Geometric Mechanics and the Dynamics of Asteroid Pairs

Hernán Cendra
Departamento de Matemática
Universidad Nacional del Sur, Av. Alem 1254
8000 Bahia Blanca and CONICET, Argentina.
uscendra@criba.edu.ar

Jerrold E. Marsden
Control and Dynamical Systems 107-81
California Institute of Technology
Pasadena, CA 91125-8100
marsden@cds.caltech.edu

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Abstract

This paper studies, using the technique of Lagrangian reduction, the geometric mechanics of a pair of asteroids in orbit about each other under mutual gravitational attraction.

1 Introduction

The binary asteroid problem is of current astrodynamic interest as recent studies suggest that up to 20% of Near-Earth Asteroids are binaries (see Margot, Nolan, Benner, et al. [2002]), and there have been many discoveries of binaries in the Main Asteroid Belt and the Kuiper Belt (Veillet, Parker, Griffin, Marsden et al. [2002]; Merline, Weidenschilling, Durda, Margo et al. [2002]). The paper Koon, Marsden, Ross, Lo, and Scheeres [2004] studied some aspects of the geometric mechanics and dynamics of asteroid pairs; that is, a pair of irregularly shaped asteroids orbiting about each other and attracted by their mutual gravitational forces. That paper focused on planar models of the restricted problem (that is, the central body is uniformly rotating, undisturbed by the second body). Already in that problem there were interesting aspects to the dynamics such as capture and ejection.

That paper reviewed parts of the literature, including studies of translational-rotational coupling, what is known about relative equilibria and their stability and related topics. See Scheeres, Ostro, Werner, et al. [2000] and Scheeres [2001, 2002,a, 2004] and references therein for further information. In this paper we focus just on the geometric mechanics aspects and take the analysis of the geometric mechanics aspects begun in Koon, Marsden, Ross, Lo, and Scheeres [2004] a little bit further. We shall focus, in particular on the general case of two irregular bodies in orbit about each other in three dimensions. We do not attempt, in this paper, to study approximations or the hierarchy of different models that have been studied.

In a sense, our program is quite simple: to carry out Lagrangian reduction, in the sense of Cendra, Marsden, and Ratiu [2001] for this problem together with some associated structures. The configuration manifold is simply two copies of the Euclidean group of three space, namely $Q = \mathrm{SE}(3) \times \mathrm{SE}(3)$ and the symmetry group is SE(3) acting by the diagonal action, which reflects the obvious overall translational and rotational symmetry of the problem. The approach we take, following this reference is to use general Lagrangian reduction methods, which rely on a specific choice of connection, whose curvature leads to a nice description of the Coriolis effects, as well as being critical for carrying out sharp stability analyses and a geometric phase analysis.

We hope that this work will lay a useful foundation for future works, such as studying relative equilibria and their stability using the energy momentum methods (as in Simo, Lewis, and Marsden [1991]), geometric phases (as in Marsden, Montgomery, and Ratiu [1990]), variational integrators (as in Marsden and West [2001]), transport calculations (as in Dellnitz, Junge, Koon, et al. [2004]), etc.

In addition to the asteroid pair problem, there are other full body problems that we expect will benefit from the present approach, such as problems involving underwater vehicles and swimming. See, for instance, Radford [2003]; Kanso, Marsden, Rowley, and Melli-Huber [2004].

One of the reasons we focus just on this specific problem is that despite earlier and very nice studies, such as Maciejewski [1995, 1999]; Goździewski and Maciejewski [1999], we have learned that it is important to lay the foundations of these sorts of problems based on general principles so that one can take full advantage of, for example, the energy-momentum method, dynamical systems ideas, how to place the problem in a hierarchy of problems, etc. This was already important in, for example, the paper Koon, Marsden, Ross, Lo, and Scheeres [2004], which reviewed some of the literature in the area.

2 Kinematics, Dynamics and Symmetry

We begin with some kinematical preliminaries.

The Configuration Manifold. This paper deals with the ideal model of asteroids as free rigid bodies in Euclidean 3-space $\mathbb{R}^3$ under mutual gravitational attraction. Therefore, the configuration space of a single asteroid is the special Euclidean group (the semidirect product of rotations and translations), which we denote by SE(3). We think of an element $g \in \mathrm{SE}(3)$ as giving the position and orientation of the asteroid relative to an inertial frame. The configuration space of the two asteroid problem is similarly given by

$$Q = \mathrm{SE}(3) \times \mathrm{SE}(3)$$

and an element $(g_1, g_2) \in Q$ represents the placement and orientation of each of the two bodies relative to a fixed inertial frame. See Figure 2.1.

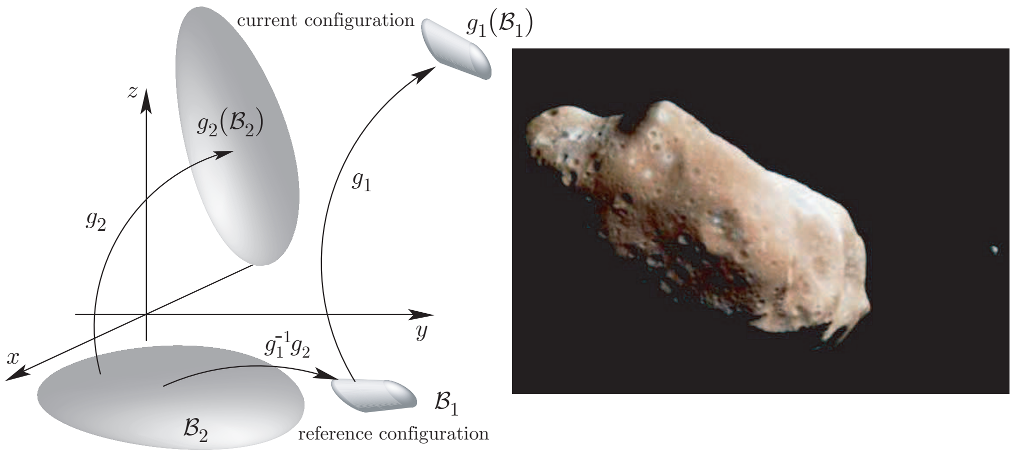

Figure 2.1: The configuration space of the asteroid pair problem is $Q = \mathrm{SE}(3) \times \mathrm{SE}(3)$. The left figure indicates how the placement and orientation of each body is given by a group element g_i. The right figure shows the asteroid pair Ida and its smaller companion Dactyl, discovered by the Galileo mission on August 28, 1993 in the asteroid belt between Mars and Jupiter.

The Lie Group SE(3)**.** As is well-known (see, for example, Marsden and Ratiu [1999], §14.7), an element of SE(3) may be represented by a 4×4 matrix,

$$g = (R, a) := \begin{bmatrix} R_1^1 & R_2^1 & R_3^1 & a^1 \\ R_1^2 & R_2^2 & R_3^2 & a^2 \\ R_1^3 & R_2^3 & R_3^3 & a^3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the matrix

$$R = \begin{bmatrix} R_1^1 & R_2^1 & R_3^1 \\ R_1^2 & R_2^2 & R_3^2 \\ R_1^3 & R_2^3 & R_3^3 \end{bmatrix}$$

is the matrix of a rotation, that is, an element of $SO(3)$, and the vector $a = (a^1, a^2, a^3)$ represents a translation.

In this notation, group multiplication is given by

$$(R, a) \cdot (S, b) = (RS, R \cdot b + a)$$

and corresponds to 4×4 matrix multiplication. The inverse of a group element (R, a) is given by $(R, a)^{-1} = (R^{-1}, -R^{-1}a)$.

If we choose an element $X \in \mathbb{R}^3$ representing a point in the reference configuration, then the current point x is given by $x = RX + a$, or, using 4×4 matrix notation,

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = g \begin{bmatrix} X \\ 1 \end{bmatrix}.$$

The Lie Algebra. An element of the Lie algebra $\mathfrak{se}(3)$ of SE(3) is represented by a tangent vector to the group SE(3) at the identity and has the form

$$\xi = (\hat{\Omega}, u) := \begin{bmatrix} 0 & -\Omega^3 & \Omega^2 & u^1 \\ \Omega^3 & 0 & -\Omega^1 & u^2 \\ -\Omega^2 & \Omega^1 & 0 & u^3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the 3×3 skew matrix $\hat{\Omega}$, an element of the Lie algebra of SO(3), corresponds to the vector $\Omega = (\Omega_1, \Omega_2, \Omega_3)$.

As in the case of ordinary rigid body mechanics (see, for example, Marsden and Ratiu [1999], the kinematic meaning of an element of $\mathfrak{se}(3)$ is easy to understand, as follows. Let $(R(t), a(t))$ be a motion of a free rigid body. That is, the motion of a particle with reference label X and current position x is given by the curve in $\mathbb{R}^3$ given by

$$x(t) = R(t)X + a(t).$$

Therefore, $\dot{a}(t) = v(t)$ is the ***translational velocity*** of the body with respect to the given inertial frame, and $R^{-1}(t)\dot{R}(t) = \hat{\Omega}(t)$ is the ***body angular velocity***. Therefore the element

$$\xi(t) = \left.\frac{d}{ds}\right|_{s=0} (R(t), a(t))^{-1} (R(t+s), a(t+s))$$

is given by $\xi(t) = \left(\hat{\Omega}(t), u(t)\right)$, where $u(t) = R(t)^{-1}v(t)$, represents the translational velocity, as viewed from the body.

The Symmetry Group. As we have seen, the configuration space for an asteroid pair is naturally given by $Q = \mathrm{SE}(3) \times \mathrm{SE}(3)$. The asteroid pair problem is obviously invariant when both asteroids are simultaneously translated and rotated by the same element of SE(3). That is, the problem has the symmetry group SE(3) acting on Q by the simultaneous diagonal action *on the left*.

Of course in particular cases, there may be additional symmetries. For instance, if one of the bodies (say the first one) has an axial symmetry, then the symmetry group is given by $S^1 \times \mathrm{SE}(3)$ with $S^1 \subset \mathrm{SO}(3)$ acting on the first body *on the right* and with SE(3) acting *on the left*. However, for the moment, we will deal with the general case in which both bodies are irregular and so the symmetry group is just SE(3).

As mentioned in the introduction (and as discussed in Koon, Marsden, Ross, Lo, and Scheeres [2004]), other specializations are also relevant, such as when one of the bodies is very small (as in the case of Dactyl) and does not affect the dynamics of the larger asteroid—the *restricted problem*. This results in much simpler problems for which a detailed dynamical analysis is possible, as indicated in Koon, Marsden, Ross, Lo, and Scheeres [2004].

The Principal Bundle of $G \times G$. We pause for a moment to consider some geometry of configuration spaces of the form $Q = G \times G$, where G is a general Lie group whose Lie algebra is denoted $\mathfrak{g}$.

To make use of the methods of Cendra, Marsden, and Ratiu [2001], we consider Q as being a principal bundle with structure group G, with G acting on Q by the diagonal action; that is, the action given by $h(g_1, g_2) = (hg_1, hg_2)$. The base of this principal bundle is $X = G$, with the projection $\pi : G \times G \to X$ given by $\pi(g_1, g_2) = g_1^{-1} g_2$, which, to conform to the notation of the general theory, we often write as simply $x = g_1^{-1} g_2$. Notice that x is unchanged under the action of G on Q.

A Connection on the Bundle. A section of the bundle $\pi : Q \to G$ is given by $\sigma(x) = (x^{-1}, e)$. Recall that having a section means that that $\pi(\sigma(x)) = x$. This shows, in particular, that π is a trivial bundle. For any $a \in G$, there is a a section $a\sigma : X \to G \times G$, given by $a\sigma(x) = (ax^{-1}, a)$. The family of all such sections of the form $a\sigma$, gives a foliation of $G \times G$. The tangent distribution to this foliation defines a principal connection by declaring this tangent distribution to be the horizontal spaces of the connection. Since these horizontal spaces are obviously integrable, this connection has zero curvature.

We shall use notation suggested by matrix groups (such as SE(3) that is realized as a group of 4×4 matrices, as explained above). This will simplify the notation somewhat, especially for those not familiar in notation for Lie groups and group actions.

Horizontal vectors have the form $\left(ax^{-1}, a, -ax^{-1}\dot{x}x^{-1}, 0\right)$, while vertical vectors (that is, tangents to the group orbits) have the form $\left(ax^{-1}, a, \dot{a}x^{-1}, \dot{a}\right)$. The associated connection 1-form $A : T(G \times G) \to \mathfrak{g}$ is given by

$$A\left(g_1, g_2, g_1\xi_1, g_2\xi_2\right) = \mathrm{Ad}_{g_2} \xi_2 = g_2 \xi_2 g_2^{-1}.$$

In the developments below, one may choose any section one wishes to obtain a connection. In particular, one may choose $\sigma_{\text{alternative}}(x) = (e, x)$, which leads to the connection one form

$$A_{\text{alternative}}\left(g_1, g_2, g_1\xi_1, g_2\xi_2\right) = g_1 \xi_1 g_1^{-1}.$$

We recall from the theory of principal connections how the connection one form is related to the horizontal and vertical spaces. See, for example, Bloch [2003] for a concise summary of principal connections suitable for people in mechanics. Namely, at each point $(g_1, g_2) \in Q$, the subspace $\ker A(g_1, g_2)$ is the horizontal space, and if $\xi \in \mathfrak{g}$ and ξ_Q is its corresponding infinitesimal generator on Q, then the span of the vectors $\xi_Q(g_1, g_2)$ as ξ ranges over $\mathfrak{g}$ is the vertical space. One readily checks that $\xi_Q(g_1, g_2) = (g_1, g_2, \xi g_1, \xi g_2))$ and also that $A(g_1, g_2) \cdot \xi_Q(g_1, g_2) = \xi$.

Facts from Rigid Body Mechanics. Now we return to the specific case of $G = \mathrm{SE}(3)$. First, we recall some facts from rigid body mechanics. The Lagrangian of a single freely spinning and isolated asteroid (that is, a freely spinning and translating rigid body) is given by its kinetic energy. It can be conveniently written using a metric, which we will denote k on the Lie algebra $\mathfrak{g}$, which is the restriction of a uniquely determined left invariant metric, also called k on G. The metric k is in fact given by

$$k(\xi, \xi) = \mathrm{tr}(K\xi^{\mathrm{T}}\xi),$$

where ξ^{T} is the transpose of ξ and the matrix K, which we shall write as

$$K = \begin{bmatrix} K_1 & 0 & 0 & 0 \\ 0 & K_2 & 0 & 0 \\ 0 & 0 & K_3 & 0 \\ 0 & 0 & 0 & K_4 \end{bmatrix},$$

satisfies

$$\begin{aligned} K_1 + K_2 &= I_3 \\ K_2 + K_3 &= I_1 \\ K_3 + K_1 &= I_2 \\ K_4 &= M, \end{aligned}$$

where I_1, I_2, and I_3, are the principal moments of inertia of the body and M is its total mass. We can easily see that

$$\frac{1}{2}k(\xi, \xi) = \frac{1}{2}\left(I_1(\Omega^1)^2 + I_2(\Omega^2)^2 + I_3(\Omega^3)^2\right) + \frac{1}{2}M\left((u^1)^2 + (u^2)^2 + (u^3)^2\right),$$

which is of course the standard expression for the kinetic energy for a free rigid body, in body coordinates.

The Two Asteroid Lagrangian. The Lagrangian for two asteroids is given by the total kinetic energy minus the mutual gravitational potential energy; it has the form

$$L(g_1, g_2, \dot{g}_1, \dot{g}_2) = \frac{1}{2}k_1(\xi_1, \xi_1) + \frac{1}{2}k_2(\xi_2, \xi_2) - V(g_1, g_2). \tag{2.1}$$

In this expression, $\xi_i = g_i^{-1}\dot{g}_i$, for $i = 1, 2$, and, for $i = 1, 2$, the metric k_i is given by

$$k_i(\xi_i, \xi_i) = \mathrm{tr}(K_i\xi_i^{\mathrm{T}}\xi_i),$$

where the matrix K_i, which we write as

$$K_i = \begin{bmatrix} K_{i1} & 0 & 0 & 0 \\ 0 & K_{i2} & 0 & 0 \\ 0 & 0 & K_{i3} & 0 \\ 0 & 0 & 0 & K_{i4} \end{bmatrix}$$

satisfies

$$\begin{aligned} K_{i1} + K_{i2} &= I_{i3} \\ K_{i2} + K_{i3} &= I_{i1} \\ K_{i3} + K_{i1} &= I_{i2} \\ K_{i4} &= M_i, \end{aligned}$$

where I_{i1}, I_{i1}, and I_{i1}, are the principal moments of inertia and M_i is the mass of the ith rigid body, $i = 1, 2$. We will sometimes write $k_i(\xi_i) = k_i(\xi_i, \xi_i)$, for $i = 1, 2$.

We can then easily see that

$$\begin{aligned} L(g_1, g_2, \dot{g}_1, \dot{g}_2) = &\frac{1}{2}\left(I_{11}(\Omega_1^1)^2 + I_{12}(\Omega_1^2)^2 + I_{13}(\Omega_1^3)^2\right) + \frac{1}{2}M_1\left((u_1^1)^2 + (u_1^2)^2 + (u_1^3)^2\right) \\ &+ \frac{1}{2}\left(I_{21}(\Omega_2^1)^2 + I_{22}(\Omega_2^2)^2 + I_{23}(\Omega_2^3)^2\right) + \frac{1}{2}M_2\left((u_2^1)^2 + (u_2^2)^2 + (u_2^3)^2\right) \\ &- V(g_1, g_2). \end{aligned}$$

Here, $\Omega_i = \left(\Omega_i^1, \Omega_i^2, \Omega_i^3\right)$ and $u_i = \left(u_i^1, u_i^2, u_i^3\right)$, represent, respectively, the body angular velocity, and the translational velocity in body coordinates of the ith rigid body, $i = 1, 2$.

Letting $x_i = g_i(X)$ denote a point in the current configuration of body i, $i = 1, 2$, and G be the gravitational constant, the gravitational potential is of course given by the standard expression

$$V(g_1, g_2) = -G\int_{g_1(\mathcal{B}_1)}\int_{g_2(\mathcal{B}_2)}\frac{\rho_1(x_1)\rho_2(x_2)}{\|x_1 - x_2\|}dx_1\, dx_2 \tag{2.2}$$

where ρ_i is the given mass density of body i.

It is easy to see from equation (2.1) that the kinetic energy for the asteroid pair is invariant under the diagonal action of G on $G \times G$. Likewise we see from (2.2) that the function $V : Q \to \mathbb{R}$ is invariant under the action of G. Thus, the asteroid pair Lagrangian is also invariant.

Momentum Map. From Noether's theorem (see Marsden and Ratiu [1999], equation 12.2.1), we then find that the conserved momentum map for the action of SE(3) on $Q = \mathrm{SE}(3) \times \mathrm{SE}(3)$ is the map $\mathbf{J} : TQ \to \mathfrak{se}(3)^*$ given by

$$\langle \mathbf{J}(g_1, g_2, \dot{g_1}, \dot{g_2}), \xi\rangle = \operatorname{tr}\left(K\xi_1^{\mathrm{T}}g_1^{-1}\xi g_1\right) + \operatorname{tr}\left(K\xi_2^{\mathrm{T}}g_2^{-1}\xi g_2\right)$$

where $\xi_i = g_i^{-1}\dot{g}_i$. This quantity represents the total *spatial* linear and angular momentum of the full body system.

3 The Lagrange–Poincaré Equations.

The Lagrange–Poincaré equations where introduced in Cendra, Marsden, and Ratiu [2001], and are a convenient representation of the reduced Euler-Lagrange equations, that is, the Euler–Lagrange equations induced on $(TQ)/G$ from those on TQ; they are called the Lagrange–Poincaré equations because in the special case of $Q = G$, they become the Euler–Poincaré equations on $\mathfrak{g}$. The general theory of these equations is given in this reference from the point of view of using connections on the bundle $Q \to X = Q/G$. The Lagrange–Poincaré equations are conveniently derived by reducing the Hamilton variational principle, rather than the Euler-Lagrange equations themselves, and the use of a principal connection gives them a particularly nice form. In what follows, we are going to use notation and results from this work, which goes by the name of *Lagrangian reduction theory*.

The Isomorphism α_A. One of the basic ingredients in this theory is an isomorphism that is obtained in a natural way from the principal connection A; in general, it is an isomorphism

$$\alpha_A : TQ/G \to TX \oplus \tilde{\mathfrak{g}},$$

where $\tilde{\mathfrak{g}}$ is the *associated bundle*, a bundle over X, which we recall is defined to be the quotient $\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$ where G acts on the first factor by the given action on Q and on the second factor by the adjoint action. The general definition is

$$\alpha_A([q, \dot{q}]_G) = T\pi(q, \dot{q}) \oplus [(q, A(q, \dot{q})]_G,$$

where $[q, \dot{q}]_G$ denotes the equivalence class of the tangent vector $(q, \dot{q})$ in the quotient TQ/G.

In our case, where $Q = G \times G$, this becomes

$$\alpha_A\left([g_1, g_2, \dot{g}_1, \dot{g}_2]_G\right) = T\pi\left(g_1, g_2, \dot{g}_1, \dot{g}_2\right) \oplus \left[(g_1, g_2), A\left(g_1, g_2, \dot{g}_1, \dot{g}_2\right)\right]_G.$$

Let us work out each term. Since $x = \pi(g_1, g_2) = g_1^{-1} g_2$, we see that $(x, \dot{x}) = T\pi\left(g_1, g_2, \dot{g}_1, \dot{g}_2\right)$, is given by

$$\begin{aligned}(x, \dot{x}) &= \left(x, -g_1^{-1}\dot{g}_1 g_1^{-1} g_2 + g_1^{-1}\dot{g}_2\right)\\ &= \left(x, -\xi_1 x + x\xi_2\right),\end{aligned}$$

where, as above, $\xi_i = g_i^{-1}\dot{g}_i$, for $i = 1, 2$. We remark for later use that using this expression, we obtain

$$\xi_1 = -\dot{x}x^{-1} + x\xi_2 x^{-1}.$$

Since $(g_1, g_2, \dot{g}_1, \dot{g}_2) = (g_1, g_2, g_1\xi_1, g_2\xi_2)$, we have

$$[(g_1, g_2), A(g_1\xi_1, g_2\xi_2)]_G = [(g_1, g_2), \mathrm{Ad}_{g_2}\xi_2]_G = \left[(x^{-1}, e), \xi_2\right]_G.$$

We shall identify $\left[(x^{-1}, e), \xi_2\right]_G \equiv (x, \xi_2)$. Under this identification the bundle $\tilde{\mathfrak{g}}$ becomes simply $\tilde{\mathfrak{g}} \equiv X \times \mathfrak{g}$, where the fiber over the point $x \in X$ is parametrized by the variable ξ_2.

The Structure of $\tilde{\mathfrak{g}} \equiv X \times \mathfrak{g}$. The geometric structure of $\tilde{\mathfrak{g}}$ can be easily derived from the formulas in Cendra, Marsden, and Ratiu [2001], and we obtain

1 Lie algebra structure on each fiber, given by

$$[(x, \xi_2), (x, \eta_2)] = (x, [\xi_2, \eta_2])$$

2 The covariant derivative on the bundle $\tilde{\mathfrak{g}}$ induced by the connection A is

$$\frac{D\left(x(t), \xi_2(t)\right)}{Dt} = \left(x(t), \dot{\xi}_2(t)\right)$$

3 The induced curvature is trivial:

$$\tilde{B}(\dot{x}, \delta x) = 0.$$

Using this structure, we will proceed to derive the Lagrange–Poincaré equations for the asteroid pair, as explained in Cendra, Marsden, and Ratiu [2001]. However, in addition, we will take advantage of the fact that the base $X = G$ is itself a group, to further simplify the reduced equations. To do so, we shall write $TX \equiv X \times \mathfrak{g}$, using *space coordinates*. That is, we represent the element $(x, \dot{x})$ by $(x, \dot{x}) \equiv (x, w)$, where $w = \dot{x}x^{-1}$. This induces an identification

$$TX \oplus \tilde{\mathfrak{g}} \equiv X \times \mathfrak{g} \times \mathfrak{g}$$

For this reason, instead of applying the general formulation of the Lagrange–Poincaré equations directly, we prefer, in this case, to work out reduced equations using reduced variations specific to this case, to take advantage of the special features of the asteroid pair problem. For this purpose, we shall first study the geometry of reduced variations.

Reduced variations. Using the previous identifications and formulas in Cendra, Marsden, and Ratiu [2001], one readily calculates reduced vertical and horizontal variations. Of course these depend on the connection A that we have chosen. Reduced vertical variations are given by

$$\delta^A(x, w, \xi_2) = ((x, w, \xi_2), (0, 0, \delta\xi_2)),$$

where $\delta\xi_2 = \dot{\eta} + [\xi_2, \eta]$, where η vanishes at the endpoints of the time interval in question; say $\eta(t_i) = 0$, for $i = 1, 2$.

Similarly, reduced horizontal variations have the form

$$\delta^A(x, w, \xi_2) = \Big((x, w, \xi_2), (\lambda x, \dot{\lambda} - [w, \lambda], 0\Big),$$

where λ satisfies $\lambda(t_i) = 0$, for $i = 1, 2$.

Reduced Lagrangian. Using the expression (2.1), and taking into account that $w = \dot{x}x^{-1}$, the reduced Lagrangian

$$l : TX \oplus \tilde{\mathfrak{g}} \to \mathbb{R},$$

is given by

$$l(x, w, \xi_2) = \frac{1}{2}\operatorname{tr}\left(K_1\xi_1^{\mathrm{T}}\xi_1 + K_2\xi_2^{\mathrm{T}}\xi_2\right) - V(e, x),$$

where $\xi_1 = -w + x\xi_2 x^{-1}$.

Vertical Lagrange–Poincaré Equations. As explained before, in the example of the asteroid pair, we choose, for computational efficiency, to derive the equations of motion from the reduced variational principle, using the reduced variations, rather than using directly the general Lagrange–Poincaré formulas.

As we have just seen, reduced vertical variations are described by $\delta(x, w, \xi_2) = (\delta x, \delta w, \delta\xi_2)$ where $\delta x = 0$, $\delta w = 0$ and $\delta\xi_2 = \dot{\eta} + [\xi_2, \eta]$, where $\eta(t_i) = 0$, for $i = 1, 2$. Since $\xi_1 = -\dot{x}_1 x^{-1} + x\xi_2 x^{-1}$, we see that $\delta\xi_1 = x\delta\xi_2 x^{-1}$. Keeping in mind the reduced variational principle

$$\delta \int_{t_0}^{t_1} l(x, w, \xi_2) dt = 0,$$

we compute

$$\begin{aligned}
\delta l(x, w, \xi_2) &= \operatorname{tr}\left(K_1 \xi_1^{\mathrm{T}} \delta\xi_1 + K_2 \xi_2^{\mathrm{T}} \delta\xi_2\right) \\
&= \operatorname{tr}\left((\xi_1 K_1)^{\mathrm{T}} \delta\xi_1 + (\xi_2 K_2)^{\mathrm{T}} \delta\xi_2\right) \\
&= \operatorname{tr}\left((\xi_1 K_1)^{\mathrm{T}} x\delta\xi_2 x^{-1} + (\xi_2 K_2)^{\mathrm{T}} \delta\xi_2\right) \\
&= \operatorname{tr}\left(\left(x^{-1}(\xi_1 K_1)^{\mathrm{T}} x + (\xi_2 K_2)^{\mathrm{T}}\right) \delta\xi_2\right).
\end{aligned}$$

Let $a = x^{-1}(\xi_1 K_1)^{\mathrm{T}} x + (\xi_2 K_2)^{\mathrm{T}}$. Then we obtain $\delta l(x, w, \xi_2) = \operatorname{tr}(a\delta\xi_2)$, and therefore,

$$\begin{aligned}
\delta \int_{t_0}^{t_1} l(x, w, \xi_2) dt &= \int_{t_0}^{t_1} \operatorname{tr}\left(a\left(\dot{\eta} + [\xi_2, \eta]\right)\right) dt \\
&= \int_{t_0}^{t_1} \operatorname{tr}\left(-\dot{a}\eta + a[\xi_2, \eta]\right) dt \\
&= \int_{t_0}^{t_1} \operatorname{tr}\left(-\dot{a}\eta + a\xi_2\eta - a\eta\xi_2\right) dt \\
&= \int_{t_0}^{t_1} \operatorname{tr}\left(-\dot{a}\eta + (a\xi_2 - \xi_2 a)\eta\right) dt \\
&= \int_{t_0}^{t_1} \operatorname{tr}\left(\left(-\dot{a} + [a, \xi_2]\right)\eta\right) dt.
\end{aligned}$$

Then we obtain the Vertical Lagrange–Poincaré Equation in the form

$$\operatorname{tr}\left(-\dot{a}\eta + (a\xi_2 - \xi_2 a)\eta\right) = 0,$$

for all $\eta \in \mathfrak{g}$. Since the metric given by the trace is nondegenerate, we can write the previous equation simply as follows

$$\frac{d}{dt} a(x, w, \xi_2) = [a(x, w, \xi_2), \xi_2] \tag{3.1}$$

However, one should keep in mind that this is really an equation in $X \times \mathfrak{g}^* \times \mathfrak{g}^*$. A more explicit expression should be worked out in order to solve the equation, which will be done later.

Horizontal Lagrange–Poincaré Equations. We must now calculate $\delta l(x, w, \xi_2)$ for horizontal variations; that is, as we saw before, variations of the form $\delta(x, w, \xi_2) = (\delta x, \delta w, \delta\xi_2)$ of the type $(\delta x, \delta w, \delta\xi_2) = (\lambda x, \dot{\lambda} - [w, \lambda], 0)$.

Using the equality $\xi_1 = -w + x\xi_2 x^{-1}$, and, also, that $\delta\xi_2 = 0$ and that $\delta w = \dot{\lambda} - [w, \lambda]$, we obtain

$$\delta\xi_1 = -\dot{\lambda} + [w, \lambda] + \lambda x\xi_2 x^{-1} - x\xi_2 x^{-1}\lambda.$$

We can always write, for an appropriate matrix φ_1 such that $\varphi_1^{\mathrm{T}} \in \mathfrak{g}$,

$$\frac{\partial V(e, x)}{\partial x}\delta x = \mathrm{tr}(\varphi_1\lambda).$$

More precisely, let

$$\lambda = \begin{bmatrix} 0 & -\lambda^3 & \lambda^2 & u^1 \\ \lambda^3 & 0 & -\lambda^1 & u^2 \\ -\lambda^2 & \lambda^1 & 0 & u^3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and seek a φ_1 of the form

$$\varphi_1(x) = \begin{bmatrix} 0 & \varphi^3(x) & -\varphi^2(x) & 0 \\ -\varphi^3(x) & 0 & \varphi^1(x) & 0 \\ \varphi^2(x) & -\varphi^1(x) & 0 & 0 \\ f^1(x) & f^2(x) & f^3(x) & 0 \end{bmatrix}.$$

Then we have

$$\mathrm{tr}\left(\varphi_1(x)\lambda\right) = \varphi_1^1(x)\lambda^1 + \varphi_1^2(x)\lambda^2 + \varphi_1^3(x)\lambda^3 + f^1(x)u^1 + f^2(x)u^2 + f^3(x)u^3,$$

showing that one can use Riesz theorem to represent the element $\partial V(e, x)/\partial x$ of $\mathfrak{g}^*$ by an element $\varphi_1(x)$ of $\mathfrak{g}$, via the nondegenerate metric given by the trace. Using this, we obtain

$$\begin{aligned}
\delta l(x, w, \xi_2) &= \mathrm{tr}\left((\xi_1 K_1)^{\mathrm{T}}\left(-\dot{\lambda} + [w, \lambda] + \lambda x\xi_2 x^{-1} - x\xi_2 x^{-1}\lambda\right)\right) - \mathrm{tr}\left(\varphi_1(x)\lambda\right) \\
&= \mathrm{tr}\left(-(\xi_1 K_1)^{\mathrm{T}}\dot{\lambda}\right) + \mathrm{tr}\left((\xi_1 K_1)^{\mathrm{T}}(w\lambda - \lambda w)\right) + \mathrm{tr}\left((\xi_1 K_1)^{\mathrm{T}}\lambda x\xi_2 x^{-1}\right) \\
&\quad - \mathrm{tr}\left((\xi_1 K_1)^{\mathrm{T}} x\xi_2 x^{-1}\lambda\right) - \mathrm{tr}\left(\varphi_1(x)\lambda\right) \\
&= \mathrm{tr}\left(-(\xi_1 K_1)^{\mathrm{T}}\dot{\lambda}\right) + \mathrm{tr}\left((\xi_1 K_1)^{\mathrm{T}} w\lambda\right) \\
&\quad - \mathrm{tr}\left(w(\xi_1 K_1)^{\mathrm{T}}\lambda\right) + \mathrm{tr}\left(x\xi_2 x^{-1}(\xi_1 K_1)^{\mathrm{T}}\lambda\right) \\
&\quad - \mathrm{tr}\left((\xi_1 K_1)^{\mathrm{T}} x\xi_2 x^{-1}\lambda\right) - \mathrm{tr}\left(\varphi_1(x)\lambda\right) \\
&= \mathrm{tr}\left(-(\xi_1 K_1)^{\mathrm{T}}\dot{\lambda} + \left[(\xi_1 K_1)^{\mathrm{T}}, w\right]\lambda + \left[x\xi_2 x^{-1}, (\xi_1 K_1)^{\mathrm{T}}\right]\lambda - \varphi_1(x)\lambda\right)
\end{aligned}$$

Using the previous calculations in the variational principle

$$\delta\int_{t_0}^{t_1} l(x, w, \xi_2)dt = 0,$$

and integrating by parts, we obtain,

$$\int_{t_0}^{t_1} \operatorname{tr}\left(\left(\frac{d}{dt}(\xi_1 K_1)^{\mathrm{T}} + \left[(\xi_1 K_1)^{\mathrm{T}}, w\right] + \left[x\xi_2 x^{-1}, (\xi_1 K_1)^{\mathrm{T}}\right] - \varphi_1(x)\right)\lambda\right) dt = 0.$$

This leads to the following form of the Horizontal Lagrange–Poincaré Equation

$$\operatorname{tr}\left(\left(\frac{d}{dt}(\xi_1 K_1)^{\mathrm{T}} + \left[(\xi_1 K_1)^{\mathrm{T}}, w\right] + \left[x\xi_2 x^{-1}, (\xi_1 K_1)^{\mathrm{T}}\right] - \varphi_1(x)\right)\lambda\right) = 0,$$

for all $\lambda \in g$. Since the metric given by the trace is nondegenerate, we can write the previous equation simply as follows

$$\frac{d}{dt}(\xi_1 K_1)^{\mathrm{T}} + \left[(\xi_1 K_1)^{\mathrm{T}}, w\right] + \left[x\xi_2 x^{-1}, (\xi_1 K_1)^{\mathrm{T}}\right] - \varphi_1(x) = 0,$$

or, equivalently,

$$\frac{d}{dt}(\xi_1 K_1)^{\mathrm{T}} = \left[(\xi_1 K_1)^{\mathrm{T}}, x\xi_2 x^{-1} - w\right] + \varphi_1(x).$$

Taking into account that $\xi_1 = x\xi_2 x^{-1} - w$, we can rewrite the Horizontal Lagrange–Poincaré Equation as follows

$$\frac{d}{dt}(\xi_1 K_1)^{\mathrm{T}} = \left[(\xi_1 K_1)^{\mathrm{T}}, \xi_1\right] + \varphi_1(x) = 0. \tag{3.2}$$

However, one should keep in mind that this is an equation in $X \times \mathfrak{g}^* \times \mathfrak{g}^*$. More explicit equations should be worked out, and we will do this shortly.

The System of Horizontal and Vertical Lagrange–Poincaré Equations Collecting what we have proven so far from equations (3.1) and (3.2), we obtain the following System of Horizontal and Vertical Lagrange–Poincaré Equations:

$$\frac{da}{dt} = [a, \xi_2] \tag{3.3}$$

$$\frac{db}{dt} = [b, \xi_1] + \varphi_1(x), \tag{3.4}$$

where

$$a = x^{-1} b x + (\xi_2 K_2)^{\mathrm{T}} \tag{3.5}$$

$$b = (\xi_1 K_1)^{\mathrm{T}} \tag{3.6}$$

$$\xi_1 = -w + x\xi_2 x^{-1} \tag{3.7}$$

$$w = \dot{x} x^{-1}. \tag{3.8}$$

Elimination of the Quantities a and b. Now we are going to transform the previous equations, in order to eliminate the auxiliary parameters a and b. First of all, differentiate equation (3.5) with respect to time to obtain

$$\begin{aligned}\dot{a} &= -x^{-1}\dot{x}x^{-1}bx + x^{-1}\dot{b}x + x^{-1}b\dot{x} + (\dot{\xi_2 K_2})^{\mathrm{T}} \\ &= -x^{-1}wbx + x^{-1}\dot{b}x + x^{-1}bwx + (\dot{\xi_2 K_2})^{\mathrm{T}} \\ &= x^{-1}[b,w]x + x^{-1}\dot{b}x + (\dot{\xi_2 K_2})^{\mathrm{T}}\end{aligned}$$

Using this and equation (3.4 we obtain

$$\begin{aligned}\dot{a} &= x^{-1}[b,w]x + x^{-1}[b,\xi_1]x + x^{-1}\varphi_1(x)x + (\dot{\xi_2 K_2})^{\mathrm{T}} \\ &= x^{-1}[b,w+\xi_1]x + x^{-1}\varphi_1(x)x + (\dot{\xi_2 K_2})^{\mathrm{T}} \\ &= x^{-1}[b,x\xi_2 x^{-1}]x + x^{-1}\varphi_1(x)x + (\dot{\xi_2 K_2})^{\mathrm{T}} \\ &= [x^{-1}bx,\xi_2] + x^{-1}\varphi_1(x)x + (\dot{\xi_2 K_2})^{\mathrm{T}}.\end{aligned}$$

Using this and equation (3.3) we obtain the equation

$$(\dot{\xi_2 K_2})^{\mathrm{T}} = \big[(\xi_2 K_2)^{\mathrm{T}},\xi_2\big] - x^{-1}\varphi_1(x)x.$$

The Set of Lagrange–Poincaré Equations. Collecting these results together, we can write the System of Horizontal and Vertical Lagrange–Poincaré Equations as follows

$$K_1\dot{\xi}_1^{\mathrm{T}} = [K_1\xi_1^{\mathrm{T}},\xi_1] + \varphi_1(x) \tag{3.9}$$

$$K_2\dot{\xi}_2^{\mathrm{T}} = [K_2\xi_2^{\mathrm{T}},\xi_2] - x^{-1}\varphi_1(x)x. \tag{3.10}$$

These Lagrange–Poincaré Equations are to be interpreted as being equations in $T^*X\oplus\tilde{\mathfrak{g}}^* \equiv X\times\mathfrak{g}^*\times\mathfrak{g}^*$, via the identifications $TX\equiv X\times\mathfrak{g}$ given by $(x,\dot{x})\equiv(x,w)$, where $w=\dot{x}x^{-1}$ and $\tilde{\mathfrak{g}}\equiv X\times\mathfrak{g}$ given by $\big((x^{-1},e),\xi\big)\equiv(x,\xi)$, as we have explained before. To this, one should add the equation

$$\dot{x} = x\xi_2 - \xi_1 x.$$

Explicit Equations of Motion. As we have explained before, equations (3.9) and (3.10), are to be interpreted as follows

$$\mathrm{tr}\left(K_1\dot{\xi}_1^{\mathrm{T}}\delta\xi_1\right) = \mathrm{tr}\left(\left([K_1\xi_1^{\mathrm{T}},\xi_1] + \varphi_1(x)\right)\delta\xi_1\right) \tag{3.11}$$

$$\mathrm{tr}\left(K_2\dot{\xi}_2^{\mathrm{T}}\delta\xi_2\right) = \mathrm{tr}\left(\left([K_2\xi_2^{\mathrm{T}},\xi_2] - x^{-1}\varphi_1(x)x\right)\delta\xi_2\right) \tag{3.12}$$

for all $\delta\xi_1,\delta\xi_2\in\mathfrak{g}$, to which one should add the equation

$$\dot{x} = x\xi_2 - \xi_1 x.$$

Now we shall transform these equations, to obtain more explicit equivalent equations, including vector equations.

To illustrate the procedure, we consider first the case of a single asteroid, that is, equations of motion of a free rigid body in Euclidean 3-space.

The Free Rigid Body in $\mathbb{R}^3$. It is clear, taking into account the last equations (3.11) and (3.12) of the previous paragraph, that equations of a free rigid body in Euclidean 3-space are given by

$$\mathrm{tr}\left(K\dot{\xi}^{\mathrm{T}}\delta\xi\right) = \mathrm{tr}\left([K\xi^{\mathrm{T}}, \xi]\delta\xi\right)$$

for all $\delta\xi \in \mathfrak{g}$. This equation essentially means that $K\dot{\xi}^{\mathrm{T}}$ and $[K\xi^{\mathrm{T}}, \xi]$ are to be interpreted as being elements of $\mathfrak{g}^*$ using the trace inner product. For any matrix $M \in L(\mathbb{R}^4, \mathbb{R}^4)$, we denote by $M|\,\mathfrak{g}$ the element of $\mathfrak{g}^*$ defined by M via the trace inner product. Thus, the equations for a free rigid body in Euclidean 3-space are equivalently written as follows

$$K\dot{\xi}^{\mathrm{T}}\Big|\,\mathfrak{g} = [K\xi^{\mathrm{T}}, \xi]\big|\,\mathfrak{g}.$$

Let

$$\xi = \begin{bmatrix} 0 & -\Omega^3 & \Omega^2 & u^1 \\ \Omega^3 & 0 & -\Omega^1 & u^2 \\ -\Omega^2 & \Omega^1 & 0 & u^3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

be an element of $\mathfrak{g}$, and let $M = (m^\alpha_\beta)$, where $\alpha, \beta \in \{1, 2, 3, 4\}$, be an element of $L(\mathbb{R}^4, \mathbb{R}^4)$. Then we have

$$\mathrm{tr}(M\xi) = (m^2_3 - m^3_2)\Omega^1 + (m^3_1 - m^1_3)\Omega^2 + (m^1_2 - m^2_1)\Omega^3 + m^4_1 u^1 + m^4_2 u^2 + m^4_3 u^3.$$

Then, the condition $\mathrm{tr}(M\xi) = 0$, for all $\xi \in \mathfrak{g}$, or, $M|\,\mathfrak{g} = 0$, is equivalent to

$$\begin{aligned} m^i_j &= m^j_i \\ m^4_j &= 0, \end{aligned}$$

where $i, j \in \{1, 2, 3\}$, and $i \neq j$. Using this, one can see that an equation of the form $A|\,\mathfrak{g} = B|\,\mathfrak{g}$, where $A = (a^\alpha_\beta)$ and $B = (b^\alpha_\beta)$ are elements of $L(\mathbb{R}^4, \mathbb{R}^4)$, is equivalent to

$$\begin{aligned} a^i_j - b^i_j &= a^j_i - b^j_i \\ a^4_j &= b^4_j, \end{aligned}$$

that is, to

$$\begin{aligned} a^i_j - a^j_i &= b^i_j - b^j_i \\ a^4_j &= b^4_j, \end{aligned}$$

where $i, j \in \{1, 2, 3\}$, and $i \neq j$. In particular, the equation for the free rigid body in Euclidean space, can be written, equivalently, as follows

$$\begin{aligned} (K\dot{\xi}^{\mathrm{T}})^i_j - (K\dot{\xi}^{\mathrm{T}})^j_i &= [K\xi^{\mathrm{T}}, \xi]^i_j - [K\xi^{\mathrm{T}}, \xi]^j_i \\ (K\dot{\xi}^{\mathrm{T}})^4_j &= [K\xi^{\mathrm{T}}, \xi]^4_j, \end{aligned}$$

where $i, j \in \{1,2,3\}$, and $i \neq j$. After carrying out the indicated calculations, one sees that these equations are equivalent to

$$\begin{aligned} I\dot{\Omega} &= I\Omega \times \Omega \\ \dot{u} &= -\Omega \times u, \end{aligned}$$

where

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix},$$

and

$$\begin{aligned} I_1 &= K_2 + K_3 \\ I_2 &= K_3 + K_1 \\ I_3 &= K_1 + K_2. \end{aligned}$$

The physical interpretation of the previous equations is clear. The first equation is simply Euler equation, where Ω is the body angular velocity, and the second represents the motion of the velocity of the center of mass u, as viewed from the body.

Equations of the Asteroid Pair in Vector Notation. We now give the explicit equations, in vector notation, for the asteroid pair problem, just as we did with one asteroid.

Let $\varphi_2(x) = -x^{-1}\varphi_1(x)x$. Equations (3.9) and (3.10) can be written as follows,

$$\begin{aligned} \operatorname{tr} K\dot{\xi}_1^{\mathrm{T}}\Big|\,\mathfrak{g} &= \operatorname{tr}[K\xi_1^{\mathrm{T}}, \xi_1]\big|\,\mathfrak{g} \\ \operatorname{tr} K\dot{\xi}_2^{\mathrm{T}}\Big|\,\mathfrak{g} &= \operatorname{tr}[K\xi_2^{\mathrm{T}}, \xi_2]\big|\,\mathfrak{g}. \end{aligned}$$

Proceeding in a similar way as we did with the case of one asteroid, we obtain the equations

$$\begin{aligned} (K_k\dot{\xi}_k^{\mathrm{T}})^i_j - (K_k\dot{\xi}_k^{\mathrm{T}})^j_i &= [K_k\xi_k^{\mathrm{T}}, \xi_k]^i_j - [K_k\xi_k^{\mathrm{T}}, \xi_k]^j_i + \varphi^i_{kj}(x) - \varphi^j_{ki}(x) \\ (K_k\dot{\xi}_k^{\mathrm{T}})^4_j &= [K_k\xi_k^{\mathrm{T}}, \xi_k]^4_j + \varphi^4_{ki}(x), \end{aligned}$$

where $i, j \in \{1,2,3\}$, $i \neq j$ and $k = 1, 2$. Using these equations, one can easily obtain, after some straightforward calculations, the following equations

$$I\dot{\Omega}_k = I_k\Omega_k \times \Omega_k + F_k(x) \tag{3.13}$$

$$\dot{u}_k = -\Omega_k \times u_k + \psi_k(x), \tag{3.14}$$

where

$$\hat{F}^i_{kj}(x) = \varphi^i_{kj} - \varphi^j_{ki},$$

and

$$\psi^i_k(x) = \frac{1}{M_k}\varphi^{\mathrm{t}\,i}_{k4}$$

for $k = 1, 2$. The complete system of equations of motion for the pair asteroid is, therefore,

$$I\dot{\Omega}_k = I_k\Omega_k \times \Omega_k + F_k(x) \tag{3.15}$$
$$\dot{u}_k = -\Omega_k \times u_k + \psi_k(x) \tag{3.16}$$
$$\dot{x} = x\xi_2 - \xi_1 x, \tag{3.17}$$

for $k = 1, 2$. These equations are in agreement with those in the literature, such as in Maciejewski [1995, 1999].

The Hamilton–Poincaré Equations. These equations were introduced in Cendra, Marsden, Pekarsky, and Ratiu [2003]. They are essentially, reduced Hamilton equations in a similar sense that the Lagrange–Poincaré equations are reduced Euler–Lagrange equations. Keep in mind that in the special case in which $Q = G$, the Hamilton-Poincaré equations reduce to the Lie-Poisson equations (see, for instance, Marsden and Ratiu [1999] for an exposition of these equations and how to realize them via *Poisson reduction* and see Cendra, Marsden, Pekarsky, and Ratiu [2003] for how to obtain them by reduction of *Hamilton's phase space principle*.

We next derive the Hamilton–Poincaré equations for the pair asteroid problem. The reduced fiber derivative (that is the reduced Legendre transformation) of the reduced Lagrangian $l(x, \dot{x}, \xi_2)$ of the pair asteroid is the map

$$\mathbb{F}l : TX \oplus \mathfrak{g} \to T^*X \oplus \mathfrak{g}^*,$$

defined by $\mathbb{F}l(x, \dot{x}, \xi_2) = (x, y, \pi^2)$, where

$$y = \frac{\partial l}{\partial \dot{x}}(x, \dot{x}, \xi_2) \tag{3.18}$$
$$\pi^2 = \frac{\partial l}{\partial \xi_2}(x, \dot{x}, \xi_2). \tag{3.19}$$

The metric k_2 induces a metric k^2 in $\mathfrak{g}^*$ in a natural way, namely, for any given $\xi_2 \in \mathfrak{g}$, we have $k^2\left(k_2(\xi_2, .), k_2(\xi_2, .)\right) = k_2(\xi_2)$. Then, for π^2 as in (3.19), we have $k^2(\pi^2) = k_2(\xi_2)$, where, by definition, $k^2(\pi^2) = k^2(\pi^2, \pi^2)$. In a similar way, let k^1 be the metric in $\mathfrak{g}^*$ induced by k_1. We now extend k^1 to a right invariant metric, also called k^1, in $T^*X \equiv T^*G$.

Using the fact that $\xi_1 = x\xi_2 x^{-1} - \dot{x}x^{-1}$, we see that y, which is given by (3.18), satisfies $y = -k_1(\xi_1, .)x$, where the right hand side is the right translation of $-k_1(\xi_1, .) \in \mathfrak{g}^*$ to the element $x \in G$. Therefore, we obtain, $k^1(y) = k_1(\xi_1)$. Collecting results, we see that the kinetic energy for the asteroid pair, namely,

$$\frac{1}{2}k_1(\xi_1) + \frac{1}{2}k_2(\xi_2),$$

can be written as follows,

$$\frac{1}{2}k^1(x)(y) + \frac{1}{2}k^2(\pi^2).$$

Finally, the reduced Hamiltonian $h : T^*X \oplus \mathfrak{g}^* \to \mathbb{R}$ is given as usual by the formula kinetic plus potential energy, so we obtain,

$$h(x, y, \pi^2) = \frac{1}{2}k^1(x)(y) + \frac{1}{2}k^2(\pi^2) + V(e, x).$$

Using the Hamiltonian h and a torsionless affine connection ∇ on G, (a specific choice is made below), we can write the Hamilton–Poincaré equations, as follows,

$$\begin{aligned}
\frac{Dy}{Dt} &= -\frac{\partial h}{\partial x} \\
\dot{x} &= k^1(x)(y, .) \\
\xi_2 &= k^2(\pi^2, .) \\
\frac{D\pi^2}{Dt} &= \operatorname{ad}^*_{\xi_2} \pi^2.
\end{aligned}$$

If one chooses the connection ∇ to be the Levi-Civita connection of the right invariant metric k^1 in X, one can give a more explicit expression for

$$\frac{\partial h}{\partial x}.$$

First of all we have

$$\frac{\partial h}{\partial x}(x, y, \pi^2) = \frac{1}{2}\frac{\partial k^1(x)(y)}{\partial x} + \frac{\partial V(x, e)}{\partial x},$$

where the partial derivative with respect to x has to be understood in a covariant sense, as explained in Cendra, Marsden, Pekarsky, and Ratiu [2003]. Since $\nabla k^1 = 0$, we see that, for any tangent vector $v_x \in T_xX$, we have

$$\frac{1}{2}\frac{\partial k^1(x)(y)}{\partial x}(v_x) = \kappa^1(x)(y, v_x),$$

which defines a tensor field $\kappa^1(x) \in \mathcal{T}^1_1(X)$, depending on k^1. This gives

$$\frac{\partial h}{\partial x}(x, y, \pi^2) = \kappa^1(x)(y, .) + \frac{\partial V(x, e)}{\partial x}$$

Collecting results, we can write the Hamilton-Poincaré equations equivalently as follows.

$$\frac{Dy}{Dt} = -\kappa^1(x)(y, .) - \frac{\partial V(x, e)}{\partial x} \tag{3.20}$$

$$\dot{x} = k^1(x)(y, .) \tag{3.21}$$

$$\frac{D\pi^2}{Dt} = \operatorname{ad}^*_{k^2(\pi^2, .)} \pi^2. \tag{3.22}$$

This system of equations has some interesting features; for instance, the last equation is not coupled to the first two. Also, the first equation is affine in y, while the second equation is linear in y.

Poisson Bracket and Symplectic Leaves in $T^*X \oplus \mathfrak{g}^*$. Following Cendra, Marsden, Pekarsky, and Ratiu [2003], and taking into account that the principal connection A we have chosen is integrable, we can easily see that the induced Poisson Bracket in $T^*X \oplus \mathfrak{g}^*$ is given simply by

$$\{f, g\}(x, y, \pi^2) = \{f, g\}_{T^*X}(x, y, \pi^2) + \{f, g\}_{\mathfrak{g}^*}(x, y, \pi^2),$$

Where $\{f, g\}_{T^*X}$ is the standard Poisson bracket in T^*X, and $\{f, g\}_{\mathfrak{g}^*}$ is the standard Poisson bracket in $\mathfrak{g}^*$. The symplectic leaves are also relatively easy to compute, following Cendra, Marsden, Pekarsky, and Ratiu [2003], and are given by $T^*X \times \mathcal{O}_\mu$, where $\mathcal{O}_\mu$ is the coadjoint orbit that contains μ. The symplectic form is given by

$$\omega(.,.) = \omega_{T^*X}(.,.) + \omega_\mu(.,.),$$

where ω_μ is the canonical symplectic form on $\mathcal{O}_\mu$. This also provides a nice example for the theory in Marsden and Perlmutter [2000].

4 Conclusions and Future Directions.

What does one gain from what appears, at first sight, to be a very complicated derivation of equations that are already in the literature? The main overall benefit is that this derivation puts the problem properly into the general framework of geometric mechanics. For instance, it answers, automatically, the following basic question: *in what sense are the equations* (3.15), (3.16), and (3.17) *variational*? For example, the answer to such questions are critical for the use of modern variational numerical integration techniques, as in, for instance, Marsden and West [2001] and Lew, Marsden, Ortiz, and West [2004].

Similarly, the Hamilton-Poincaré formulation gives a very nice setting in which one can derive the Poisson bracket formulation of the problem. For instance, it answers, in a systematic way, the sense in which equations (3.20), (3.21), and (3.22) are Hamiltonian. Many of the previous approaches were quite ad hoc in this regard.

We have also mentioned another benefit, namely that the geometric mechanics setting is the first step that is needed to make use of the energy-momentum method, which has not been heretofore employed for this problem. It is the most powerful stability method known for mechanical systems with symmetry and so its use in future researches will be very important. This is also essential for carrying out the dynamical systems analysis of tube and lobe dynamics, ejection calculations, capture probabilities, etc. There is much current interest in these techniques at present; see, for example, Astakhov, Burbanks, Wiggins, and Farrelly [2003]; Koon, Marsden, Ross, Lo, and Scheeres [2004]; Dellnitz, Junge, Koon, et al. [2004] and references therein. It is clear that future applications and development of these techniques are going to depend on the structures and the variational computational algorithms that geometric mechanics provides and the present article is, we believe, a useful contribution towards these goals.

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References

Astakhov, S. A., A. D. Burbanks, S. Wiggins, and D. Farrelly [2003], Chaos-assisted capture of irregular moons, *Nature* **423**, 264–267.

Bloch, A. [2003], *Nonholonomic Mechanics and Control*, (with the collaboration of J. Ballieul, P. Crouch, and J.E. Marsden). Interdisciplinary Applied Mathematics, volume 24, Springer-Verlag.

Cendra, H., J. E. Marsden, S. Pekarsky, and T. S. Ratiu [2003], Reduction of Hamilton's phase space principle, *Moscow Mathematical Journal* (special volume in honor of V. Arnold), **3**, 833–867.

Cendra, H., J. E. Marsden, and T. S. Ratiu [2001], *Lagrangian reduction by stages*, volume 152 of *Memoirs*. American Mathematical Society, Providence, R.I.

Dellnitz, M., O. Junge, W. S. Koon, F. Lekien, M. W. Lo, J. E. Marsden, K. Padberg, R. Preis, S. Ross, and B. Thiere [2004], Transport in dynamical astronomy and multibody problems, *(preprint)*. preprint.

Goździewski, K. and A. J. Maciejewski [1999], Unrestricted planar problem of a symmetric body and a point mass. Triangular libration points and their stability, *Celest. Mech. and Dyn. Astron.* **75**, 251–285.

Kanso, E., J. E. Marsden, C. W. Rowley, and J. Melli-Huber [2004], Locomotion of articulated bodies in a perfect fluid, *(submitted)*.

Koon, W.-S., J. E. Marsden, S. Ross, M. Lo, and D. J. Scheeres [2004], Geometric mechanics and the dynamics of asteroid pairs, *NY Acad of Sciences* **1017**, 11–38.

Lew, A., J. E. Marsden, M. Ortiz, and M. West [2004], Variational time integration for mechanical systems, *Intern. J. Num. Meth. in Engin.* , **60**, 153–212.

Maciejewski, A. [1995], Reduction, relative equilibria and potential in the two rigid bodies problem, *Celest. Mech. and Dyn. Astron.* **63**, 1–28.

Maciejewski, A. J. [1999], The two rigid bodies problem. Reduction and relative equilibria. In *Hamiltonian systems with three or more degrees of freedom (S'Agaró, 1995)*, volume 533 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 475–479. Kluwer Acad. Publ., Dordrecht.

Margot, J. L., M. C. Nolan, L. A. M. Benner, S. J. Ostro, R. F. Jurgens, J. D. Giorgini, M. A. Slade, and D. B. Campbell [2002], Binary asteroids in the near-earth object population. *Science*, **296**, 1445–1448.

Marsden, J. E., R. Montgomery, and T. S. Ratiu [1990], *Reduction, symmetry and phases in mechanics*, *Memoirs of the AMS*, vol **436**. Amer. Math. Soc., Providence, RI.

Marsden, J. E. and M. Perlmutter [2000], The orbit bundle picture of cotangent bundle reduction, *C. R. Math. Acad. Sci. Soc. R. Can.* **22**, 33–54.

Marsden, J. E. and T. S. Ratiu [1999], *Introduction to Mechanics and Symmetry*, volume 17 of *Texts in Applied Mathematics, vol. 17; 1994, Second Edition, 1999.* Springer-Verlag.

Marsden, J. E. and M. West [2001], Discrete mechanics and variational integrators, *Acta Numerica* **10**, 357–514.

Merline, W. J., S. J. Weidenschilling, D. D. Durda, J. L. Margot, P. Pravec, and A. D. Storrs [2002], Asteroids do have satellites. In W.M. Bottke, A. Cellino, P. Paolicchi, and R. P. Binzel, editors, *Asteroids III*, pages 289–312. Univ. of Arizona Press, Tucson.

Radford, J. [2003], *Symmetry, Reduction and Swimming in a Perfect Fluid*, PhD thesis, California Institute of Technology.

Scheeres, D. J. [2001], Changes in rotational angular momentum due to gravitational interactions between two finite bodies. *Celest. Mech. Dyn. Astron.*, **81**,39–44.

Scheeres, D. J. [2002], Stability in the full two-body problem. *Celest. Mech. Dyn. Astron.*, **83**, 155–169. 2002.

Scheeres, D. J. [2002a], Stability of binary asteroids. *Icarus*, **159**, 271–283.

Scheeres, D. J. [2004], Stability of relative equilibria in the full two-body problem. *Annals of the New York Academy of Science*, **1017**.

Scheeres, D. J., Ostro S. J., Werner R. A., Asphaug E., and Hudson R. S. [2000], Effects of gravitational interactions on asteroid spin states, *Icarus*, **147**, 106–118.

Simo, J. C., D. R. Lewis, and J. E. Marsden [1991], Stability of relative equilibria I: The reduced energy momentum method, *Arch. Rational Mech. Anal.* **115**, 15–59.

Veillet, C., J.W. Parker, I. Griffin, B. Marsden, A. Doressoundiram, M. Buie, D. J. Tholen, M. Connelley, and M. J. Holman [2002], The binary kuiper-belt object 1998 ww31. *Nature*, **416**, 711–713.