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FACTOR–ANALYSIS OF NONLINEAR MAPPINGS: p–REGULARITY THEORY

Alexey Tret'yakov

Computing Center of the Russian Academy of Sciences
Vavilova 40, Moscow, GSP-1, Russia
and
University of Podlasie in Siedlce, 3 Maja, Siedlce, Poland

Jerrold E. Marsden

Control and Dynamical Systems 107-81
California Institute of Technology, Pasadena, CA 91125

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Ever since we began to prove things that were obviously true,
many of them turned out to be false, Bertrand Russell

Abstract. The paper presents recent advances in p-regularity theory, which has been developing successfully for the last twenty years. The main result of this theory gives a detailed description of the structure of the zero set of an irregular nonlinear mapping. We illustrate the theory with an application to degenerate problems in different fields of mathematics, which substantiates the general applicability of the class of p-regular problems. Moreover, the connection between singular problems and nonlinear mappings is shown. Amongst the applications, the structure of p-factor-operators is used to construct numerical methods for solving degenerate nonlinear equations and optimization problems.

1. Introduction.

This paper concerns the problem of solving a nonlinear equation of the form

$$F(x) = 0, \tag{1.1}$$

where $F : X \to Y$ is a sufficiently smooth mapping from a Banach space X to a Banach space Y. Of course, the solution to many interesting nonlinear problems can be cast in this form and there have been many works devoted to this problem. The purpose of this paper is to present some of our own work and that of others in this area in a coherent way, which has hitherto been scattered throughout various references, as well as giving a number of new results.

We separate nonlinear mappings F and problems of the form (1.1) into two classes, called *regular* and *irregular*. Roughly speaking, regular problems are those to which implicit function theorem arguments can be applied and the irregular ones are those to which it cannot, at least not directly.

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In the history of mathematics, there have been several examples in which fundamental results were obtained independently in the same general time period. One such example concerns theorems about the structure of the zero set of an irregular mapping satisfying a special higher–order regularity condition. The result that we are referring to was simultaneously obtained in Buchner, Marsden and Schecter [11] and Tret'yakov [31], and it was closely related to the results of Magnus [27], Szulkin [29], and others. In Fink and Rheinboldt [15], it was noted that the theorem in [11] was a powerful generalization of Morse Lemma and some interesting counterexamples for a naive approach to the Morse Lemma were found. This theorem is one of the basic results for the p–regularity theory.

It is perhaps worth noting that the results of Buchner, Marsden and Schecter [11] were motivated by the problem of abstracting known results for the concrete problem of the solution structure of the Einstein equations of general relativity. The solution set of this important set of equations, despite their highly nonlinear character, turn out to have only quadratic singularities at metric tensors with symmetry. A general result along these lines, which may be viewed as a special infinite dimensional Morse lemma for certain types of vector-valued functions, was given in Arms, Marsden and Moncrief [2]. Other papers that motivated this theory are those of Tromba [35], Golubitsky and Marsden [16] and Buchner, Marsden and Schecter [12], which deal with an infinite dimensional Morse lemma.

Goal of the Present Paper. In this paper, we show how to apply *p-regularity theory*, also known as *factor-analysis of nonlinear mappings* to the description and investigation of singular mappings and, in addition, to develop methods for finding solutions to related singular problems. In particular, we show how these ideas apply to some specific situations, such as optimization and bifurcation problems.

General Notation. Let $\mathcal{L}(X, Y)$ be the space of all continuous linear operators from X to Y and for a given linear operator $\Lambda : X \to Y$, we denote its kernel and image by $\operatorname{Ker}\Lambda = \{x \in X \mid \Lambda x = 0\}$ and $\operatorname{Im}\Lambda = \{y \in Y \mid y = \Lambda x \text{ for some } x \in X\}$. Also, $\Lambda^* : Y^* \to X^*$ denotes the adjoint of Λ, where X^* and Y^* denote the dual spaces of X and Y, respectively.

Let p be a natural number and let $B : X \times X \times \ldots \times X$ (with p copies of X) $\to Y$ be a continuous symmetric p-multilinear mapping. The p-form ***associated to*** B is the map $B[\cdot]^p : X \to Y$ defined by

$$B[x]^p = B(x, x, \ldots, x),$$

for $x \in X$. Alternatively, we may simply view $B[\cdot]^p$ as a homogeneous polynomial $Q : X \to Y$ of degree p, i.e., $Q(\alpha x) = \alpha^p Q(x)$. The space of continuous homogeneous polynomials $Q : X \to Y$ of degree p will be denoted by $\mathcal{Q}^p(X, Y)$.

If $F : X \to Y$ is differentiable, its derivative at a point $x \in X$ will be denoted $F'(x) : X \to Y$. If $F : X \to Y$ is of class C^p, we let $F^{(p)}(x)$ be the pth derivative of F at the point x (a symmetric multilinear map of p copies of X to Y) and the associated p-form, also called the ***p*th–order mapping**, is

$$F^{(p)}(x)[h]^p = F^{(p)}(x)(h, h, \ldots, h).$$

Furthermore, we use the following key notation,

$$\operatorname{Ker}^p F^{(p)}(x) = \{h \in X \mid F^{(p)}(x)\, [h]^p = 0\}$$

is the ***p-kernel*** of the p-order mapping.

2. **Singularity and Essential Nonlinearity.**

The Regular Case. Fix a point $x^* \in X$ and suppose that $F : X \to Y$ is C^1. It is well known that if F is ***regular*** at x^*, i.e.,

$$\operatorname{Im} F'(x^*) = Y, \tag{2.1}$$

then the properties of the linear approximation of F locally correspond to the properties of the mapping F, since the mapping F can be locally linearized by a local diffeomorphism; that is, by a nondegenerate transformation of coordinates. Namely, there exist a neighborhood U of the point 0 and a C^1 mapping $\varphi : U \to X$ such that $\varphi(0) = x^*$, $\varphi'(0) = I_X$, (the identity map on X), and

$$F(\varphi(x)) = F(x^*) + F'(x^*)x \tag{2.2}$$

for all $x \in U$. See any standard reference for this fact, such as [1]. If the regularity condition (2.1) is not satisfied, then there is no such correspondence in general.

Essential Nonlinearity and Singular Maps. There exist numerous problems where the linear approximation of F is not enough to describe the properties of the mapping. For example, there are essential nonlinear mappings, i.e., mappings whose local linearization does not give a good approximation. We formalize this as follows.

Definition 2.1. *Let V be a neighborhood of x^* in X. A C^2 mapping $F : V \to Y$ is referred to as* ***essentially nonlinear at the point*** *x^*, if there exists a perturbation of the form*

$$\widetilde{F}(x^* + x) = F(x^* + x) + \omega(x), \quad \text{where} \quad \|\omega(x)\| = o(\|x\|),$$

such that there does not exist any C^1 nondegenerate transformation of coordinates $\varphi(x) : U \to X$ such that $\varphi(0) = x^*$, $\varphi'(0) = I_X$ and (2.2) holds with φ and $\widetilde{F}$.

Definition 2.2. *We say the mapping F is* ***singular (or degenerate, abnormal)*** *at x^* if it fails to be regular; that is, its derivative is not onto:*

$$\operatorname{Im} F'(x^*) \neq Y. \tag{2.3}$$

The following Theorem establishes the relationship between these two notions.

Theorem 2.3. *Suppose $F : V \to Y$ is C^2 and that x^* is a solution of* (1.1)*. Then F is essentially nonlinear at the point x^* if and only if F is singular at the point x^*.*

Proof. Suppose that F is singular at the point x^*, i.e., Im $F'(x^*) \neq Y$, so there exists a nonzero element $\xi \in Y$ such that

$$\xi \notin \text{Im } F'(x^*); \tag{2.4}$$

we may suppose that $\|\xi\| = 1$. Suppose that F is not essentially nonlinear at x^*.

Define the mapping $\tilde{F}$ as

$$\tilde{F}(x^* + x) = F(x^*) + F'(x^*)x + \xi\|x\|^2. \tag{2.5}$$

Note that $\xi\|x\|^2 \notin \text{Im } F'(x^*)$.

By virtue of the above assumptions, (2.2) and (2.5), there exists a C^1 mapping $\varphi : U \to X$ such that $\varphi(0) = x^*$, $\varphi'(0) = I_X$ and

$$\tilde{F}(\varphi(x)) = \tilde{F}(x^*) + \tilde{F}'(x^*)x = F(x^*) + F'(x^*)x \tag{2.6}$$

for all $x \in U$. Since $F(x^*) = 0$ and $F'(x^*)x \in \operatorname{Im} F'(x^*)$, then from (2.6) we have

$$\tilde{F}(\varphi(x)) \in \operatorname{Im} F'(x^*). \tag{2.7}$$

However, using $F(x^*) = 0$, $\varphi(0) = x^*$ and $\varphi'(0) = I_X$, we obtain

$$\begin{aligned}\tilde{F}(\varphi(x)) &= F(x^* + (\varphi(x) - x^*)) \\ &= F(x^*) + F'(x^*)(\varphi(x) - x^*) + \xi\|\varphi(x) - x^*\|^2 \\ &= F'(x^*)(\varphi(x) - x^*) + \xi\|\varphi(0) + \varphi'(0)x + \omega_1(x) - x^*\|^2 \\ &= F'(x^*)(\varphi(x) - x^*) + \xi\|x + \omega_1(x)\|^2,\end{aligned} \tag{2.8}$$

where $\|\omega_1(x)\| = o(\|x\|)$. Thus, for small x,

$$\xi\|x + \omega_1(x)\|^2 \neq 0.$$

Taking into account (2.4), (2.8) and the fact that $F'(x^*)(\varphi(x) - x^*) \in \operatorname{Im} F'(x^*)$, we conclude from this that

$$\tilde{F}(\varphi(x)) \notin \operatorname{Im} F'(x^*). \tag{2.9}$$

This contradicts (2.7) and therefore F is essentially nonlinear at x^*.

To prove the converse, suppose that F is essential nonlinear at x^*, but that F is not singular; i.e., is regular at this point. Then by persistence of the regularity condition, for any perturbation

$$\widetilde{F}(x^* + x) = F(x^* + x) + \omega(x),$$

where $\|\omega(x)\| = o(\|x\|)$, the map $\widetilde{F}(x^* + x)$ is regular at x^* and $F'(x^*) = \widetilde{F}'(x^*)$. Hence, by virtue of a Theorem concerning the representation of a regular mapping (see Izmailov and Tret'yakov [22])[1], $\widetilde{F}(x^* + x)$ is represented as

$$\tilde{F}(\varphi(x)) = \tilde{F}(x^*) + \tilde{F}'(x^*)x,$$

where $\varphi(0) = x^*$ and $\varphi'(0) = I_X$. It contradicts to the definition of essential nonlinearity of the mapping F. □

Destruction of the Structure of Solution Sets. We shall be interested in the ***solution set***

$$M(x^*) = \{x \in U \mid F(x) = F(x^*)\}$$

for the nonlinear equation (1.1). As we shall see later, this is the feasible set for an associated optimization problem (see (5.1)). In particular, if x^* is a zero of F, the set $M(x^*)$ is the zero-set of F.

If F is an essentially nonlinear mapping at the point x^*, then *the structure of the solution set of the problem* (1.1) *need not be preserved under a small perturbation of* F. For example, there might be a small perturbation of F such that the solution set $M(x^*)$ reduces to the single point x^*, as in the following example.

Example. Let $X = \mathbb{R}^2$, whose points are denoted $x = (x_1, x_2) \in \mathbb{R}^2$ and let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x) = (x_1 - x_2)^2$. The solution set is obviously defined by the condition $x_1 = x_2$. This problem is singular; note that for the perturbation $\widetilde{F} = (x_1 - x_2)^2 + x_1^4$, the zero set consists of only the single solution $x_1 = x_2 = 0$.

This phenomena of the destruction of the solution set is, as we shall see in §5, relevant for optimization problems in which $F(x) = 0$ is a constraint set and for which $M(x^*)$ is a feasible set. The destruction of the feasible set is an important

[1]Under additional splitting assumptions, which are not made here, this result would be a standard consequence of the implicit function theorem, as in, for example, [1], §2.5.

fact for optimization problems because most results in that subject depend on the structure of the feasible set.

In particular, if F is an essentially nonlinear constraint in the optimization problem, one cannot not guarantee the preservation of the structure of the feasible set of the problem under a small perturbation of F. Hence, the classical theoretical results become non-informative or false for the problem. Essentially nonlinear problems need, therefore, new theoretical results and some such results are given in the present paper in the following sections.

3. **The p-factor Operator.** For the purpose of describing essentially nonlinear problems, the concept of p-regularity was introduced by Tret'yakov [30, 31, 33] using the notion of a p-factor operator.

We construct the p-factor operator under the *assumption* that the space Y is decomposed into the direct sum

$$Y = Y_1 \oplus \ldots \oplus Y_p, \tag{3.1}$$

where $Y_1 = \mathrm{cl}\,(\mathrm{Im}\ F'(x^*))$, the closure of the image of the first derivative of F evaluated at x^*, and the remaining spaces are defined as follows. Let Z_2 be a closed complementary subspace to Y_1 (we are *assuming* that such a closed complement exists) and let $P_{Z_2} : Y \to Z_2$ be the projection operator onto Z_2 along Y_1. Let Y_2 be the closed linear span of the image of the quadratic map $P_{Z_2} F^{(2)}(x^*)[\cdot]^2$. More generally, define inductively,

$$Y_i = \mathrm{cl}\,(\mathrm{span}\ \mathrm{Im}\, P_{Z_i} F^{(i)}(x^*)[\cdot]^i) \subseteq Z_i, \quad i = 2, \ldots, p-1,$$

where Z_i is a choice of closed complementary subspace for $(Y_1 \oplus \ldots \oplus Y_{i-1})$ with respect to Y, $i = 2, \ldots, p$, and $P_{Z_i} : Y \to Z_i$ is the projection operator onto Z_i along $(Y_1 \oplus \ldots \oplus Y_{i-1})$ with respect to Y, $i = 2, \ldots, p$. Finally, let $Y_p = Z_p$.

Define the following mappings (see Tret'yakov [33])

$$f_i(x) : U \to Y_i, \quad f_i(x) = P_{Y_i} F(x), \quad i = 1, \ldots, p,$$

where $P_{Y_i} : Y \to Y_i$ is the projection operator onto Y_i along $(Y_1 \oplus \ldots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \ldots \oplus Y_p)$ with respect to Y, $i = 1, \ldots, p$.

Definition 3.1. *The linear operator $\Psi_p(h) \in \mathcal{L}(X, Y_1 \oplus \ldots \oplus Y_p)$, for $h \in X$, is defined by*

$$\Psi_p(h) = f_1'(x^*) + \frac{1}{2!} f_2''(x^*)[h] + \ldots + \frac{1}{p!} f_p^{(p)}(x^*)[h]^{p-1},$$

*and is called the **p-factor operator**.*

Note that in the ***completely degenerate case***, i.e., in the case that

$$F^{(r)}(x^*) = 0, \quad r = 1, \ldots, p-1,$$

then the p-factor operator is simply $F^{(p)}(x^*)[h]^{p-1}$.

Definition 3.2. *We say that the mapping F is **p-regular at** x^* **along an element** h, if $\mathrm{Im}\,\Psi_p(h) = Y$.*

Definition 3.3. *We say the mapping F is **p-regular at** x^* if it is p-regular along any h from the set*

$$H_p(x^*) = \left\{ \bigcap_{i=1}^{p} \mathrm{Ker}^i\, f_i^{(i)}(x^*) \right\} \Big\backslash \{0\}.$$

Definition 3.4. *A mapping $F \in C^p$ is called* ***strongly*** *p***–regular** *at a point x^* if there exists $\alpha > 0$ such that*

$$\sup_{h\in H_\alpha} \|\{\Psi_p(h)\}^{-1}\| < \infty,$$

where

$$H_\alpha = \left\{ h \in X \mid \|f_i^{(i)}(x^*)[h]^i\|_{Y_i} \le \alpha \quad \text{for all} \quad i = 1, \dots, p, \quad \|h\|_X = 1 \right\}.$$

Remark. Not only Ψ_p but also every Y_i in $Y_1 \oplus \ldots \oplus Y_p$ depends on the choice of the element h. To simplify our notations we write Y_i instead of $Y_i(h)$, $i = 1, \dots, p$.

4. **A Generalization of the Lyusternik Theorem.** It will be useful to recall the following definition of tangent vectors and tangent cones (see, for instance, Ioffe and Tikhomirov [17] or Clarke [13]).

Definition 4.1. *We call h a* ***tangent vector*** *to a set $M \subseteq X$ at $x^* \in M$ if there exist $\varepsilon > 0$ and a function $r : [0, \varepsilon] \to X$ with the property that for $t \in [0, \varepsilon]$, we have $x^* + th + r(t) \in M$ and*

$$\lim_{t\to 0} \frac{\|r(t)\|}{t} = 0.$$

The collection of all tangent vectors at x^* is called the ***tangent cone*** *to M at x^* and it is denoted by $T_1M(x^*)$.*

It is well known that to solve nonlinear problems in the regular case, one may use classical results such as the implicit function theorem, the Lagrange and Euler theory of optimality conditions, the Lyusternik theorem (see Ioffe and Tikhomirov [17]) and others. We recall the latter now.

Theorem 4.2 (Lyusternik Theorem). *Let X and Y be Banach spaces and U be a neighborhood of x^* in X. Suppose $F : U \to Y$ is Fréchet differentiable on V, and the mapping $F' : U \to \mathcal{L}(X, Y)$ is continuous at x^*. Suppose further that F is regular at x^*. Then the tangent cone to the set $M(x^*) = \{x \in U \mid F(x) = F(x^*)\}$ is the linear space that is the kernel of $F'(x^*)$:*

$$T_1M(x^*) = \operatorname{Ker} F'(x^*). \tag{4.1}$$

In Buchner, Marsden and Schecter [11] and Tret'yakov [30, 31], a generalization of the classical Lyusternik theorem for p-regular mappings was first derived and proved. It may be applied to describe the zero set of a p-regular mapping.

Theorem 4.3 (Generalized Lyusternik Theorem). *Let X and Y be Banach spaces, and U be a neighborhood of a point $x^* \in X$. Assume that $F : X \to Y$ is a p–times continuously Fréchet differentiable mapping in U and is p–regular at x^*. Then*

$$T_1M(x^*) = H_p(x^*).$$

We now give another version of the theorem that was formulated in Tret'yakov [33] and [34]. See also Buchner, Marsden and Schecter [11] and Izmailov and Tret'yakov [22] for additional results along these lines.

To state the result, we shall denote by $\operatorname{dist}(x, M)$, the ***distance function*** from a point $x \in X$ to a set M:

$$\operatorname{dist}(x, M) = \inf_{y\in M} \|x - y\|, \quad x \in X.$$

Theorem 4.4. *Let X and Y be Banach spaces, and U be a neighborhood of a point $x^* \in X$. Assume that $F : X \to Y$ is a p–times continuously Fréchet differentiable mapping in U and satisfies the condition of strong p–regularity at x^*. Then there exist a neighborhood $U' \subseteq U$ of x^*, a mapping $\xi \mapsto x(\xi) \ : \ U' \to X$, and constants $\delta_1 > 0$ and $\delta_2 > 0$ such that*

$$F(\xi + x(\xi)) = F(x^*),$$

$$\|x(\xi)\|_X \le \delta_1 \sum_{i=1}^{p} \frac{\|f_i(\xi) - f_i(x^*)\|_{Y_i}}{\|\xi - x^*\|^{i-1}}, \tag{4.2}$$

and

$$\|x(\xi)\|_X \le \delta_2 \sum_{i=1}^{p} \|f_i(\xi) - f_i(x^*)\|_{Y_i}^{1/i}$$

for all $\xi \in U'$.

5. **p-order Conditions for Optimality.** Let $f : X \to \mathbb{R}$ be a (sufficiently smooth, real valued) function and $F : X \to Y$ be a (sufficiently smooth) mapping. Consider the following nonlinear constrained optimization problem:

$$\begin{array}{c} \displaystyle\min_{x \in X} \ f(x) \\ \text{subject to } F(x) = 0, \end{array} \tag{5.1}$$

Let the solution set to this problem in a neighborhood of a given solution x^* be denoted by $M(x^*)$. The ***p-factor-Lagrange function*** is defined as follows:

$$\mathcal{L}_p(x, h, \lambda_0(h), y(h)) = \lambda_0(h) f(x) + \sum_{i=1}^{p} \left\langle y_i(h),\ f_i^{(i-1)}(x)[h]^{i-1} \right\rangle. \tag{5.2}$$

Here, the functions $y(h)$ play the role of higher-order Lagrange multipliers.

Theorem 5.1 (Necessary and Sufficient Conditions for Optimality). *Let X and Y be Euclidean spaces, and U be a neighborhood of the point x^*. Suppose that $f \in C^2(U, \mathbb{R})$ and that $F \in C^{p+1}(U, Y)$. Suppose that for $h \in H_p(x^*)$ the set $\operatorname{Im} \Psi_p(h)$ is closed in $Y_1 \oplus \ldots \oplus Y_p$.*

1. *If x^* is a local solution to problem (5.1), then there exist $\lambda_0(h) \in \mathbb{R}$ and multipliers $y_i(h) \in Y_i^*$, $i = 1, \ldots, p$, such that they do not all vanish , and*

   $$\mathcal{L}'_{p\,x}(x^*, h, \lambda_0(h), y(h)) = \lambda_0(h)\, f'(x^*) + \sum_{i=1}^{p} (f_i^{(i)}(x^*)[h]^{i-1})^* y_i(h) = 0.$$

   If, moreover, $\operatorname{Im}\ \Psi_p(h) = Y_1 \oplus \ldots \oplus Y_p$, *then* $\lambda_0(h) \neq 0$.
2. *Moreover suppose that the set $\operatorname{Im} \Psi_p(h)$ is closed in $Y_1 \oplus \ldots \oplus Y_p$ for any element $h \in H_p(x^*)$ and $\operatorname{Im}\ \Psi_p(h) = Y_1 \oplus \ldots \oplus Y_p$. If there exist $\alpha > 0$ and multipliers $y_i(h) \in Y_i^*$, $i = 1, \ldots, p$, such that*

   $$\mathcal{L}'_{p\,x}(x^*, h, 1, y(h)) = 0 \tag{5.3}$$

   and

   $$\mathcal{L}''_{p\,xx}(x^*, h, 1, y(h))[h]^2 \ge \alpha \|h\|^2, \tag{5.4}$$

   for all $h \in H_p(x^*)$ then x^* is an isolated solution to problem (5.1).

The proof of the necessary conditions for optimality is given in Izmailov and Tret'yakov [22], and the sufficient conditions for p–regular mappings are derived in Brezhneva and Tret'yakov [10].

6. **p-regularity and p-majorizability.** The results of this subsection are adapted from Izmailov and Tret'yakov [22].

DEFINITION 6.1. *The mapping $F : U \to Y$ is said to be $p-$**majorizable** at the point x^*, if there exist a neighborhood $U' \subseteq U$ and a constant $\delta > 0$ such that*

$$\operatorname{dist}(x, M(x^*)) \leq \delta \|F(x)\|_Y^{1/p},$$

for all $x \in U'$.

In the case of p-regular mappings the structure of the zero set does not change in an essential way, but it may have some small perturbations. This is derived from the following results.

THEOREM 6.2. *Let X be Banach space and Y be Hilbert space, U be a neighborhood of x^* in X. Suppose $F \in C^p(U, Y)$ and $F^{(i)}(x^*) = 0$, $i = 1, \ldots, p-1$. Then F is p–regular at x^* if and only if for any continuous mapping $\Omega : U \to Y$ such that $\Omega(x^*) = 0$ and $\|\Omega(x)\|_Y = o(\|x - x^*\|^p)$, the mapping $\widetilde{F} = F + \Omega$ is $p-$majorizable at the point x^*;*

COROLLARY 6.3. *The tangent cone to the solution set of the mapping F coincides with the tangent cone to the solution set of the mapping $\widetilde{F}$ up to order $o(\|x - x^*\|)$. Hence, in this sense, the zero set changes only a small bit under small perturbations.*

7. **The Representation Theorem.** The results of this section follow the work of Izmailov and Tret'yakov [22].

THEOREM 7.1 (Representation Theorem). *Let X and Y be Banach spaces, and V be a neighborhood of x^* in X. Suppose $F : V \to Y$ is of class C^{p+1} and $F^{(i)}(x^*) = 0$, $i = 1, \ldots, p-1$. Suppose further that for a constant $C > 0$,*

$$\sup_{\|h\|_X = 1} \|\{F^{(p)}(x^*)[h]^{p-1}\}^{-1}\| \leq C.$$

Then there exist a neighborhood U of 0 and a neighborhood V of x^* in X and mappings $\varphi : U \to X$ and $\psi : V \to X$ such that φ and ψ are Fréchet-differentiable at 0 and x^*, respectively, and

(a) $\varphi(0) = x^*, \quad \psi(x^*) = 0$;
(b) $F(\varphi(x)) = F(x^*) + \frac{1}{p!} F^{(p)}(x^*)[x]^p \quad \text{for all} \quad x \in U$;
(c) $F(x) = F(x^*) + \frac{1}{p!} F^{(p)}(x^*)[\psi(x)]^p \quad \text{for all} \quad x \in V$;
(d) $\varphi'(0) = \psi'(x^*) = I_X$.

COROLLARY 7.2 (Generalized Morse lemma). *Let X and Y be Hilbert spaces, V be a neighborhood of x^* in X. Suppose $F : V \to Y$ is of class C^{p+1} and the set $\operatorname{Im} F^{(p)}(x^*)$ is closed in Y. Furthermore, assume that*

$$\sup_{\|h\|_X = 1} \|(F^{(p)}(h))^{-1}\| < \infty.$$

Then there exists a neighborhood U of 0 in X and a mapping $\varphi : U \to X$ such that $\varphi(0) = x^*$, $\varphi(U) \subset V$, φ is diffeomorphism from U onto $\varphi(U)$, $\varphi'(0)$ is the identity operator onto X and

$$\varphi(T_1 \cap U) = \varphi(H_p \cap U) = F^{-1}(F(x^*)) \cap \varphi(U).$$

A proof of this fact directly follows from the results in Buchner, Marsden and Schecter [11] and the definition of p–regular mapping. For the case $p = 2$ this result is given in Avakov, Agrachyov, and Arutyunov [3].

8. **Methods for Solving Singular Nonlinear Equations.**
The 2-factor Method. This method was derived in Belash and Tret'yakov [5] and Tret'yakov [32, 33]. Suppose that F is 2-regular at x^* with respect to some h. The scheme of the 2-factor-method can, in this case, be written in the form

$$x_{k+1} = x_k - [F'(x_k) + P_k^{\perp} F''(x_k)h_k]^{-1} F(x_k), \quad k = 0, 1, \ldots, \tag{8.1}$$

where $h_k \in \operatorname{Ker} F'(x_k)$, $\|h_k\| = 1$ and where $P_k^{\perp}$ is the orthogonal projector onto $(\operatorname{Im}\, F'(x_k))^{\perp}$, $k = 0, 1, \ldots$. Note that the scheme (8.1) is similar to the Newton method.

We consider the simplest case in which we assume that

$$\dim \operatorname{Ker} F'(x_k) = \dim \operatorname{Ker} F'(x^*), \quad k = 0, 1, \ldots.$$

Theorem 8.1. *Let x^* be a solution of (1.1), and U be a neighborhood of the point x^* in $\mathbb{R}^n$. Suppose that F is 2-regular at x^* with respect to $h^* \in \operatorname{Ker} F'(x^*)$. Then there exists a neighborhood $U' \subseteq U$ of x^* in $\mathbb{R}^n$ such that for any point $x_0 \in U'$ the sequence defined by (8.1) converges to x^* and*

$$\|x_{k+1} - x^*\| \le R\,\|x_k - x^*\|^2 \quad \forall k = 0, 1, \ldots,$$

with a constant $R > 0$.

The development of the 2-factor-method and its application to a wide range of problems are given in Izmailov and Tret'yakov [24]. A survey of various methods for finding singular solutions to nonlinear problems and a general scheme for solving singular nonlinear equations are presented in Brezhneva and Izmailov [6].

9. **Genericity of the Class of p-regular Problems.** Consider the problem of genericity of the notion of p-regularity of a mapping F. For simplicity, we restrict our considerations to the case of absolute degeneration in the spaces $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, $n, m \in \mathbb{N}$. By virtue of the definition of a p-regular mapping, the problem of genericity reduces to the problem of genericity of the p-regularity property of mappings $Q(x) = Q[x]^p = Q[\underbrace{x, \ldots, x}_{p}]$, $x \in X$ in the space $\mathcal{Q}^p(\mathbb{R}^n, \mathbb{R}^m)$. This problem is solved by an application of the transversality theory and the following Theorem (see Buchner, Marsden and Schecter [11], Ortega and Rheinboldt [28], and Izmailov and Tret'yakov [24]).

Theorem 9.1. *Let $h \in \mathbb{R}^n$, $h \neq 0$. Then the set of the mappings $Q \in \mathcal{Q}^p(\mathbb{R}^n, \mathbb{R}^m)$, which are p-regular at the element h, is open and dense in $\mathcal{Q}^p(\mathbb{R}^n, \mathbb{R}^m)$ for $p, n, m \in \mathbb{N}$, $n \ge m$,.*

10. **Implicit Function Theorem.**

Theorem 10.1 (Tret'yakov [33], [34]). *Suppose that X, Y and Z are Euclidean spaces, W is a neighborhood of a point (ξ_0, η_0) in $X \times Y$, and assume that $F : W \to Z$ is of class C^2. Suppose $F(\xi_0, \eta_0) = 0$ and the following conditions hold:*

1. *the **singularity condition**:*

$$f_i^{(r)}\underbrace{\xi \ldots \xi}_{q}\,\underbrace{\eta \ldots \eta}_{r-q}(\xi_0, \eta_0) = 0, \quad r = 1, \ldots, i-1,\ q = 0, \ldots, r-1,\ i = 1, \ldots, p;$$

2. *the* p*-**regularity condition** at the point* (ξ_0, η_0)*: there is a neighborhood* $U(\xi_0)$ *of the point* ξ_0 *in* X *such that*
$$\Psi_{p\,\eta}(\xi_0, \eta_0, h)Y = Z$$
for all
$$h \in \{\Psi_{p\,\eta}(\xi_0, \eta_0)\}^{-1}(-F(\xi, \eta_0))$$
and all $\xi \in U(\xi_0)$ *such that* $F(\xi, \eta_0) \neq 0$,
3. *the **Banach condition**: for any* $z \in Z$, $\|z\| = 1$, *there exists* $\eta \in Y$ *such that*
$$\Psi_{p\,\eta}(\xi_0, \eta_0, \eta)\eta = z, \quad \|\eta\| \leq c,$$
where $c > 0$ *is independent of* z *constant;*
4. *the **elliptic condition** with respect to the independent variable* ξ*:*
$$\|f_i(\xi, \eta_0)\|_{Z_i} \geq m\|\xi - \xi_0\|_X^i$$
for all $\xi \in U$ *and for all* $i = 1, \ldots, p$, *where* $m > 0$ *is some number and* U *is a neighborhood of the point* ξ_0 *in* X.

Then for any $\varepsilon > 0$ *there exist* $\delta > 0$, $K > 0$ *and a map* $\varphi : U(\xi_0, \delta) \to U(\eta_0, \varepsilon)$ *such that*

(a) $\varphi(\xi_0) = \eta_0$*;*
(b) $F(\xi, \varphi(\xi)) = 0$ *for all* $\xi \in U(\xi_0, \delta)$*;*
(c)
$$\|\varphi(\xi) - \eta_0\|_Y \leq K \sum_{i=1}^{p} \|f_i(\xi, \eta_0)\|_{Z_i}^{1/i}$$
for all $\xi \in U(\xi_0, \delta)$.

11. **Generalization of Banach's Open Mapping Theorem.** A mapping $Q : X \to Y$ will be called ***quadratic*** if there exists a bilinear mapping $B : X \times X \to Y$ such that
$$Q(x) := Q[x]^2 = B(x,\, x)$$
for all $x \in X$. In other words, as in the introduction, a quadratic mapping is a homogeneous polynomial mapping of degree 2.

Denote by Q^{-1} the right inverse operator (in general, it is many-valued) for the mapping Q. The right inverse operator has norm given by
$$\|Q^{-1}\| = \sup_{\|y\|_Y = 1} \inf \left\{ \|x\|_X \mid x \in X, \quad Q[x]^2 = y \right\}.$$

Theorem 11.1. *Assume that* $Q : \mathbb{R}^n \to \mathbb{R}^m$ *is a continuous quadratic mapping satisfying*
$$\operatorname{Im} Q[x]^2 = \mathbb{R}^m.$$
Then
$$\|Q^{-1}\| < \infty.$$

For $p > 2$ this result does not hold in general, as is illustrated by the following example.

Example. (Izmailov and Tret'yakov [22]). Consider the mapping
$$Q \in \mathcal{Q}^5(\mathbb{R}^2,\, \mathbb{R}^2), \quad Q[x]^5 = \begin{bmatrix} x_1^5 - x_1^2\, x_2^3 \\ x_1^3\, x_2^2 \end{bmatrix}$$
We have $\operatorname{Im} Q = \mathbb{R}^2$, but $\|Q^{-1}\| = \infty$.

12. **Application to Differential Equations.** We consider the Cauchy problem for the following partial differential equation of first order

$$a(x, y, u)p + b(x, y, u)q = c(x, y, u), \tag{12.1}$$

with initial data for $s = 0$ on a curve l,

$$x = x(t), \quad y = y(t), \quad u = u(t), \tag{12.2}$$

where x and y are independent variables; $u = u(x, y)$ is unknown function; and where p and q are, following the Monge notations, $p \equiv \partial u/\partial x$, $q \equiv \partial u/\partial y$; and where $a(x, y, u)$, $b(x, y, u)$, $c(x, y, u)$ are given continuous functions.

Let us apply the preceding implicit function theorem to investigate the issue of existence of the solution of the Cauchy problem in the singular case, i.e., when the Jacobian is equal to zero; that is,

$$\Delta(s, t) = x_s y_t - x_t y_s = 0. \tag{12.3}$$

Letting

$$\xi = (x, y), \quad \eta = (s, t),$$

we write

$$F(\xi, \eta) = \begin{pmatrix} x - x(s, t) \\ y - y(s, t) \end{pmatrix} = 0.$$

Consider the case $p = 2$. Suppose that all parameters of (12.1) and functions $x(t), y(t), u(t)$ are sufficiently smooth and

$$s_1 < s_0 < s_2, \quad t_1 < t_0 < t_2.$$

$x(t_0) = x_0$; $y(t_0) = y_0$; $u(t_0) = u_0$.

Without loss of generality suppose that

$$x(s_0, t_0) = 0, \quad y(s_0, t_0) = 0,$$

so that

$$F(\xi, \eta_0) = \xi.$$

Theorem 12.1. (Brezhneva and Tret'yakov [9].) *Let $a(x, y, u)$, $b(x, y, u)$, $c(x, y, u)$ be continuously differentiable functions, Ω be a neighborhood of the point t_0, $l \in C^1(\Omega)$. Suppose $M = M(x_0, y_0, u_0)$ belongs to l and under some value $s_0 \in (s_1, s_2)$ for the point $\eta_0 \equiv (s_0, t_0)$ the following conditions hold.*

1. *the **singularity condition**:*

$$\Delta(\eta_0) = x_s(\eta_0)y_t(\eta_0) - x_t(\eta_0)y_s(\eta_0) = 0;$$

2. *the p-**regularity condition**:*

$$\Psi_{p\,\eta}(\xi_0, \eta_0, h)\, Y = Z$$

for all

$$h \in \{\Psi_{p\,\eta}(\xi_0, \eta_0)\}^{-1}(-\xi)$$

and all ξ with $\|\xi\| = 1$;

3. *the **Banach condition**: for any $z \in Z$, $\|z\| = 1$, there exists $\eta \in Y$ such that*

$$\Psi_{p\,\eta}(\xi_0, \eta_0, \eta)\eta = z, \quad \|\eta\| \le c,$$

where $c > 0$ is a constant independent of z.

Then there exists a continuous solution $u = u(x, y)$ to the equation (12.1) in some neighborhood U of the point (x_0, y_0) in $X \times Y$. This solution contains some part of the curve l passing through the point M.

13. **Application to Bifurcation Theory.** We will give a sufficient condition for bifurcation at a multiple eigenvalue for a typical nonlinear eigenvalue problem. Namely, we consider the following equation,

$$f(u,\lambda) = \Delta u - (\lambda + \lambda_0)g(u) = 0 \quad \text{in } \Omega$$

with boundary conditions

$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded region in $\mathbb{R}^n$, $f : H_0^{s+2} \times \mathbb{R} \to H^s$, $\lambda \in \mathbb{R}$, λ_0 is an eigenvalue of Δ with multiplicity $n > 1$, and where $g(0) = 0$, $g'(0) = 1$, $g''(0) = 1$.

Let $\mathrm{Ker}\,(D_u f(0,0)) = \mathrm{Ker}\,(\Delta - \lambda_0 I)$ be spanned by the L^2 orthonormal functions $u_1, \ldots, u_n$. Consider an element $u = z_1 u_1 + \ldots + z_n u_n$ and introduce the mapping

$$F(x) = F(z_1, \ldots, z_n, \lambda) = f(u, \lambda), \quad x = (u, \lambda).$$

Since $F'(x)$ is a Fredholm and self-adjoint operator,

$$\mathrm{Ker}\,(F'(x^*)) = (\mathrm{Im}\; F'(x^*))^{\perp}.$$

Using the last relation, p-regularity theory and the generalized Lyusternik theorem, we obtain the following: *if the mapping F is p–regular at the point x^* with respect to element $h = (u,\lambda) \neq (0,\lambda)$ then $x^* = (0,0)$ is the bifurcation point of the problem under consideration.*

To see why this is so, consider the case $p = 2$. Let $P^{\perp}$ be the orthogonal projection onto $Y_2 = (\mathrm{Im}\; F'(x^*))^{\perp}$. The equality

$$P^{\perp} F''(x^*)[h]^2 = 0, \quad h = (u, \lambda)$$

is equivalent to the inclusion

$$F''(x^*)[h]^2 \in \mathrm{Im}\; F'(x^*).$$

Hence,

$$\langle F''(x^*)[h]^2, u_i \rangle = 0, \quad i = 1, \ldots, n. \tag{13.1}$$

If the system (13.1) has a nonzero solution $z_1, \ldots, z_n$, then we get an element $h = z_1 u_1 + \ldots + z_n u_n$ as desired, and the point $x^* = (0,0)$ is the bifurcation point to the problem under consideration.

Example. (See Buchner, Marsden and Schecter [11].) Consider

$$f(u,\lambda) = \Delta u - (\lambda - 10)g(u) = 0$$

on $\Omega = [0,\pi] \times [0,\pi]$ in $\mathbb{R}^2$ with $u = 0$ on $\partial\Omega$. Assume $g(0) = 0$, $g'(0) = 1$ and $g''(0) = 1$. Then $u = 0$, $\lambda = 0$ is a bifurcation point.

To be specific, take $n = 2$, and then $u_1 = \sin 3x \sin y$, $u_2 = \sin x \sin 3y$. It is easy to prove that the mapping F is 2-regular at the point $x^* = (0,0)$ and the element $h = (1, 1, 5(a + 3b))$ is the solution to the system (13.1), where $a = \frac{16}{27}$ and $b = \frac{68}{75}$. The proof of 2-regularity of the mapping F reduces to the test of the non-degeneration of the matrix $F''(x^*)[h]$ where $F''(x^*)[h]^2 = 0$.

14. **The Poincaré-Andronoff-Hopf Bifurcation and p-regularity Theory.** Consider the following problem

$$\dot{x}(t) = g(\mu, x), \qquad x(0) = x(\tau), \tag{14.1}$$

where $g(\mu, x^*) = 0$ for all $\mu \in W_1$, $g : W_1 \times W_2 \to \mathbb{R}^2$ is of class $C^{(p+2)}$, W_1 is an open set in $\mathbb{R}$, W_2 is an open set in $\mathbb{R}^n$, and $\mu \in W_1$ is a parameter.

Let M be a set of points $(\tau, \mu, x) \in \mathbb{R} \times W_1 \times W_2$ such that the solution $x(t)$ of (14.1) passes through the point (τ, μ, x), where $x(0) = x(\tau)$, $\dot{x}(t) = g(\mu, x(t))$.

We can represent the set M as $M = S \cup P$, where $S = R \times W_1 \times \{x_*\}$ is a simple two-dimensional solution branch to (14.1) and $P = M \backslash S$.

The problem is to describe the structure of the set P of periodic solutions of (14.1) in the vicinity of (τ_*, μ_*, x_*).

Let us denote $F(\tau, \mu, x) \overset{\text{def}}{=} x - \varphi(\tau, \mu, x)$, $\quad \mathbb{R}^l \overset{\text{def}}{=} \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ and consider the equation

$$F(\tau, \mu, x) = x - \varphi(\tau, \mu, x) = 0. \tag{14.2}$$

It is well known that for equation (14.1) there exists a solution

$$\varphi(\tau, \mu, x) \in C^{p+1}([-\delta, \tau_* + \delta] \times V_1 \times V_2, \mathbb{R}^n),$$

where V_1 is the vicinity of the point μ_*, such that jointly with (14.2), for all $\tau \in [-\delta, \tau_* + \delta]$, $\mu \in V_1$, and $x \in V_2$ we have

$$\begin{aligned} &\dot{\varphi}(\tau, \mu, x) = g(\mu, \varphi(\tau, \mu, x)), \\ &\varphi(0, \mu, x) = x, \\ &\varphi(\tau, \mu, x_*) = x_*. \end{aligned}$$

This means that the solution $\varphi(\cdot, \mu, x)$ is satisfied to our problem (14.1) and the set $M = S \cup P$ coincides with the solution set $F^{-1}(0)$ in the neighborhood O of the point (τ_*, μ_*, x_*).

Using Corollary 7.2 of the Representation Theorem, we get the following.

Theorem 14.1. *Let V be a neighborhood of the point $z_* \in \mathbb{R}^l$ and suppose that $F \in C^{p+1}(V, \mathbb{R}^n)$ is p–regular at the point z_*, and $F(z_*) = 0$. Then there exist a neighborhood $\mathcal{O} = (U_1 \times U_2 \times \mathcal{O}^n)$ of the point $0 \in \mathbb{R}^l$ and a mapping $\rho : \mathcal{O} \to \mathbb{R}^n$ such that $\rho(0) = z_*$, $\rho(\mathcal{O}) \subset V$ is a neighborhood of the point $z_* \in \mathbb{R}^l$, where ρ is a diffeomorphism from $\mathcal{O}$ onto $\rho(\mathcal{O})$, $\rho'(\mathcal{O}) = I_{\mathbb{R}^l}$ and*

$$\rho(T_1 \cap \mathcal{O}) = F^{-1}(0) \cap \rho(\mathcal{O}), \quad T_1(z^*) = \bigcap_{r=1}^{p} \operatorname{Ker}^r f^{(r)}(z^*).$$

In another words, the set M can be described in the following way

$$M \cap \rho(\mathcal{O}) = \rho(\bigcap_{r=1}^{p} \operatorname{Ker}^r f^{(r)}(z^*) \cap \mathcal{O}^n).$$

Note that testing the assumption that F is p–regular mapping is the independent and interesting problem.

Consider the specific case $p = 2$ (see Izmailov [20]) and denote by L_2 the eigen–subspace of the linear operator $\partial_x g(\mu_*, x_*)$ that corresponds to the eigenvalue $\pm i\omega$ and by P_2 the orthogonal projection onto L_2. Then, by virtue of the Corollary 7.2, we immediately obtain the ***Andronoff–Hopf Theorem***, since 2-regularity condition is, in fact, equivalent to the Andronoff–Hopf condition

$$\operatorname{tr} P_2 \partial_x g(\mu_*, x_*) \neq 0.$$

This follows from

$$T_1(z^*) = (\mathbb{R} \times \mathbb{R} \times \{0\}) \cup (\{0\} \times \{0\} \times L_2).$$

In fact, we have

$$S \cap \rho(\mathcal{O}) = \rho(U_1 \times U_2 \times \{0\})$$

and

$$(P \cup \{(2\pi/\omega, \mu_*, x_*)\}) \cap \rho(\mathcal{O}) = \rho(\{0\} \times \{0\} \times (L_2 \cap \mathcal{O}^n)).$$

15. **Singular Problems in the Calculus of Variations.** In this section we apply the p–regularity theory to the isoperimetric problem and discuss the case $p = 2$. One may view this as an instance of obtaining the 2-factor-Euler-Lagrange equation for optimality.

We consider a functional of the form

$$J_0[y] = \int_{x_1}^{x_2} F(x, y, y')dx, \tag{15.1}$$

where $y \in C^1([x_1, x_2], \mathbb{R})$ and $F(x, y, y')$ is a function with continuous first and second derivatives with respect to its arguments.

The isoperimetric problem in calculus of variations can be formulated as follows.

Isoperimetric problem. *Minimize the functional* J_0 *in* (15.1) *subject to the subsidiary condition*

$$J_1[y] = \int_{x_1}^{x_2} G(x, y, y')dx = l \tag{15.2}$$

and boundary conditions

$$y(x_1) = y_1, \quad y(x_2) = y_2. \tag{15.3}$$

In accordance with Lauwerier [26] and other books on the calculus of variations, if the regularity condition $\operatorname{Im} J_1'[y^*] \neq 0$ is fulfilled, then a necessary condition that y is an extremum to the isoperimetric problem can be written as

$$\frac{d}{dx}(F_{y'} + \lambda G_{y'}) = F_y + \lambda G_y, \tag{15.4}$$

where λ is a constant. Notice that equation (15.4) is just the Euler-Lagrange equations for the functional $J_0 + \lambda J_1$.

The ***increment*** of the functional J_1 is defined as

$$\Delta J_1 = J_1[y(x) + \delta y] - J_1[y(x)] = L[y(x), \delta y] + \Psi[y(x), \delta y]\, \|\delta y\|,$$

where $|\Psi[y(x), \delta y]| \to 0$ as $\delta y \to 0$, $L[y(x)]$ is a linear functional, and $L[y(x), \delta y] = L[y(x)]\delta y$ is called the ***variation*** of the functional J_1. Moreover, we can write

$$L[y(x) + \delta y]\delta y - L[y(x)]\delta y = A[y(x)](\delta y)^2 + \Psi[y(x), \delta y]\, |\delta y|^2,$$

and $A[y(x)](\delta y)^2$ is called the ***second variation*** of the functional J_1.

Define the functionals $J_1'[y^*]$ and $J_1''[y^*]$ as

$$J_1'[y^*] = L[y^*], \quad J_1''[y^*] = A[y^*].$$

In accordance with Korneva and Tret'yakov [25], we have:

$$J_1'[y^*]h(x) = G_{y'}h(x)\big|_{x=x_1}^{x=x_2} + \int_{x_1}^{x_2} \left[G_y - \frac{d}{dx}G_{y'}\right] h(x)dx,$$

and

$$J_1''[y^*]h_1(x)h_2(x) = h_1'(x)G_{y'y'}h_2(x)|_{x=x_1}^{x=x_2} + \int_{x_1}^{x_2} \Big[G_{yy}h_1(x) + G_{yy'}h_1'(x) - \frac{d}{dx}[G_{y'y}h_1(x) + G_{y'y'}h_1'(x)]\Big] h_2(x)dx.$$

Choose a function $h(x)$ such that $h(x) \in \operatorname{Ker}^2 J_1''[y^*]$, i.e.,

$$J_1''(y^*)h^2(x) = \int_{x_1}^{x_2} [G_{yy}h^2(x) + 2G_{yy'}h(x)h'(x) + G_{y'y}(h(x))^2]dx = 0.$$

Next, we make the following definition.

Definition 15.1. *The isoperimetric problem is called* ***singular*** *or* ***irregular*** *if*

$$\operatorname{Im} J_1'[y^*] = 0. \tag{15.5}$$

The equality (15.5) implies that y^* is an extremal of J_1. Note that there are singular isoperimetric problems with nontrivial extremum, and the necessary condition (15.4) is empty for these problems, i.e., the equation (15.4) has no solution with corresponding F and G.

We now are able to introduce new necessary conditions for optimality that are similar to (15.4) and which are useful for the singular isoperimetric problem.

Theorem 15.2. *Let a curve y^* be an extremum of the functional* (15.1) *under the condition* (15.2) *and boundary conditions* (15.3)*, and y^* be an extremal of the functional* (15.2)*, i.e.,* (15.5) *holds. Suppose there is a function $h(x) \in \operatorname{Ker}^2 J_1''[y^*]$ such that $J_1''[y^*]h(x) \neq 0$. Then, there is a constant λ such that y^* is an extremal of the functional*

$$\int_{x_1}^{x_2} H(x, y, y')dx,$$

where $H = F + \lambda(G_y h + G_{y'} h')$ is a 2*-factor-Euler-Lagrange function, and the following* 2*-factor-Euler-Lagrange equation holds,*

$$F_y + \lambda(G_{yy}h + G_{yy'}h') - \frac{d}{dx}[F_{y'} + \lambda(G_{yy'}h + G_{y'y'}h')] = 0. \tag{15.6}$$

It should be noted that (15.6) is similar to (15.4) and that the condition $J_1''[y^*]h(x) \neq 0$ is the 2–regularity condition for the isoperimetric problem.

Example. Consider the problem of minimizing the functional

$$J_0[y] = \int_{-1}^{1} y^2 dx \tag{15.7}$$

subject to the constraints

$$y(-1) = y(1) = \frac{5}{4}, \quad J_1[y] = \int_{-1}^{1} \Big(y' - \frac{x}{2}\Big)^2 dx = 0. \tag{15.8}$$

Let show that the curve $y^* = \frac{1}{4}x^2 + 1$ is an extremal of J_1 and y^* does not satisfy the classical necessary optimality condition (15.4), but satisfies necessary optimality conditions that are formulated in theorem 15.2.

First of all, a necessary condition for a curve $y(x)$ to be an extremal of J_1 is that $y(x)$ satisfies the corresponding Euler-Lagrange equation:

$$\frac{d}{dx}\left(2\left(y' - \frac{x}{2}\right)\right) = 0.$$

The above equation implies that

$$2y'' - 1 = 0.$$

Integrating, we have

$$y = \frac{1}{4}x^2 + C_1 x + C_2.$$

Using the boundary conditions (15.8), we obtain

$$y^* = \frac{1}{4}x^2 + 1. \tag{15.9}$$

Using (15.8) and (15.9) we have

$$J_1[y^*] = \int_{-1}^{1} \left(\frac{1}{2}x - \frac{x}{2}\right)^2 dx = 0,$$

and hence, y^* is an extremal of J_1.

According to (15.4), the classical necessary conditions for y^* to be an extremum to the problem (15.7)–(15.8) is that y^* satisfies the equation:

$$\frac{d}{dx}\left(2\lambda\left(y' - \frac{x}{2}\right)\right) = 2y,$$

that is

$$2\lambda y'' + \lambda = 2y,$$

Putting (15.9) into the last equation, we get

$$\frac{1}{2}x^2 + 2 - 2\lambda\frac{1}{2} + \lambda = 0,$$

so that

$$\frac{1}{2}x^2 + 2 = 0 \quad \forall x.$$

The last equation has no solution. Hence, *the necessary condition* (15.4) *is an empty condition for this example.*

We will prove that the function y^* from (15.9) satisfies the hypotheses of Theorem 15.2. First of all, we will show that the function

$$h(x) = x^4 + 24x^2 - 25 \tag{15.10}$$

satisfies the conditions of Theorem 15.2. The first condition of the theorem, namely $h(x) \in \operatorname{Ker}^2 J_1''[y^*]$, is equivalent to

$$\int_{-1}^{1} 2(h'(x))2dx = 0.$$

Putting (15.10) into this equation, we get

$$\int_{-1}^{1} (4x^3 + 48x)dx = (x^4 + 24x^2)|_{-1}^{1} = 1 + 24 - 1 - 24 = 0.$$

The second condition of the theorem, namely $J_1''[y^*]h(x) \neq 0$, implies that

$$\frac{d}{dx}(2h') \neq 0,$$

that is

$$\frac{d}{dx}(4x^3 + 48x) = 12x^2 + 48 \neq 0.$$

Hence the both conditions of the theorem are fulfilled. Now construct the 2-factor-Euler-Lagrange function H for this example:

$$H = y^2 + \lambda \left(2 \left(y' - \frac{x}{2} \right) (4x^3 + 48x) \right).$$

Suppose that y^* from (15.9) is an extremum to the problem (15.7)–(15.8) and find a constant λ such that the 2-factor-Euler-Lagrange equation (15.6) holds for y^* and λ. Putting (15.9) into (15.6), we get

$$2y - \frac{d}{dx}(2\lambda(4x^3 + 48x)) = 0,$$

that is,

$$2y - 24\lambda x^2 - 96\lambda = 0,$$

so that

$$\frac{1}{2}x^2 + 2 - \lambda(24x^2 + 96) = 0.$$

Our equation for λ thus reduces to

$$(1 - 48\lambda) \left(\frac{1}{2}x^2 + 2 \right) = 0.$$

The last equality implies that $\lambda = 1/48$.

Hence, the theorem 15.2 gives us nonempty necessary optimality conditions for irregular isoperimetric problem.

16. **A Method for Nonlinear Equality–Constrained Optimization Problems.** In this section, we present a method for solving smooth nonlinear equality–constrained optimization problems for which the constraint matrix at the solution is irregular. The method of construction is based on the optimality conditions for 2–regular problems.

We consider the problem

$$\begin{array}{c} \min_{x \in \mathbb{R}^n} f(x), \\ \text{subject to } F(x) = 0, \end{array} \tag{16.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a sufficiently smooth function and $F : \mathbb{R}^n \to \mathbb{R}^m$ is a sufficiently smooth mapping, written as

$$F(x) = \begin{pmatrix} F_1(x) \\ \dots \\ F_m(x) \end{pmatrix} = 0.$$

Suppose that $F'(x^*)$ is irregular at the solution x^*; namely:

$$\operatorname{Rank} F'(x^*) = r < m.$$

Without loss of generality, assume that

$$F'_{r+1}(x^*) = 0, \dots, F'_m(x^*) = 0. \tag{16.2}$$

The subspace $\text{Im}F'(x^*)$ has an orthogonal complementary subspace $(\text{Im}F'(x^*))^\perp$ in $\mathbb{R}^m$. Under the assumption (16.2), we have:

$$(\text{Im}F'(x^*))^\perp = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m \mid x_1 = \dots = x_r = 0\}.$$

Remark. Note that implementation of the proposed method does not require a knowledge of the number r. It is possible to apply the algorithm described in Brezhneva, Izmailov, Tret'yakov, and Khmura [7] for the determination of r, without x^* being known precisely.

In §5 we considered the optimality conditions for p–regular constrained optimization problems. In the following, we will need necessary optimality conditions for $p = 2$.

THEOREM 16.1. *Let x^* be a solution to* (16.1), *and F be 2-regular at x^* on $h^* \in H_2(x^*)$. Then, there exists $y^* \in \mathbb{R}^m$ such that*

$$f'(x^*) + (F'(x^*) + P^{\perp} F''(x^*)h^*)^T y^* = 0, \tag{16.3}$$

where $P^{\perp}$ is the orthogonal projector onto $(\mathrm{Im} F'(x^*))^{\perp}$.

Recall from Definition 3.3 that

$$H_2(x^*) = \mathrm{Ker}\, F'(x^*) \cap \mathrm{Ker}^2\, P^{\perp} F''(x^*).$$

We also use the following notation:

$$z = (x, y) \in \mathbb{R}^{n+m}, \quad z^* = (x^*, y^*),$$

where x^* is a solution to (16.1) and $y^* \in \mathbb{R}^m$ satisfies (16.3). In this section, we also let $P_M h_0$ denote the orthogonal projection of the element h_0 onto the set M.

The basic idea of our method for solving nonlinear equality–constrained optimization problems is to construct a mapping $R : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ such that z^* is a regular solution to the system

$$R(z) = 0. \tag{16.4}$$

Namely, we consider the following method. Let x^* be a solution to (16.1), and let F be 2–regular at x^* on $h^* = P_{H_2(x^*)} h_0$, where $h_0 \in \mathbb{R}^n$ is some arbitrary element, $\|h_0\| = 1$, $\|h_0 - h^*\| \le \varepsilon$, and $\varepsilon \ge 0$ is sufficiently small.

Define the mapping:

$$\widehat{F} : \mathbb{R}^n \to \mathbb{R}^m, \quad \widehat{F}(x) = (F_1(x), \ldots, F_r(x), 0, \ldots, 0)^T, \tag{16.5}$$

Note that for $\widehat{F}$, the following equality holds:

$$\dim \mathrm{Ker}\, \widehat{F}\,'(x) = \dim \mathrm{Ker}\, F\,'(x^*)$$

for all $x \in U_\varepsilon(x^*)$, and

$$P^{\perp} = P_{(\mathrm{Im}\, F\,'(x^*))^{\perp}} = P_{(\mathrm{Im}\, \widehat{F}\,'(x))^{\perp}}$$

for all $x \in U_\varepsilon(x^*)$, where $U_\varepsilon(x^*)$ is a sufficiently small ε–neighborhood of x^* in $\mathbb{R}^n$.

Introduce the mapping

$$G : \mathbb{R}^n \to \mathbb{R}^m, \quad G(x) = F(x) + P^{\perp} F'(x) h(x), \tag{16.6}$$

where

$$h(x) = P_{H_2(x)} h^0, \quad \|h_0\| = 1, \tag{16.7}$$

and

$$H_2(x) = \mathrm{Ker} \widehat{F}\,'(x) \cap \mathrm{Ker}\,^2 \{P^{\perp}\, F\,''(x)\} \backslash \{0\}. \tag{16.8}$$

Note that from the condition of 2–regularity of the mapping F, it follows that for any $x \in U_\varepsilon(x^*)$ the set $H_2(x) \ne \{0\}$.

Define the function

$$\bar{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad \bar{L}(x, y) = f(x) + \langle y, G(x) \rangle.$$

In accordance with Brezhneva and Tret'yakov [8], the following theorem is pivotal for this method.

THEOREM 16.2. *Let x^* be a solution to* (16.1), *F be 2-regular at x^* on $h^* = P_{H_2(x^*)}h_0$, G be defined by* (16.6)–(16.8), *and let there exist $y^* \in \mathbb{R}^m$ such that* (16.3) *holds and such that*

$$\langle \bar{L}''_{xx}(x^*, y^*)h, h\rangle > 0 \quad \forall h \in \ker G'(x^*)\backslash\{0\}.$$

Then, there exists $\varepsilon > 0$ such that $z^* = (x^*, y^*)$ is an isolated regular solution to the system of nonlinear equations

$$R(z) = \begin{pmatrix} f'(x) + (G'(x))^T y \\ G(x) \end{pmatrix} = 0. \tag{16.9}$$

Using theorem 16.2, it follows that the original problem of finding a singular solution to problem (16.1) is reduced to finding an isolated solution to system (16.9). To find such a solution, any traditional numerical method can be applied, for example, the Newton's method.

17. **Conclusion.** In this paper, we have presented the key constructions and results of p–regularity theory, a subject that has been actively developed for the last fifteen years. One of the basic results of the theory, the theorem about the structure of the zero set of an irregular mapping satisfying a special higher–order regularity condition, was simultaneously obtained in Buchner, Marsden and Schecter [11] and Tret'yakov [31]. In this paper we have showed how to apply p–regularity theory to the description and investigation of singular mappings in different fields of mathematics. We believe that it is very promising to apply the theory to other fields of mathematics and to new nonlinear objects.

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E-mail address: tret@ap.siedlce.pl
E-mail address: marsden@cds.caltech.edu