff [2003], where generalizations to noncompact symmetry groups are studied.

The case when one drops the requirement that μ_e be regular was, to our knowledge, first studied in Montaldi [1997]. A persistence result for *extremal* relative equilibria based on topological arguments can be found in the same reference; this approach has been continued in Montaldi and Tokieda [2003]. In this context, z_e is allowed to have a nontrivial isotropy subgroup.

We will say that a point is ***symmetric*** if its isotropy subgroup is nontrivial. The study of periodic orbits and relative equilibria around symmetric points in equivariant dynamical systems is of interest because it is usually in this context that some interesting bifurcation phenomena occur, an observation that can be traced back to [44]. In Hamiltonian systems with symmetry, the structure of the conical singularities of the momentum map at symmetric points (including the infinite-dimensional case) was first developed in Arms, Marsden, and Moncrief [1981].

In the more general setting, including that of proper group actions, Ortega and Ratiu [1997] and Lerman and Singer [1998] extended the persistence results of Patrick and Montaldi. In this case the persistent surface of relative equilibria lies in a symplectic strata of $\mathbf{J}^{-1}(\mu)/G_\mu$ (thought of as a Poisson variety) corresponding to a fixed orbit type. The use of a stratification point of view alone, however, does not seem to be adequate for the purpose of obtaining branches of relative equilibria that break the symmetry.

In the context of (non-Hamiltonian) equivariant dynamical systems, Krupa [1990] studied the problem of bifurcation of relative equilibria from symmetric ones, following a method that consists of the decomposition of the vector field in equivariant components, one in the direction along the group orbit, and another in a transverse direction, along a slice. The bifurcation analysis was then carried out by looking at the bifurcations associated with the flow induced on the slice. In the case of Hamiltonian vector fields, the strategy followed by Krupa has been adapted to take advantage of the symplectic structure, something that is achieved by the use of the Marle-Guillemin-Sternberg (MGS) normal form. Using this normal form, Chossat, Lewis, Ortega, and Ratiu [2002] have studied the structure of relative equilibria close to symmetric orbits in the context of general Hamiltonian G-systems, where G is a Lie group acting properly, giving a method for finding some types of symmetry-breaking branches of relative equilibria bifurcating from a symmetric relative equilibrium. One of their results, obtained using the equivariant branching lemma, states that if z_e is a relative equilibrium of the Hamiltonian system $(P, \Omega, H, G, \mathbf{J})$ and there is a point $\xi \in \mathfrak{g}$, and a subspace $V_0 \subset T_{z_e}P$ such that $\ker D^2(H - \mathbf{J}^\xi)(z_e) = \mathfrak{g}_\mu \cdot z_e \oplus V_0$, then, generically, for all $K \subset G_\xi \cap G_{z_e}$ for which $\dim(V_0^K) = 1$ (where V_0^K is the fixed point set of K in V_0) and which satisfies some extra technical conditions, there is a branch of relative equilibria with isotropy subgroup equal to K bifurcating from z_e. Related ideas can be found in Roberts and de Sousa Dias [1997], Ortega and Ratiu [1999], and Roberts, Wulff, and Lamb [2002]. One may view some of these results as an extension and abstraction of those in Lewis, Marsden, and Ratiu [1987] and Lewis, Ratiu, Simo, and Marsden [1992].

The approaches to the bifurcation theory of relative equilibria mentioned in the preceding paragraph, which make use of slice theorems, are complementary to the method followed in this paper. The application of the slice theorem, aided by the MGS normal form, essentially reduces the analysis of bifurcations from a general relative equilibrium to the analysis of bifurcations from an equilibrium of a system that has only the isotropy subgroup as a symmetry group. That approach still leaves open how the bifurcations of the slice reduced system should be handled. In some cases the blowing-up procedure described in the present paper is appropriate, while in other cases the method of restricting to fixed point spaces and using the equivariant branching lemma (as already mentioned) may be appropriate.

Our blowing-up technique is designed to deal with the fact that as the point in phase space and μ vary in a neighborhood of the relative equilibrium in question, the amended potential is unbounded (and so is singular in a very real sense).

There are some other results in the literature about predicting the existence of periodic orbits or relative equilibria around a given equilibrium or relative equilibrium but which, in contrast with the present paper and the results that we have mentioned above, deal with nearby energy (instead of momentum) level sets. The interested reader can consult the theorems of Weinstein [1973] and Moser [1976] in the context of general Hamiltonian systems and Montaldi, Roberts, and Stewart [1988], Ortega and Ratiu [2002b], and Lerman and Tokieda [1999] in the context of Hamiltonian systems with symmetry. Also, we mention that symmetry-breaking bifurcations have been studied in Montaldi and Roberts [1999] in the context of discrete isotropy subgroups.

In future work, we wish to study not only persistence and bifurcation, but also stability. In fact, the methods here, which rely on the blowup of the amended potential, are relevant for such a study since the amended potential is such a basic tool in stability theory, as in Simo, Lewis, and marsden [1991], as we shall discuss in the next section and in Sections 1.4 and 2.1. See also Patrick [1992], [2002] and references therein for an important study of the stability of relative equilibria when μ is not a generic value.

1.2. Summary of the Results Obtained

This paper gives specific results for the existence and description of symmetry-breaking branches of relative equilibria bifurcating from a given set $\mathcal{E}$ of symmetric relative equilibria, provided that $\mathcal{E}$ is of the type described below. We do this in the context of simple mechanical systems with symmetry, where the symmetry group G is a compact Lie group.

More precisely, we consider the problem of finding the branches of relative equilibria in TQ that emanate from a set $\mathcal{E}$ of relative equilibria of the form $\mathcal{E} = \mathfrak{t} \cdot q_e$, where q_e is a critical point of the potential and $\mathfrak{t} \subset \mathfrak{g}$ is a maximal Abelian subalgebra containing $\mathfrak{g}_{q_e}$. We assume that $G_{q_e} \cong S^1$, although much of what is said in this paper only uses (in a crucial way) that G_{q_e} is Abelian. Furthermore, we assume that G acts freely on a neighborhood around but excluding $G \cdot q_e$. (With these conditions, notice that every relative equilibrium in $\mathcal{E}$ is a symmetric point.) We believe that the statements where we do use that the symmetry group of q_e is S^1 can in fact be generalized to the assumption that G_{q_e} is a torus, as is explained in the Conclusion. (This was also pointed out to us by Tudor Ratiu and Răzvan Tudoran, whom we thank.)

The basic idea of our strategy is a simultaneous rescaling of directions in configuration space along a slice of the action, constructed at q_e, and certain directions in $\mathfrak{g}^*$, the dual of the Lie algebra of G. This allows one to regularize or "blow-up" the amended potential V_μ (we recall the relevant definitions at the end of Section 1.4) around q_e and then apply the implicit function theorem to the blown-up variables in order to find branches of relative equilibria. This regularization is needed because the amended potential is singular at symmetric points.

It is natural to use the amended potential V_μ in this analysis if one recalls that (away from singular points) relative equilibria are critical points of V_μ and, according to the reduced energy-momentum method (see Simo, Lewis, and Marsden [1991]), points where its second variation is positive are stable (this is a generalization of the classical Routh method for stability).

A different technique for applying the amended potential around symmetric configurations to study relative equilibria can be found in Karapetyan [2000]. One important difference between the technique in this reference and the one that we describe in this paper is that by blowing-up the configuration coordinates we are able to consider terms coming from the *amendment* of the amended potential that can not be recovered using the technique in Karapetyan [2000].

1.3. Simple Mechanical System with Symmetry

In this section we introduce some standard facts and notation. We refer the reader to Abraham and Marsden [1978], Marsden [1992], and Marsden and Ratiu [1999] for more details.

Recall that a ***simple mechanical system*** consists of a Riemannian manifold Q together with a potential function $V : Q \to \mathbb{R}$. These elements define a Hamiltonian system on T^*Q (the cotangent bundle on Q) with Hamiltonian given by $H : T^*Q \to \mathbb{R}$, $H(p_q) = \langle\!\langle p_q, p_q \rangle\!\rangle/2 + V(q)$, where $\langle\!\langle\, , \rangle\!\rangle$ is the naturally induced metric on $T^*_q Q$. The Hamiltonian vector field X_H, which determines Hamilton's canonical equations, is defined by the interior product relation $\mathbf{i}_{X_H}\Omega = dH$, where $\Omega = \sum dq^i \wedge dp_i$ is the canonical symplectic form on T^*Q.

Since we want to talk about systems with symmetry, we need to recall some standard notions about group actions. Let G be a Lie group acting on a set M. The ***orbit*** of an element $x \in M$ is the set $\{y \in M \mid y = g \cdot x \text{ for some } g \in G\}$. The subgroup $G_x := \{g \in G \mid g \cdot x = x\} \subset G$ is the ***isotropy subgroup*** of x. If the action is proper, then G_x is compact. As we mentioned before, we say that a point $x \in M$ is ***symmetric*** if $G_x \neq \{e\}$. We say that the action of G is locally free at $x \in M$ if $\mathfrak{g}_x = \{0\}$, where $\mathfrak{g}_x$ denotes the Lie algebra of G_x. If H is a subgroup of G, the ***fixed point set*** for H in M is the set $M^H = \mathrm{Fix}(H, M) := \{y \in M \mid h \cdot y = y \text{ for all } h \in H\}$. If the action of G is (globally) free and proper, then M/G, the space of G-orbits, has the structure of a smooth manifold.

We say that $\xi \in \mathfrak{g}$ (respectively, $\mu \in \mathfrak{g}^*$) is a ***regular element*** if the adjoint orbit of ξ (respectively, the coadjoint orbit of μ) is of maximal dimension.

Let G act on the configuration manifold Q of a simple mechanical system and assume that the metric on Q is G-invariant. Then G acts on T^*Q symplectically by the cotangent lift. This action has an associated momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ given by

$$\left\langle \mathbf{J}(p_q), \xi \right\rangle = \left\langle p_q, \xi_Q(q) \right\rangle,$$

where $\xi \in \mathfrak{g}$, $p_q \in T^*Q$. This means that $J^\xi := \langle \mathbf{J}(\cdot), \xi \rangle$ is the Hamiltonian function of the vector field ξ_{T^*Q}, for every $\xi \in \mathfrak{g}$. We will also make use of the Lagrangian form of the momentum map given by

$$\mathbf{J}_L := \mathbf{J} \circ \mathbb{F}L, \tag{1.1}$$

where $\mathbb{F}L$ denotes the fiber derivative induced by the Lagrangian $L(v_q) = \frac{1}{2} \langle\langle v_1, v_1 \rangle\rangle - V(q)$. Thus, $\left\langle \mathbf{J}_L(v_q), \xi \right\rangle = \left\langle\!\left\langle v_q, \xi_Q(q) \right\rangle\!\right\rangle$.

The ***locked inertia tensor*** is the map $\mathbb{I} : Q \to \mathcal{L}(\mathfrak{g}, \mathfrak{g}^*)$, where $\mathcal{L}(\mathfrak{g}, \mathfrak{g}^*)$ denotes the set of linear transformations from the Lie algebra to its dual, given by

$$\langle \mathbb{I}(q)\xi, \eta \rangle = \left\langle\!\left\langle \xi_Q(q), \eta_Q(q) \right\rangle\!\right\rangle,$$

so that if the action is locally free at q, then $\mathbb{I}(q)$ defines an inner product on $\mathfrak{g}$. From the definition of $\mathbf{J}_L$, it is easy to see that $\mathbf{J}_L(\xi_Q(q)) = \mathbb{I}(q)\xi$.

From G-invariance of the metric and the formula (cf. Marsden and Ratiu [1999, lemma 9.3.7])

$$(\mathrm{Ad}_g \xi)_Q(q) = g \cdot \xi_Q(g^{-1} \cdot q), \tag{1.2}$$

it is easy to show that for all $q \in Q$,

$$\mathbb{I}(g \cdot q) = \mathrm{Ad}^*_{g^{-1}} \circ \mathbb{I}(q) \circ \mathrm{Ad}_{g^{-1}}. \tag{1.3}$$

From this, one gets the following:

Proposition 1.1. *For all $q \in Q$,*

$$d \left\langle \mathbb{I}(\cdot)\xi, \eta \right\rangle (q) \cdot \zeta_Q(q) = \left\langle \mathbb{I}(q)[\xi, \zeta], \eta \right\rangle + \left\langle \mathbb{I}(q)\xi, [\eta, \zeta] \right\rangle .$$

We will also need to use an infinitesimal version of equation (1.3). Multiplying both sides of 1.3 on the right by Ad_g we get $\mathrm{Ad}^*_{g^{-1}} \mathbb{I}(q) = \mathbb{I}(g \cdot q)\mathrm{Ad}_g$. Differentiating with respect to g, we obtain

$$- \mathrm{ad}^*_\xi \circ \mathbb{I}(q) = [D\mathbb{I}(q) \cdot \xi_Q(q)] + \mathbb{I}(q) \circ \mathrm{ad}_\xi . \tag{1.4}$$

1.4. Relative Equilibria

Consider a general Hamiltonian system on a symplectic manifold (P, Ω) with Hamiltonian H and suppose that a Lie group G acts symplectically on P. We say that $z_e \in P$ is a ***relative equilibrium*** if the projection on P/G of the dynamical trajectory that passes through z_e consists of a single point. Equivalently, z_e is a relative equilibrium if $X_H(z_e) \in T_{z_e}(G \cdot z_e)$. Examples of relative equilibria are a rigid body moving around one of its principal axes ($G = SO(3)$) and circular motion in the planar Kepler problem ($G = S^1$).

In this paper we are concerned with symmetry-breaking bifurcations of relative equilibria. The following definition, which formalizes the concept of bifurcation adequately for our purposes, is consistent with the standard notion of bifurcation as in Chow and Hale [1982]. We say that a family of relative equilibria $\mathcal{F} \subset TQ$ ***bifurcates*** (respectively, ***persists***) from a given set of relative equilibria $\mathcal{E} \subset \mathcal{F}$ if there is an open set Λ in a Banach space (the *parameter* space), a connected $\tilde{\mathcal{F}} \subset \Lambda \times TQ$, and a $\lambda_0 \in \Lambda$ such that, denoting $\tilde{\mathcal{F}}_\lambda := \{v \in TQ \mid (\lambda, v) \in \tilde{\mathcal{F}}\}$, we have that $\mathcal{F} = \cup_{\lambda \in \Lambda} \tilde{\mathcal{F}}_\lambda$, $\mathcal{E} = \tilde{\mathcal{F}}_{\lambda_0}$ and there is a neighborhood $V \subset \Lambda$ of λ_0 and an open set $W \subset TQ$, $W \cap \mathcal{E} \neq \varnothing$, with the property that if $\lambda \in V$, $\lambda \neq \lambda_0$, then $\tilde{\mathcal{F}}_\lambda \cap W$ is not homeomorphic (respectively, not equal) to $\tilde{\mathcal{F}}_{\lambda_0} \cap W$. Moreover, if there is a subgroup $K \subset G$ such that the isotropy subgroup of every element in $\mathcal{E}$ is conjugate to K, then we say that $\mathcal{F}$ is a ***symmetry-breaking*** family of relative equilibria bifurcating from $\mathcal{E}$ if the isotropy subgroup of every element in $\mathcal{F} \setminus \mathcal{E}$ is conjugate to a proper subgroup of K.

It can be shown that z_e is a relative equilibrium if there exists a $\xi \in \mathfrak{g}$ such that $dH_\xi(z_e) = 0$, where $H_\xi := H(\cdot) - \langle \mathbf{J}(\cdot) - \mu, \xi \rangle$ (where $\mu = J(z_e)$) is the ***augmented Hamiltonian***. For simple mechanical systems with symmetry, with Hamiltonian $H(p_q) = \langle\!\langle p_q, p_q \rangle\!\rangle/2 + V(q)$, the augmented Hamiltonian criterion for relative equilibria translates into two equivalent criteria: (a) a point $z_e = (q_e, p_e) \in T^*Q$ is a relative equilibrium iff $p_e = \mathbb{F}L(\xi_Q(q_e))$ and q_e is a critical point of V_ξ for some $\xi \in \mathfrak{g}$, where $V_\xi(q) := V(q) - \langle \mathbb{I}(q)\xi, \xi \rangle /2$ is the ***augmented potential***; (b) a point $(q_e, p_e) \in T^*Q$ is a relative equilibrium iff $p_e = \mathcal{A}_\mu(q_e)$ and q_e is a critical point of V_μ for some $\mu \in \mathfrak{g}^*$, where $V_\mu(q) := H\left(\mathcal{A}_\mu(q)\right) = V(q) + \frac{1}{2}\left\langle \mu, \mathbb{I}^{-1}(q)\mu \right\rangle$ is the ***amended potential***. Here $\mathcal{A}_\mu$ is the one-form defined by

$$\langle \mathcal{A}_\mu, v_q \rangle := \langle \mu, \mathcal{A}(v_q) \rangle, \tag{1.5}$$

where $\mathcal{A} : TQ \longrightarrow \mathfrak{g}$ is the ***mechanical connection*** defined by $\mathcal{A}(v_q) := \mathbb{I}^{-1}(q) J_L(v_q)$. The difference between the amended potential and the potential is sometimes called the ***amendment***. Of course, the amended potential criterion for relative equilibrium is only valid at points where the locked inertia tensor is invertible.

The mechanical connection $\mathcal{A}$ introduced above is a connection on the principal bundle $Q \longrightarrow Q/G$, that is to say, $\mathcal{A}$ is G-equivariant and $\mathcal{A}\left(\xi_Q(q)\right) = \xi$. It is easy to see that G-invariance of $\mathcal{A}$ implies that V_μ is G_μ-invariant.

Remark. As we have already explained in Section 1.2, the strategy of this paper is to study the existence of branches of relative equilibria by blowing up the amended potential around configurations where the locked-inertia is singular. It may also be possible to do a bifurcation analysis using the *augmented potential* V_ξ, which is defined even at points where the locked-inertia-tensor is not invertible. We choose, however, to work with the amended potential V_μ because the *blown-up* amended potential that we obtain in the process of regularizing the condition $dV_\mu = 0$ can be related to the (standard) amended potential close to the singular point. This is advantageous because V_μ plays in some sense a more fundamental role than V_ξ for simple mechanical systems on cotangent bundles. For example, V_μ appears as one of the terms in the reduced Hamiltonian (cf. Marsden [1992, chap. 3]). Moreover, positive definiteness of the blown-up amended potential implies, at least in the abelian case, positive definiteness of the second variation of the standard amended potential evaluated at the bifurcating relative equilibria, close to the singular point. This is relevant to the application of the energy momentum method of stability analysis (cf. Marsden [1992, chap. 5]). And, as is shown in Simo, Lewis, and Marsden [1991], the amended potential gives sharper stability conditions than the augmented potential. Consistent with this, the converse of the energy momentum method (in the sense of dissipation-induced instabilities (cf. Bloch, Krishnaprasad, Marsden, and Ratiu [1994]) uses, in a crucial way, the amended potential and not the augmented potential.

Also, V_μ is related to the Routhian of reduced Lagrangian systems (which can be viewed as a generalization of Routh's method for reducing the degrees of freedom of systems with cyclic coordinates). Indeed, one can see this relation by noticing that the Routhian at momentum μ induced by the Lagrangian L is given by (cf. Marsden, Ratiu and Scheurle [2000])

$$R^\mu : TQ \longrightarrow \mathbb{R}, \qquad v_q \mapsto L(v_q) - \left\langle \mu, \mathbb{I}^{-1}(q) J_L(v_q) \right\rangle,$$

and therefore the second term in $R^\mu(v_q)$ restricted to $J_L^{-1}(\mu)$ equals minus twice the amendation term in the amended potential. More importantly, at least in some cases, one can use the technique of blowing up the amended potential described in this paper to blow up the Routhian. This blown-up Routhian defines Euler-Lagrange equations that describe the dynamics close to the bifurcation point (cf. the remark at the end of Section 2.1).

2. Motivation: The Double Spherical Pendulum

In this section we illustrate with an example the blowing-up method that will be discussed in Section 3. The example consists of finding the branches of relative equilibria emanating from the straight-down configuration of the double spherical pendulum. These branches have been studied previously by direct calculation in Marsden and Scheurle [1993], and our purpose in this section is only to illustrate the general method with a simple but nontrivial example.

The double spherical pendulum is the mechanical system depicted in Figure 1 and consists of two point masses m_1, m_2 in $\mathbb{R}^3$ subject to the presence of a constant gravitational field pointing in the negative vertical direction. The mass m_1 is constrained to move on a sphere of radius l_1, and m_2 moves on a sphere of radius l_2 centered around m_1. Thus, the configuration space of the system is $Q := S^2 \times S^2$.

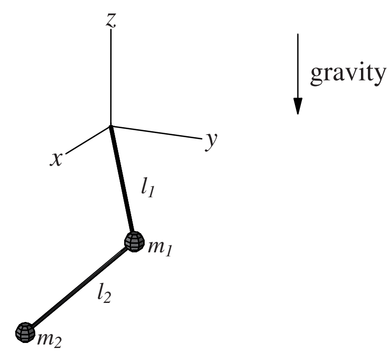

Fig. 1. The double spherical pendulum.

The Lagrangian is given by

$$L(\mathbf{q}_1, \mathbf{q}_2;\ \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2) = \frac{1}{2}\left(m_1\|l_1\dot{\mathbf{q}}_1\|^2 + m_2\|l_1\dot{\mathbf{q}}_1 + l_2\dot{\mathbf{q}}_2\|^2\right) - g\,(m_1 l_1 \mathbf{q}_1 + m_2(l_1\mathbf{q}_1 + l_2\mathbf{q}_2))\cdot\mathbf{k},$$

where $(\mathbf{q}_1, \mathbf{q}_2) \in Q$ and g is the gravitational constant.

This Lagrangian is invariant with respect to the tangent lift of the action of S^1 on Q given by rotations around the vertical axis. The corresponding infinitesimal generator for this action is given by

$$\xi_Q(\mathbf{q}_1, \mathbf{q}_2) = \left(\xi\, l_1(-q_{1y}, q_{1x}, 0), \xi\, l_2(-q_{2y}, q_{2x}, 0)\right)_{(\mathbf{q}_1, \mathbf{q}_2)},$$

where $\mathbf{q}_i = (q_{ix}, q_{iy}, q_{iz})$ $(i = 1, 2)$ and $\xi \in \mathbb{R}$. Thus the locked inertia tensor (which in this case is just a scalar) is given by

$$\begin{aligned}\mathbb{I}(\mathbf{q}_1, \mathbf{q}_2) &= \|(-q_{1y}, q_{1x}, 0), (-q_{2y}, q_{2x}, 0)\|_K^2 \\ &= m_1\|l_1\mathbf{q}_1^\perp\|^2 + m_2\|l_1\mathbf{q}_1^\perp + l_2\mathbf{q}_2^\perp\|^2,\end{aligned}$$

where $\|\ \|_K$ is the norm associated with the metric induced by the kinetic energy (which can be read off from the Lagrangian), "$\perp$" denotes projection onto the horizontal plane, and $\|\ \|$ denotes the usual norm in $\mathbb{R}^2$. The momentum map for the S^1-action on Q is computed to be given by

$$J_L(\mathbf{q}_i, \dot{\mathbf{q}}_i) = \mathbf{k}\cdot[m_1 l_1^2(\mathbf{q}_1 \times \dot{\mathbf{q}}_1) + m_2(l_1\mathbf{q}_1 + l_2\mathbf{q}_2)\times(l_1\dot{\mathbf{q}}_1 + l_2\dot{\mathbf{q}}_2)].$$

2.1. Blowing up the Amended Potential

In preparation to writing down the amended potential, introduce the polar coordinates $\{r_i, \theta_i\}$ defined by

$$\mathbf{q}_i = \frac{1}{l_i}\left(r_i\cos\theta_i, r_i\sin\theta_i, -\sqrt{l_i^2 - r_i^2}\right), \quad i = 1, 2,$$

with $0 \leq r_i \leq l_i$ and $\theta_i \in S^1$. The potential then takes the form

$$\begin{aligned} V &= -g(m_1 l_1 \mathbf{q}_1 + m_2(l_1 \mathbf{q}_1 + l_2 \mathbf{q}_2)) \cdot \mathbf{k} \\ &= -m_1 g \sqrt{l_1^2 - r_1^2} - m_2 g \left(\sqrt{l_1^2 - r_1^2} + \sqrt{l_2^2 - r_2^2} \right) \\ &= V_0 + \frac{(m_1 + m_2) g}{2 l_1} r_1^2 + \frac{m_2 g}{2 l_2} r_2^2 + \text{ h.o.t.}, \end{aligned}$$

where $V_0 = -g(m_1 l_1 + m_2(l_1 + l_2))$ is the value of the potential at the straight-down configuration. The locked inertia tensor becomes, with $\varphi := \theta_2 - \theta_1$,

$$\mathbb{I} = m_1 r_1^2 + m_2(r_1^2 + r_2^2 + 2 r_1 r_2 \cos\varphi).$$

Therefore, the amended potential is given by

$$V_\mu = V + \frac{\mu^2}{2 m_1 r_1^2 + m_2(r_1^2 + r_2^2 + 2 r_1 r_2 \cos\varphi)},$$

which of course is not defined at $(r_1, r_2) = (0, 0)$.

In order to find the relative equilibria whose configuration is close to the two pendulae pointing downwards, we introduce the following rescaling to blow up the singular straight-down configuration. (The strategy for finding the relative equilibria around the other three symmetric states, i.e., one arm resting upwards and the other one resting upwards or downwards, is analogous.)

Assuming that $\mu \geq 0$, introduce the variable τ through scaling

$$\mu = \tau^2, \qquad r_1 = \tau s_1, \qquad r_2 = \tau s_2.$$

The variables s_1, s_2 may be assumed to be bounded away from zero as $\tau \to 0$. Then the amended potential takes the form

$$V_\mu = V_0 + \tau^2 W(\tau, s_1, s_2, \varphi),$$

where

$$\begin{aligned} W(\tau, s_1, s_2, \varphi) &= \frac{1}{2}\left(\frac{g(m_1 + m_2)}{l_1} s_1^2 + \frac{g m_2}{l_2} s_2^2 \right. \\ &\qquad \left. + \frac{1}{(m_1 + m_2) s_1^2 + m_2 s_2^2 + 2 m_2 s_1 s_2 \cos\varphi} \right) + O(\tau^2). \end{aligned}$$

Notice that W is smooth, even at $\tau = 0$.

It is clear that, for $\tau \neq 0$ fixed, the point $(\tau s_1, \tau s_2, \varphi)$ is a critical point of V_μ if and only if (s_1, s_2, φ) is a critical point of

$$W_\tau := W(\tau, \cdot).$$

Also, if $(\tilde{s}_1, \tilde{s}_2, \tilde{\varphi})$ is a nondegenerate critical point of W_0, then, by the implicit function theorem, there are functions $s_1(\tau), s_2(\tau), \varphi(\tau)$ defined on some interval $[0, \varepsilon]$ such that, for each $\tau \in [0, \varepsilon]$, $(s_1(\tau), s_2(\tau), \varphi(\tau))$ is a critical point of W_τ. Therefore, to each nondegenerate critical point of W_0 we can associate a branch of relative equilibria, parametrized by τ, of the form

$$\mathcal{A}_{\tau^2}\left[\psi\left(\tau s_1(\tau), \tau s_2(\tau), \varphi(\tau)\right)\right] \in T^*Q, \tag{2.1}$$

where $\mathcal{A}_\mu$ is the associated one-form to the mechanical connection defined in (1.5) and

$$\psi(r_1, r_2, \varphi) := \left(\frac{1}{l_1}\left(r_1, 0, -\sqrt{l_1^2 - r_1^2}\right), \frac{1}{l_2}\left(r_2 \cos\varphi, r_2 \sin\varphi, -\sqrt{l_2^2 - r_2^2}\right)\right) \in Q.$$

In other words, to each nondegenerate critical point of W_0 we can associate a symmetry-breaking branch of relative equilibria emanating from the straight-down configuration. Notice that the image of ψ restricted to $\{(r_1, r_2, \varphi) \mid r_1 \geq 0\}$ consists of configurations belonging to distinct group-orbits so that, keeping only the critical points $(\tilde{s}_1, \tilde{s}_2, \tilde{\varphi})$ of W_0 satisfying $\tilde{s}_1 > 0$, the curve parametrized by τ given in (2.1) can be regarded as a curve of equivalence classes of relative equilibria in T^*Q/G.

We can think of W_0 as the *blown-up* amended potential. We now proceed to obtain its critical points.

A computation shows that

$$\frac{\partial W_0}{\partial \varphi} = \frac{m_2 s_1 s_2 \sin\varphi}{(m_1 s_1^2 + 2m_2 s_1 s_2 \cos\varphi + m_2(s_1^2 + s_2^2))^2}.$$

If we assume that $s_1 s_2 \neq 0$, then equating the right-hand side to zero gives $\varphi = 0$ or $\varphi = \pi$, which corresponds to $\mathbf{q}_1^\perp$ and $\mathbf{q}_2^\perp$ being colinear. By allowing s_1 and s_2 to have opposite signs, we need to consider only the case

$$\varphi = 0. \tag{2.2}$$

Furthermore,

$$\left.\frac{\partial W_0}{\partial s_1}\right|_{\varphi=0} = -\frac{m_1 + (1+\rho)m_2}{(m_1 + m_2(1+\rho)^2)^2 s_1^3} + \frac{g(m_1+m_2)s_1}{l_1},$$

$$\left.\frac{\partial W_0}{\partial s_2}\right|_{\varphi=0} = m_2\left(\frac{g\rho s_1}{l_2} - \frac{1+\rho}{(m_1 + m_2(1+\rho)^2)^2 s_1^3}\right),$$

where $\rho = s_2/s_1$. Equating the above expressions to zero gives

$$l\,\bar{m}(1+\rho) - \rho(\bar{m} + \rho) = 0 \tag{2.3}$$

and

$$s_1^4 = \frac{l_1}{m_2^2}\,\frac{l\,(1+\rho)}{g\,\rho(\bar{m} + \rho(2+\rho))^2}, \tag{2.4}$$

where $\bar{m} := (m_1 + m_2)/m_2$ and $l := l_2/l_1$. Equations (2.2), (2.3), and (2.4) give the critical points of W_0.

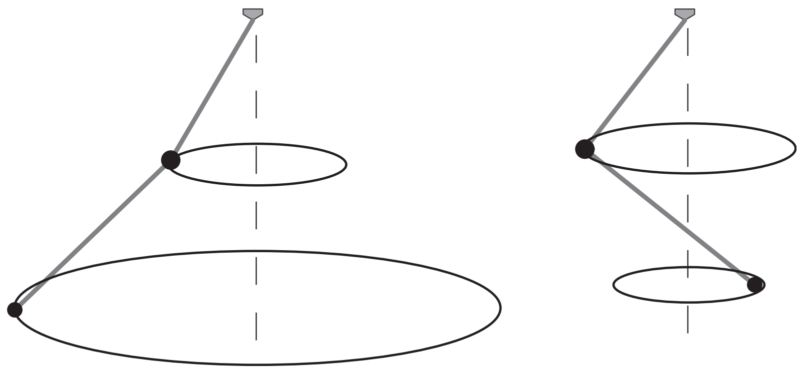

Fig. 2. Relative equilibria configurations for the DSP.

Since equation (2.4) is already explicit once we know ρ, to obtain the critical points of W_0 it only remains to consider equation (2.3). This is a quadratic equation, thus giving two possible branches of relative equilibria. One easily verifies that the two roots $\rho_\pm$ of this equation lie in the ranges

$$\begin{aligned} -\bar{m} \ &< \ \rho_- < -1 \quad \text{(cowboy)}, \\ 0 \ &< \ \rho_+ < \bar{m}\, l \quad \text{(stretched-out)}, \end{aligned}$$

which correspond to the "cowboy" and "stretched-out" types of relative equilibria. (These ranges agree with the ones obtained in Marsden and Scheurle [1993]). These two types are illustrated in Figure 2.

One verifies that for physical values of the system parameters, that is to say, $\bar{m} > 1$, $l > 0$, the critical points of W_0 obtained above are always nondegenerate. Therefore we have obtained all the branches of relative equilibria emanating from the straight-down configuration for the double spherical pendulum.

Remark. The blown-up amended potential W_0 can also give information about the stability of the relative equilibria bifurcating from the equilibrium state. For example, one can check that the second variation of W_0, evaluated at the solution of $dW_0 = 0$ that corresponds to the *stretched-out* relative equilibrium branch, is positive definite for all physically meaningful values of the system parameters $\{l, \bar{m}\}$. It is easy to see that this implies that $\delta^2 V_\mu$ evaluated at a stretched-out relative equilibrium close to the straight-down equilibrium is also positive definite. From the energy momentum method of stability analysis (cf. Simo, Lewis, and Marsden [1991], which in this simple case is nothing but Routh's method), this implies that the straight-down relative equilibria close to the straight-down equilibrium are nonlinearly (orbitally) stable. This agrees with the results in Marsden and Scheurle [1993], but it is easier to compute the second variation working with W_0 than with V_μ. For relative-equilibria bifurcating from the straight-down equilibrium in the *cowboy* branch, the analysis of $\delta^2 W_0$ is inconclusive. In this case, stability information can be obtained from the linearization of the Euler-Lagrange vector field of the blown-up Routhian, obtained using a rescaling analogous to the one we have used to blow up the amended potential.

3. Regularization of the Amended Potential Criterion

In this section we generalize the blowing-up procedure that we applied to the double spherical pendulum. One of the main things that we generalize is that, instead of having an isolated symmetric point, we will consider the situation where G acts freely in some G-invariant neighborhood of the G-orbit of a symmetric equilibrium configuration q_e, excluding the orbit itself. We will continue to restrict ourselves to the case when G is a connected compact Lie group and $G_{q_e} \cong S^1$. (However, up to Section 3.4, we only assume that G_{q_e} is a torus.)

In this setting we are interested in the following problem. Given a set of relative equilibria $\mathcal{E} \subset TQ$ that intersects each G-orbit only once, (a) give sufficient conditions that guarantee the existence of a branch of (classes of) relative equilibria emanating from $\bar{\mathcal{E}} = \pi_G(\mathcal{E})$, where $\pi_G : TQ \to (TQ)/G$ is the canonical projection, and (b) give a criteria for enumerating distinct branches of relative equilibria emanating from $\bar{\mathcal{E}}$.

In this paper we restrict the consideration of these questions when $\mathcal{E}$ satisfies some further conditions, as described in the next subsection.

3.1. Setting of the Problem

Let $(Q, \langle\langle\, , \rangle\rangle, V, G)$ be a simple mechanical G-system, with G a compact connected Lie group with Lie algebra $\mathfrak{g}$. Let $q_e \in Q$ be a symmetric configuration such that G acts freely in some G-invariant neighborhood of the G-orbit of q_e excluding the orbit itself. Assume that $\mathfrak{g}_{q_e}$ contains at least one regular element (which is generically true) and that $\mathfrak{g}_{q_e}$ is abelian. Let $\mathfrak{t}$ be the maximal Abelian subalgebra containing $\mathfrak{g}_{q_e}$.

With these assumptions, let $\mathcal{E} := \mathfrak{t} \cdot q_e$. Then $\mathcal{E}$ is a set of relative equilibria contained in $\operatorname{Fix}(G_{q_e}, TQ)$, as we show in the following remarks.

Remarks. Some immediate consequences of our assumptions are

1. The configuration q_e is a critical point of the potential V. Indeed, $0_{q_e} \in \mathfrak{t} \cdot q_e$ is a relative equilibrium and thus (from the augmented potential criterion) an equilibrium.
2. It is easy to see that the assumption $\mathfrak{g}_{q_e} \subset \mathfrak{t}$ implies that $\mathfrak{t} \cdot q_e \subset \operatorname{Fix}\left(G_{q_e}, TQ\right)$. Thus $G_{v_e} = G_{q_e}$ for every $v_e \in \mathfrak{t} \cdot q_e$. Hence $\mathcal{E}$ intersects each G-orbit only once. Indeed, if $g \in G$ and $v_e, g \cdot v_e \in \mathcal{E}$, then $q_e = g \cdot q_e$, so that $g \in G_{q_e} = G_{v_e}$ and therefore $v_e = g \cdot v_e$.
3. The set $\mathcal{E}$ consists of relative equilibria. Indeed, from the augmented potential criterion, if $\xi' \in \mathfrak{t}$, then $\xi'_Q(q_e)$ is a relative equilibrium if and only if there is a $\xi \in \xi' + \mathfrak{g}_{q_e} \subset \mathfrak{t}$ such that q_e is a critical point of $q \mapsto \langle \mathbb{I}(q), \xi, \xi \rangle$. We now show that this is indeed the case for all $\xi \in \mathfrak{t}$. Since G_{q_e} acts trivially on $\mathfrak{t}$, it follows from (1.3) that the function $\mathbb{I}_{\xi\xi} := \left\langle \mathbb{I}\left(\exp_{q_e}(\cdot)\right) \xi, \xi \right\rangle$ defined on $(\mathfrak{g} \cdot q_e)^{\perp}$ is G_{q_e}-invariant. Since 0 is isolated in $\operatorname{Fix}(G_{q_e}, (\mathfrak{g} \cdot q_e)^{\perp})$, it follows that $d\,\mathbb{I}_{\xi\xi} = 0$. Hence $d\,\langle \mathbb{I}(\cdot)\xi, \xi \rangle\,(q_e) \cdot \delta v_{q_e} = 0$ for every variation δv_{q_e} in $(\mathfrak{g} \cdot q_e)^{\perp}$. It remains to show

that the same is true for every variation δv_{q_e} in $\mathfrak{g} \cdot q_e$. But from Proposition 1.1 and Lemma 3.3 below,

$$d \langle \mathbb{I}(\cdot)\xi, \xi \rangle (q_e) \cdot \eta_Q(q_e) = 2 \langle \mathbb{I}(q_e)\xi, [\xi, \eta] \rangle = 0,$$

for all $\eta \in \mathfrak{g}$. This proves the claim.

3.2. Relative Equilibria in the Associated Bundle

Let $\mathbf{J}_L : TQ \to \mathfrak{g}^*$ be the momentum map as defined in the introduction.

Proposition 3.1. *The map from TQ to $Q \times \mathfrak{g}^*$ given by $(q, v) \mapsto (q, \mathbf{J}_L(q, v))$ restricted to the set of relative equilibria is one-to-one.*

Proof. Since $\left\langle \mathbf{J}_L \left(\xi_Q(q)\right), \xi \right\rangle = \left\langle\!\left\langle \xi_Q(q), \xi_Q(q) \right\rangle\!\right\rangle$, then $\mathbf{J}_L(\xi_Q(q)) = 0$ iff $\xi_Q(q) = 0$. Therefore $\ker \left(\mathbf{J}_L|_{\mathfrak{g}\cdot q}\right) = \{0\}$. Let $\left(q, \xi_{1\,Q}(q)\right), \left(q, \xi_{2\,Q}(q)\right)$ be two relative equilibria with the same momentum. Then $\mathbf{J}_L(q, (\xi_1 - \xi_2)_Q(q)) = 0$, and it follows that $(\xi_1 - \xi_2)_Q(q) = 0$. Thus $\left(q, \xi_{1\,Q}(q)\right) = \left(q, \xi_{2\,Q}(q)\right)$. This shows that the map under consideration is one-to-one when restricted to the set of vertical vectors (i.e., vectors of the form $\xi_Q(q)$, $\xi \in \mathfrak{g}$) and, in particular, when restricted to the set of relative equilibria. □

Therefore the map $(q, v) \mapsto (q, \mathbf{J}_L(q, v))$ identifies the set of relative equilibria in TQ with a subset of $Q \times \mathfrak{g}^*$. From the formula for the action of G on infinitesimal generators (cf. equation (1.2)), it follows that the set of relative equilibria drops to $(Q \times \mathfrak{g}^*)/G$.

For the rest of Section 3 we set the following:

Notation. Let $H := G_{q_e}$ and $N := (\mathfrak{g} \cdot q_e)^\perp \subset T_{q_e}Q$. Let $r_0 \in \mathbb{R}^+$ be such that $\exp_{q_e}$ restricted to $B_{r_0}(0)$, the ball of radius r_0 around 0 in $T_{q_e}Q$, is a diffeomorphism onto its image. Let $N' = N \cap B_{r_0}(0)$. Let $I = (-1, 1) \subset \mathbb{R}$. Abusing notation, we will frequently write $\mathbb{I}$ instead of $\mathbb{I} \circ \exp_{q_e}$, and similarly with the potential V and the amended potential V_μ.

From our assumptions it follows that H acts freely on $N \setminus \{0\}$. Thus $Q' := G \cdot \exp_{q_e}(N')$ is a G-invariant neighborhood of $G \cdot q_e$, and G acts freely on $Q' \setminus (G \cdot q_e)$.

It is easy to see that $N' \times \mathfrak{g}^*$ can be identified with a slice at $(q_e, 0)$ with respect to the diagonal action of G on $Q' \times \mathfrak{g}^*$. Therefore (see for example Duistermaat and Kolk [2000, Section 2.3]),

Proposition 3.2. *The map from $(N' \times \mathfrak{g}^*)/G_{q_e}$ to $(Q' \times \mathfrak{g}^*)/G$ given by*

$$[v, \mu]_{G_{q_e}} \mapsto [\exp_{q_e}(v), \mu]_G$$

is a homeomorphism. Moreover, it is a diffeomorphism when restricted to $\left((N' \setminus \{0\}) \times \mathfrak{g}^*\right)/G_{q_e}$.

As a corollary we get that there is a one-to-one correspondence between equivalence classes in T^*Q/G that correspond to relative equilibria (i.e., equivalence classes of the form $[p_q]$ with $X_H(p_q) \in \mathfrak{g} \cdot p_q$) and the set

$$\{[(v, \mu)] \in (N' \times \mathfrak{g}^*)/G_{q_e} \mid dV_\mu(v) = 0\}.$$

Under this correspondence, a sequence of equivalence classes of relative equilibria in T^*Q/G that approaches $\mathbb{F}L(\mathfrak{t} \cdot q_e)/G$ maps to a sequence $\{[(v_i, \mu_i)]\}$ in $(N \times \mathfrak{g}^*)/G_{q_e}$ with the property that $v_i \to 0$ and μ_i approaches $\mathbb{I}(q_e) \cdot \mathfrak{t}$.

3.3. Regularizing the Group Velocity

The problem arises that, since the locked inertia tensor is not invertible at $v = 0$, the amended potential $V_\mu(v) = V(v) + \frac{1}{2}\left\langle \mu, \mathbb{I}^{-1}(v) \cdot \mu \right\rangle$ is not defined at $v = 0$. In order to regularize the term $\mathbb{I}^{-1}(v)\mu$, we will propose as an ansatz a particular blowing-up of v and μ. For this purpose, let us first introduce a particular splitting of $\mathfrak{g}$ which in turn will induce a splitting of $\mathfrak{g}^*$.

For notational convenience, let $\mathfrak{k}_0 := \mathfrak{g}_{q_e} = \ker \mathbb{I}(q_e)$. Choose $\mathfrak{k}_1 \subset \mathfrak{g}$ a complementary G_{q_e}-invariant subspace of $\mathfrak{k}_0$ in $\mathfrak{t}$. Let $\mathfrak{k}_2 = [\mathfrak{g}, \mathfrak{t}]$. Since $\mathfrak{t}$ is a maximal abelian subalgebra, it follows (see, e.g., Section IV.1 of Bröcker and Dieck [1985]) that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}_2 = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2$.

On the dual of the Lie algebra, let

$$\mathfrak{m}_i := (\mathfrak{k}_j \oplus \mathfrak{k}_k)^\circ,$$

where (i, j, k) is a cyclic permutation of $(0, 1, 2)$. It follows that $\mathfrak{g}^* = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$.

Lemma 3.3. *Let* $\mathfrak{t}^\sharp := [\mathfrak{g}, \mathfrak{t}]^\circ$. *Then* $\mathbb{I}(q_e) \cdot \mathfrak{t} \subset \mathfrak{t}^\sharp$.

Proof. Let $\eta \in \mathfrak{g}_{q_e}$ be a regular element of the adjoint action. Since $\eta \in \mathfrak{t}$, it follows (see, e.g., Bröcker and Dieck [1985]) that $\mathfrak{g}_\eta = \mathfrak{t}$. Hence $\ker \operatorname{ad}_\eta = \mathfrak{t}$. Recall (see ibidem) that $\mathfrak{t}^\perp := [\mathfrak{g}, \mathfrak{t}]$ is a subspace orthogonal to $\mathfrak{t}$ with respect to any G-invariant inner product on $\mathfrak{g}$. Thus $\operatorname{ad}_\eta\big|_{\mathfrak{t}^\perp} : \mathfrak{t}^\perp \longrightarrow \mathfrak{t}^\perp$ is an isomorphism.

From equation (1.4), and the fact that $\eta_Q(q_e) = 0$, we get

$$\operatorname{ad}^*_\eta \circ \mathbb{I}(q_e) = -\mathbb{I}(q_e) \circ \operatorname{ad}_\eta.$$

Let $\xi \in \mathfrak{t}$, $\zeta \in \mathfrak{t}^\perp$. Then $\zeta = \operatorname{ad}_\eta \zeta'$ for some $\zeta' \in \mathfrak{t}^\perp$, and

$$\begin{aligned} \langle \mathbb{I}(q_e)\xi, \zeta \rangle &= \left\langle \mathbb{I}(q_e)\xi, \operatorname{ad}_\eta \zeta' \right\rangle = \left\langle \operatorname{ad}^*_\eta \mathbb{I}(q_e)\xi, \zeta' \right\rangle \\ &= -\left\langle \mathbb{I}(q_e)\operatorname{ad}_\eta \xi, \zeta' \right\rangle = 0, \end{aligned}$$

because $\operatorname{ad}_\eta \xi = 0$. □

Lemma 3.4. *For* $i = 1, 2$, $\mathfrak{m}_i = \mathbb{I}(q_e)\mathfrak{k}_i$.

Proof. Since $\mathfrak{k}_0 = \ker \mathbb{I}(q_e)$, it follows that $\langle \mathbb{I}(q_e)\mathfrak{g}, \mathfrak{k}_0\rangle = \langle \mathbb{I}(q_e)\mathfrak{k}_0, \mathfrak{g}\rangle = \{0\}$, hence $\mathbb{I}(q_e)\mathfrak{g} \subset \mathfrak{k}_0^\circ$. Since $\dim \mathbb{I}(q_e)\mathfrak{g} = \dim \mathfrak{g} - \dim \mathfrak{k}_0 = \dim \mathfrak{k}_0^\circ$, we have that

$$\mathbb{I}(q_e)\mathfrak{g} = \mathfrak{k}_0^\circ. \tag{3.1}$$

In particular, $\mathbb{I}(q_e)\mathfrak{k}_1 \subset \mathfrak{k}_0^\circ$. From Lemma 3.3 we have that $\mathbb{I}(q_e)\mathfrak{t} \subset \mathfrak{k}_2^\circ = \mathfrak{t}^\sharp$. Since $\mathfrak{t} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$, then $\mathbb{I}(q_e)\mathfrak{k}_1 = \mathbb{I}(q_e)\mathfrak{t}$, and it follows that $\mathbb{I}(q_e)\mathfrak{k}_1 \subset \mathfrak{k}_2^\circ$. Therefore $\mathbb{I}(q_e)\mathfrak{k}_1 \subset \mathfrak{k}_0^\circ \cap \mathfrak{k}_2^\circ = (\mathfrak{k}_0 \oplus \mathfrak{k}_2)^\circ = \mathfrak{m}_1$. Since $\dim \mathfrak{k}_1 = \dim(\mathfrak{k}_0 \oplus \mathfrak{k}_2)^\circ$, we conclude that $\mathbb{I}(q_e)\mathfrak{k}_1 = \mathfrak{m}_1$.

From equation (3.1) we have in particular that $\mathbb{I}(q_e)\mathfrak{k}_2 \subset \mathfrak{k}_0^\circ$. Since $\mathbb{I}(q_e)\mathfrak{k}_1 \subset \mathfrak{k}_2^\circ$ and $\mathbb{I}(q_e)$ is symmetric, it follows that $\mathbb{I}(q_e)\mathfrak{k}_2 \subset \mathfrak{k}_1^\circ$. Therefore $\mathbb{I}(q_e)\mathfrak{k}_2 \subset \mathfrak{k}_0^\circ \cap \mathfrak{k}_1^\circ = (\mathfrak{k}_0 \oplus \mathfrak{k}_1)^\circ = \mathfrak{m}_2$. As above, a dimension count shows that the contention is indeed an equality. □

Remark. It follows that $\mathbb{I}(q_e)\cdot\mathfrak{t} = \mathfrak{t}^\sharp \cap \operatorname{ann}(\mathfrak{g}_{q_e})$, which makes Lemma 3.3 more precise. Indeed, using the definitions of $\mathfrak{k}_i$, $i = 0, 1, 2$, $\mathbb{I}(q_e)\cdot\mathfrak{t} = \mathbb{I}(q_e)(\mathfrak{k}_0 \oplus \mathfrak{k}_1) = \mathbb{I}(q_e)\mathfrak{k}_1 = \mathfrak{m}_1 = \operatorname{ann}(\mathfrak{k}_2 \oplus \mathfrak{k}_0) = \operatorname{ann}\mathfrak{k}_2 \cap \operatorname{ann}\mathfrak{k}_0 = \mathfrak{t}^\sharp \cap \mathfrak{g}_{q_e}$.

With this splitting of the dual of the Lie algebra, we are ready to introduce a blowing-up that regularizes $\mathbb{I}^{-1}(v)\mu$. Let $\hat{\mathfrak{m}}_0$ be the unit sphere in $\mathfrak{m}_0$ with respect to some G_{q_e}-invariant inner product, and let

$$\begin{aligned} \nu : \mathbb{R} \times \hat{\mathfrak{m}}_0 \times \mathfrak{m}_1 \times \mathfrak{m}_2 &\longrightarrow \mathfrak{g}^*, \\ (\tau; \hat{\mu}_0, \mu_1, \mu_2) &\mapsto \mu_1 + \tau\mu_2 + \tau^2\hat{\mu}_0. \end{aligned}$$

In this manner, for $v \in (\mathfrak{g}\cdot q_e)^\perp$ and $\mu = (\hat{\mu}_0, \mu_1, \mu_2) \in \hat{\mathfrak{m}}_0 \times \mathfrak{m}_1 \times \mathfrak{m}_2$ fixed, the curve $(\tau v, \nu(\tau, \mu))$ approaches a point in the set $\{0\} \times J_L(\mathfrak{t}\cdot q_e)$, which we have identified with $\mathcal{E}$.

Let $\mathfrak{k} := \mathfrak{k}_1 \oplus \mathfrak{k}_2$. Since G_{q_e} acts trivially on $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is the complement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to a G_{q_e}-invariant inner product, then $\mathfrak{k}$ is a G_{q_e}-invariant complement of $\mathfrak{k}_0 = \mathfrak{g}_{q_e}$ in $\mathfrak{g}$. Let $\widehat{\mathfrak{g}^*} := \hat{\mathfrak{m}}_0 \times \mathfrak{m}_1 \times \mathfrak{m}_2$ and define

$$\begin{aligned} \Phi &: I \times N' \times \widehat{\mathfrak{g}^*} \times \mathfrak{g}_{q_e} \times \mathfrak{k} \to \mathfrak{g}^*, \\ \Phi &\ (\tau, v, \mu, \xi, \eta) := \mathbb{I}(\tau v)\cdot(\xi + \eta) - \nu(\tau, \mu), \end{aligned} \tag{3.2}$$

where $I := (-1, 1)$. We want to show that we can solve $\Phi(\tau, v, \mu, \xi, \eta) = 0$ for ξ, η as smooth functions of τ, v, μ so that $\mathbb{I}^{-1}(\tau v)\nu(\tau, \mu)$ is equal to $\xi + \eta$ and thus is a smooth function of τ, v, μ.

To solve $\Phi = 0$, we apply a Lyapunov-Schmidt type of analysis as follows. Let

$$\Pi : \mathfrak{g}^* \longrightarrow \mathbb{I}(0)\cdot\mathfrak{g}$$

be the projection induced by the splitting $\mathfrak{g}^* = \mathbb{I}(0)\cdot\mathfrak{g} + \operatorname{ann}\mathfrak{k}$. (Notice that $\mathbb{I}(0)\cdot\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and $\operatorname{ann}\mathfrak{k} = \mathfrak{m}_0$.)

Step 1. Solve $\Pi \circ \Phi = 0$ for η in terms of τ, v, μ, ξ: Let

$$\begin{aligned} \hat{\mathbb{I}}(v) &:= \Pi \circ \mathbb{I}(v)|_{\mathfrak{k}}, \\ \tilde{\mathbb{I}}(v) &:= \Pi \circ \mathbb{I}(v)|_{\mathfrak{g}_{q_e}}, \end{aligned}$$

so that $\hat{\mathbb{I}}(\tau v)$ is an isomorphism even when $\tau = 0$. Recalling that $\mu = (\hat{\mu}_0, \mu_1, \mu_2)$, observe that

$$\begin{aligned}\Pi \circ \Phi(0, v, \mu, \xi, \eta) &= \Pi[\mathbb{I}(0) \cdot (\xi + \eta) - \nu(0, \mu)] \\ &= \hat{\mathbb{I}}(0) \cdot \eta - \mu_1.\end{aligned}$$

Thus, letting $\eta_\mu := \hat{\mathbb{I}}(0)^{-1}\mu_1$, we get that $\Pi \circ \Phi(0, v, \mu, \xi, \eta_\mu) = 0$. Moreover, it is easy to see that

$$\frac{\partial}{\partial \eta}(\Pi \circ \Phi)(0, v, \mu, \xi, \eta_\mu) = \hat{\mathbb{I}}(0),$$

which is not singular. Therefore, by the implicit function theorem, there exists a smooth function $\eta(\tau, v, \mu, \xi)$ such that $\eta(0, v, \mu, \xi) = \eta_\mu$ and

$$\Pi \circ \Phi\,(\tau, v, \mu, \xi, \eta(\tau, v, \mu, \xi)) \equiv 0.$$

The function η is defined in some open set in $\mathbb{R} \times N' \times \widehat{\mathfrak{g}^*} \times \mathfrak{g}_{q_e}$ containing $\{0\} \times N' \times \widehat{\mathfrak{g}^*} \times \mathfrak{g}_{q_e}$.

Observe that

Proposition 3.5. *The expression* $\eta_\mu := \hat{\mathbb{I}}(0)^{-1}\mu_1$ *belongs to* $\mathfrak{t}$.

Proof. Notice that

$$\hat{\mathbb{I}}(0)(\mathfrak{t} \cap \mathfrak{k}) = \Pi \circ \mathbb{I}(0)(\mathfrak{t} \cap \mathfrak{k}) = \mathbb{I}(0)(\mathfrak{t} \cap \mathfrak{k}) = \mathbb{I}(0)(\mathfrak{t}) = \mathfrak{m}_1,$$

where in the second-to-last equality we have used that $\mathfrak{t} = \ker \mathbb{I}(0) + (\mathfrak{t} \cap \mathfrak{k})$. Since $\hat{\mathbb{I}}(0)$ is not singular, it follows that $\hat{\mathbb{I}}(0)|_{(\mathfrak{t}\cap\mathfrak{k})}$ is a nonsingular linear transformation onto $\mathfrak{m}_1$. Therefore $\eta_\mu \in \mathfrak{t} \cap \mathfrak{k} \subset \mathfrak{t}$. □

Step 2. Let

$$\begin{aligned}&\varphi\; : I \times N' \times \widehat{\mathfrak{g}^*} \times \mathfrak{g}_{q_e} \to \mathfrak{g}^*, \\ &\varphi\; (\tau, v, \mu, \xi) := (\mathrm{Id} - \Pi) \cdot \Phi\,(\tau, v, \mu, \xi, \eta(\tau, v, \mu, \xi))\,.\end{aligned}$$

In particular, $\varphi(0, v, \mu, \xi) = (\mathrm{Id} - \Pi)(\mathbb{I}(0)(\xi + \eta_\mu) - \mu_1)$. Since both $\mathrm{Im}\,\mathbb{I}(0)$ and $\mathfrak{m}_1$ are contained in $\mathrm{Im}\,\Pi$, it follows that $\varphi(0, v, \mu, \xi) \equiv 0$. Therefore, the information that we can extract from the equation $\varphi = 0$ will only be revealed by the first or a higher derivative of φ with respect to τ, evaluated at $\tau = 0$. We have that

Proposition 3.6. $\partial\varphi/\partial\tau(0, v, \mu, \xi) \equiv -(\mathrm{Id} - \Pi)\mu_2$.

For the proof of this proposition, we need the following lemmas. Recall that G_{q_e} acts linearly on $T_{q_e}Q$. For $\xi \in \mathfrak{g}_{q_e}$, $v \in T_{q_e}Q$, let $\xi \cdot v$ denote the infinitesimal generator of ξ at v.

Lemma 3.7. *Let $\xi, \eta \in \mathfrak{g}$ and $q \in Q$. Suppose that $dV_\xi(q) = 0$, where V_ξ is the augmented potential, and suppose that both η and $[\xi, \eta]$ belong to $\mathfrak{g}_q$. Then $d\,\langle \mathbb{I}(\cdot)\xi, \eta\rangle\,(q) = 0$.*

Proof. Since $dV_\xi(q) = 0$, then $\xi_Q(q)$ is a relative equilibrium, that is to say, $X_H(z_q) = \xi_P(z_q)$, where $z_q = \mathbb{F}L(\xi_Q(q))$ and $P = T^*Q$. Now, suppose that both $\eta, [\eta, \xi] \in \mathfrak{g}_q$. Then

$$\eta_P(z_q) = \left.\frac{d}{dt}\right|_{t=0} \mathbb{F}L(\exp(t\eta)\cdot \xi_Q(q)) = \mathbb{F}L([\eta, \xi]_Q(q)) = 0,$$

where we have used that $g\cdot(\xi_Q(q)) = (\mathrm{Ad}_g\xi)_Q(g\cdot q)$. It follows that $(\xi+\eta)_P(z_q) = X_H(z_q)$, and hence that $0 = dV_{\xi+\eta}(q) = dV_\xi(q) + d\,\langle \mathbb{I}(\cdot)\xi, \eta\rangle\,(q) + \frac{1}{2}d\|\eta_Q(\cdot)\|^2(q)$. The first term in the latter expression vanishes by assumption and the last one vanishes by noticing that, since $\eta_Q(q) = 0$, $\|\eta_Q\left(\exp_q(\tau v)\right)\|^2 = O(\tau^2)$ for every $v \in T_qQ$. Therefore $\langle \mathbb{I}(0)\xi, \eta\rangle = 0$. □

Lemma 3.8. *For all $\xi \in \mathfrak{g}_{q_e}$, $\eta \in \mathfrak{t}$, $d\,\langle \mathbb{I}(\cdot)\xi, \eta\rangle\,(q_e) = 0$.*

Proof. Since $\mathfrak{g}_{q_e} \subset \mathfrak{t}$ and $\mathfrak{t}$ is a maximal abelian subalgebra, then $[\xi, \eta] = 0 \in \mathfrak{g}_{q_e}$. Therefore the claim follows from the previous lemma. □

Proof of Proposition 3.6. Observe that

$$\begin{aligned}\frac{\partial\varphi}{\partial\tau}(0, v, \mu, \xi) \;=\; &(\mathrm{Id}-\Pi)\big[\,(D\mathbb{I}(0)\cdot v)\,(\xi+\eta_\mu)\\ &\qquad + \mathbb{I}(0)\frac{\partial\eta}{\partial\tau}(0, v, \mu, \xi) - \frac{\partial\nu}{\partial\tau}(0, \mu)\big].\end{aligned}$$

As above, $(\mathrm{Id}-\Pi)\circ\mathbb{I}(0) = 0$ because $\mathrm{Im}\,\mathbb{I}(0) = \mathrm{Im}\,\Pi$. From Lemma 3.8,

$$(D\mathbb{I}(0)\cdot v)(\mathfrak{t}) \subset \mathrm{ann}(\mathfrak{g}_{q_e}) = \mathrm{Im}\,\Pi.$$

But, by Proposition 3.5 and since $\xi \in \mathfrak{g}_{q_e} \subset \mathfrak{t}$, $\xi + \eta_\mu \in \mathfrak{t}$. Therefore $(\mathrm{Id}-\Pi)(D\mathbb{I}(0)\cdot v)(\xi+\eta_\mu) = 0$. Since $\partial\nu/\partial\tau(0, \mu) = \mu_2 \in \mathrm{Im}\,\Pi$, the proposition follows. □

Since $\varphi(0, v, \mu, \xi) = \partial\varphi/\partial\tau(0, v, \mu, \xi) \equiv 0$, it follows that $\varphi(\tau, v, \mu, \xi) = \tau^2\varphi_2(\tau, v, \mu, \xi)$ for some smooth function φ_2. We now wish to investigate under what conditions the equation $\varphi_2 = 0$ defines ξ as a function of τ, v, μ. This can be the case only if the equation

$$\varphi_2(0, v, \mu, \xi) = 0 \tag{3.3}$$

can be solved for ξ as a function of v, μ. In order to be able to apply the implicit function theorem to continue the solution for $\tau \neq 0$, we need to have that

$$\frac{\partial\varphi_2}{\partial\xi}\,(0, v, \mu, \xi(v, \mu)) = \frac{\partial}{\partial\xi}\frac{\partial^2\varphi}{\partial\tau^2}\,(0, v, \mu, \xi(v, \mu))$$

be nonsingular, where $\xi(v, \mu)$ is the solution to (3.3).

From the definition of φ,

$$\frac{\partial^2\varphi}{\partial\tau^2} = (\mathrm{Id}-\Pi)\Big[(D^2\mathbb{I}(\tau v)\cdot(v,v))(\xi+\eta)+2(D\mathbb{I}(\tau v)\cdot v)\cdot\frac{\partial\eta}{\partial\tau} + \mathbb{I}(\tau v)\cdot\frac{\partial^2\eta}{\partial\tau^2}-\frac{\partial^2\nu}{\partial\tau^2}\Big].$$

Using the fact that $(\mathrm{Id}-\Pi)\circ\mathbb{I}(0)=0$, we get

$$\left.\frac{\partial^3\varphi}{\partial\xi\partial\tau^2}\right|_{\tau=0} = (\mathrm{Id}-\Pi)\Big[(D^2\mathbb{I}(0)\cdot(v,v))\cdot\left(\mathrm{Id}_{\mathfrak{g}_{q_e}}+\left.\frac{\partial\eta}{\partial\xi}\right|_{\tau=0}\right) + 2(D\mathbb{I}(0)\cdot v)\cdot\left.\frac{\partial^2\eta}{\partial\xi\partial\tau}\right|_{\tau=0}\Big],$$

where $\mathrm{Id}_{\mathfrak{g}_{q_e}}$ is the identity operator in $\mathfrak{g}_{q_e}$. Using that η is the solution to the equation $\Pi\circ\Phi=0$ given by the implicit function theorem, we get that

$$\frac{\partial\eta}{\partial\xi} = -\hat{\mathbb{I}}(\tau v)^{-1}\tilde{\mathbb{I}}(\tau v),$$

and

$$\frac{\partial^2\eta}{\partial\tau\partial\xi} = -\hat{\mathbb{I}}(\tau v)^{-1}(D\hat{\mathbb{I}}(\tau v)\cdot v)\hat{\mathbb{I}}(\tau v)^{-1}\tilde{\mathbb{I}}(\tau v)-\hat{\mathbb{I}}(\tau v)^{-1}(D\tilde{\mathbb{I}}(\tau v)\cdot v).$$

Hence, using $\tilde{\mathbb{I}}(0)=0$, we get

$$\left.\frac{\partial\eta}{\partial\xi}\right|_{\tau=0}=0 \quad\text{and}\quad \left.\frac{\partial^2\eta}{\partial\xi\partial\tau}\right|_{\tau=0} = -\hat{\mathbb{I}}(0)^{-1}\left(D\tilde{\mathbb{I}}(0)\cdot v\right).$$

Therefore,

$$\left.\frac{\partial^3\varphi}{\partial\xi\partial\tau^2}\right|_{\substack{\tau=0\\ v,\,\mu}} : \mathfrak{g}_{q_e}\to\operatorname{ann}\mathfrak{k}$$

is given by

$$\left.\frac{\partial^3\varphi}{\partial\xi\partial\tau^2}\right|_{\substack{\tau=0\\ v,\,\mu}} = (\mathrm{Id}-\Pi)\Big[\left.\left(D^2\mathbb{I}(0)\cdot(v,v)\right)\right|_{\mathfrak{g}_{q_e}} - 2(D\mathbb{I}(0)\cdot v)\hat{\mathbb{I}}(0)^{-1}\left(D\tilde{\mathbb{I}}(0)\cdot v\right)\Big],$$

and we observe that it does not depend on μ. Next we note that there is a nondegenerate pairing between $\operatorname{ann}\mathfrak{k}$ and $\mathfrak{g}_{q_e}$. This means that $\dim(\operatorname{ann}\mathfrak{k})=\dim(\mathfrak{g}_{q_e})$ and that if $\mu\in\operatorname{ann}\mathfrak{k}$ is such that $\langle\mu,\xi\rangle=0$ for all $\xi\in\mathfrak{g}_{q_e}$, then $\mu=0$, which is clear because if $\mu\in\operatorname{ann}\mathfrak{k}$, then $\langle\mu,\xi\rangle=0$ for all $\xi\in\mathfrak{k}$, and by definition $\mathfrak{g}=\mathfrak{g}_{q_e}\oplus\mathfrak{k}$.

It follows that $(\partial^3\varphi/\partial\xi\partial\tau)\big|_{\tau=0,v}$ is degenerate at v if and only if the quadratic form on $\mathfrak{g}_{q_e}$ given by

$$\xi \mapsto \|\xi \cdot v\|^2 - \left\langle \left(D\tilde{\mathbb{I}}(0) \cdot v\right) \xi, \hat{\mathbb{I}}(0)^{-1} \left(D\tilde{\mathbb{I}}(0) \cdot v\right) \xi \right\rangle \tag{3.4}$$

is degenerate. Next, let

$$\mathcal{Z} = \{v \in N \mid \text{ the quadratic form in (3.4) is degenerate}\}.$$

Notice that $\mathcal{Z}$ is closed and invariant with respect to multiplication by scalars; in particular, $0 \in \mathcal{Z}$. (In the two examples that we study in this paper, the double spherical pendulum and the symmetric coupled rigid bodies, one has that $\mathcal{Z} = \{0\}$.)

Proposition 3.9. *The set $\mathcal{Z}$ is G_{q_e}-invariant.*

Proof. Let $\xi \in \mathfrak{g}_{q_e}$. From G-invariance of the metric it is clear that $v \mapsto \|\xi \cdot v\|^2$ is G_{q_e}-invariant. Thus it suffices to show that

$$v \mapsto \left\langle \left(D\tilde{\mathbb{I}}(0) \cdot v\right) \xi, \hat{\mathbb{I}}(0)^{-1} \left(D\tilde{\mathbb{I}}(0) \cdot v\right) \xi \right\rangle$$

is G_{q_e}-invariant. Since $\mathfrak{k}$ is G_{q_e}-invariant, we have that $\mathbb{I}(q_e) \cdot \mathfrak{g} \oplus \operatorname{ann}\mathfrak{k}$ is a G_{q_e}-invariant decomposition of $\mathfrak{g}^*$ and therefore Π is G_{q_e}-equivariant. From formula (1.3) and G-invariance of the metric, it follows that $\mathbb{I}(h \cdot v)\xi = h \cdot \mathbb{I}(v)\xi$ and thus $\tilde{\mathbb{I}}(h \cdot v)\xi = h \cdot \tilde{\mathbb{I}}(v)\xi$ for all $\xi \in \mathfrak{g}_{q_e}$, $h \in G_{q_e}$, and $v \in N'$. Therefore $(D\tilde{\mathbb{I}}(0) \cdot v)\xi$ is G_{q_e}-invariant as a function of v, for all $\xi \in \mathfrak{g}_{q_e}$. Since $\langle \cdot, \hat{\mathbb{I}}(0)^{-1} \cdot \rangle$ is a G_{q_e}-invariant inner product on $\mathbb{I}(0)\mathfrak{g}$, the claim follows. □

Let

$$N'' = N' \setminus \mathcal{Z} = N \cap B_{r_0}(0) \setminus \mathcal{Z}. \tag{3.5}$$

We then have the following:

Proposition 3.10. *Restricting φ_2 to the domain $I \times N'' \times \widehat{\mathfrak{g}^*} \times \mathfrak{g}_{q_e}$, the equation*

$$\varphi_2(\tau, v, \mu, \cdot) = 0 \tag{3.6}$$

has a unique solution $\xi(\tau, v, \mu) \in \mathfrak{g}_{q_e}$, which is smooth.

Proof. From the previous discussion we know that if $v \neq \mathcal{Z}$, then $\partial\varphi_2/\partial\xi$ is nonsingular and hence, by the implicit function theorem, there is an open neighborhood $V_0 \subset I \times N'' \times \widehat{\mathfrak{g}^*}$ containing $\{0\} \times N'' \times \widehat{\mathfrak{g}^*}$ such that there exists a smooth function $\xi : V_0 \to \mathfrak{g}_{q_e}$ satisfying (3.6). Outside the set defined by the condition $\tau v = 0$, the equation $\varphi_2(\tau, v, \mu, \xi) = 0$ yields a unique solution for ξ, namely the $\mathfrak{g}_{q_e}$-component of $\mathbb{I}(\tau v)^{-1}\nu(\tau, \mu)$, which is a smooth function of the τ, v, μ parameters. Therefore, ξ is smooth and uniquely defined outside $\{0\} \times N'' \times \widehat{\mathfrak{g}^*}$ and hence on all of $I \times N'' \times \widehat{\mathfrak{g}^*}$. □

Remark. The previous proposition says that if we define

$$\xi(\tau, v, \mu) := \mathbb{I}(\tau v)^{-1}\nu(\tau, \mu)$$

and v is bounded away from $\mathcal{Z}$, then $\xi(\tau, v, \mu)$ is smooth even in a neighborhood of $\tau = 0$.

3.4. Decomposition of the Relative Equilibrium Condition

In this subsection we show that near q_e the amended potential criterion for relative equilibria is equivalent to two conditions, equations (3.10). These are criticality conditions for the amended potential evaluated along slice and group directions, respectively. In the next section we will use the blowing-up that we introduced above in order to regularize these conditions.

Recall that Proposition 3.2 allows us to reduce the problem of finding the set of equivalence classes of relative equilibria whose configuration is near $[q_e]$ to the problem of finding the set of relative equilibria $[v, \mu] \in (N \times \mathfrak{g}^*)/H$ with v close to (but different from) zero. The regularization of the amended potential criterion that we will discuss involves working with a convenient parametrization of $(N\setminus\{0\} \times \mathfrak{g}^*)/H \cong (N\setminus\{0\})/H \times \mathfrak{g}^*$. Our analysis, however, will be local, in the sense that instead of dealing with all of $(N \setminus \{0\})/H$, we will only deal with an open set in $(N \setminus \{0\})/H$ admitting a smooth local section. This means that we will work with a slice on $N \setminus \{0\}$, according to the following:

Definition 3.11. Let G be a Lie group acting freely and properly on a manifold M. We say that $S \subset M$ is a ***slice*** for the action of G on M if S is a connected submanifold of M and

1. $T_sM = \mathfrak{g} \cdot s \oplus T_sS \quad \forall s \in S$;
2. S intersects each G-orbit at most once.

Let S be a slice for the action of H on $\hat{N}$, where $\hat{N}$ is the unit sphere in N. It is guaranteed to exist because H is an isotropy subgroup; thus it is compact, and proper actions always admit a slice (cf. Duistermaat and Kolk [2000, Section 2.3]). Then

$$U := \{\rho s \mid \rho \in (0, r_0), s \in S\} \subset N\setminus\{0\} \tag{3.7}$$

is a slice for the action of H on $N \setminus \{0\}$. The slice U has the property that if $\tau \in (0, 1)$, $u \in U$, then $\tau u \in U$. Let $\tilde{N} = H \cdot U$. Then $\tilde{N} \subset N\setminus\{0\}$ is H-invariant and $(\tilde{N} \times \mathfrak{g}^*)/H \cong U \times \mathfrak{g}^*$.

For the rest of Section 3 we will work with the slice U defined by (3.7) for a fixed choice of a slice S for the action of H on $\hat{N}$. (Beginning with Section 3.5 we will also assume that S does not intersect the degeneracy set of Proposition 3.9.)

The decomposition of the amended potential criterion is based on the splitting of the tangent space at configuration points near q_e, as stated in the following:

Proposition 3.12. *Let $\nu(\tau)$ be a curve in $\mathfrak{g}^*$ such that $\mathfrak{g}_{\nu(0)} = \mathfrak{t}$. Let $\iota : U \hookrightarrow T_{q_e}Q$ be the inclusion map and $\varsigma := \exp_{q_e} \circ\, \iota$. Then there exists $\varepsilon > 0$ such that for all $0 < \tau < \varepsilon$,*

$$T_{\varsigma(\tau u)}Q = \mathfrak{g}_{\nu(\tau)} \cdot \varsigma(\tau u) \oplus \varsigma_*(T_{\tau u}U) \oplus T_{\iota(\tau u)} \exp_{q_e} (\mathfrak{k}_2 \cdot q_e).$$

For the proof of this proposition we need the following lemma, which is a special case of the stability of the transversality of smooth maps. (See e.g. Guillemin and Pollack [1974].)

Lemma 3.13. *Let G be a Lie group acting on a Riemannian manifold Q, $q \in Q$, and let $\mathfrak{k} \subset \mathfrak{g}$ (subspace) such that $\mathfrak{k} \cap \mathfrak{g}_q = \{0\}$. Let M be a subspace of T_qQ such that $\mathfrak{k} \cdot q \oplus M = T_qQ$. Then there is an $\varepsilon > 0$ such that if $\|v\| < \varepsilon$,*

$$T_{\exp_q(v)}Q = \mathfrak{k} \cdot \exp_q(v) \oplus T_v \exp_q \cdot M.$$

Proof of Proposition 3.12.. By definition, $T_{q_e}Q = \mathfrak{k}_1 \cdot q_e \oplus \mathfrak{k}_2 \cdot q_e \oplus N$. Using Lemma 3.13 (with $\mathfrak{k} = \mathfrak{k}_1$ and $M = \mathfrak{k}_2 \cdot q_e \oplus N$), we see that there is an $\varepsilon > 0$ such that if $0 \le \tau < \varepsilon$,

$$T_{\exp_{q_e}(\iota(\tau u))}Q = \mathfrak{k}_1 \cdot \exp_{q_e}(\iota(\tau u)) \oplus T_{\iota(\tau u)} \exp_{q_e}(N \oplus \mathfrak{k}_2 \cdot q_e). \tag{3.8}$$

Since U is a slice for the action of H on N, then

$$N \approx T_{\tau u}N = T_{\tau u}U \oplus \mathfrak{k}_0 \cdot (\tau u).$$

Since $\exp_{q_e}$ is a (local) diffeomorphism and $T_v \exp_{q_e}(\xi \cdot v) = \xi \cdot (\exp_{q_e}(v))$ for all $\xi \in \mathfrak{k}_0$, $v \in T_{q_e}Q$, it follows that

$$\begin{aligned} T_{\iota(\tau u)} \exp_{q_e}(N) &= T_{\iota(\tau u)} \exp_{q_e}(T_{\tau u}U) \oplus T_{\iota(\tau u)} \exp_{q_e}(\mathfrak{k} \cdot (\tau u)) \\ &= (\exp_{q_e})_* \iota_*(T_{\tau u}U) \oplus \mathfrak{k}_0 \circ \exp_{q_e}(\iota(\tau u)). \end{aligned}$$

Using this expression and the fact that $\mathfrak{t} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$, we get from equation (3.8) that

$$T_{\varsigma(\tau u)}Q = \mathfrak{t} \cdot \varsigma(\tau u) \oplus \varsigma_*(T_{\tau u}U) \oplus T_{\iota(\tau u)} \exp_{q_e}(\mathfrak{k}_2 \cdot q_e). \tag{3.9}$$

Since $\mathfrak{g}_{\nu(\tau)}$ tends to $\mathfrak{t}$ as τ tends to zero, we can substitute the first summand in the right-hand side of equation (3.9) by $\mathfrak{g}_{\nu(\tau)} \cdot \exp_{q_e}(\tau u)$, for τ small enough. □

As a consequence of Proposition 3.12, we have the following corollary, which gives the desired decomposition of the amended potential criterion for relative equilibria:

Corollary 3.14. *Let $\nu(t)$ be as in Proposition 3.12 and let $u \in U$. Then there is an $\varepsilon \in (0, 1)$ such that $dV_{\nu(\tau)}(\varsigma(\tau u)) = 0$ iff*

$$d(\varsigma^* V_{\nu(\tau)})(\tau u) = 0 \quad \textit{and} \quad d\left(\left(\exp_{q_e}\right)^* V_{\nu(\tau)}\right)(\iota(\tau u))\Big|_{\mathfrak{k}_2 \cdot q_e} = 0, \tag{3.10}$$

for $0 < \tau < \varepsilon$.

Proof. Since $V_{\nu(\tau)}$ is $\mathfrak{g}_{\nu(\tau)}$-invariant then, by splitting given by Proposition 3.12, $dV_{\nu(\tau)}(\varsigma(\tau u)) = 0$ iff $dV_{\nu(t)}(\varsigma(\tau u)) \cdot \varsigma_* \cdot \delta u = d\left(\varsigma^* V_{\nu(\tau)}\right)(\tau u) \cdot \delta u = 0$ for all $\delta u \in T_{\tau u}U$ and $dV_{\nu(\tau)}(\varsigma(\tau u)) \cdot T_{\iota(\tau u)} \exp_{q_e} \cdot \xi_Q(q_e) = \delta\left((\exp_{q_e})^* V_{\nu(\tau)}\right)(\iota(\tau u)) \cdot \xi_Q(q_e)$ for all $\xi \in \mathfrak{k}_2$. □

3.5. Regularization of the Relative Equilibrium Condition

In this subsection we regularize the relative equilibria conditions of the previous section, equations (3.10). Theorem 3.15 regularizes the first condition and Theorem 3.16 the second one.

For the remainder of Section 3 we restrict ourselves to the case $\dim \mathfrak{g}_{q_e} = 1$. (In Section 5 we outline the idea for extending the analysis to the case when $\dim \mathfrak{g}_{q_e} > 1$.) Then $\dim \mathfrak{m}_0 = 1$, and the rescaling given in Section 3.3 becomes

$$\begin{aligned} \nu : \mathbb{R} \times (\mathfrak{m}_1 \oplus \mathfrak{m}_2) &\longrightarrow \mathfrak{g}^*, \\ (\tau, \mu_1 + \mu_2) &\mapsto \mu_1 + \tau\mu_2 + \tau^2\lambda_0, \end{aligned} \tag{3.11}$$

where $\mu_i \in \mathfrak{m}_i$, $i = 1, 2$, and λ_0 is a generator of $\mathfrak{m}_0$. From Section 3.3 we have that $(\tau, v, \mu) \mapsto \mathbb{I}(\tau v)^{-1}\nu(\tau, \mu)$ is a smooth function on $I \times N'' \times (\mathfrak{m}_1 \oplus \mathfrak{m}_2)$, where N'' was defined in equation (3.5) and $I = (-1, 1)$.

Assumption. For the rest of this subsection and the next one, we assume that the slice S chosen in the definition of U given by equation (3.7) does not intersect the degeneracy set $\mathcal{Z}$ of Proposition 3.9. It follows that $U \cap \mathcal{Z} = \varnothing$ and thus $U \subset N''$.

We will continue using the notation $\iota : U \longrightarrow T_{q_e}Q$ (the inclusion map) and $\varsigma = \exp_{q_e} \circ\, \iota$.

Theorem 3.15. *Let $W' : (I \setminus \{0\}) \times U \times \mathbf{J}_L(\mathfrak{g} \cdot q_e) \to \mathbb{R}$ be given by*

$$W'(\tau, u, \mu) = \varsigma^* V_{\nu(\tau,\mu)}(\tau u).$$

Then W' can be extended to a smooth function on $I \times U \times \mathbf{J}_L(\mathfrak{g} \cdot q_e)$ and

$$W'(\tau, u, \mu) = W_0(\mu) + \tau^2 W(\tau, u, \mu)$$

for some smooth real-valued functions W_0, W defined over $\mathbf{J}_L(\mathfrak{g}\cdot q_e)$ and $I \times U \times \mathbf{J}_L(\mathfrak{g}\cdot q_e)$, respectively.

Proof. Let $v = \iota(u) \in N$. Note that

$$\varsigma^* V_{\nu(\tau,\mu)}(\tau u) = V(\tau v) + \frac{1}{2}\left\langle \nu(\tau, \mu), \mathbb{I}^{-1}(\tau v)\nu(\tau, \mu)\right\rangle. \tag{3.12}$$

We have already shown that $\mathbb{I}(\tau v)^{-1}\nu(\tau, \mu)$ is smooth, so that the same is true for the left-hand side of (3.12). The remaining assertion follows from the following straightforward, albeit lengthy, computations.

Let ξ_0 be a generator of $\mathfrak{g}_{q_e}$. Then $\mathbb{I}(\tau v)^{-1}\nu(\tau, \mu) = \alpha(\tau, v, \mu)\xi_0 + \eta(\tau, v, \alpha(\tau, v, \mu), \mu)$ where α and η are smooth functions which, in the notation of equation (3.2), correspond to $\xi = \alpha(\tau, v, \mu)\xi_0$ and $\eta = \eta(\tau, v, \alpha(\tau, v, \mu), \mu)$. Fix $\mu \in \mathbf{J}_L(\mathfrak{g} \cdot q_e)$, $v \in N''$, and for brevity write $\alpha(\tau) = \alpha(\tau, v, \mu)$, $\eta(\tau, \alpha) = \eta(\tau, v, \alpha, \mu)$. Letting $\Pi_i : \mathfrak{g}^* \to \mathfrak{m}_i$ be the projections induced by the splitting $\mathfrak{g}^* = \bigoplus_{i=0}^2 \mathfrak{m}_i$, we will use the notation

$\mu_i := \Pi_i \mu$. As before, let $\Pi : \mathfrak{g}^* \to \mathfrak{m}_1 \oplus \mathfrak{m}_2$. From equation (3.2),

$$\begin{aligned}
\frac{\partial \eta}{\partial \tau}(0,\alpha) &= -D_{\xi_1}(\Pi\Phi)^{-1} \cdot D_\tau(\Pi\Phi)|_{\tau=0} \\
&= -\alpha\, \hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0)\cdot v)\xi_0 + (D\hat{\mathbb{I}}(0)^{-1}\cdot v)\mu_1 + \hat{\mathbb{I}}(0)^{-1}\mu_2, \\
\frac{\partial \eta}{\partial \alpha}(0,\alpha) &= -D_{\xi_1}(\Pi\Phi)^{-1} \cdot D_\alpha(\Pi\Phi)|_{\tau=0} \\
&= -\hat{\mathbb{I}}(0)^{-1}\tilde{\mathbb{I}}(0)\xi_0 = 0,
\end{aligned}$$

since $\tilde{\mathbb{I}}(0)\xi_0 = 0$.

From (3.12) and since $\xi_1(0,\alpha) = \hat{\mathbb{I}}(0)^{-1}\mu_1$ and $\mathfrak{m}_1$ annihilates $\ker \mathbb{I}(0)$, we get

$$V_{\nu(\tau,\mu)}(\tau v)|_{\tau=0} = V(0) + \frac{1}{2}\left\langle \mu_1, \hat{\mathbb{I}}(0)^{-1}\mu_1 \right\rangle, \tag{3.13}$$

which is independent of v. Now, differentiating (3.12) with respect to τ, we get

$$\begin{aligned}
\left.\frac{\partial}{\partial \tau}\right|_{\tau=0} V_{\nu(\tau,\mu)}(\tau v) &= dV(0)\cdot v + \frac{1}{2}\langle \mu_2, \alpha\,\xi_0 + \eta(0,\alpha)\rangle \\
&\quad + \frac{1}{2}\left\langle \mu_1, \frac{\partial \alpha}{\partial \tau}\left(\xi_0 + \frac{\partial \eta}{\partial \alpha}(0,\alpha)\right) + \frac{\partial \eta}{\partial \tau}(0,\alpha)\right\rangle,
\end{aligned}$$

because $\nu(0,\mu) = \mu_1$ and $\partial\nu/\partial\tau(0,\mu) = \mu_2$. In the right-hand side, the first term vanishes because we have assumed that $dV(0) = 0$. The second term vanishes because $\alpha\xi_0 + \eta(0,\alpha) \in \mathfrak{t}$ and $\mathfrak{m}_2 = \mathfrak{t}^\circ$. Using the expressions for $\partial\eta/\partial\alpha$ and $\partial\eta/\partial\tau$ obtained above, and the fact that $\mathfrak{m}_1$ annihilates ξ_0, we see that the third term is equal to one half of

$$\begin{aligned}
\left\langle \mu_1, \frac{\partial \xi_1}{\partial \tau}(0,\alpha)\right\rangle &= -\alpha\left\langle \mu_1, \hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0)\cdot v)\xi_0\right\rangle \\
&\quad + \left\langle \mu_1, (D\hat{\mathbb{I}}(0)^{-1}\cdot v)\mu_1\right\rangle + \left\langle \mu_1, \hat{\mathbb{I}}(0)^{-1}\mu_2\right\rangle.
\end{aligned}$$

Now we check that each of the terms in the right-hand side of this expression vanishes: Let $\zeta := \hat{\mathbb{I}}(0)^{-1}\mu_1 \in \mathfrak{k}_1 \subset \mathfrak{t}$. Then

$$\left\langle \mu_1, \hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0)\cdot v)\xi_0\right\rangle = \left\langle (D\tilde{\mathbb{I}}(0)\cdot v)\xi_0, \zeta\right\rangle = \langle (D\mathbb{I}(0)\cdot v)\xi_0, \zeta\rangle = 0,$$

because of Lemma 3.7,

$$\begin{aligned}
\left\langle \mu_1, (D\hat{\mathbb{I}}(0)^{-1}\cdot v)\mu_1\right\rangle &= \left\langle \mu_1, -\hat{\mathbb{I}}(0)^{-1}(D\hat{\mathbb{I}}(0)\cdot v)\hat{\mathbb{I}}(0)^{-1}\mu_1\right\rangle \\
&= -\left\langle (D\hat{\mathbb{I}}(0)\cdot v)\zeta, \zeta\right\rangle = -\langle (D\mathbb{I}(0)\cdot v)\zeta, \zeta\rangle = 0,
\end{aligned}$$

since $\zeta \in \mathfrak{t}$ and thus $\zeta_Q(q_e)$ is a relative equilibrium. Finally,

$$\left\langle \mu_1, \hat{\mathbb{I}}(0)^{-1}\mu_2\right\rangle = \langle \mu_2, \zeta\rangle = 0,$$

because $\mathfrak{m}_2$ annihilates $\mathfrak{t}$. We conclude that $\partial/\partial\tau|_{\tau=0} V_{\nu(\tau,\mu)}(\tau v) = 0$. Thus,

$$\varsigma^* V_{\nu(\tau,\mu)}(\tau u) = W_0(\mu) + \tau^2 W(\tau, u, \mu),$$

where $W_0(\mu)$ is equal to the right-hand side of (3.13) and W is some smooth function. □

Theorem 3.16. *Let $X' : (I \setminus \{0\}) \times U \times \mathbf{J}_L(\mathfrak{g} \cdot q_e) \to \mathfrak{k}_2^*$ be given by*

$$\left\langle X'(\tau, u, \mu), \zeta \right\rangle = d\left((\exp_{q_e})^* V_{\nu(\tau,\mu)} \right)(\iota(\tau u)) \cdot \zeta_Q(q_e).$$

Then X' can be extended to a smooth function on $I \times U \times \mathbf{J}_L(\mathfrak{g} \cdot q_e)$ and

$$X'(\tau, u, \mu) = \tau X(\tau, u, \mu)$$

for some smooth function $X : I \times U \times \mathbf{J}_L(\mathfrak{g} \cdot q_e) \to \mathfrak{k}_2^*$.

Proof. It suffices to show that $X'(\tau, u, \mu)$ is a smooth function at $\tau = 0$ and that $X'(0, u, \mu) = 0$. Let $v = \iota(u)$. Then

$$\begin{aligned} \left\langle X'(\tau, u, \mu), \zeta \right\rangle &= d\left((\exp_{q_e})^* V_{\nu(\tau,\mu)} \right)(\tau v) \cdot \zeta_Q(q_e) \\ &= dV(\tau v) \cdot \zeta_Q(q_e) + \frac{1}{2} \left\langle \nu(\tau, \mu), (D\mathbb{I}^{-1}(\tau v) \cdot \zeta_Q(q_e))\nu(\tau, \mu) \right\rangle \\ &= dV(\tau v) \cdot \zeta_Q(q_e) - \frac{1}{2} \left\langle (D\mathbb{I}(\tau v) \cdot \zeta_Q(q_e))\xi(\tau, v, \mu), \xi(\tau, v, \mu) \right\rangle, \end{aligned} \tag{3.14}$$

where $\xi(\tau, v, \mu) = \mathbb{I}^{-1}(\tau v)\nu(\tau, \mu)$. Since $\xi(\tau, v, \mu)$ is smooth at $\tau = 0$, then so is $\left\langle X'(\tau, u, \mu), \zeta \right\rangle$. Using Proposition 1.1 we see that

$$\left\langle X'(0, u, \mu), \zeta \right\rangle = dV(0) \cdot \zeta_Q(0) - 2 \left\langle \mathbb{I}(0)[\xi(0, v, \mu), \zeta], \xi(0, v, \mu) \right\rangle.$$

Since V is G-invariant, then $dV(0) \cdot \zeta_Q(q_e) = 0$. Since $\xi(0, v, \mu) \in \mathfrak{t}$, then $[\xi(0, v, \mu), \zeta] \in \mathfrak{k}_2$. Since $\mathbb{I}(0)\mathfrak{t} \subset \mathfrak{k}_2^\circ$, then $\left\langle X'(0, u, \mu), \zeta \right\rangle = 0$. □

The expression for $X(0, u, \mu)$ is relatively simple and it is worthwhile to include it here. Recall that $\xi(\tau, u, \mu) := \mathbb{I}^{-1}(\varsigma(\tau u))\nu(\tau, \mu)$.

Proposition 3.17. *With $u \in U$, $\mu \in \mathbf{J}_L(\mathfrak{g} \cdot q_e)$, $\zeta \in \mathfrak{k}_2$,*

$$\begin{aligned} \langle X(0, u, \mu), \zeta \rangle &= D^2 V(q_e) \cdot (u, \zeta_Q(q_e)) \\ &\quad - \left\langle \left(D^2\mathbb{I}(q_e) \cdot (u, \zeta_Q(q_e)) \right) \xi, \xi \right\rangle \\ &\quad - 2\left(\left\langle \mathbb{I}(q_e) \left[\frac{\partial \xi}{\partial \tau}, \zeta \right], \xi \right\rangle + \left\langle \mathbb{I}(q_e) \frac{\partial \xi}{\partial \tau}, [\xi, \zeta] \right\rangle \right), \end{aligned}$$

where $\xi = \xi(0, u, \mu)$ and $\partial\xi/\partial\tau = \partial\xi/\partial\tau(0, u, \mu)$.

Proof. We have that $\langle X(0, u, \mu), \zeta \rangle = \frac{d}{d\tau}\big|_{\tau=0} \langle X'(\tau, u, \mu), \zeta \rangle$. Differentiating (3.14), we get

$$\begin{aligned} \frac{d}{d\tau}\bigg|_{\tau=0} \left\langle X'(\tau, u, \mu), \zeta \right\rangle &= D^2 V(q_e) \cdot (\zeta_Q(q_e), u) \\ &\quad - \left\langle \left(D^2\mathbb{I}(q_e) \cdot (u, \zeta_Q(q_e)) \right) \xi(0, v, \mu), \xi(0, v, \mu) \right\rangle \\ &\quad - 2\left\langle (D\mathbb{I}(q_e) \cdot \zeta_Q(q_e)) \frac{\partial \xi}{\partial \tau}(0, v, \mu), \xi(0, v, \mu) \right\rangle. \end{aligned}$$

Applying Proposition 1.1 to the last term in the right-hand side, which can be rewritten as

$$\left\langle \left(D\mathbb{I}(q_e)\cdot\zeta_Q(q_e)\right)\frac{\partial\xi}{\partial\tau},\xi\right\rangle = d\left\langle \mathbb{I}(\cdot)\frac{\partial\xi}{\partial\tau},\xi\right\rangle(q_e)\cdot\zeta_Q(q_e),$$

gives the desired result. □

3.6. Bifurcating Branches of Relative Equilibria

In this section we apply the implicit function theorem to obtain branches of relative equilibria bifurcating from $\mathcal{E} = \mathfrak{t}\cdot q_e$ (or rather, the corresponding problem in TQ/G). We continue to assume that $\dim\mathfrak{m}_0 = 1$.

Definition 3.18. For $(\tau, u, \mu_1, \mu_2) \in I\times U\times\mathfrak{m}_1\times\mathfrak{m}_2$, let $\Delta_{(\tau,u,\mu_1,\mu_2)}$ be the bilinear form given by

$$\begin{aligned}\Delta_{(\tau,u,\mu_1,\mu_2)}\left((\delta u,\nu),(\overline{\delta u},\overline{\nu})\right) \;=\; &\frac{\partial^2 W}{\partial u^2}\cdot(\delta u,\overline{\delta u}) + \frac{\partial^2 W}{\partial\mu_2\partial u}\cdot(\delta u,\overline{\nu})\\ &+\frac{\partial\left\langle X,\hat{\mathbb{I}}(0)^{-1}\nu\right\rangle}{\partial u}\cdot\overline{\delta u} + \frac{\partial\left\langle X,\hat{\mathbb{I}}(0)^{-1}\nu\right\rangle}{\partial\mu_2}\cdot\overline{\nu}\end{aligned}$$

with the partial derivatives evaluated at (τ, u, μ_1, μ_2).

We now show that for every $(\bar{u}, \bar{\mu}_1, \bar{\mu}_2) \in U\times\mathfrak{m}_1\times\mathfrak{m}_2$ satisfying the nondegenerate criticality conditions

$$\left.\begin{array}{l} d\,W_{(0,\bar{\mu}_1,\bar{\mu}_2)}(u) = 0,\\ X_{(0,\bar{\mu}_1,\bar{\mu}_2)}(u) = 0,\\ \Delta_{(0,\bar{u},\bar{\mu}_1,\bar{\mu}_2)} \text{ is nondegenerate,}\end{array}\right\}, \tag{3.15}$$

we have a branch of relative equilibria bifurcating from $\bar{\mathcal{E}} := \pi_G(\mathcal{E})$, where $\pi_G : TQ \longrightarrow TQ/G$ is the canonical projection. Here $W_{(0,\mu_1,\mu_2)}(u) = W(0, u, \mu_1 + \mu_2)$ and similarly for X.

Theorem 3.19. *Suppose that $(\bar{u}, \bar{\mu}_1, \bar{\mu}_2) \in U\times\mathfrak{m}_1\times\mathfrak{m}_2$ are data satisfying (3.15). Then there exists an open neighborhood $\mathfrak{m}_1' \subset \mathfrak{m}_1$ containing $\bar{\mu}_1$ and a continuous map*

$$\sigma^{(\bar{u},\bar{\mu}_2)} : [0,1)\times\mathfrak{m}_1' \longrightarrow TQ/G$$

consisting of classes of relative equilibria such that

$$\left(\sigma^{(\bar{u},\bar{\mu}_2)}\right)^{-1}\left(\bar{\mathcal{E}}\right) = \{0\}\times\mathfrak{m}_1'.$$

A value $\sigma^{(\bar{u},\bar{\mu}_2)}(0,\mu_1)$ is the unique class of relative equilibria in TQ/G corresponding to $\xi_Q(q_e)$, where $\mathbb{I}(q_e)\cdot\xi = \mu_1$. Furthermore, if $(\bar{u},\bar{\mu}_2) \neq (\bar{\bar{u}},\bar{\bar{\mu}}_2)$, then $\operatorname{Im}\sigma^{(\bar{u},\bar{\mu}_2)} \cap \operatorname{Im}\sigma^{\bar{\bar{u}},\bar{\bar{\mu}}_2)} \subset \bar{\mathcal{E}}$.

Proof. Let $\tilde{J} : \mathfrak{g} \cdot Q \subset TQ \longrightarrow Q \times \mathfrak{g}^* : v_q \mapsto (q, J_L(v_q))$, $\psi_1 := \exp_{q_e} \times \mathrm{Id} : N' \times \mathfrak{g}^* \longrightarrow Q \times \mathfrak{g}^*$ and $i : \bar{U} \times \mathfrak{g}^* \hookrightarrow N' \times \mathfrak{g}^*$ be the inclusion map, where $\bar{U} = \{\tau u \mid -1 < \tau < 1, u \in U\}$ and U is defined by equation 3.7. Let $U_0 := U \cup \{\mathbf{0}\}$. By G-equivariance of the momentum map and H-equivariance of $\exp_{q_e}$, it follows that the primed maps in the commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{g} \cdot Q \subset TQ & \xrightarrow{\tilde{J}} & Q \times \mathfrak{g}^* & \xleftarrow{\psi_1} & N' \times \mathfrak{g}^* & \xleftarrow{i} & \bar{U} \times \mathfrak{g}^* \\
\downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_H} & & \| \\
(\mathfrak{g} \cdot Q)/G \subset TQ/G & \xrightarrow{\tilde{J}'} & Q \times_G \mathfrak{g}^* & \xleftarrow{\psi_1'} & N' \times_H \mathfrak{g}^* & \xleftarrow{i'} & \bar{U} \times \mathfrak{g}^*
\end{array}$$

are well defined. Let $A = (\psi_1 \circ i)^{-1}(\mathrm{Im}(\psi_1 \circ i) \cap \mathrm{Im}\,\tilde{J})$. Since $\tilde{J}$ is one-to-one, then it has an inverse $\tilde{J}^{-1} : \mathrm{Im}(\tilde{J}) \longrightarrow \mathfrak{g} \cdot Q$. Redefining i to be restricted to $A \subset \bar{U} \times \mathfrak{g}^*$, we get the commutative diagram

$$\begin{array}{ccc}
\mathfrak{g} \cdot Q \subset TQ & \xleftarrow{\psi_2} & A \subset \bar{U} \times \mathfrak{g}^* \\
\downarrow{\scriptstyle \pi_G} & & \| \\
(\mathfrak{g} \cdot Q)/G \subset TQ/G & \xleftarrow{\psi_2'} & A \subset \bar{U} \times \mathfrak{g}^*
\end{array}, \tag{3.16}$$

where $\psi_2 := \tilde{J}^{-1} \circ \psi_1 \circ i$ and $\psi_2' := (\tilde{J}')^{-1} \circ \psi_1' \circ i'$. It is easy to check that ψ_2 is a continuous injection and that ψ_2', restricted to $A \cap (U_0 \times \mathfrak{g}^*)$, is a homeomorphism onto its image.

For a given $\mu_1' \in \mathfrak{m}_1'$, let

$$\mathcal{B}^{(\mu_1')} := \{(\bar{u}, \bar{\mu}_2) \in U \times \mathfrak{m}_2 \mid (\bar{u}, \mu_1', \bar{\mu}_2) \text{ satisfies (3.15)}\}. \tag{3.17}$$

It follows from the implicit function theorem that there is an $\varepsilon > 0$, an open neighborhood $\mathfrak{m}_1' \subset \mathfrak{m}_1$ of μ_1', and, for every $b := (\bar{u}, \bar{\mu}_2) \in \mathcal{B}^{(\mu_1')}$, continuous functions $u^{(b)}$, $\mu_2^{(b)}$,

$$\left(u^{(b)}, \mu_2^{(b)}\right) : (-\varepsilon, \varepsilon) \times \mathfrak{m}_1' \longrightarrow U \times \mathfrak{m}_2,$$

such that $(u^{(b)}(0, \mu_1'), \mu_2^{(b)}(0, \mu_1')) = (\bar{u}, \bar{\mu}_2)$ and, for all $(\tau, \mu_1) \in (-\varepsilon, \varepsilon) \times \mathfrak{m}_1'$, both $\partial W/\partial u$ and X equal zero when evaluated at $(\tau, u^{(b)}(\tau, \mu_1), \mu_1 + \mu_2^{(b)}(\tau, \mu_1))$.

Given $b \in \mathcal{B}^{(\mu_1')}$, let

$$\begin{aligned}
\tilde{\sigma}^{(b)} : [0, \varepsilon) \times \mathfrak{m}_1' &\longrightarrow A \cap (U_0 \times \mathfrak{g}^*), \\
(\tau, \mu_1) &\mapsto \left(\tau u^{(b)}(\tau, \mu_1), \nu(\tau, \mu_1, \mu_2^{(b)}(\tau, \mu_1))\right).
\end{aligned}$$

Let $\sigma^{(b)} := \psi_2' \circ \tilde{\sigma}^{(b)}$. Then, by the commutativity of (3.16) and the injectivity of ψ_2' restricted to $A \cap (U_0 \times \mathfrak{g}^*)$, to prove the theorem it is enough to check that (a) $\mathrm{Im}(\psi_2 \circ \tilde{\sigma}^{(b)})$ lies within the set of relative equilibria in TQ; (b) $(\psi_2 \circ \tilde{\sigma}^{(b)})^{-1}(\mathcal{E}) = \{0\} \times \mathfrak{m}_1'$ and $\psi \circ \tilde{\sigma}^{(b)}(0, \mu_1) = \xi_Q(q_e)$ implies that $\mathbb{I}(q_e) \cdot \xi = \mu_1$; and (c) if $b_1 \neq b_2$, then

$\operatorname{Im}\tilde{\sigma}^{(b_1)} \cap \operatorname{Im}\tilde{\sigma}^{(b_2)} = \{0\} \times \mathfrak{m}_1' \subset U_0 \times \mathfrak{g}^*$. Notice that by a reparametrization we may assume that the domain of $\sigma^{(b)}$ is $[0, 1) \times \mathfrak{m}_1'$.

It is easy to check that $\operatorname{Im}\tilde{\sigma}^{(b)} \subset A$. Theorems 3.15 and 3.16 imply that for all $(\tau, \mu_1) \in [0, \varepsilon) \times \mathfrak{m}_1'$ the relative equilibria conditions (3.10) are both satisfied at $\operatorname{Im}\left(\psi_2 \circ \tilde{\sigma}^{(b)}\right) \subset TQ$; this shows (a).

From the definition of $\tilde{J}$ it follows that if $\psi_2(u, \mu) = \xi_Q(q_e)$, $\xi \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$, then $u = 0$ and $\mathbb{I}(q_e) \cdot \xi = \mu$. Hence $\psi_2^{-1}(\mathcal{E}) = \{0\} \times \mathfrak{m}_1' = \tilde{\sigma}^{(b)}(\{0\} \times \mathfrak{m}_1')$. Recall that, by definition of U, if $b = (\bar{u}, \bar{\mu}_2) \in \mathcal{B}^{(\mu_1')}$, then $\bar{u} \neq 0$. Thus, by taking ε smaller and shrinking $\mathfrak{m}_1'$ if necessary, we have that $\tau u^{(b)}(\tau, \mu_1) = 0$ only if $\tau = 0$. Hence $\tilde{\sigma}^{(b)}(\tau, \mu_1) \in \psi_2^{-1}(\mathcal{E})$ only if $\tau = 0$, and we conclude that $\left(\psi_2 \circ \tilde{\sigma}^{(b)}\right)^{-1}(\mathcal{E}) = \{0\} \times \mathfrak{m}_1'$. This settles (b).

Finally, to show (c), take $(\tau_1, y_1), (\tau_2, y_2) \in [0, \varepsilon) \times \mathfrak{m}_1'$. From the definition of $\tilde{\sigma}$, it follows that if $\tilde{\sigma}^{(b_1)}(\tau_1, y_1) = \tilde{\sigma}^{(b_2)}(\tau_2, y_2)$, then $\tau_1^2 \hat{\mu}_0 = \tau_2^2 \hat{\mu}_0$, hence $\tau_1 = \tau_2$, and $y_1 = y_2$. As $(\tau_1, y_1) \to (0, \mu_1')$ we have that $u^{(b_i)} \to b_i$. Since $b_1 \neq b_2$ then, by taking ε smaller and shrinking $\mathfrak{m}_1'$ if necessary, we have that $u^{(b_1)}(\tau_1, y_1) \neq u^{(b_2)}(\tau_1, y_1)$. Hence, $\tilde{\sigma}^{(b_1)}(\tau_1, y_1) = \tilde{\sigma}^{(b_2)}(\tau_1, y_1)$ implies that $\tau_1 = 0$ and $\tilde{\sigma}^{(b_1)}(\tau_1, y_1) = (0, y_1)$. □

Remark. Theorem 3.19 can be interpreted as saying that every $(\bar{u}, \bar{\mu}_2) \in \mathcal{B}^{(\mu_1')}$ labels a symmetry-breaking branch of classes of relative equilibria in TQ/G bifurcating from $\bar{\mathcal{E}}$ and parametrized by τ and μ_1 (with μ_1 in a neighborhood of μ_1'). Branches corresponding to different labels are distinct, since (locally) their intersection outside $\bar{\mathcal{E}}$ is empty.

Remark. From Theorem 3.19 it follows that the dimension of the bifurcating branches of classes of relative equilibria in TQ/G, away from the set of symmetric states, is equal to $1 + \dim(\mathfrak{m}_1) = \dim \mathfrak{g}_{q_e} + \dim(\mathfrak{k}_1) = \dim \mathfrak{t} = \operatorname{Rank} G$. Thus, viewing these branches as sets in TQ, their dimension is equal to $\dim G + \operatorname{Rank} G$. This agrees with the generic dimension of the set of relative equilibria when the action is free, as discussed in Patrick [1995].

Remark. A direct consequence of Theorem 3.19 is that if the discrete set $\mathcal{B}^{(\mu_1')}$ of nondegenerate critical points defined in (3.17) has more than one element, then we can conclude the existence of branches of relative equilibria bifurcating from $\bar{\mathcal{E}}$, in the sense explained in Section 1.4. This can be seen as follows. With $Q' := G \cdot \exp_{q_e}(N')$, given $[q] \in Q'/G$, let $\operatorname{dist}_{G\cdot q_e}([q]) := \|n\|$, where $q = g \cdot n$ for some $g \in G, n \in N'$. (Since N' is a slice at q_e for the G-action on Q', this notion of *distance* from the orbit $G \cdot q_e$ is well defined.) For $\lambda \in \mathbb{R}, \lambda \geq 0$, let $\mathcal{R}_\lambda := \cup_{b\in\mathcal{B}^{(\mu_1')}} \operatorname{dist}_{G\cdot q_e}^{-1}(\lambda) \cap \operatorname{Im}\sigma^{(b)}$. Then $\mathcal{R}_0 = \{[\xi_Q(q_e)] \in TQ/G \mid \mathbb{I}(q_e) \cdot \xi \in \mathfrak{m}_1', \xi \in \mathfrak{g}\}$. Without loss of generality we can assume that $\mathfrak{m}_1'$ is connected and hence so is $\mathcal{R}_0$. On the other hand, if $\lambda \neq 0$ and $\mathcal{B}^{(\mu_1')}$ contains more than one point, then $\mathcal{R}_\lambda$ is disconnected. This follows from the fact that $\operatorname{dist}_{G\cdot q_e}(\bar{\mathcal{E}}) = \{0\}$ and $\operatorname{Im}\sigma^{(b_1)} \cap \operatorname{Im}\sigma^{(b_2)} \subset \bar{\mathcal{E}}$ if $b_1, b_2 \in \mathcal{B}^{(\mu_1')}, b_1 \neq b_2$. Thus $\mathcal{R}_0$ is not homeomorphic to $\mathcal{R}_\lambda$, and we can conclude that the family of relative equilibria $\mathcal{R} := \cup_{b\in\mathcal{B}^{(\mu_1')}} \operatorname{Im}\sigma^{(b)} = \cup_\lambda \mathcal{R}_\lambda$ bifurcates from $\mathcal{R}_0 \subset \bar{\mathcal{E}}$. If the set $\mathcal{B}^{(\mu_1)}$ contains only one point, then Theorem 3.19 only implies persistence of relative equilibria from $\mathcal{R}_0$.

Remark. In Hamiltonian problems of the sort considered in this paper, it is difficult to establish that a given method locally captures all of the solutions; that is, all of the relative equilibria. Relative to our method, we have proved the existence of a family of relative equilibria bifurcating from a given set of relative equilibria $\bar{\mathcal{E}}$. If one could prove (as we conjecture is true) that the method captures all the relative equilibria locally, then the preceding remark would say that the set of relative equilibria near $\bar{\mathcal{E}}$ literally bifurcates from $\bar{\mathcal{E}}$.

4. Example: Symmetric Coupled Rigid Bodies

Here we will illustrate the application of the theory developed in the previous section with the example of the two symmetric coupled rigid bodies moving in three-dimensional space with zero potential. In contrast with the double spherical pendulum, the example studied in Section 2, the set of symmetric states in the symmetric coupled rigid bodies from which branches of relative equilibria bifurcate is not discrete.

There has been extensive mathematical study of the symmetric coupled rigid bodies. Patrick [1990], [1989] studied the relative equilibria in this example using the augmented potential criterion together with an explicit classification of all the group orbits, thus achieving a complete enumeration of the relative equilibria. With a different approach, Mittagunta [1996], [1994] gave a lower bound on the number of relative equilibria in momentum level sets based on a Morse theoretic analysis of the topology of the reduced spaces. Roberts and de Sousa Dias [1997] used the Marle-Guillemin-Sternberg slice decomposition to study the bifurcation of relative equilibria nearby symmetric states in a system consisting also of symmetric rigid bodies but requiring the presence of a potential (to ensure a certain nondegeneracy condition).

In the analysis of the symmetric coupled rigid bodies presented in this section, we do not attempt to obtain new results. Our objective is only to illustrate how our theory can be applied to an example with symmetric states where the group is non-abelian and relatively large. Applying the technique of the previous section, we are able to recover the relative equilibria that bifurcate from the class of symmetric relative equilibria consisting of the states where the axis of symmetry of the two bodies are aligned, each body is rotating around its axis of symmetry with independent velocity, and the total angular momentum of the system is different from zero. The branches of relative equilibria thus obtained (see Proposition 4.3) break the symmetry.

4.1. Description and Preliminaries

Consider the mechanical system formed by two symmetric rigid bodies with equal moments of inertia coupled by an ideal spherical joint along their axes of symmetry and such that the distance from the center of mass of either body to the joint is the same. (See Figure 3.) From this description it follows that we can attach to each rigid body a coordinate system with respect to which its inertia matrix is equal to $I = \begin{pmatrix} I^{xy} & & \\ & I^{xy} & \\ & & I^{z} \end{pmatrix}$, for some positive real numbers I^{xy}, I^{z}. The Lagrangian of the system consists purely of

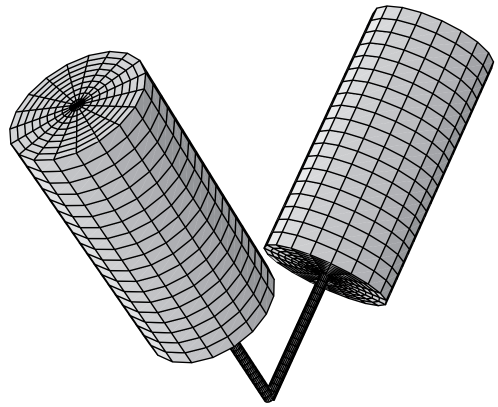

Fig. 3. Symmetric coupled rigid bodies.

kinetic energy. After reducing by translations, the configuration space becomes

$$Q := SO(3) \times SO(3).$$

The physical interpretation of this configuration space is that a given $(A_1, A_2) \in Q$ represents the configuration obtained by applying the rotation A_i to body i, where the initial *reference configuration* consists of the two bodies aligned on top of each other, the common center of mass lying at the origin and the common axis of symmetry being aligned with the $\mathbf{e}_3$-axis.

It is convenient to express elements in $TQ = T(\mathrm{SO}(3)^2)$ in terms of the **body coordinates**. These are defined by the diffeomorphism $\cap : (\mathrm{SO}(3) \times \mathbb{R}^3)^2 \to T(\mathrm{SO}(3)^2)$ given by

$$(A_i; \Omega_i)^{\cap} := (A_i; A_i\hat{\Omega}_i).$$

Here $(A_i; \Omega_i)$ represents the point $(A_1, A_2; \Omega_1, \Omega_2) \in \mathrm{SO}(3)^2 \times (\mathbb{R}^3)^2$, $(A_i; \dot{A}_i)$ represents the point $(A_1, A_2; \dot{A}_1, \dot{A}_2) \in T((\mathrm{SO}(3))^2)$, and $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ is the standard isomorphism $\hat{X} = \begin{pmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{pmatrix}$. The inverse diffeomorphism is given by

$$(A_i; \dot{A}_i)^{\cup} := (A_i; (A_i^{-1}\dot{A}_i)^{\vee}),$$

where $\vee$ denotes the inverse of $\wedge$.

One verifies (cf. Patrick [1989]) that the Lagrangian of the system after reduction by translations is given by

$$L(A_1, A_2; \dot{A}_1, \dot{A}_2) = \frac{1}{2}\Omega_1^t \tilde{J}\Omega_1 + \frac{1}{2}\Omega_2^t \tilde{J}\Omega_2 - \beta A_1(\mathbf{e}_3 \times \Omega_1) \cdot A_2(\mathbf{e}_3 \times \Omega_2),$$

where $\dot{A}_i = A_i\hat{\Omega}_i$, $\Omega_i \in \mathbb{R}^3$, $\tilde{J} := \begin{pmatrix} 1 & & \\ & 1 & \\ & & \alpha \end{pmatrix}$, and the system parameters α, β are given by

$$\alpha = \frac{2I^{xy}}{I^{xy} + I^z + \epsilon}, \qquad \beta = \frac{\epsilon}{I^{xy} + I^z + \epsilon},$$

with $\epsilon = (m_1 m_2)_{|S|^2}/(m_1 + m_2)$, where $|S|$ denotes the distance from the center of mass of either body to the joint. β is called the *coupling constant*. Observe that $0 \leq \beta < 1$.

A straightforward computation shows that the fiber derivative corresponding to this Lagrangian is given (in body coordinates) by

$$\begin{aligned}\langle \mathbb{F}L(A_1, A_2)(\Omega_1, \Omega_2), (W_1, W_2)\rangle \; = \; & \Omega_1^T \tilde{J} W_1 + \Omega_2^T \tilde{J} W_2 \\ & -\beta \left(A_1(\mathbf{e}_3 \times W_1) \cdot A_2(\mathbf{e}_3 \times \Omega_2)\right. \\ & \left.+A_1(\mathbf{e}_3 \times \Omega_1) \cdot A_2(\mathbf{e}_3 \times W_2)\right). \end{aligned} \qquad (4.1)$$

Now, consider the action of $G := SO(3) \times S^1 \times S^1$ on Q given by

$$(B, \theta_1, \theta_2) \cdot (A_1, A_2) = \left(BA_1 \exp(-\theta_1 \hat{\mathbf{e}}_3), BA_2 \exp(-\theta_2 \hat{\mathbf{e}}_3)\right).$$

Physically this corresponds to rotating each body around its axis of symmetry by angles θ_1, θ_2 and then applying the rotation B to the system as a whole. One verifies that the Lagrangian is invariant with respect to the tangent lift of this action. Thus, G is the symmetry group of the system.

The Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ with the Lie bracket given by

$$[(\mathbf{x}, y_1, y_2), (\mathbf{x}', y_1', y_2')] = (\mathbf{x} \times \mathbf{x}', 0, 0).$$

For every $\xi = (\mathbf{x}, y_1, y_2) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \cong \mathfrak{g}$, the infinitesimal generator associated with the given action of G on Q is computed to be

$$\xi_Q(A_1, A_2) = (A_1, A_2; A_1^T \mathbf{x} - y_1 \mathbf{e}_3, A_2^T \mathbf{x} - y_2 \mathbf{e}_3)^{\cap}. \qquad (4.2)$$

Identifying T^*Q with $\mathrm{SO}(3)^2 \times (\mathbb{R}^3)^2$ via the standard inner product on $(\mathbb{R}^3)^2$ and identifying $\mathfrak{g}^*$ with $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ via the standard inner product on $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$, one computes that the *momentum map* $J : T^*Q \to \mathfrak{g}^*$ associated with the cotangent lift of the action is given by

$$J(A_1, A_2, \Pi_1, \Pi_2) = (A_1\Pi_1 + A_2\Pi_2, -\Pi_1 \cdot \mathbf{e}_3, -\Pi_2 \cdot \mathbf{e}_3).$$

4.2. Fiber over a Symmetric Point

We will now study the branches of relative equilibria emanating from a subspace of symmetric relative equilibria in the fiber over a symmetric point. The configurations with nontrivial isotropy are the ones in which the axis of symmetry of the two bodies are aligned, so that the two bodies lie on top of each other or they point in opposite directions. We will treat only the former case, since the latter is analogous.

Let $q_e = (\,\mathrm{Id},\ \mathrm{Id}) \in Q$. This corresponds to a configuration consisting of the two bodies on top of each other. Let $\{\mathbf{e}_i\}_{i=1}^3$ be the canonical basis in $\mathbb{R}^3$. The isotropy subgroup of q_e is

$$G_{q_e} = \{(\exp(t\hat{\mathbf{e}}_3), t, t)\} \cong S^1. \qquad (4.3)$$

Its Lie algebra is $\mathfrak{g}_{q_e} = \mathrm{span}\{(\mathbf{e}_3, 1, 1)\}$.

We now want to obtain the set $\mathcal{R}_{q_e}$ defined as the set of relative equilibria inside the fiber $T_{q_e}Q$.

The augmented potential for the SCRB is given by

$$V_\xi(A_1, A_2) = \left\langle \mathbb{F}L(A_1, A_2) \cdot \xi_Q(A_1, A_2), \xi_Q(A_1, A_2) \right\rangle,$$

where $\xi_Q(A_1, A_2)$ is given by (4.2). Therefore $V_\xi(A_1, A_2)$ is given by (4.1) with

$$(\Omega_1, \Omega_2) = (W_1, W_2) = (A_1^T \mathbf{x} - y_1\, \mathbf{e}_3, A_2^T \mathbf{x} - y_2\, \mathbf{e}_3).$$

A computation shows that, for $i = 1, 2$,

$$\Omega_i^T \tilde{J} \Omega_i = (A_i^T \mathbf{x}) \cdot \tilde{J}(A_i^T \mathbf{x}) - 2\alpha y_i (A_i^T \mathbf{x}) \cdot \mathbf{e}_3 + y_i^2 \alpha,$$

and

$$A_1(\mathbf{e}_3 \times \Omega_1) \cdot A_2(\mathbf{e}_3 \times \Omega_2) = (A_1 \mathbf{e}_3 \times \mathbf{x}) \cdot (A_2 \mathbf{e}_3 \times \mathbf{x}).$$

Collecting terms, we get that

$$\begin{aligned} V_\xi(A_1, A_2) \;=\; & (A_1^T \mathbf{x}) \cdot \tilde{J}(A_1^T \mathbf{x}) + (A_2^T \mathbf{x}) \cdot \tilde{J}(A_2^T \mathbf{x}) \\ & -2\alpha \left(y_1\, A_1^T \mathbf{x} + y_2\, A_2^T \mathbf{x} \right) \cdot \mathbf{e}_3 + \alpha (y_1^2 + y_2^2) \\ & -2\beta\, ((A_1 \mathbf{e}_3) \times \mathbf{x}) \cdot ((A_2 \mathbf{e}_3) \times \mathbf{x}) \,. \end{aligned}$$

For $i = 1, 2$, let $A_i = \exp(t \hat{\mathbf{w}}_i)$, $\mathbf{w}_i \in \mathbb{R}^3$. A computation shows that

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} (A_i^T \mathbf{x}) \cdot \tilde{J}(A_i^T \mathbf{x}) \;&=\; x^T [\hat{\mathbf{w}}_i, \tilde{J}] \mathbf{x} \\ &=\; 2(1-\alpha)(\mathbf{x} \cdot \mathbf{e}_3)\, (\mathbf{x} \times \mathbf{e}_3) \cdot \mathbf{w}_i, \\ \left. \frac{d}{dt} \right|_{t=0} \left[-2\alpha \left(y_1\, A_1^T \mathbf{x} + y_2\, A_2^T \mathbf{x} \right) \cdot \mathbf{e}_3 \right] \;&=\; 2\alpha (\mathbf{x} \times \mathbf{e}_3) \cdot (y_1 \mathbf{w}_1 + y_2 \mathbf{w}_2), \\ \left. \frac{d}{dt} \right|_{t=0} [2\beta\, ((A_1 \mathbf{e}_3) \times \mathbf{x}) \cdot ((A_2 \mathbf{e}_3) \times \mathbf{x})] \;&=\; 2\beta (\mathbf{x} \cdot \mathbf{e}_3)\, (\mathbf{x} \times \mathbf{e}_3) \cdot (\mathbf{w}_1 + \mathbf{w}_2). \end{aligned}$$

Collecting terms, we obtain

$$\left. \frac{d}{dt} \right|_{t=0} V_\xi(A_1, A_2) = 2 \sum_{i=1}^{2} \{ [(1 - \alpha - \beta)(\mathbf{x} \cdot \mathbf{e}_3) + \alpha y_i]\, (\mathbf{x} \times \mathbf{e}_3) \} \cdot \mathbf{w}_i .$$

It follows that $dV_\xi(\,\mathrm{Id},\ \mathrm{Id}) = 0$ if and only if either

$$(1 - \alpha - \beta)(\mathbf{x} \cdot \mathbf{e}_3) + \alpha\, y_i = 0 \quad \text{(for both } i = 1 \text{ and } i = 2\text{)},$$

or

$$\mathbf{x} \times \mathbf{e}_3 = 0.$$

From this computation and the augmented potential criterion (which was recalled in Section 1.4), it follows that the relative equilibria inside $T_{q_e} Q$ are given by

$$\mathcal{R}_{q_e} = \{ \xi_Q(\,\mathrm{Id},\ \mathrm{Id}) \mid \xi \in \mathfrak{l}_1 \cup \mathfrak{l}_2 \},$$

where

$$\mathfrak{l}_1 := \operatorname{span}\{(\mathbf{e}_3, 0, 0), (\mathbf{0}, 1, 0), (\mathbf{0}, 0, 1)\},$$
$$\mathfrak{l}_2 := \operatorname{span}\left\{(\mathbf{e}_1, 0, 0), (\mathbf{e}_2, 0, 0), \left(\frac{\alpha}{1-\alpha-\beta}\mathbf{e}_3, -1, -1\right)\right\}.$$

Notice that for every $v \in \mathfrak{l}_1 \cdot q_e$ we have that $G_v = G_{q_e}$, with G_{q_e} as in equation (4.3). In contrast, the relative equilibria corresponding to $\mathfrak{l}_2 \cdot q_e$ have trivial symmetry, i.e., $\forall v \in \mathfrak{l}_2 \cdot q$, $\mathfrak{g}_v = \{0\}$. (Notice that we can not adjust α so that $\mathfrak{g}_{q_e} \subset \mathfrak{l}_2$ because we would need $\alpha/(1-\alpha-\beta) = -1$, which in turn would imply $\beta = 1$, which is not possible unless the two bodies degenerate to point masses.) Let

$$\mathcal{E} := \mathcal{R}_{q_e} \cap (TQ)^{G_{q_e}} = \mathfrak{l}_1 \cdot q_e. \tag{4.4}$$

Thus $\mathcal{E}$ corresponds to the states in which the two bodies are rotating around their common axis of symmetry, each one with independent arbitrary constant angular velocity.

For the remainder of our discussion, we will study only the relative equilibria bifurcating from $\mathfrak{l}_1 \cdot q_e$.

4.3. Regularization of the Amended Potential

Recall that $\mathbb{I} : Q \to L(\mathfrak{g}, \mathfrak{g}^*)$ is the locked inertia tensor induced by the metric on Q, as defined in Section 1.3. Consider the basis $\mathcal{B} = \{\xi_i\}_{i=1}^5$ for $\mathfrak{g}$ given by $\xi_1 = (\mathbf{e}_3, 1, 1)$, $\xi_2 = (\mathbf{0}, 1, 0)$, $\xi_3 = (\mathbf{0}, 0, 1)$, $\xi_4 = (\mathbf{e}_1, 0, 0)$, and $\xi_5 = (\mathbf{e}_2, 0, 0)$. Then $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2$, where

$$\mathfrak{k}_0 := \ker \mathbb{I}(q_e) = \operatorname{span}\{\xi_1\},$$
$$\mathfrak{k}_1 := \operatorname{span}\{\xi_2, \xi_3\},$$
$$\mathfrak{k}_2 := [\mathfrak{g}, \mathfrak{k}_0 \oplus \mathfrak{k}_2] = \operatorname{span}\{\xi_4, \xi_5\}.$$

Notice that $\mathfrak{l}_1 = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ is a maximal abelian Lie subalgebra and that $\mathfrak{l}_1 \cdot q_e = \mathfrak{k}_1 \cdot q_e$.

Denote with $\mathcal{B}^*$ the dual basis of $\mathcal{B}$. Identify $\mathfrak{g}^*$ with $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ via the natural inner product $\langle (\mathbf{x}, y_1, y_2), (\mathbf{x}', y_1', y_2') \rangle := \mathbf{x} \cdot \mathbf{x}' + y_1 y_1' + y_2 y_2'$ (where "$\cdot$" denotes the standard inner product on $\mathbb{R}^3$). Then $\mathcal{B}^* = \{\nu^i\}_{i=1}^5$, where

$$\nu^1 = (\mathbf{e}_3, 0, 0), \qquad \nu^2 = (-\mathbf{e}_3, 1, 0), \qquad \nu^3 = (-\mathbf{e}_3, 0, 1),$$
$$\nu^4 = (\mathbf{e}_1, 0, 0), \qquad \nu^5 = (\mathbf{e}_2, 0, 0). \tag{4.5}$$

A calculation shows that the matrix of the locked inertia tensor at q_e with respect to the basis $\mathcal{B}, \mathcal{B}^*$ is given by

$$[\mathbb{I}(q_e)]_{\mathcal{B},\mathcal{B}^*} = \begin{pmatrix} 0 & & & & \\ & \alpha & & & \\ & & \alpha & & \\ & & & 2(1-\beta) & \\ & & & & 2\,(1-\beta) \end{pmatrix}. \tag{4.6}$$

Thus we see that $\mathbb{I}(q_e) \cdot \xi_i = \alpha\, \nu^i$ for $i = 2, 3$. Hence, $\mathbb{I}(q_e) \cdot (\mathfrak{k}_0 \oplus \mathfrak{k}_1) = \mathbb{I}(q_e) \cdot \mathfrak{k}_1 \subset \operatorname{span}(\nu^1, \nu^2, \nu^3) = \mathfrak{k}_2^\circ$. Therefore all the conditions of Section 3.1 hold and we can follow the procedure described in Section 3.4 for splitting and rescaling the dual of the Lie algebra in order to blow up the amended potential.

As in Section 3.4, the splitting $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2$ induces the dual splitting $\mathfrak{g}^* = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where

$$\begin{aligned} \mathfrak{m}_0 &:= (\mathfrak{k}_1 \oplus \mathfrak{k}_2)^\circ = \operatorname{span}(\nu^1), \\ \mathfrak{m}_1 &:= (\mathfrak{k}_0 \oplus \mathfrak{k}_2)^\circ = \operatorname{span}(\nu^2, \nu^3), \\ \mathfrak{m}_2 &:= (\mathfrak{k}_0 \oplus \mathfrak{k}_1)^\circ = \operatorname{span}(\nu^4, \nu^5). \end{aligned}$$

Notice that $\mathfrak{m}_2 = (\mathfrak{l}_1)^\circ$.

Now consider the map $\nu : \mathbb{R} \times (\mathbb{I}(q_e) \cdot \mathfrak{g}) \to \mathfrak{g}^*$ defined in equation (3.11). For our example it is explicitly given as follows. Let $\mu = \mu_1 + \mu_2$ with $\mu_1 = x_2\nu^2 + x_3\nu^3$, $\mu_3 = x_4\nu^4 + x_5\nu^5$, so that $\mu_i \in \mathfrak{m}_i$ $(i = 1, 2)$. Then

$$\begin{aligned} \nu(\tau, \mu) &= \nu(\tau; x_2, x_3, x_4, x_5) = \mu_1 + \tau\mu_2 + \tau^2\nu^1 \\ &= \tau^2\nu^1 + x_2\nu^2 + x_3\nu^3 + \tau(x_4\nu^4 + x_5\nu^5). \end{aligned}$$

Since we want to consider directions transversal to the group action at q_e, we define

$$N := (\mathfrak{g} \cdot q_e)^\perp = \operatorname{span}\{(\mathbf{e}_1, -\mathbf{e}_1)^\wedge, (\mathbf{e}_2, -\mathbf{e}_2)^\wedge\}.$$

Then G_{q_e} acts irreducibly on N. It is clear that

$$U_0 := \{\rho\, (\mathbf{e}_1, -\mathbf{e}_1)^\wedge \mid \rho > 0\}$$

is a (global) section of the principal bundle $(N \setminus \{0\}) \to (N \setminus \{0\})/G_{q_e}$. Let

$$\mathbb{I}_0(\rho) := \mathbb{I}\left(\exp(\rho\, (\mathbf{e}_1, -\mathbf{e}_1)^\wedge)\right). \tag{4.7}$$

A computation shows that the matrix representation of $\mathbb{I}_0$ with respect to the basis $\mathcal{B}, \mathcal{B}^*$ is given by

$$[\mathbb{I}_0(\rho)]_{\mathcal{B},\mathcal{B}^*} = \begin{pmatrix} 2\Big[\begin{smallmatrix}\alpha(-1+\cos\rho)^2 \\ +(1+\beta)\sin^2\rho\end{smallmatrix}\Big] & \alpha(1-\cos\rho) & \alpha(1-\cos\rho) & 0 & 0 \\ \alpha(1-\cos\rho) & \alpha & 0 & 0 & \alpha\sin\rho \\ \alpha(1-\cos\rho) & 0 & \alpha & 0 & -\alpha\sin\rho \\ 0 & 0 & 0 & 2(1-\beta\cos(2\rho)) & 0 \\ 0 & \alpha\sin\rho & -\alpha\sin\rho & 0 & 2\Big[(1-\beta)\cos^2\rho+\alpha\sin^2\rho\Big] \end{pmatrix}.$$

Let $\xi(\tau, \rho;\, x_i) := \mathbb{I}_0(\tau\rho)^{-1}\nu(\tau;\, x_i)$. From the remark following Proposition 3.10, we know that $\xi(\tau, \rho;\, x_i)$ is a smooth function even in a neighborhood of $\tau = 0$, provided that ρ is away from zero. A computation shows that $\xi(\tau, \rho;\, x_i) = \xi_0 + \tau\,\xi_1 + O(\tau^2) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ with

$$\begin{aligned} \xi_0 &= \left(\frac{2 - (x_2 + x_3)\ \rho^2}{4\ (1+\beta)\ \rho^2}, \frac{x_2}{\alpha}, \frac{x_3}{\alpha};\ 0, 0\right), \\ \xi_1 &= \left(0, 0, 0;\ \frac{x_4}{2 - 2\beta}, \frac{x_5 - x_2\,\rho + x_3\,\rho}{2 - 2\beta}\right). \end{aligned} \tag{4.8}$$

4.4. Relative Equilibria Bifurcating from Symmetric States

We now consider relative equilibria that bifurcate from the subspace of symmetric states in the fiber over the point q_e. Theorem 3.16 guarantees the existence of a $\mathfrak{k}_2^*$-valued smooth function X such that

$$dV_{\nu(\tau,\mu)}\left(\exp(\tau\rho(\mathbf{e}_1,-\mathbf{e}_1)^\wedge)\right)\cdot\eta_Q(q_0)=\tau\left\langle X(\tau,\rho,\mu),\eta\right\rangle,$$

where $\eta\in\mathfrak{k}_2$ and $\mu:=(x_2,x_3,x_4,x_5)$. For the symmetric coupled rigid bodies, the formula for $\langle X,\eta\rangle|_{\tau=0}$ given in Proposition 3.17 reduces to

$$\langle X(0,\rho,\mu),\eta\rangle=\langle\mathbb{I}_0(0)[\xi_1,\eta],\xi_0\rangle+\langle\mathbb{I}_0(0)\xi_1,[\xi_0,\eta]\rangle,$$

where ξ_0 and ξ_1 are given in (4.3) and $\mathbb{I}_0(0)$ is equal to the right-hand side of (4.6).

Using that $\mathfrak{k}_2=\operatorname{span}\{(\mathbf{e}_1,0,0),(\mathbf{e}_2,0,0)\}$, we get, after a computation, that the condition $X(0,\rho,\mu)=0$ is equivalent to the pair of equations

$$\left.\begin{aligned}\langle X(0,\rho,\mu),(\mathbf{e}_1,0,0)\rangle=(x_5-\rho\,y_-)\,f(\rho,x_2,x_3)=0\\ \langle X(0,\rho,\mu),(\mathbf{e}_2,0,0)\rangle=x_4\,f(\rho,x_2,x_3)=0\end{aligned}\right\},\tag{4.9}$$

where $f(\rho,x_2,x_3)=\Delta/\left(4(\beta^2-1)\rho^2\right)$ with

$$\Delta=2(1-\beta)+(1+3\beta)\rho^2y_+,\tag{4.10}$$

and where we have introduced the linear change of variables

$$y_+:=x_2+x_3,\qquad y_-:=x_2-x_3.$$

The rescaled amended potential restricted to U_0 is given by

$$V_{\nu(\tau,\mu)}\left(\exp\left(\tau\,\rho(\mathbf{e}_1,-\mathbf{e}_1)^\wedge\right)\right)=\left\langle\nu(\tau,\mu),\mathbb{I}_0^{-1}(\tau\rho)\nu(\tau,\mu)\right\rangle.$$

By Theorem 3.15 we know that $V_{\nu(\tau,\mu)}\left(\exp\left(\tau\,\rho(\mathbf{e}_1,-\mathbf{e}_1)^\wedge\right)\right)$ is a smooth function, even in a neighborhood of $\tau=0$, provided that ρ is away from zero. A computation shows that its Taylor expansion with respect to τ is given by

$$V_{\nu(\tau,\mu)}\left(\exp\left(\tau\,\rho(\mathbf{e}_1,-\mathbf{e}_1)^\wedge\right)\right)=\frac{y_+^2+y_-^2}{2\alpha}+\tau^2W(\tau,\rho,\mu),$$

with

$$\begin{aligned}W(\tau,\rho,\mu)\;=\;&\frac{(1+\beta)(x_4^2+x_5^2)-(1-\beta)y_+}{2(1-\beta^2)}+\frac{1}{2(1+\beta)\rho^2}\\&-\frac{\rho\,x_5\,y_-}{1-\beta}+\frac{\left((1-\beta)\,y_+^2+4\,(1+\beta)\,y_-^2\right)\rho^2}{8(1-\beta^2)}+O(\tau^2).\end{aligned}$$

Therefore,

$$\frac{\partial W}{\partial\rho}(0,\rho,\mu)=\frac{4(1-\beta)+4(1+\beta)\rho^3x_5\,y_--\left((1-\beta)y_+^2+4(1+\beta)y_-^2\right)\rho^4}{4(\beta^2-1)\rho^3}.\tag{4.11}$$

From the pair of equations in (4.9) we see that $X(0, \rho, \mu) = 0$ if and only if either $\Delta = 0$ or

$$x_5 - \rho\, y_- = 0 \quad \text{and} \quad x_4 = 0. \tag{4.12}$$

If we assume that (4.12) holds, then, after substituting in (4.11), we see that the equation $\partial W/\partial\rho(0, \rho, \mu) = 0$ is equivalent to

$$4 - \rho^4\, y_+^2 = 0, \tag{4.13}$$

and thus $\rho = \sqrt{2/|x_2 + x_3|}$. In summary, we have shown the following:

Proposition 4.1. *Given x_2, x_3 such that $x_2 + x_3 \neq 0$, let*

$$(\tilde{\rho}, \tilde{x}_4, \tilde{x}_5) = \sqrt{2/\,|x_2 + x_3|}\ (1, 0, x_2 - x_3)\,, \tag{4.14}$$

and let $\tilde{\mu} = (x_2, x_3, \tilde{x}_4, \tilde{x}_5)$. Then

$$X(0, \tilde{\rho}, \tilde{\mu}) = 0 \quad \textit{and} \quad \frac{\partial W}{\partial \rho}(0, \tilde{\rho}, \tilde{\mu}) = 0.$$

Eliminating $\rho^2 y_+$ from equations (4.13) and $\Delta = 0$ (with Δ given by (4.10)), we get

$$\beta(\beta^2 - 1) = 0.$$

Therefore:

Lemma 4.2. *If $0 < \beta < 1$, then the conditions $\rho = \sqrt{2/|x_2 + x_3|}$ and $\Delta = 0$ are mutually exclusive.*

Expressing $X(\tau, \rho, \mu)$ in terms of the basis of $\mathfrak{k}_2$ dual to $\{(e_1, 0, 0), (e_2, 0, 0)\}$, we compute from (4.9) and (4.11) that

$$\frac{\partial(X, \partial W/\partial\rho)}{\partial(\rho, x_4, x_5)}(0, \tilde{\rho}, \tilde{\mu}) = \begin{pmatrix} \Delta\, \tilde{\rho}\, y_- & 0 & -\Delta\, \tilde{\rho} \\ 0 & \Delta\, \tilde{\rho} & 0 \\ -4[(1-\beta)y_+^2 + (1+\beta)y_-^2]\tilde{\rho}^3 & 0 & 4(1+\beta)\tilde{\rho}^3 y_- \end{pmatrix}.$$

The determinant is computed to be

$$\left|\frac{\partial(X, \partial W/\partial\rho)}{\partial(\rho, x_4, x_5)}(0, \tilde{\rho}, \tilde{\mu})\right| = \frac{16\sqrt{2}(\beta - 1)\Delta^2}{\sqrt{|x_2 + x_3|}}.$$

Therefore, if $0 < \beta < 1$, then it follows from Lemma 4.2 and the implicit function theorem that the equations

$$\left.\begin{aligned} X(\tau, \rho; x_2, x_3, x_4, x_5) = 0 \\ \frac{\partial W}{\partial \rho}(\tau, \rho; x_2, x_3, x_4, x_5) = 0 \end{aligned}\right\} \tag{4.15}$$

implicitly define the parameters (ρ, x_4, x_5) as smooth functions of (τ, x_2, x_3). More precisely, for every bounded open region $V \subset \mathbb{R}^2 \setminus \{(x_2, x_3) \mid x_2 + x_3 = 0\}$, there exists an $\epsilon > 0$ and smooth functions (ρ, x_4, x_5) defined on $(-\varepsilon, \varepsilon) \times V$ such that (ρ, x_4, x_5) evaluated at $(0, x_2, x_3)$ is equal to the right-hand side of (4.14) and for all $(\tau, x_2, x_3) \in (-\varepsilon, \varepsilon) \times V$,

$$(\tau, \rho(\tau, x_2, x_3);\ x_2, x_3, x_4(\tau, x_2, x_3), x_5(\tau, x_2, x_3))$$

is a solution of (4.15).

Recall that $\mathcal{E} := \mathfrak{l}_1 \cdot q_e$ is the set of relative equilibria in $T_{q_e}Q$ with symmetry group equal to G_{q_e} (cf. equation (4.4)) and that $\mathfrak{m}_2 := (\mathfrak{l}_1)^\circ \subset \mathfrak{g}^*$. Recall also that $\mathcal{B}^* = \{\nu^i\}_{i=1}^5$ is the basis for $\mathfrak{g}^*$ given by equation (4.5). To facilitate notation, let $\tilde{\mathcal{A}} : Q \times \mathfrak{g}^* \to T^*Q$ be given by $\tilde{\mathcal{A}}(q, \mu) := \mathcal{A}_\mu(q)$, where $\mathcal{A}_\mu$ is the associated one-form of the mechanical connection introduced in Section 1.3. From the preceding remarks we conclude with the following:

Proposition 4.3. *Suppose that* $0 < \beta < 1$. *For every* $\mu_1 = x_2\nu^2 + x_3\nu^3 \in \mathbf{J}_L(\mathcal{E})$ *such that* $x_2 + x_3 \neq 0$, *there exist an* $\epsilon > 0$ *and a curve* $(\rho^{(\mu_1)}, \mu_2^{(\mu_1)}) : [0, \epsilon] \to \mathbb{R} \times \mathfrak{m}_2$ *such that*

$$\left(\rho^{(\mu_1)}(0), \mu_2^{(\mu_1)}(0)\right) = \sqrt{2/|x_2 + x_3|}\,(1, (x_2 - x_3)\nu^5),$$

and such that, for $\tau \in [0, \epsilon]$, *the curve*

$$\alpha^{(\mu_1)}(\tau) := \left(\exp(\tau\rho^{(\mu_1)}(\tau)(\mathbf{e}_1, -\mathbf{e}_1)^\wedge), \tau^2\nu^1 + \mu_1 + \tau\mu_2^{(\mu_1)}(\tau)\right) \in Q \times \mathfrak{g}^*$$

satisfies that $\tilde{\mathcal{A}}\left(\alpha^{(\mu_1)}(\tau)\right)$ *is a symmetry-breaking branch of relative equilibria emanating from* $\tilde{\mathcal{A}}(q_e, \mu_1)$.

Remark. In Patrick [1990] (see also Patrick [1989]), where an exhaustive classification of the relative equilibria for the SCRB is offered, it is shown that there is a family of classes of relative equilibria persisting from $\mathcal{E}$ represented by the states described in body coordinates by

$$\mathcal{E}' = \left\{(\exp(\theta\,\mathbf{e}_1), \exp(-\theta\,\mathbf{e}_1); t_2\,\mathbf{e}_3, t_3\,\mathbf{e}_3) \in SO(3)^2 \times (\mathbb{R}^3)^2\right\},$$

where the angle θ and the spin velocities t_2, t_3 are arbitrary. (Physically, states in $\mathcal{E}'$ correspond to each body rotating with arbitrary angular velocity around their axis of symmetry with these forming an arbitrary angle at the joint, and with no further overall rotation.) The momenta of states in $\mathcal{E}'$ is given by $t_2\mathbb{I}_0(\tau)\xi_2 + t_3\mathbb{I}_0(\tau)\xi_3$, with $\theta = \tau\,\rho(\tau, \mu_1)$, $\mathbb{I}_0$ given by (4.7) and ξ_2, ξ_3 the basis elements in $\mathcal{B}$ introduced above. It is straightforward to see that $\mathbb{I}_0(\theta)\xi_K = O(\tau^2)\nu^1 + \alpha\nu^k + O(\tau)\nu^5$. This shows that the existence of the family of relative equilibria $\mathcal{E}'$ is consistent with Proposition 4.3. Thus our method predicts one of the types of relative equilibria computed *by hand* in Patrick [1990].

5. Conclusions

In the context of simple mechanical systems with symmetry, we have given a method for predicting the existence of symmetry-breaking branches of relative equilibria bifurcating from a given set $\mathcal{E}$ of relative equilibria with nontrivial isotropy. Although there are several results concerning bifurcation of relative equilibria in the literature, most of them are in the context of general Hamiltonian systems with symmetry, while in this paper we exploit the more detailed structure that we get from working with simple mechanical systems with symmetry. In fact, we directly relate the bifurcating branches to a study of the blown-up amended potential, a fundamental object in the study of simple mechanical systems on cotangent bundles.

The methods we use assume that $\mathcal{E} = \mathfrak{t} \cdot q_e \in TQ$ for some maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and some equilibrium configuration q_e such that $\mathfrak{g}_{q_e} \subset \mathfrak{t}$, where $\mathfrak{g}$ is the Lie algebra of the symmetry group which is assumed to be compact. We have also assumed that q_e is an isolated point with nontrivial isotropy in a slice through q_e for the action of the symmetry group.

The method in this paper is based on the regularization of the *amended potential criterion for relative equilibria* around q_e. This regularization was achieved through a rescaling (or *blowing-up*) of certain directions in the dual of the Lie algebra of the symmetry group and the directions in configuration space along a slice for the action of the symmetry group at q_e. After regularization, the bifurcating branches were obtained by an application of the implicit function theorem. Then our bifurcation analysis was done in shape space.

Although most of the analysis in Section 3 is done assuming that G_{q_e} (the isotropy group of q_e) is an n-torus, the regularization of the relative equilibrium condition and the final bifurcation analysis (Sections 3.5 and 3.6) are done in the context of $G_{q_e} \cong S^1$. We believe, however, that the methods of these sections can easily be adapted to the more general case of G_{q_e} being an n-torus. Presumably, within this generalization, the nondegeneracy condition of Section 3.6 will still involve checking that a certain $\mathbb{R}$-valued bilinear map analogous to the map defined in Definition 3.18 is nondegenerate but $\lambda \in \hat{\mathfrak{m}}_0$ will appear as an extra parameter in the bifurcating surfaces of relative equilibria of Theorem 3.19. Here $\hat{\mathfrak{m}}_0$ is the unit sphere in the subspace $\mathfrak{m}_0 \subset \mathfrak{g}^*$ with respect to some G_{q_e}-invariant inner product, as defined in Section 3.3. (See Birtea, Puta, Ratiu, Tudoran [2004] for an alternative approach to this generalization.)

In the example of the double spherical pendulum, one can deduce stability of the relative equilibria in a bifurcating branch (at least in a neighborhood of the equilibrium configuration) from the positive definiteness of the second variation of the blown-up amended potential; cf. the remark at the end of Section 2.1. This follows from an application of the energy-momentum method (cf. Marsden [1992, Chap. 5]) when the symmetry group is Abelian. Presumably a similar stability analysis can be adapted to the general case studied in Section 3. (This idea has been successfully implemented for toral actions in Birtea, Puta, Ratiu, Tudoran [2004].)

For non-abelian groups the stability analysis requires that we look at the second variation of the so-called Arnold form (cf. op. cit.). It would be of interest to study whether there is a blown-up version of this form.

The example of the double spherical pendulum also suggests that one can try to generalize the method of using the linearized vector field associated with the blown-up Routhian to determine the stability type of the bifurcating branches of relative equilibria (cf. the remark in Section 2.1). To this end one would look at the eigenvalues of the linearized vector field of the Euler-Lagrange equations associated with the blown-up Routhian, where the blowing-up would take place in shape space.

It would also be of interest to generalize the method of this paper to the case when q_e is not an isolated fixed point of the action of G_{q_e} on the slice. It seems likely that one could implement a series of slice decompositions that would be related with the lattice of isotropy subgroups.

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