THE EQUIVALENCE OF CONTROLLED LAGRANGIAN AND CONTROLLED HAMILTONIAN SYSTEMS

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Abstract. The purpose of this paper is to show that the method of controlled Lagrangians and its Hamiltonian counterpart (based on the notion of passivity) are equivalent under rather general hypotheses. We study the particular case of simple mechanical control systems (where the underlying Lagrangian is kinetic minus potential energy) subject to controls and external forces in some detail. The equivalence makes use of almost Poisson structures (Poisson brackets that may fail to satisfy the Jacobi identity) on the Hamiltonian side, which is the Hamiltonian counterpart of a class of gyroscopic forces on the Lagrangian side.

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Dedicated to the Memory of J.L. Lions.

1. Introduction

Brief History and Background. Hamiltonian control systems were introduced by Brockett, Willems, and van der Schaft in [?], [?], [?,?,?]. They have been considerably developed since that time; see for example [?], [?], and [?,?], which extended the van der Schaft method to include systems with symmetry. One of the key methods introduced in these works is that of *potential shaping* to stabilize mechanical systems. Potential shaping refers to the use of feedback control to realize a modification to the potential energy of a given mechanical system, i.e., the control law is the force that derives from the potential energy modification. The controlled dynamics then resembles the original system dynamics, with a modification in the potential energy. The potential energy of the controlled system can be shaped for performance within the limitations of the available control authority. An important feature is that analysis tools for mechanical systems such as use of the energy as a Lyapunov

function, can be applied to the controlled system since the closed-loop system dynamics describe a modified mechanical system,.

The idea behind the *method of controlled Lagrangians* was introduced by Krishnaprasad, Bloch, Marsden and Sanchez, starting with [**?**], and [**?**]. Interestingly, [?] emphasized the Hamiltonian and non-canonical Poisson bracket approach. This was put into a systematic Lagrangian context in a series of papers of Bloch, Leonard and Marsden, starting with [**?**]. These works introduced the important notion of *kinetic shaping* which is analogous to potential shaping except that the control law is selected to realize a desired modification to kinetic energy, i.e., the control law is the force that derives from the kinetic energy modification. In this case the controlled system dynamics resemble the original dynamics but with a modified kinetic energy. In these early papers it is shown how kinetic shaping can be applied to underactuated systems (such as a spacecraft stabilized by an internal rotor and the inverted pendulum on a cart), for which potential shaping methods are not applicable. Since that time, this approach has matured and developed considerably and has been applied to many systems, such as spherical pendula and underwater vehicles, as in for example, [**?**,**?**] and references therein. The addition of damping was studied in [**?**,**?**] and some preliminary tracking results were given in [?]. We shall refer to this class of systems as *controlled Lagrangian systems*, or *CL systems* for short.

In tandem with the development of energy shaping methodology, possibilities for structure modification have been investigated. For example, in the papers [?] and [?], the authors demonstrated, in the context of the spacecraft dynamics problem, the concept of using feedback control to modify the system's Poisson structure. Structure-modifying control laws were designed to stabilize an underwater vehicle with internal rotors in the works [**?**,**?**] of Woolsey and Leonard. In the underwater vehicle work, the modified structure is almost Lie-Poisson, i.e., the Poisson structure for the controlled system does not satisfy the Jacobi identity. The idea of structure modification is somewhat analogous and certainly complementary to the notion of energy shaping. In structure modification, the closed-loop system is a mechanical system but the structure (e.g., the Poisson structure) is different from the structure of the original dynamics. Modifying both structure and energy by means of feedback control gives the opportunity for more flexibility (i.e., a larger class of control inputs is considered) as compared to energy shaping alone, while the advantages of providing a closed-loop system that is equivalent to a mechanical system are retained. Recent developments on controlled Lagrangian systems with modified structure were presented in [**?**].

For purposes of the present work, two important recent references on the Hamiltonian side are those of Ortega, Spong, Gómez-Estern, Blankenstein and van der Schaft, [**?**] and [**?**]. We shall refer to the Hamiltonian versions of systems introduced and studied in these papers as *controlled Hamiltonian systems* or *CH systems* for short. The paper [?] develops a formulation of *passivity-based control* using *interconnection and damping assignment*, or IDA-PBC. See [**?**] and [**?**] for additional applications of the passivity-based control approach. The idea, as in the controlled Lagrangian approach, is that it applies to the problem of stabilizing *underactuated mechanical systems* using both *potential and kinetic shaping*. In [?], the authors also discuss structure modification, including modifications of the Poisson structure.

Purpose of this Paper. The main purpose of this paper is to show that *the CL system method is equivalent to the CH system method*; we shall focus on simple mechanical control systems (where the Lagrangian has the form of kinetic minus potential energy), although this is true more generally, as will be evident in the body of the paper. One of the interesting features is that we identify the Lagrangian analog of the failure of Jacobi's identity on the Hamiltonian side, namely in terms of gyroscopic forces, which is a natural Lagrangian concept. This then provides a framework on the Lagrangian as well as the Hamiltonian side for considering the problem of general structure modification.

In addition to showing this equivalence, in a companion paper we shall extend both the Lagrangian and Hamiltonian sides of this theory to include systems with symmetry and the relevant reduction theory (Lagrangian reduction, where one reduces variational principles and Hamiltonian reduction, where one reduces symplectic and Poisson structures). This extension is critical for examples like spacecraft control, underwater vehicle control, etc. In fact, this class of reduced systems was recognized early as an important one on both the Hamiltonian and Lagrangian sides; see [?] and [?].

Notation. We shall use fairly standard notation, as follows: the configuration manifold for the mechanical system under consideration will be denoted Q, the Lagrangian will be denoted $L : TQ \to \mathbb{R}$ and Hamiltonians will be denoted $H : T^*Q \to \mathbb{R}$. We will assume that Q is finite dimensional and denote the dimension by n. The second order tangent bundle is denoted $T^{(2)}Q$ and consists of second derivatives of curves in Q. See [**?**] and [**?**] for additional details. Coordinates on Q will be denoted $(q^1, \dots, q^n)$. Likewise, coordinates on TQ are denoted $(q^1, \dots, q^n, \dot{q}^1, \dots \dot{q}^n)$ and coordinates on T^*Q are denoted $(q^1, \dots, q^n, p_1, \dots, p_n)$.

Strategy. The procedure introduced in [?] (and used in the more recent works of [?], [**?**] and [?] as well), can be described in general terms, as follows. One starts with an underactuated n-degree of freedom mechanical control system of the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = u_i, \quad i = 1, \dots, k \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad i = k+1, \dots, n. \tag{2}$$

Thus, the degrees of freedom represented by $q^1, \dots, q^k$ are the actuated degrees of freedom and the remaining are unactuated. One then seeks a new *controlled* Lagrangian L_c and a special class of feedback controls $u_1, \dots, u_k$ such that the system (**??**)-(**??**) is equivalent to the Euler-Lagrange equations for L_c with control u_c of rank k:

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{q}} - \frac{\partial L_c}{\partial q} = u_c. \tag{3}$$

The requirement that these two systems be equivalent leads to the problem of *matching* for which many specific techniques, including potential and kinetic shaping were developed. Some important general matching techniques were developed by Auckly, Kapitanski and White in [**?**] and by Hamberg in [**?, ?**]; these were important inspirations for the approach in this paper.

The matching process usually has some free parameters which can be interpreted as control gains. The advantage of the form (**??**) is explained as follows: One chooses a set of appropriate parameters such that the energy E_c corresponding to the Lagrangian L_c has a minimum at the equilibrium of interest. The control u_c is chosen to be a conrol that is a dissipative, possibly together with a gyroscopic, force. One then uses the energy E_c as a Lyapunov function to which LaSalle's invariance principle can be applied to demonstrate stabilization. One often has a demonstrably large region of attraction.

In the present paper, we view this matching problem in a different way that turns out to be quite crucial. We consider the problem of determining the *equivalence* of two possibly underactuated and/or *externally forced* CL systems. Then the matching problem becomes one of determining when (**??**)-(**??**) and (**??**) are CL-equivalent. We shall also develop the notion of CH-equivalent systems and likewise use this to deal with the matching problem on the Hamiltonian side. While this may seem at this point to be a rather superficial reformulation of the problem, as we shall see, it is a powerful point of view that enables us to elegantly achieve our goals.

There is another important point here; in our previous papers (see [?, ?, ?]), we developed methods that make control design *systematic and constructive* for certain classes of mechanical systems, which avoided having to deal with solving PDE's for each new application. In this context, these papers also dealt with the important issues of *asymptotic stabilization* and gave an indication that the methods are also useful for *tracking*. These are obviously important points from the view of applications. However, fully exploiting and generalizing all of these results is not the purpose of the present paper; rather it is to develop the general notion of equivalence and to show that the CL method is equivalent to the CH method. We shall, however, see how the constructive methods mentioned can be derived in the general context of the present work.

Motivating Example. Consider a model of an inverted pendulum on a cart, an example we will return to in the next section (see Figure **??**). We use it now to motivate the discussion and illustrate some of the ideas.
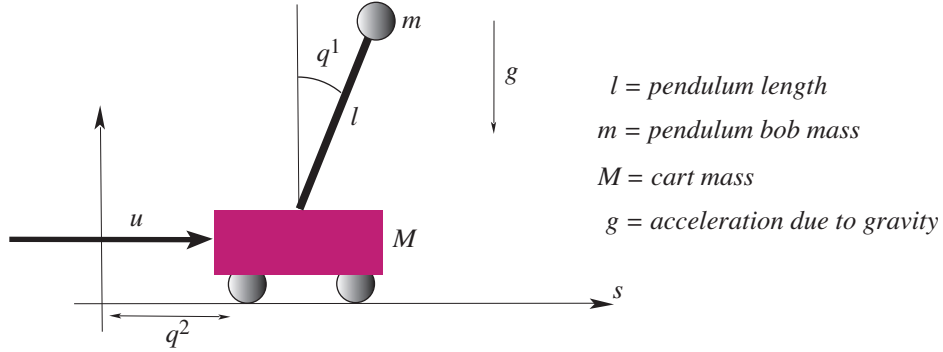
FIGURE 1. The pendulum on a cart.

The Lagrangian L of the system is given by

$$L(q^1, q^2, \dot{q}^2, \dot{q}^2) = \frac{1}{2}(\dot{q}^1)^2 + \cos(q^1)\dot{q}^1\dot{q}^2 + (\dot{q}^2)^2 - \cos(q^1), \tag{4}$$

where q^1 is the angle of the pendulum from the vertical position and q^2 is the displacement of the cart from the origin. Here, all the parameter values (i.e., those identified in Figure **??**) are taken to have unit value. The control u is exerted along the q^2 variable as shown in the figure. The equations of motion with control u is given by

$$\ddot{q}^1 + \cos(q^1)\ddot{q}^2 - \sin(q^1) = 0, \qquad \cos(q^1)\ddot{q}^1 + 2\ddot{q}^2 - \sin(q^1)(\dot{q}^1)^2 = u. \tag{5}$$

The goal is to design a feedback control law u such that the point $(q^1, q^2, \dot{q}^1, \dot{q}^2) = (0, 0, 0, 0)$ becomes an asymptotically stable equilibrium for the dynamics (**??**). The energy E of the Lagrangian L is given by

$$E(q^1, q^2, \dot{q}^1, \dot{q}^2) = \frac{1}{2}(\dot{q}^1)^2 + \cos(q^1)\dot{q}^1\dot{q}^2 + (\dot{q}^2)^2 + \cos(q^1).$$

One cannot directly use the energy E as a Lyapunov function because $(0, 0, 0, 0)$ is a saddle point of E.

The method of controlled Lagrangians leads one to the following (non-obvious and, at the moment, mysterious) form of control u:

$$u = \frac{199((\dot{q}^1)^2 - \cos(q^1))\sin(q^1) + (2 - \cos^2(q^1))\left(v - \frac{1}{500}\sin(q^1) - \frac{1}{100000}q^2\right)}{1 - 100\cos^2(q^1)} \tag{6}$$

where v is to be chosen. In (**??**), the somewhat strange numerical parameters were chosen to get a large region of attraction, which will be discussed later. Then, one can show either by a lengthy direct calculation, or by use of the theory, that the dynamics of the closed-loop system, (**??**) + (**??**), is the same as the dynamics of the Lagrangian

$$L_c = \frac{20000\cos^2(q^1) - 100}{2}(\dot{q}^1)^2 + 100\cos(q^1)\dot{q}^1\dot{q}^2 + \frac{1}{2}\dot{q}^2 + 100\cos(q^1) - \frac{1}{200000}(q^2 + 200\sin(q^1))^2 \tag{7}$$

with the control force

$$u_c = v \begin{bmatrix} 200\cos(q^1) \\ 1 \end{bmatrix} \tag{8}$$

where the parameter value $1/200000$ in the potential energy of L_c corresponds to a slow driving of the cart to the origin as compared to a rapid stabilization of the pendulum to the upright position. The energy E_c of the Lagrangian L_c is given by

$$E_c = \frac{20000\cos^2(q^1) - 100}{2}(\dot{q}^1)^2 + 100\cos(q^1)\dot{q}^1\dot{q}^2 + \frac{1}{2}\dot{q}^2 - 100\cos(q^1) + \frac{1}{200000}(q^2 + 200\sin(q^1))^2$$

and has a *minimum* at $(0, 0, 0, 0)$ in the set

$$S := \{(q^1, q^2, \dot{q}^1, \dot{q}^2) \mid 100\cos^2(q^1) > 1\} = \left(-0.9365 \times \frac{\pi}{2}, 0.9365 \times \frac{\pi}{2}\right) \times \mathbb{R}^3. \tag{9}$$

The minimum value of E_c is (-100), and the kinetic energy of E_c is positive definite in the set S. We are now in a position to use E_c as a Lyapunov function for the dynamics of the system (L_c, u_c), (or, equivalently, (L, u)). The time derivative of E_c is given by

$$\frac{dE_c}{dt} = \langle(\dot{q}^1, \dot{q}^2), u_c\rangle = v(200\dot{q}^1\cos(q^1) + \dot{q}^2).$$

We choose $v = -(200\dot{q}^1\cos(q^1) + \dot{q}^2)$ so that u_c becomes dissipative, i.e.,

$$\frac{dE_c}{dt} = -(200\dot{q}^1\cos(q^1) + \dot{q}^2)^2 \leq 0.$$

Hence, $(0, 0, 0, 0)$ becomes a Lyapunov stable equilibrium of the closed-loop system. By applying LaSalle's theorem (see, for example, [**?**]), one can also show asymptotic stability of $(0, 0, 0, 0)$ in a region of attraction, $R := E_c^{-1}([-100, -11]) \bigcap S$. Notice that R is a large region of attraction because such points as $(0, 4000, 0, 0)$ and $\left(0.908 \times \frac{\pi}{2}, 0, 0, 0\right)$ are contained in it. But it is still a conservative choice of a region of attraction and by adjusting the parameters one can find an even larger region of attraction.

Recap of the Discussion. First, we find a Lagrangian L_c and a control u_c so that the Lagrangian control system (L_c, u_c) is *equivalent* to the original system, (L, u). The energy E_c of L_c has a minimum at $(0, 0, 0, 0)$, whereas the energy E of the original L does not. Hence, we are able to use E_c for a Lyapunov function and, to achieve asymptotic stability of $(0, 0, 0, 0)$, we choose dissipation for u_c. A large region of attraction is achievable.

Clearly, this technique depends on a good theory for the equivalence of two Euler–Lagrange systems with control forces. In this paper we provide such a theory; in particular, we show how to find systems equivalent to a given system. We also go beyond this to our second main point, namely a proof of the equivalence of the CL point of view and the CH point of view.

2. Controlled Lagrangian Systems

To formally define a controlled Lagrangian system, we need to first recall a few things about the Euler-Lagrange equations from basic geometric mechanics (see, for example, [**?**]).

Euler-Lagrange Equations. The Euler-Lagrange operator $\mathcal{EL}$ assigns to a Lagrangian $L : TQ \to \mathbb{R}$, a bundle map $\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$ which may be written in local coordinates (and with the summation convention in force) as

$$\mathcal{EL}(L)_i(q, \dot{q}, \ddot{q})dq^i = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) - \frac{\partial L}{\partial q^i}(q, \dot{q})\right)dq^i$$

in which it is understood that one regards the first term on the right hand side as a function on the second order tangent bundle $T^{(2)}Q$ by formally applying the chain rule and then replacing everywhere dq/dt by $\dot{q}$ and

$d\dot{q}/dt$ by $\ddot{q}$. Hence the Euler-Lagrange equations of a Lagrangian L may be written as

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q}) = 0.$$

Controlled Lagrangian Systems. The view of a controlled Lagrangian system given by (**??**)-(**??**) will now be generalized to include external forces and also will be made intrinsic (independent of a specific coordinate representation). This definition is fundamental to the Lagrangian side of this work.

Definition 2.1. *A **controlled Lagrangian (CL) system** is a triple (L, F, W) where the function $L : TQ \to \mathbb{R}$ is the Lagrangian, the fiber-preserving map $F : TQ \to T^*Q$ is an external force and $W \subset T^*Q$ is a subbundle of T^*Q, called the **control bundle**, representing the **actuation directions**.*

Sometimes, we will identify the subbundle W with the set of bundle maps from TQ to W. The fact that W may be smaller than the whole space corresponds to the system being *underactuated*. The equations of motion of the system (L, F, W) may be written as

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q}) = F(q, \dot{q}) + u \tag{10}$$

with a control u selected from W. When we choose a specific control map $u : TQ \to W$ (so that u is a function of $(q^i, \dot{q}^i)$), then we call the triple (L, F, u) a **closed-loop Lagrangian system**. We will typically be interested in such feedback controls in this paper.

In the special case when W is integrable (that is, its annihilator $W^\circ \subset TQ$ is integrable in the usual Frobenius sense) and we choose coordinates appropriately, then a CL system, that is the equations (**??**), can be locally written in coordinates as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_i + u_i, \quad i = 1, \dots, k \tag{11}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_i, \qquad i = k+1, \dots, n. \tag{12}$$

Here the coordinates $q^1, \dots, q^k$ are chosen so that $dq^1, \dots dq^k$ span W, so W is k dimensional in this case. The external forces can include gyroscopic forces, friction forces, etc.

Aside: Interconnection Structures and Graphs. If one has a collection of CL systems (L^i, F^i, W^i), $i = 1, \dots, N$ (as occurs in the coordinated control of a collection of N satellites or N underwater vehicles, for example), then one can keep track of the **interconnection structure** by using a directed graph with the individual CL system representing the nodes of the graph and the interconnection structure representing the edges. If, for example, in this graph, there is a directed link from the ith subsystem (L^i, F^i, W^i) to the jth subsystem (L^j, F^j, W^j), then we would have a control link $u : TQ_i \to W_j$. We intend to pursue this point of view in future publications for purposes of coordinated control of many subsystems. For simplicity of the exposition in the present paper, we assume that we have lumped everything together into one system.

Simple CL Systems. Now we are ready to embark on the discussion of the matching problem in the general context of CL systems. We shall begin with the case of simple Lagrangians.

Definition 2.2. *A CL system (L, F, W) is called a **simple CL system** if the Lagrangian L has the form of kinetic minus potential energy:*

$$L(q, \dot{q}) = \frac{1}{2}m(q)(\dot{q}, \dot{q}) - V(q),$$

where m is a non-degenerate symmetric $(0, 2)$-tensor (the mass matrix, or Riemannian structure).

We will sometimes omit the q-dependence of m in the notation, but this dependence will be understood. When L is a simple Lagrangian, then the Euler-Lagrange operator is written in a matrix form as

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q}) = m\ddot{q} + (\mathbf{d}m[\dot{q}])\dot{q} - \frac{\partial L}{\partial q} \tag{13}$$

where $(\mathbf{d}m[\dot{q}])_{ij} = \mathbf{d}(m_{ij})(\dot{q})$, or in coordinates,

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q})_j = m_{ij}\ddot{q}^i + \frac{\partial m_{ij}}{\partial q^k}\dot{q}^i\dot{q}^k - \frac{1}{2}\frac{\partial m_{ik}}{\partial q^j}\dot{q}^i\dot{q}^k + \frac{\partial V}{\partial q^j}.$$

See, for example, [?] for how to relate these expressions to Christoffel symbols and geodesics. We make this assumption of a simple CL system only for the sake of simplicity and to make the exposition more concrete. One can readily generalize the results to more general forms of Lagrangians (see Remark 1 below).

Notation. As is standard, we represent the two tensor m in coordinates with indices down: m_{ij}. This corresponds to the fact that at each point $q \in Q$, m is a bilinear map $m_q : T_qQ \times T_qQ \to \mathbb{R}$. If m is non-degenerate, it defines an isomorphism $m_q^\flat : T_qQ \to T_q^*Q$ by $m_q^\flat(v_q) \cdot u_q = m_q(v_q, u_q)$. The matrix representation of m and $m^\flat$ are the same and correspond to simply writing the mass matrix as an $n \times n$ matrix. The inverse of the linear map $m_q^\flat$ is denoted $m_q^{-1} : T_q^*Q \to T_qQ$. Its matrix is denoted m^{ij}; it is the inverse of the standard mass matrix and satisfies $m_{ij}m^{jk} = \delta_i^k$. To simplify the notation, we will let m denote both the bilinear form on T_qQ and the associated linear map of T_qQ to T_q^*Q.

The Matching Conditions and CL-Equivalence. Consider now two *simple* Lagrangian systems (L_1, F_1, W_1) and (L_2, F_2, W_2) with

$$L_1(q, \dot{q}) = \frac{1}{2}m_1(\dot{q}, \dot{q}) - V_1(q) \quad \text{and} \quad L_2(q, \dot{q}) = \frac{1}{2}m_2(\dot{q}, \dot{q}) - V_2(q). \tag{14}$$

The $\ddot{q}$ equation of the closed-loop system $(L_\alpha, F_\alpha, u_\alpha)$ with $\alpha = 1, 2$ is given in coordinates as follows.

$$\ddot{q}_{L_\alpha} = m_\alpha^{-1}\left[-(\mathbf{d}m_\alpha[\dot{q}])\dot{q} + \frac{\partial L_\alpha}{\partial q} + F_\alpha + u_\alpha\right]. \tag{15}$$

Here we use matrix and vector-style notation, suppressing the indices on q^i to avoid cluttering the equation with too many indices. We can then formally define matching conditions between these two systems.

Definition 2.3. *Given the two CL systems* (L_1, F_1, W_1) *and* (L_2, F_2, W_2), *the **Euler-Lagrange matching conditions** are*

    ELM-1 : $W_1 = m_1 m_2^{-1}(W_2)$,
    ELM-2 : $\text{Im}\left[(\mathcal{EL}(L_1) - F_1) - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2)\right] \subset W_1$,
where Im *means the pointwise image of the map in brackets.*

    We say that the two simple CL Lagrangian systems (L_1, F_1, W_1) *and* (L_2, F_2, W_2) *are* **CL-equivalent** *if* **ELM-1** *and* **ELM-2** *hold. We use the symbol,* $\overset{L}{\sim}$ *for this equivalence relation.*

One can show that CL-equivalence is indeed an equivalence relation. The reflexivity and the transitivity are obvious. The symmetry follows if we multiply both sides of **ELM-1** and **ELM-2** by $m_2 m_1^{-1}$.

In coordinates and in terms of concrete mass matrices, the condition **ELM-1** means that the matrix product $m_1 m_2^{-1}$ takes the subspace associated to the actuation directions of the second system to the subspace associated to the actuation directions of the first system; this may of course be viewed as a linear algebra condition. The condition **ELM-2** means that the given combination of Euler-Lagrange expressions and forces lies in the actuation direction of the first system. If (L_1, F_1, W_1) and (L_2, F_2, W_2) are equivalent, then for any choice of

control $u_2 : TQ \to W_2$, there is a control $u_1 : TQ \to W_1$ such that such that the two closed-loop systems (L_1, F_1, u_1) and (L_2, F_2, u_2) produce the same equations of motion, and vice versa. The relation between such u_1 and u_2 is given by (??).

One can easily check by coordinate computation that the map

$$\left[(\mathcal{EL}(L_1) - F_1) - m_1 m_2^{-1} (\mathcal{EL}(L_2) - F_2) \right] \tag{16}$$

in **ELM-2** can be regarded as a map defined on TQ because the acceleration terms from the two Euler-Lagrange expressions cancel each other.

Equations of Motion. Now we can give the main result on when two CL systems give the same equations of motion.

Proposition 2.4. *Suppose two simple controlled Lagrangian systems (L_1, F_1, W_1) and (L_2, F_2, W_2) with Lagrangians in (??) are CL-equivalent. Then the two closed-loop systems (L_1, F_1, u_1) and (L_2, F_2, u_2) produce the same equations of motion if and only if the two feedback controls $u_1 : TQ \to W_1$ and $u_2 : TQ \to W_2$ satisfy*

$$u_1 = (\mathcal{EL}(L_1) - F_1) - m_1 m_2^{-1} (\mathcal{EL}(L_2) - F_2) + m_1 m_2^{-1} u_2. \tag{17}$$

Proof. By (??) and (??) one can compute

$$m_1 (\ddot{q}_{L_1} - \ddot{q}_{L_2}) = u_1 - m_1 m_2^{-1} u_2 - \left[(\mathcal{EL}(L_1) - F_1) - m_1 m_2^{-1} (\mathcal{EL}(L_2) - F_2) \right]$$

The conditions **ELM-1** and **ELM-2** imply that (??) holds if and only if $\ddot{q}_1 = \ddot{q}_2$. Also recall that the map in (??) can be regarded as a map defined on TQ. $\qquad\square$

Coordinate Expressions. We express the Euler-Lagrange matching conditions in coordinates. Let (L_1, F_1, W_1) and (L_2, F_2, W_2) be two simple CL systems with Lagrangians as in (??). Suppose that we are given a decomposition of the forces F_i as

$$F_i = F_i^v + F_i^q$$

with $i = 1, 2$ where F_i^q is independent of the velocity $\dot{q}$. Then **ELM-2** can be written in coordinates as

$$W_1^\circ \left[(\mathbf{d} m_1 [\dot{q}]) \dot{q} - \frac{\partial}{\partial q} \left(\frac{1}{2} \dot{q}^T m_1 \dot{q} \right) - F_1^v (q, \dot{q}) \right.$$
$$\left. - m_1 m_2^{-1} \left((\mathbf{d} m_2 [\dot{q}]) \dot{q} - \frac{\partial}{\partial q} \left(\frac{1}{2} \dot{q}^T m_2 \dot{q} \right) - F_2^v (q, \dot{q}) \right) \right] = 0 \tag{18}$$

and

$$W_1^\circ \left[\frac{\partial V_1}{\partial q} - F_1^q - m_1 m_2^{-1} \left(\frac{\partial V_2}{\partial q} - F_2^q \right) \right] = 0 \tag{19}$$

where W_1° is a matrix whose rows span the annihilator of W_1, given by

$$\{ v \in TQ \mid \langle v, \alpha \rangle = 0 \text{ for all } \alpha \in W_1 \}.$$

CL-Inclusion. There is a more general concept than the CL-equivalence relation. We will give this definition for simple Lagrangian systems, but it can be readily generalized for general Lagrangian systems (see Remark 1 below).

Definition 2.5. *We say that a simple CL system (L_1, F_1, W_1) **includes** the simple CL system (L_2, F_2, W_2) if the following holds:*

ELI-1 : $W_1 \supset m_1 m_2^{-1}(W_2)$,
ELI-2 : $\mathrm{Im}\left[(\mathcal{EL}(L_1) - F_1) - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2)\right] \subset W_1$.

If (L_1, F_1, W_1) includes (L_2, F_2, W_2), then for any choice of control $u_2 : TQ \to W_2$, there is a control $u_1 : TQ \to W_1$ satisfying (??) such that the two closed-loop systems (L_1, F_1, u_1) and (L_2, F_2, u_2) produce the same equations of motion. This introduces a partial order in the class of simple controlled Lagrangian systems. We call this partial order **CL-inclusion**. One can check that *two simple CL systems include each other if and only if they are CL-equivalent.*

Remarks. 1. In this paper we always assume that Lagrangians are regular, i.e., $\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0$. For simple CL systems this means of course that the mass matrix is nonsingular. We can generalize the Euler-Lagrange matching conditions for general Lagrangians which are not necessarily simple as follows. Let $L : TQ \to \mathbb{R}$ be a Lagrangian. It induces a globally well–defined map $m_L : TQ \to \mathrm{Sym}_2(T^*Q)$ (symmetric two-tensors with indices down) given in tangent bundle charts as follows:

$$m_L(q, \dot{q}) = \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^i \partial \dot{q}^j} dq^i \otimes dq^j.$$

Then the Euler-Lagrange matching conditions for general Lagrangian systems (L_1, F_1, W_1) and (L_2, F_2, W_2) are given by

ELM-1′ : $W_1 = m_{L_1} m_{L_2}^{-1}(W_2)$,
ELM-2′ : $\mathrm{Im}\left[(\mathcal{EL}(L_1) - F_1) - m_{L_1} m_{L_2}^{-1}(\mathcal{EL}(L_2) - F_2)\right] \subset W_1$.

The equation in (??) is replaced by

$$u_1 = (\mathcal{EL}(L_1) - F_1) - m_{L_1} m_{L_2}^{-1}(\mathcal{EL}(L_2) - F_2) - m_{L_1} m_{L_2}^{-1} u_2$$

so that the two closed-loop systems (L_1, F_1, u_1) and (L_2, F_2, u_2) produce the same equations of motion.

2. One can approach the Euler-Lagrange equations using the Lagrangian two-form (see §7.3 of [?] for details). However, in this paper, we do not take this approach but rather take the variational approach since the variational approach is more natural for many purposes, including Lagrangian reduction; we will use this variational approach to discuss reduced CL and CH systems in a companion paper (see [?] for an overview of the theory of Lagrangian reduction). Since §7.3 of [?] does not consider external forces, we give a formula for the Lagrangian vector field X_E for a given Lagrangian system (L, F, W) with the energy function E defined by

$$E(q) = \langle \mathbb{F}L(q, \dot{q}), (q, \dot{q})\rangle - L(q, \dot{q}). \tag{20}$$

We recall that the Legendre transformation $\mathbb{F}L$ is defined as follows: for $v_q, w_q \in T_qQ$

$$\langle \mathbb{F}L(v_q), w_q\rangle = \left.\frac{d}{ds}\right|_{s=0} L(v_q + sw_q). \tag{21}$$

In coordinates,

$$(q^i, p_i) = \mathbb{F}L(q, \dot{q}) = \left(q^i, \frac{\partial L}{\partial \dot{q}^i}\right).$$

If $L(q, \dot{q}) = \frac{1}{2}m(\dot{q}, \dot{q}) - V(q)$ is simple, then

$$\mathbb{F}L = m : TQ \to T^*Q$$

is a linear isomorphism on each fiber.

The Lagrangian vector field X_E on TQ of the system (L, F, W) and with with controls $u : TQ \to W$ is verified (using computations similar to those in, for example [?]), to be given by

$$X_E = \Omega_L^\sharp \mathbf{d}E + T\mathbb{F}L^{-1}(\text{vlift}_m(F)) + T\mathbb{F}L^{-1}(\text{vlift}_m(u)).$$

The notation in this equation is as follows: $T\mathbb{F}L^{-1} : T(T^*Q) \to T(TQ)$ is the tangent map of the inverse Legendre transformation $\mathbb{F}L^{-1} : T^*Q \to TQ$, and Ω_L is the Lagrangian two form on TQ defined by $\Omega_L = \mathbb{F}L^*\Omega_{\text{can}}$ with Ω_{can} the canonical 2-form on T^*Q. Also, the *vertical lift* of a vector $w_q \in T_qQ$ along the vector $v_q \in T_qQ$ is the vector $\text{vlift}_{v_q}(w_q) \in T_{v_q}(TQ)$ defined by

$$\text{vlift}_{v_q}(w_q) = \left.\frac{d}{ds}\right|_{s=0} (v_q + sw_q).$$

In coordinates $(q^i, v^i, \delta q^i, \delta v^i)$ on TTQ, the lift of the vector with components w^i and with base point q^i along the vector v^i, also with base point q^i, is the vector with base point (q^i, v^i) with components given by $(0, w^i)$. Likewise, the vertical lift of a one form $\beta_q \in T_q^*Q$ along a one form $\alpha_q \in T_q^*Q$ is the vector $\text{vlift}_{\alpha_q}(\beta_q) \in T_{\alpha_q}(T^*Q)$ defined by

$$\text{vlift}_{\alpha_q}(\beta_q) = \left.\frac{d}{ds}\right|_{s=0} (\alpha_q + s\beta_q).$$

In coordinates $(q^i, \alpha_i, \delta q^i, \delta\alpha_i)$ on TT^*Q, the lift of the one form with components β_i and with base point q^i along the one form α_i, also with base point q^i, is the vector with base point (q^i, α_i) with components given by $(0, \beta_i)$. For a force map $F : TQ \to T^*Q$, the vertical lift $\text{vlift}_m(F)$ along $m = \mathbb{F}L$ is defined pointwise at $v_q \in T_qQ$ to be $\text{vlift}_m(F)(v_q) = \text{vlift}_{m(v_q)} F(v_q)$. The notation $\text{vlift}_m u$ is defined in the same way. See [?] for additional properties of Lagrangian vector fields and [**?**] for more on the forced case.

3. For a given physical system, the force F in the triple (L, F, W) is an external force. For the purpose of *controller design*, the force F can also be used as a design parameter. As an example of what this means, we consider a gyroscopic force F_{gr}. By definition, gyroscopic forces preserve the energy E of the system. Hence, the following should be satisfied:

$$\frac{d}{dt}E = \langle \dot{q}, F_{\text{gr}} \rangle = 0.$$

Specifically, this holds if F_{gr} can be written in the following form:

$$F_{\text{gr}}(q, \dot{q}) = S(q, \dot{q})\dot{q}$$

where $S(q, \dot{q})$ is a linear map of T_qQ to T^*Q that satisfies $S(q, \dot{q})^T = -S(q, \dot{q})$. In the special case that $S = (S_{ij})$ depends only on q and satisfies the following integrability condition,

$$\frac{\partial S_{ik}(q)}{\partial q^l} + \frac{\partial S_{kl}(q)}{\partial q^i} + \frac{\partial S_{li}(q)}{\partial q^k} = 0,$$

then there are functions $I_i(q)$ satisfying

$$S_{ik}(q) = \frac{\partial I_i(q)}{\partial q^k} - \frac{\partial I_k(q)}{\partial q^i}$$

such that we can incorporate the gyroscopic force $F(q, \dot{q}) = S(q, \dot{q})\dot{q}$ into the Lagrangian as follows: $(L, F_{\mathrm{gr}} + F, W) \overset{L}{\sim} (L + I_i(q)\dot{q}^i, F, W)$. However, one does not have to restrict to such special S's because the skew-symmetry of S is the only property we need for the gyroscopic force from the energy-conservation point of view. This is essentially the same as using almost Poisson structures by not enforcing the Jacobi-identity condition, which will be discussed in the next section on controlled Hamiltonian systems.

4. Notice that any CL system (L, F, W) is CL-equivalent to $(-L, -F, W)$. The equation (**??**) becomes $u_1 = -u_2$. If an equilibrium is a minimum point of the energy of (L, F, W), then it will be a maximum point of the energy function of $(-L, -F, W)$. Therefore, it is not crucial whether the energy function has a maximum or a minimum at the equilibrium as far as whether or not we can stabilize the equilibrium.

Control Synthesis via CL system. We now discuss how one may apply the concept of CL-equivalence to designing control laws for mechanical system control problems.

The general idea is as follows. Suppose we are given a CL system (L_1, F_1, W_1) and a given set of desired performance requirements. In general, it is difficult to directly find a control law to meet these performance requirements. Instead, the strategy is that we first find a CL-equivalent (or CL-included) system (L_2, F_2, W_2) for which one can more easily find a control $u_2 \in W_2$ such that the performance requirements are met for the closed-loop Lagrangian system (L_2, F_2, u_2). The control u_1 can then be calculated according to (**??**).

For example, the problem of asymptotic stabilization of a desired equilibrium of a given system (L_1, F_1, W_1) can be stated as follows. Given the CL system (L_1, F_1, W_1), find a CL-equivalent (or CL system-included) system (L_2, F_2, W_2) for which one can find a control $u_2 \in W_2$ such that the equilibrium is asymptotically stable for the closed-loop Lagrangian system (L_2, F_2, u_2). Then, by Proposition **??**, the equilibrium is also asymptotically stable in the closed-loop system (L_1, F_1, u_1) with u_1 given by (**??**).

We might also want to ask for a parameterized family of equivalent systems (L_2, F_2, W_2) satisfying the above so that there is room for tuning, increasing region of attraction, or otherwise improving performance.

The Inverted Pendulum on a Cart. We now apply the theory to the example of an inverted pendulum on a cart and re-derive the result in §VI of [?] (here, we assume the inclination angle of the slope is zero). The purpose is not to extend in any way these previous results, but rather to use this example in a pedagogical way to illustrate the ideas that have been presented so far in the context of a known example; see Figure **??**.

To conform to the notation in the present paper, we shall use $q = (q^1, q^2)$ in place of (θ, s) in [?]. Let $(L_1, F_1 = 0, W_1)$ be the inverted pendulum system with the Lagrangian

$$L_1(q, \dot{q}) = \frac{1}{2}\alpha(\dot{q}^1)^2 + \beta\cos(q^1)\dot{q}^1\dot{q}^2 + \frac{1}{2}\gamma(\dot{q}^2)^2 - D\cos(q^1)$$

where α, β, γ, D are positive real numbers (given in terms of the notation of the figure by $\alpha = ml^2, \beta = ml, \gamma = M + m$ and $D = mgl$). Here, W_1 is spanned by $\mathbf{d}q^2$; that is, the matrix $[0 \ 1]^T$. Let (L_2, F_2, W_2) be a candidate CL system equivalent to $(L_1, 0, W_1)$, where the Lagrangian is

$$L_2(q, \dot{q}) = \frac{1}{2}m_2(\dot{q}, \dot{q}) - V_2(q)$$

with

$$m_2(q) = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}$$

and the gyroscopic force

$$F_2(q, \dot{q}) = \begin{bmatrix} 0 & -f_1(q)\dot{q}^1 - f_2(q)\dot{q}^2 \\ f_1(q)\dot{q}^1 + f_2(q)\dot{q}^2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}^1 \\ \dot{q}^2 \end{bmatrix}.$$

It is reasonable to make the gyroscopic force F_2 be quadratic in $\dot{q}$ because every term containing $\dot{q}$ in $\mathcal{EL}(L_2)$ is quadratic in $\dot{q}$. Notice $W_1^\circ = [1\ 0]$. The equation (**??**) becomes the following:

$$0 = A(q)\frac{\partial m_{11}}{\partial q^1} + B(q)\left(2\frac{\partial m_{12}}{\partial q^1} - \frac{\partial m_{11}}{\partial q^2} - 2f_1\right) \tag{22}$$

$$0 = A(q)\left(\frac{\partial m_{11}}{\partial q^2} + f_1\right) + B(q)\left(\frac{\partial m_{22}}{\partial q^1} - f_2\right) \tag{23}$$

$$0 = A(q)\left(2\frac{\partial m_{12}}{\partial q^2} - \frac{\partial m_{22}}{\partial q^1} + 2f_2\right) + B(q)\frac{\partial m_{22}}{\partial q^2} \tag{24}$$

with $A = (\alpha m_{22} - \beta\cos(q^1)m_{12})$ and $B = (\beta\cos(q^1)m_{11} - \alpha m_{12})$. We can regard f_1 and f_2 as free parameters. In order to reproduce the results of [?], we set $f_1 = f_2 = 0$. However, in such an example as the ball-and-beam system, the usage of this gyroscopic force is crucial (see [?] with the main result of the current paper in mind). One can try to directly solve PDE's in (**??**)–(**??**) as in [?]. However, it is sometimes easier to make some assumptions to reduce the PDE's to a set of ODE's as follows. We assume that m_{11} depends on q^1 only and m_{12} and m_{22} are of the following form just as in the original system:

$$m_{12}(q^1) = b\cos(q^1); \qquad m_{22} = d,$$

with $b, d \in \mathbb{R}$. Then, (**??**) and (**??**) are automatically satisfied and (**??**) becomes

$$(\alpha d - \beta b\cos^2(q^1))m'_{11}(q^1) - (\beta m_{11}(q^1) - \alpha b)2b\cos(q^1)\sin(q^1) = 0$$

which can be solved for $m_{11}(q^1)$ as follows:

$$m_{11}(q^1) = \alpha(ad + b/\beta) - ab\beta\cos^2(q^1)$$

with $a \in \mathbb{R}$. With this solution, the equation (**??**) can be written as

$$\frac{-D(b + ad\beta)}{\beta}\sin(q^1) = \frac{\partial V_2}{\partial q^1} + a\beta\cos(q^1)\frac{\partial V_2}{\partial q^2}$$

which can be solved for V_2 as follows:

$$V_2(q^1, q^2) = D(ad + b/\beta)\cos(q^1) + V_\epsilon(q^2 - a\beta\sin(q^1)),$$

with V_ϵ an arbitrary function. The control subbundle W_2 is computed as follows:

$$W_2 = \text{span}\{(\alpha d - \beta b\cos^2(q^1))(\mathbf{d}q^2 - a\beta\cos(q^1)\mathbf{d}q^1)\}.$$

For simplicity, let us choose a quadratic function for V_ϵ such that V_2 becomes

$$V_2(q^1, q^2) = D(ad + b/\beta)\cos(q^1) + \frac{1}{2}\epsilon(q^2 - a\beta\sin(q^1))^2.$$

The total energy of the CL system (L_2, F_2, W_2) is given by

$$E_2 = \frac{1}{2}(\alpha(ad + b/\beta) - ab\beta\cos^2(q^1))(\dot{q}^1)^2 + b\cos(q^1)\dot{q}^1\dot{q}^2 + \frac{1}{2}d(\dot{q}^2)^2$$
$$+ D(ad + b/\beta)\cos(q^1) + \frac{1}{2}\epsilon(q^2 - a\beta\sin(q^1))^2.$$

One can check that the energy E_2 has a minimum at $(0,0,0,0)$ in the set

$$R := \{(q^1, q^2, \dot{q}^1, \dot{q}^2) \mid \alpha d < \beta b \cos^2(q^1)\} \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^3 \tag{25}$$

if the following holds:

$$d > 0, \quad \epsilon > 0, \quad a < 0, \quad \alpha d/\beta < b < -ad\beta.$$

Notice that the condition $\alpha d/\beta < b$ guarantees the constant nonzero rank of W_2. One can achieve asymptotic stabilization of the equilibrium by choosing a dissipative input $u_2 \in W_2$ as follows:

$$u_2 = -c(\mathbf{d}q^2 - a\beta \cos(q^1)\mathbf{d}q^1) \otimes (\mathbf{d}q^2 - a\beta \cos(q^1)\mathbf{d}q^1)$$

with $c > 0$. In coordinates,

$$u_2(q^1, q^2, \dot{q}^1, \dot{q}^2) = -c(\dot{q}^2 - a\beta \cos(q^1)\dot{q}^1) \begin{bmatrix} -a\beta \cos(q^1) \\ 1 \end{bmatrix}$$

The asymptotic stabilization can be straightforwardly proved by the application of LaSalle's theorem (see, for example [?]). By choosing a, b, ϵ with $|a|, |b| \gg 1, |\epsilon| \ll 1$, one can get a large region of attraction.[1] Notice that we can make the set R in (??) as close to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^3$ as possible, with a large value of b.

By Proposition ??, the closed-loop system $(L_1, 0, u_1)$ with u_1 obeying (??) has the origin as an asymptotically stable equilibrium in a large region of attraction. Explicitly, u_1 is given by

$$u_1 = \frac{(b\gamma - d\beta)(\alpha(\dot{q}^1)^2 - D\cos(q^1))\sin(q^1) + (\alpha\gamma - \beta^2\cos^2(q^1))(c(a\beta\cos(q^1)\dot{q}^1 - \dot{q}^2) + \epsilon(a\beta\sin(q^1) - q^2))}{d\alpha - b\beta\cos^2(q^1)} \mathbf{d}q^2.$$

This control law u_1 is essentially the same as the control law (48) in [?], with $\psi = 0$.

Simplified Matching. We review some of the results in [?] in the framework of the current paper. Let $Q = S \times G$ be the configuration space where G is an Abelian Lie group, and let $q = (x, \theta) = (x^\alpha, \theta^a) \in S \times G$ be a chosen set of coordinates.

Let (x_e, θ_e) be the equilibrium of interest. Since we will look at the dynamics in a neighborhood of an equilibrium, one may assume that S and G are just Euclidean spaces (a neighborhood doesn't necessarily mean that it is small in size).

Consider a given CL system of the form $(L, 0, W)$ (so there are no external forces), where L has the form

$$L = \frac{1}{2}g_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta + g_{\alpha a}(x)\dot{x}^\alpha\dot{\theta}^a + \frac{1}{2}g_{ab}\dot{\theta}^a\theta^b - V(x)$$

where we assume that $g_{\alpha\beta}, g_{\alpha a}, V$ depend on x only and g_{ab} is constant. Assume also that the control subbundle is such that the control forces are in the group direction; that is, W is given by $W = 0 \oplus T^*G \subset T^*Q$, i.e.,

$$W(q) = \text{ the subspace of } T_q^*Q \text{ spanned by the columns of } \begin{bmatrix} O \\ I_G \end{bmatrix}$$

where I_G is the identity matrix of dimension $\dim G$.

---

[1] In the motivating example in §??, we chose $a = -200$, $b = 100$, $\epsilon = 10^{-5}$, $c = 1$.

Motivated by the *constructive approach* for this class of systems given in [?] and references therein, we consider a second CL system $(L_{\tau,\sigma,\rho,\epsilon}, 0, W_{\tau,\sigma,\rho,\epsilon})$ given by

$$L_{\tau,\sigma,\rho,\epsilon} = \frac{1}{2}\left(g_{\alpha\beta} + \rho\left(1 - \frac{1}{\sigma}\right)\left(1 - \frac{1}{\sigma} - \frac{1}{\rho}\right)g_{\alpha a}g^{ab}g_{b\beta}\right)\dot{x}^\alpha \dot{x}^\beta$$
$$+ \rho\left(1 - \frac{1}{\sigma}\right)g_{\alpha a}\dot{x}^\alpha\dot{\theta}^a + \frac{1}{2}\rho g_{ab}\dot{\theta}^a\dot{\theta}^b - V(x^\alpha) - V_\epsilon(x^\alpha, \theta^a),$$

and

$$W_{\tau,\sigma,\rho,\epsilon} = \left\langle \mathbf{d}\theta^a + \left(1 - \frac{1}{\sigma} - \frac{1}{\rho}\right)g^{ac}g_{c\alpha}\mathbf{d}x^\alpha \mid a = 1, \ldots, \dim G \right\rangle$$
$$= \text{ the subbundle spanned by the columns of } \begin{bmatrix} \left(1 - \frac{1}{\sigma} - \frac{1}{\rho}\right)g_{\alpha d}g^{da} \\ I_G \end{bmatrix} \qquad (26)$$

where $\sigma, \rho \in \mathbb{R}$ are free parameters.

To follow the notation in §??, let $(L_1, 0, W_1) = (L_{\tau,\sigma,\rho,\epsilon}, 0, W_{\tau,\sigma,\rho,\epsilon})$ and $(L_2, 0, W_2) = (L, 0, W)$. First we examine the condition **ELM-1**. One computes

$$m_2 m_1^{-1}\begin{bmatrix} O \\ I_G \end{bmatrix} = \begin{bmatrix} \left(1 - \frac{1}{\sigma} - \frac{1}{\rho}\right)g_{\alpha d}g^{da} \\ I_G \end{bmatrix}\left[g_{ab} - \left(1 - \frac{1}{\sigma}\right)g_{a\alpha}g^{\alpha\beta}g_{\beta b}\right]\left[\rho B^{be}\right] \qquad (27)$$

where B^{ab} is the inverse matrix of $B_{ab} := g_{ab} - g_{a\alpha}g^{\alpha\beta}g_{\beta b}$. For **ELM-1** to hold, the following should hold: $\rho \neq 0$ and

$$\left[g_{ab} - \left(1 - \frac{1}{\sigma}\right)g_{a\alpha}g^{\alpha\beta}g_{\beta b}\right] \text{ is invertible .}$$

It is an easy linear algebra exercise to show that the matrix $[g_{ab} - hg_{a\alpha}g^{\alpha\beta}g_{\beta b}]$ is invertible for $h \in \mathbb{R}$ if and only if the matrix $[g_{\alpha\beta} - hg_{\alpha a}g^{ab}g_{b\beta}]$ is invertible. In [?], we chose σ and ρ satisfying

$$1 - \frac{1}{\sigma} > \max\left\{\lambda \left|\det[g_{\alpha\beta} - \lambda g_{\alpha a}g^{ab}g_{b\beta}]\right|_{x=x_e} = 0\right\}$$

and $\rho < 0$ in order to achieve stability of the closed-loop system under the assumption that the matrix $[g_{a\alpha}(x_e)]$ is one to one. With this choice of σ and ρ, one can see from (??) that $m_2 m_1^{-1}(W_1) = W_2$, i.e., **ELM-1** holds. Notice that the annihilator W_1° of W_1 is spanned by the row vectors of the matrix

$$K := \begin{bmatrix} I_G, & -\left(1 - \frac{1}{\sigma} - \frac{1}{\rho}\right)g_{\alpha d}g^{da} \end{bmatrix}.$$

In this case **ELM-2** can be written as

$$K\left[\mathcal{EL}(L_1) - m_1 m_2^{-1}\mathcal{EL}(L_2)\right] = O,$$

which holds if and only if

$$-\left(\frac{\partial V}{\partial \theta^a} + \frac{\partial V_\epsilon}{\partial \theta^a}\right)\left(1 - \frac{1}{\sigma} - \frac{1}{\rho}\right)g^{ad}g_{d\alpha} + \frac{\partial V_\epsilon}{\partial x^\alpha} = 0$$

which is equation (16) in [?].

Letting $u_1 = u^{\text{diss}}$ times the basis vector given in (**??**), equation (**??**) can then be solved for u_2, which is the control for the originally given system $(L, 0, W)$. The expression for u_2 derived from (**??**) is the same as equation (22) in [?]. In the same paper, the energy function of the CL system $(L_{\tau,\sigma,\rho,\epsilon}, 0, W_{\tau,\sigma,\rho,\epsilon})$ has a maximum at the equilibrium point and in that paper there is a detailed computation showing the asymptotic stability of the equilibrium in the closed-loop system. Also, in [?] use is made of the integrability of W_2 to choose another set of local coordinates convenient for an asymptotic stability analysis.

3. Controlled Hamiltonian Systems

In this section we formally define a controlled Hamiltonian (CH) system analogously to a controlled Lagrangian system. First we discuss almost Poisson structures.

Almost Poisson Structure. Following [?], we define an **almost Poisson tensor** B on T^*Q to be a skew-symmetric $(2, 0)$-tensor on T^*Q. Its **almost Poisson bracket** $\{\,,\,\} : \mathcal{F}(T^*Q) \times \mathcal{F}(T^*Q) \to \mathcal{F}(T^*Q)$ (where $\mathcal{F}(M)$ is the space of smooth real-valued functions on the manifold M) is defined as

$$\{F, G\} = B(\mathbf{d}F, \mathbf{d}G)$$

for $F, G \in \mathcal{F}(T^*Q)$. Then $\{\,,\,\}$ satisfies the following properties:

(i) $\{F, G\} = -\{G, F\}$,
(ii) $\{F + G, H\} = \{F, H\} + \{G, H\}$,
(iii) $\{FG, H\} = F\{G, H\} + G\{F, H\}$

for $F, G, H \in \mathcal{F}(T^*Q)$. It is not necessary that it satisfy the Jacobi identity:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$$

for $F, G, H \in \mathcal{F}(T^*Q)$. In coordinates, the almost Poisson tensor B can be written in terms of its action on the coordinate functions:

$$B(q, p) = \begin{bmatrix} \{q^i, q^j\} & \{q^i, p_j\} \\ \{p_i, q^j\} & \{p_i, p_j\} \end{bmatrix}.$$

The induced map $B^\sharp : T^*T^*Q \to TT^*Q$ is defined as

$$B(\alpha_z, \beta_z) = \langle \alpha_z, B^\sharp(z)\beta_z \rangle$$

for $\alpha_z, \beta_z \in T_z^*T^*Q$.

It is well-known that almost Poisson structures arise in nonholonomic mechanics; see, eg, [?,?] and references therein. The geometric way of expressing the Jacobi identity of the $\{\,,\,\}$ in terms of B is to use the Schouten bracket (see Chapter 10 of [?] for more details).

Controlled Hamiltonian Systems. The Hamiltonian analog of a CL system is defined as follows.

Definition 3.1. *A **controlled Hamiltonian system (CH system)** is a quadruple (H, B, F, W) where $H : T^*Q \to \mathbb{R}$ is the Hamiltonian function, B is an almost Poisson tensor, $F : T^*Q \to T^*Q$ is the (external force) fiber-preserving map, and $W \subset T^*Q$ is the (control) subbundle.*

Sometimes, W denotes the set of bundle maps from T^*Q to W. As on the Lagrangian side, when we choose a specific control $u : T^*Q \to W$, then we call the quadruple (H, B, F, u) a **closed-loop Hamiltonian system**. The vector field $X_{(H,B,F,u)}$ of the closed-loop system (H, B, F, u) is given by

$$X_{(H,B,F,u)} = B^\sharp \mathbf{d}H + \text{vlift}(F) + \text{vlift}(u)$$

with a control $u : T^*Q \to W$, where the vertical lift operator vlift $: T^*Q \times T^*Q \to TT^*Q$ is defined as follows: for $\alpha_q, \beta_q \in T_q^*Q$

$$T_{\alpha_q} T^*Q \ni \mathrm{vlift}(\beta_q, \alpha_q) = \left. \frac{d}{ds} \right|_{s=0} (\alpha_q + s\beta_q)$$

and vlift$(F) : T^*Q \to TT^*Q$ is defined by

$$\mathrm{vlift}(F)(\alpha_q) = \mathrm{vlift}(F(\alpha_q), \alpha_q)$$

for $\alpha_q \in T_q^*Q$ and vlift(u) is defined in a similar manner. For a subbundle W of T^*Q, define

$$\mathrm{vlift}(W) = \bigcup_{q \in Q} \{ \mathrm{vlift}(\beta_q, \alpha_q) \mid \alpha_q \in T_qQ, \ \beta_q \in W_q \}.$$

See also Remark 2 in §2.

Hamiltonian Matching. Suppose we have two controlled Hamiltonian systems (H_i, B_i, F_i, W_i), $i = 1, 2$.

Definition 3.2. *We say that these systems satisfy the **Hamiltonian matching conditions** if*

    HM-1 : $W_1 = W_2$,
    HM-2 : $\mathrm{Im}[(B_1^\sharp \mathbf{d}H_1 + \mathrm{vlift}(F_1)) - (B_2^\sharp \mathbf{d}H_2 + \mathrm{vlift}(F_2))] \subset \mathrm{vlift}(W_1)$.

*In addition we say that two Hamiltonian systems are **CH-equivalent** if **HM-1** and **HM-2** hold for the systems. We use the symbol, $\overset{H}{\sim}$ for this equivalence relation.*

Proposition 3.3. *Suppose that the two CH systems (H_1, B_1, F_1, W_1) and (H_2, B_2, F_2, W_2) are CH-equivalent. Then the two closed-loop systems (H_1, B_1, F_1, u_1) and (H_2, B_2, F_2, u_2) produce the same vector fields if and only if the controls u_1 and u_2 satisfy*

$$\mathrm{vlift}(u_2) = \mathrm{vlift}(u_1) + (B_1^\sharp \mathbf{d}H_1 + \mathrm{vlift}(F_1)) - (B_2^\sharp \mathbf{d}H_2 + \mathrm{vlift}(F_2)) \tag{28}$$

Proof. Just consider the following equation:

$$X_{(H_1, B_1, F_1, u_1)} = X_{(H_2, B_2, F_2, u_2)}.$$

$\square$

One can also define a partial order, CH-inclusion in the class of CH systems on T^*Q as follows.

Definition 3.4. *A CH system (H_1, B_1, F_1, W_1) is said to **include** another CH system (H_2, B_2, F_2, W_2) if the following holds:*

    HI-1 : $W_1 \supset W_2$,
    HI-2 : $\mathrm{Im}[(B_1^\sharp \mathbf{d}H_1 + \mathrm{vlift}(F_1)) - (B_2^\sharp \mathbf{d}H_2 + \mathrm{vlift}(F_2))] \subset \mathrm{vlift}(W_1)$.

If (H_1, B_1, F_1, W_1) includes (H_2, B_2, F_2, W_2), then for any choice of control $u_2 : T^*Q \to W_2$, there exists a control $u_1 : T^*Q \to W_1$ satisfying (**??**) such that the two closed-loop systems with these controls produce the same equations of motion.

Simple CH Systems. The definition of a simple Hamiltonian control system is slightly more subtle than its Lagrangian counterpart.

Definition 3.5. *A CH system (H, B, F, W) is called **simple** when the Hamiltonian function has the form kinetic plus potential energy:*

$$H(q, p) = \frac{1}{2}\langle p, m^{-1}(q)p \rangle + V(q) \tag{29}$$

where m is a nondegenerate symmetric $(0, 2)$-tensor and the almost Poisson tensor B is nondegenerate and has the form:

$$B(q, p) = \begin{bmatrix} O & K(q)^T \\ -K(q) & J(q, p) \end{bmatrix} \tag{30}$$

in cotangent coordinates (q, p) on T^*Q where K, J are $n \times n$ matrices with $n = \dim Q$.

One can check that the statement that B has the form (**??**) is independent of the choice of cotangent bundle coordinates for T^*Q. We call almost Poisson tensors of form (**??**) with K invertible **simple**.

Decomposition of Simple Almost Poisson Tensors. Now, we define a decomposition of simple almost Poisson tensors in the following way. Let B be a given simple almost Poisson tensor. The following relation

$$\text{vlift}(\psi_B) = (B^\sharp)^* \circ \Theta \tag{31}$$

defines a unique $\psi_B \in \Gamma(\text{Aut}(T^*Q))$. Here, $\Gamma(\text{Aut}(T^*Q))$ denotes sections of the automorphism bundle; that is, and element of $\Gamma(\text{Aut}(T^*Q))$ assigns to each $q \in Q$, an invertible linear transformation of T_q^*Q to itself. Also, Θ denotes the canonical one form on T^*Q, and $(B^\sharp)^* : T^*T^*Q \to TT^*Q$ is the dual of $B^\sharp : T^*T^*Q \to TT^*Q$. Suppose B is given by (**??**) in coordinates. Then,

$$B^\sharp(q, p) = K_{ji}(q)\frac{\partial}{\partial p_j} \otimes \frac{\partial}{\partial q^i} - K_{ij}(q)\frac{\partial}{\partial q^j} \otimes \frac{\partial}{\partial p_i} + J_{ij}(q, p)\frac{\partial}{\partial p_j} \otimes \frac{\partial}{\partial p_i}.$$

Since $(B^\sharp)^*(q, p) = -B^\sharp(q, p)$, we have

$$(B^\sharp)^* \circ \Theta(q, p) = -B^\sharp(q, p)(p_k \mathrm{d}q^k) = K_{ij}(q)p_j\frac{\partial}{\partial p_i}$$

which is the vertical lift of $K_{ij}(q)p_j \mathrm{d}q^i$ at (q, p). Hence, (**??**) defines a unique $\psi_B \in \Gamma(\text{Aut}(T^*Q))$ and its local expression is given by

$$\psi_B(q, p) = (q, K(q)\, p). \tag{32}$$

with B given by (**??**) in coordinates.

Given a simple almost Poisson tensor B, we can uniquely decompose B into the two almost Poisson tensors B_r and B_gr as follows:

$$B = B_\mathrm{r} + B_\mathrm{gr}$$

where

$$B_\mathrm{r} = (\psi_B^{-1})^* B_\mathrm{can}; \quad B_\mathrm{gr} = B - (\psi_B^{-1})^* B_\mathrm{can} \tag{33}$$

and B_{can} is the canonical Poisson tensor on T^*Q. When B is given by (??) in coordinates, we have the following coordinate expressions:

$$
B_{\mathrm{r}}(q,p) \;=\; \begin{bmatrix} O & K(q)^T \\ -K(q) & C_K(q,p) \end{bmatrix} \tag{34}
$$

$$
B_{\mathrm{gr}}(q,p) \;=\; \begin{bmatrix} O & O \\ O & J(q,p) - C_K(q,p) \end{bmatrix} \tag{35}
$$

where

$$
\begin{aligned}
(C_K)_{ij}(q,p) &= -\left\langle K(q)^{-1}p, [(K(q)^T)_i, (K(q)^T)_j] \right\rangle \tag{36} \\
&= \left(\frac{\partial K_{is}(q)}{\partial q^r} K_{jr}(q) - K_{ir}(q) \frac{\partial K_{js}(q)}{\partial q^r} \right) K^{sl}(q) p_l
\end{aligned}
$$

where $(K(q)^T)_i$ is the i-th column of the matrix $K(q)^T$ and $[\,,\,]$ is the Lie bracket. The formula (??) is essentially the same as the equation (19) in [?]. By (??), (??) and (??), we have

$$
\psi_B = \psi_{B_{\mathrm{r}}}. \tag{37}
$$

Notice that the Poisson tensor B_{r} satisfies the Jacobi-identity because it is a pull-back of the canonical Poisson bracket.

Construction of Gyroscopic Forces. Given a simple Hamiltonian system $(H, B = B_{\mathrm{r}}+B_{\mathrm{gr}}, F, W)$, the simple (degenerate) almost Poisson tensor B_{gr} and the Hamiltonian H defines a **_gyroscopic force_** $F_{\mathrm{gr}} : T^*Q \to T^*Q$ by the following relation:

$$
\mathrm{vlift}(F_{\mathrm{gr}}) = (B_{\mathrm{gr}})^{\sharp}\mathrm{d}H. \tag{38}
$$

By (??), in coordinates,

$$
(B_{\mathrm{gr}})^{\sharp}\mathrm{d}H(q,p) = \left((J - C_K)_{ij}(q,p) \frac{\partial H(q,p)}{\partial p_j} \right) \frac{\partial}{\partial p_i}
$$

which is the vertical lift of $\left((J - C_K)_{ij}(q,p)\frac{\partial H(q,p)}{\partial p_j} \right) \mathrm{d}q^i$ at (q,p). Hence, (??) defines the unique force $F_{\mathrm{gr}} : T^*Q \to T^*Q$, which is locally written as

$$
F_{\mathrm{gr}}(q,p) = \left(q, (J - C_K)_{ij}(q,p)\frac{\partial H(q,p)}{\partial p_j} \right).
$$

The reason we call F_{gr} the gyroscopic force is that it does not change the Hamiltonian H in the following sense

$$
\mathrm{vlift}(F_{\mathrm{gr}})[H] = \mathrm{d}H((B_{\mathrm{gr}})^{\sharp}\mathrm{d}H) = B_{\mathrm{gr}}(\mathrm{d}H, \mathrm{d}H) = 0
$$

by the skew-symmetry of B_{gr}. The dynamics with gyroscopic forces still gives conservation of energy.

CH-Equivalence for Simple Systems. This decomposition of simple almost Poisson tensors simplifies the class of simple Hamiltonian systems under the CH-equivalence relation. Suppose that we are given a simple Hamiltonian system $(H, B = B_{\mathrm{r}} + B_{\mathrm{gr}}, F, W)$. Then (??) implies

$$
B^{\sharp}\mathbf{d}H + \mathrm{vlift}(F) = (B_{\mathrm{r}})^{\sharp}\mathbf{d}H + \mathrm{vlift}(F_{\mathrm{gr}} + F).
$$

Therefore the simple Hamiltonian system $(H, B = B_{\mathrm{r}} + B_{\mathrm{gr}}, F, W)$ is CH-equivalent to the simple Hamiltonian system $(H, B_{\mathrm{r}}, F_{\mathrm{gr}} + F, W)$ where F_{gr} is given by (??). By (??) and (??),

$$\psi_{B_{\mathrm{r}}}^* B_{\mathrm{r}} = \psi_B^* B_{\mathrm{r}} = B_{\mathrm{can}}. \tag{39}$$

This proves the following result.

Proposition 3.6. *A given simple CH system $(H, B = B_{\mathrm{r}} + B_{\mathrm{gr}}, F, W)$ is CH-equivalent to the CH system $(H, B_{\mathrm{r}}, F_{\mathrm{gr}} + F, W)$ where $B = B_{\mathrm{r}} + B_{\mathrm{gr}}$ is the decomposition of B into the regular part and the gyroscopic part and $F_{\mathrm{gr}} : T^*Q \to T^*Q$ is determined by the relation $\mathrm{vlift}(F_{\mathrm{gr}}) = B_{\mathrm{gr}}^\sharp \mathrm{d}H$. In particular, B_r satisfies (??).*

A consequence is

Corollary 3.7. *An arbitrary simple CH system is CH-equivalent to a simple Hamiltonian system (H, B, F, W) with the Poisson tensor B satisfying the Jacobi identity and $\psi_B^* B = B_{\mathrm{can}}$. Equivalently, one can say $B = \phi^* B_{\mathrm{can}}$ for some $\phi \in \Gamma(\mathrm{Aut}(T^*Q))$. In coordinates, B is always of the form*

$$B(q, p) = \begin{bmatrix} O & K(q)^T \\ -K(q) & C_K(q, p) \end{bmatrix}$$

with C_K in (??) when ϕ (or, ψ_B^{-1}) is in coordinates given by $\phi(q, p) = (q, K(q)^{-1} p)$.

Proof. A direct consequence of Proposition ?? and (??). □

We now consider the opposite direction of Proposition ??. Let us consider a simple CH system $(H, B, F_{\mathrm{gr}}, W)$ with $H = \frac{1}{2}\langle p, m^{-1}p \rangle + V(q)$ and F_{gr} a gyroscopic force. By definition of the gyroscopic force, $F_{\mathrm{gr}} = (F_{\mathrm{gr}})_i \mathrm{d}q^i$ satisfies

$$0 = \mathrm{vlift}(F_{\mathrm{gr}})[H] = (F_{\mathrm{gr}})_i m^{ij} p_j$$

where $m^{-1} = (m^{ij})$. Hence, $(F_{\mathrm{gr}})_i$ should be of the following form

$$(F_{\mathrm{gr}})_i(q, p) = p_l S^{lk}(q, p) m_{ki}(q), \quad S^{lk} = -S^{kl}.$$

Hence, we have proved the following:

Proposition 3.8. *Given a simple CH system $(H, B, F + F_{\mathrm{gr}}, W)$ with F_{gr} a gyroscopic force, the following holds:*

1. *F_{gr} has the form:*

$$F_{\mathrm{gr}}(q, p) = p_l S^{lk}(q, p) m_{ki}(q) \mathrm{d}q^i, \quad S^{lk} = -S^{kl}$$

   for some functions $S^{ij}(q, p)$, where m_{ij} is the mass matrix of H.
2. *We have*

$$(H, B, F + F_{\mathrm{gr}}, W) \overset{H}{\sim} (H, B + \tilde{B}, F, W),$$

   where

$$\tilde{B}(q, p) = m_{ik}(q) S^{kl}(q, p) m_{lj}(q) \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial p_j}.$$

Proof. One needs to check that $B + \tilde{B}$ is a simple Poisson tensor, which is readily done. □

Proposition ?? and Proposition ?? show that the failure of the Jacobi identity of the Poisson bracket of a Poisson bracket of a CH system is directly related to gyroscopic forces.

Pull-back Systems. The concept of pull-back systems will be useful in the next section in showing the equivalence of the CL system method and the CH system method. This notion is a technical device that is needed for the proofs later and can be omitted on first reading. Consider a CH system (H, B, F, W) and $\phi \in \Gamma(\mathrm{Aut}(T^*Q))$. Then, the pull-back system $\phi^*(H, B, F, W)$ is defined to be the associated CH system $(\phi^*H, \phi^*B, \phi^*F, \phi^*W)$, where

$$\phi^*H = H \circ \phi \quad \text{and} \quad (\phi^*B)(z)(\alpha_z, \beta_z) = B(w)(T_w^*(\phi^{-1})\alpha_z, T_w^*(\phi^{-1})\beta_z)$$

for $\phi(w) = z$ and $\alpha_z, \beta_z \in T_z^*(T^*Q)$, and $\phi^*F = \phi^{-1} \circ F \circ \phi$. Here, the pull-back notation in ϕ^*F should be regarded as an action of $\Gamma(\mathrm{Aut}(T^*Q))$ on the set of fiber-preserving maps on T^*Q. Notice also that in the current paper, $\phi^*W = \phi^{-1}(W)$ by definition. The notation ϕ^*W should not be confused with the standard notation of pull-back bundles. When we regard W as the set $\{u : T^*Q \to W\}$, then ϕ^*W reads $\phi^*W = \{\phi^*u = \phi^{-1} \circ u \circ \phi \mid u \in W\}$. Hence, we will write ϕ^*W as $\phi^{-1}W \circ \phi$ to respect both interpretations. We claim

$$\phi^*X_{(H,B,F,u)} = X_{\phi^*(H,B,F,u)}.$$

It is well-known (or a straightforward computation) that $(\phi^*B)^\sharp \mathbf{d}(\phi^*H) = \phi^*(B^\sharp \mathbf{d}H)$. We have only to show $\phi^*(\mathrm{vlift}(F)) = \mathrm{vlift}(\phi^*F)$ where one should be careful that pull-back notation in the left hand side is the usual pull-back of a vector field by a diffeomorphism ϕ, and the pull-back notation on the right hand side should be understood as $\phi^{-1} \circ F \circ \phi$ as we mentioned before. Indeed, for $w \in T^*Q$, we have

$$(\phi^*(\mathrm{vlift}(F)))(w) = T_{\phi(w)}\phi^{-1} \cdot \mathrm{vlift}(F)(\phi(w))$$

$$= T_{\phi(w)}\phi^{-1} \frac{d}{ds}\bigg|_{s=0} (\phi(w) + sF \circ \phi(w))$$

$$= \frac{d}{ds}\bigg|_{s=0} \phi^{-1}(\phi(w) + sF \circ \phi(w))$$

$$= \frac{d}{ds}\bigg|_{s=0} (w + s(\phi^*F)(w))$$

$$= \mathrm{vlift}(\phi^*F)(w).$$

The same relation holds for u. One can readily show the following:

Proposition 3.9. *Let $\phi \in \Gamma(\mathrm{Aut}(T^*Q))$. Then the following hold:*

1. *The pull-back system via ϕ of a simple CH system is also simple.*
2. *Two CH systems (H_1, B_1, F_1, W_1) and (H_2, B_2, F_2, W_2) are CH-equivalent if and only if the corresponding pull-back systems $\phi^*(H_1, B_1, F_1, W_1)$ and $\phi^*(H_2, B_2, F_2, W_2)$ are CH-equivalent.*

In particular, it is useful to have a coordinate expression for ϕ^*B when B satisfies $\psi_B^*B = B_{\mathrm{can}}$ for $\psi \in \Gamma(\mathrm{Aut}(T^*Q))$. The almost Poisson tensor B is written in coordinates as in (**??**). Consider $\phi \in \Gamma(\mathrm{Aut}(T^*Q))$ with the local coordinate expression $\phi(q, p) = (q, D(q)^{-1}p)$. Then the pull-back tensor ϕ^*B is expressed in coordinates as

$$(\phi^*B)(q, p) = \begin{bmatrix} O & (D(q)K(q))^T \\ -D(q)K(q) & C_{DK} \end{bmatrix}$$

since

$$\phi^*B = ((\psi_B)^{-1} \circ \phi)^*(\psi_B)^*B = ((\psi_B)^{-1} \circ \phi)^*B_{\mathrm{can}}. \tag{40}$$

Here $\psi_B^{-1} \circ \phi(q, p) = (q, (D(q)K(q))^{-1}p)$ and we use the formula in Corollary **??**. The equation (**??**) implies that $\psi_{\phi^*B} = \phi^{-1} \circ \psi_B$ and $(\psi_{\phi^*B})^*B = B_{\mathrm{can}}$. This proves that a simple CH system (H, B, F, W) with $\psi_B^*B = B_{\mathrm{can}}$ is pulled back by $\phi \in \Gamma(\mathrm{Aut}(T^*Q))$ to the simple CH system $\phi^*(H, B, F, W)$ satisfying $(\psi_{\phi^*B})^*(\phi^*B) = B_{\mathrm{can}}$ and $\psi_{\phi^*B} = \phi^{-1} \circ \psi_B$.

4. Equivalence of CL Systems and CH Systems: Simple Mechanical Systems

The goal of this section is to show that for simple mechanical systems, the method of controlled Lagrangians and that of controlled Hamiltonians are equivalent. A more detailed statement is contained in Theorem **??** and Corollary **??**. For this purpose, we will first review the Legendre transformations and then get to the problem of the equivalence of the two methods of CL systems and CH systems.

Frequently, a Hamiltonian system on T^*Q induces a Hamiltonian vector field through a canonical symplectic structure (or, canonical Poisson structure) on T^*Q before any reduction processes. This is because a Hamiltonian system on T^*Q often comes from a Lagrangian system on TQ via a Legendre transformation associated to a given Lagrangian function. Hence if there is more than one Lagrangian function, there can be multiple transformations between TQ and T^*Q. We will carefully deal with this in this section, too.

Legendre Transformations. Recall that in this paper all the Lagrangians L on TQ and all the Hamiltonians H on T^*Q are regular, i.e., $\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0$ and $\det\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right) \neq 0$ for $(q, \dot{q}) \in TQ$ and $(q, p) \in T^*Q$. This assumption implies that the Legendre transformation $\mathbb{F}L : TQ \to T^*Q$ (defined in (**??**)) and the inverse Legendre transformation $\mathbb{F}H : T^*Q \to TQ$ (defined below) are both locally invertible. In fact, we shall assume that the systems are **hyperregular**; that is, these transformations are globally invertible. This is automatic for simple systems.

It is well-known (see, for example, [?]) that a given Lagrangian system (L, F^L, W^L) is transformed by the Legendre transformation $\mathbb{F}L$ to the Hamiltonian system $(H, B_{\text{can}}, F^H, W^H)$ where

$$H(\alpha) = \langle \alpha, \mathbb{F}L^{-1}(\alpha) \rangle - L \circ \mathbb{F}L^{-1}(\alpha) \quad \text{for } \alpha \in T^*Q \tag{41}$$

$$F^H = F^L \circ \mathbb{F}L^{-1} \tag{42}$$

$$W^H = W^L \circ \mathbb{F}L^{-1} \tag{43}$$

where $W^L \circ \mathbb{F}L^{-1}$ is understood as W^L as a subbundle of T^*Q and also understood as the set $\{u \circ \mathbb{F}L^{-1} | u : TQ \to W^L\}$ when we regard $W^L \circ \mathbb{F}L^{-1}$ as a set of bundle maps from T^*Q to W. Namely, the Euler-Lagrange equation

$$\mathcal{EL}(L) = F^L + u^L$$

with $u^L : TQ \to W^L$ is *equivalent* to CH vector field

$$X_{(H, B_{\text{can}}, F^H, u^H)} = B_{\text{can}}^\sharp \mathbf{d}H + \text{vlift}(F^L \circ \mathbb{F}L^{-1}) + \text{vlift}(u^L \circ \mathbb{F}L^{-1}).$$

We now suppose that we are given a Hamiltonian function H on T^*Q. Then it induces the inverse Legendre transformation $\mathbb{F}H : T^*Q \to TQ$ as follows:

$$\langle \mathbb{F}H(\alpha_q), \beta_q \rangle = \frac{d}{ds}\Big|_{s=0} H(\alpha_q + s\beta_q) \tag{44}$$

for $\alpha_q, \beta_q \in T_q^*Q$, where the word *inverse* will be justified later. It is well-known (or easy to check) that a given CH system $(H, B_{\text{can}}, F^H, W^H)$ is transformed by the inverse Legendre transformation $\mathbb{F}H$ to the CL system $(L, F^L = F^H \circ \mathbb{F}H^{-1}, W^L = W^H \circ \mathbb{F}H^{-1})$ where $L(v) = \langle v, \mathbb{F}H^{-1}(v) \rangle - H \circ \mathbb{F}H^{-1}(v)$ for $v \in TQ$. The CH vector field $X_{(H, B_{\text{can}}, F^H, u^H)} = B_{\text{can}}^\sharp \mathbf{d}H + \text{vlift}(F^H) + \text{vlift}(u^H)$ is equivalent to the Euler-Lagrange equation $\mathcal{EL}(L) = F^L + u^H \circ \mathbb{F}H^{-1}$.

Suppose that a given CL system (L, F^L, W^L) is transformed by the Legendre transformation $\mathbb{F}L$ to the CH system $(H, B_{\text{can}}, F^H, W^H)$. Then $(H, B_{\text{can}}, F^H, W^H)$ is transformed back to (L, F^L, W^L) by the inverse Legendre transformation $\mathbb{F}H$ since $\mathbb{F}H = \mathbb{F}L^{-1}$ in this case by Proposition 7.4.2 of [?]. One can also start this argument from the Hamiltonian system.
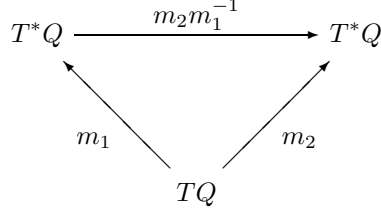
FIGURE 2. Diagram of Legendre transformations

CH-Equivalence Proves CL-Equivalence. We first show that the matching conditions of simple CL systems can be derived from those of simple CH systems; the computation involved in this direction is simpler than that involved in showing the other way around. However, the computation carried out here will be also used in the paragraph entitled **CL-Equivalence Proves CH-Equivalence**. A special case of **CH-Equivalence Proves CL-Equivalence** was dealt with in [?] and [?] but without the external force F. We also improve the foundational setting for the controlled Hamiltonian method. This improved setting will pay off when we consider systems with symmetry and reduction.

Suppose we have two simple CL systems (L_1, F_1^L, W_1^L) and (L_2, F_2^L, W_2^L) with $L_1(q, \dot{q}) = \frac{1}{2} m_1(\dot{q}, \dot{q}) - V_1(q)$ and $L_2(q, \dot{q}) = \frac{1}{2} m_2(\dot{q}, \dot{q}) - V_2(q)$. They define two Legendre transformations $\mathbb{F}L_1, \mathbb{F}L_2 : TQ \to T^*Q$ as follows

$$(q, p) = \mathbb{F}L_1(q, \dot{q}) = (q, m_1(q)\dot{q}), \tag{45}$$

$$(q, \tilde{p}) = \mathbb{F}L_2(q, \dot{q}) = (q, m_2(q)\dot{q}). \tag{46}$$

The CL system (L_1, F_1^L, W_1^L) is transformed via $\mathbb{F}L_1$ to the CH system $(H_1, B_1 = B_{\mathrm{can}}, F_1^H = F_1^L \circ \mathbb{F}L^{-1}, W_1^H = W_1^L \circ \mathbb{F}L^{-1})$, and the second CL system (L_2, F_2^L, W_2^L) is transformed via $\mathbb{F}L_2$, to $(\tilde{H}_2, \tilde{B}_2 = B_{\mathrm{can}}, \tilde{F}_2^H = F_2^L \circ \mathbb{F}L_2^{-1}, \tilde{W}_2^H = W_2^L \circ \mathbb{F}L_2^{-1})$, where

$$H_1(q, p) = \frac{1}{2} \langle p, m_1(q)^{-1} p \rangle + V_1(q),$$

$$\tilde{H}_2(q, \tilde{p}) = \frac{1}{2} \langle \tilde{p}, m_2(q)^{-1} \tilde{p} \rangle + V_2(q).$$

We now would like to show that checking the CL-equivalence of (L_1, F_1^L, W_1^L) and (L_2, F_2^L, W_2^L) is the same as checking the CH-equivalence of their transformed CH systems. Thereby, we show that CH-equivalence proves CL-equivalence. Since the two Legendre transformations in (**??**) and (**??**) are not the same in general, we need to pull-back the system $(\tilde{H}_2, \tilde{B}_2, \tilde{F}_2^H, \tilde{W}_2^H)$ via $\mathbb{F}L_2 \circ \mathbb{F}L_1^{-1} = m_2 \circ m_1^{-1} \in \Gamma(\mathrm{Aut}(T^*Q))$, as in the commutative diagram in Figure **??**.

Let $(H_2, B_2, F_2^H, W_2^H) = (\mathbb{F}L_2 \circ \mathbb{F}L_1^{-1})^*(\tilde{H}_2, \tilde{B}_2, \tilde{F}_2^H, \tilde{W}_2^H)$ where one computes

$$H_2(q, p) = \frac{1}{2} \langle p, m_1(q)^{-1} m_2(q) m_1(q)^{-1} p \rangle + V_2(q)$$

$$B_2(q, p) = \begin{bmatrix} O & (m_1(q) m_2(q)^{-1})^T \\ -m_1(q) m_2(q)^{-1} & C_{m_1 m_2^{-1}}(q, p) \end{bmatrix}$$

$$W_2^H = (\mathbb{F}L_2 \circ \mathbb{F}L_1^{-1})^*(W_2^L \circ \mathbb{F}L_2^{-1})$$

$$= (\mathbb{F}L_2 \circ \mathbb{F}L_1^{-1})^{-1}(W_2^L \circ \mathbb{F}L_2^{-1}) \circ \mathbb{F}L_2 \circ \mathbb{F}L_1^{-1}$$

$$= (\mathbb{F}L_1 \circ \mathbb{F}L_2^{-1})(W_2^L \circ \mathbb{F}L_1^{-1}).$$

We will now show the following

$$(H_1, B_1, F_1^H, W_1^H) \overset{H}{\sim} (H_2, B_2, F_2^H, W_2^H) \iff (L_1, F_1^L, W_1^L) \overset{L}{\sim} (L_2, F_2^L, W_2^L) \tag{47}$$

First, **HM-1** reads

$$W_1^H = W_2^H \iff W_1^L \circ \mathbb{F}L_1^{-1} = m_1 m_2^{-1}(W_2^L \circ \mathbb{F}L_1^{-1})$$
$$\iff W_1^L = m_1 m_2^{-1} W_2^L$$

whose right hand side is exactly **ELM-1**. Hence **HM-1** for (H_1, B_1, F_1^H, W_1^H) and (H_2, B_2, F_2^H, W_2^H) is equivalent to **ELM-1** for (L_1, F_1^L, W_1^L) and (L_2, F_2^L, W_2^L). Second, since $u_1^H, u_2^H \in W_1^H = W_2^H$, **HM-2** can be *equivalently* written in coordinates as

$$\left(\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix}_{H_1} - \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix}_{H_2} \right) \in \text{vlift}(W_1^H(q)) \simeq O \oplus W_1^H(q) \tag{48}$$

where

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix}_{H_i} = X_{(H_i, B_i, F_i^H, u_i^H)} = B_i^\sharp dH_i + \text{vlift}(F_i^H) + \text{vlift}(u_i^H)$$

for each $i = 1, 2$ where the subscript H_i denotes the CH system (H_i, B_i, F_i^H, W_i^H) for simplicity.

Since $W_1^H = W_2^H = W_1^L \circ \mathbb{F}L_1^{-1} = m_1 m_2^{-1} W_2^L \circ \mathbb{F}L_2^{-1}$ under **HM-1**, we can write the controls $u_i^H \in W_1^H$ as

$$u_1^H = u_1^L \circ \mathbb{F}L_1^{-1}$$
$$u_2^H = m_1 m_2^{-1} u_2^L \circ \mathbb{F}L_2^{-1}$$

for $u_i^L \in W_i^L$ which can be considered via the Legendre transformations $\mathbb{F}L_1$ and $\mathbb{F}L_2$ as the controls for (L_i, F_i^L, W_i^L) for $i = 1, 2$, respectively.

Recall from (**??**) and (**??**) that $p = m_1(q)m_2(q)^{-1}\tilde{p}$. Using this, we find that

$$\dot{q}_{H_1} - \dot{q}_{H_2} = m_1(q)^{-1}p - m_2(q)^{-1}\tilde{p} = m_1(q)^{-1}p - m_2(q)^{-1}m_2(q)m_1(q)^{-1}p = 0.$$

The equation for $\dot{p}_{H_1}$ can be written in terms of $(q, \dot{q})$ as follows:

$$\dot{p}_{H_1} = \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}} = \frac{\partial L_1}{\partial q} + F_1^L + u_1^L.$$

The equation $\dot{p}_{H_2}$ can be written in terms of $(q, \dot{q})$ as follows:

$$\begin{aligned}
\dot{p}_{H_2} &= \frac{d}{dt}(m_1 m_2^{-1}\tilde{p}) \\
&= (\mathbf{d}(m_1 m_2^{-1})[\dot{q}])\tilde{p} + m_1 m_2^{-1}\frac{d}{dt}\tilde{p}_{\tilde{H}_2} \\
&= (\mathbf{d}(m_1 m_2^{-1})[\dot{q}])m_2\dot{q} + m_1 m_2^{-1}\frac{d}{dt}\frac{\partial L_2}{\partial \dot{q}} \\
&= (\mathbf{d}m_1[\dot{q}])\dot{q} - m_1 m_2^{-1}(\mathbf{d}m_2[\dot{q}])\dot{q} + m_1 m_2^{-1}\left(\frac{\partial L_2}{\partial q} + F_2^L + u_2^L \right) \\
&= (\mathbf{d}m_1[\dot{q}])\dot{q} - m_1 m_2^{-1}\left((\mathbf{d}m_2[\dot{q}])\dot{q} - \frac{\partial L_2}{\partial q} - F_2^L - u_2^L \right)
\end{aligned}$$

Hence,

$$
\begin{aligned}
\dot{p}_{H_2} - \dot{p}_{H_1} &= \left((\mathbf{d}m_1[\dot{q}])\dot{q} - \frac{\partial L_1}{\partial q} - F_1^L - u_1^L \right) - m_1 m_2^{-1} \left((\mathbf{d}m_2[\dot{q}])\dot{q} - \frac{\partial L_2}{\partial q} - F_2^L - u_2^L \right) \\
&= \left(m_1 \ddot{q} + (\mathbf{d}m_1[\dot{q}])\dot{q} - \frac{\partial L_1}{\partial q} - F_1^L - u_1^L \right) - m_1 m_2^{-1} \left(m_2 \ddot{q} + (\mathbf{d}m_2[\dot{q}])\dot{q} - \frac{\partial L_2}{\partial q} - F_2^L - u_2^L \right) \\
&= (\mathcal{EL}(L_1) - F_1^L) - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2^L) - u_1^L + m_1 m_2^{-1} u_2^L.
\end{aligned}
\tag{49}
$$

Therefore, (**??**) holds if and only if

$$
\mathrm{Im}[(\mathcal{EL}(L_1) - F_1^L) - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2^L)] \subset W_2^L
$$

since $u_1^L, m_1 m_2^{-1} u_2^L \in W_1^L = m_1 m_2^{-1} W_2^L$. Therefore, we have shown (**??**). Finally, one can show from (**??**) and (**??**) that condition (**??**) on the inputs u_i^L's is equivalent to condition (**??**) on the inputs u_i^H's.

We make a remark on an alternative way of comparison of **HM-2** and **ELM-2**. One can show by a brute-force coordinate computation that

$$
\begin{aligned}
[(B_2^\sharp \mathbf{d}H_2 &+ \mathrm{vlift}(F_2^H)) - (B_1^\sharp \mathbf{d}H_1 + \mathrm{vlift}(F_1^H))] \\
&\simeq O \oplus \left[(\mathcal{EL}(L_1) - F_1^L) - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2^L) \right].
\end{aligned}
$$

This computation is much more complicated and it does not directly lead to the equivalence of (**??**) and (**??**). This is why we did not choose this brute-force computational method in this paper.

CL-Equivalence Proves CH-Equivalence. We now show that the Hamiltonian matching conditions of simple CH systems can be derived from those of simple CL systems. Consider two simple CH systems (H_1, B_1, F_1^H, W_1^H) and (H_2, B_2, F_2^H, W_2^H) with $H_i(q, p) = \frac{1}{2}\langle p, m_i^{-1}(q)p \rangle + V_i(q)$ for $i = 1, 2$. By Proposition **??** and Proposition **??**, without loss of generality, we may assume that

$$
B_1 = B_{\mathrm{can}}; \qquad \psi_{B_2}^* B_2 = B_{\mathrm{can}}.
$$

In coordinates, we write B_2 and ψ_{B_2} as follows:

$$
B_2(q, p) = \begin{bmatrix} O & K_2(q)^T \\ -K_2(q) & C_{K_2}(q, p) \end{bmatrix}
$$

and

$$
(q, p) = \psi_{B_2}(q, \tilde{p}) = (q, K_2(q)\tilde{p}).
$$

Consider the pull-back system $(\tilde{H}_2, \tilde{B}_2 = B_{\mathrm{can}}, \tilde{F}_2^H, \tilde{W}_2^H) := (\psi_{B_2})^*(H_2, B_2, F_2^H, W_2^H)$ with

$$
H_2(q, \tilde{p}) = \frac{1}{2}\langle \tilde{p}, \tilde{m}_2^{-1}(q)\tilde{p} \rangle + V_2(q); \quad \tilde{m}_2 := \psi_{B_2}^{-1} m_2 (\psi_{B_2}^*)^{-1}
\tag{50}
$$

where $\psi_{B_2}^* \in \Gamma(\mathrm{Aut}(TQ))$ is the dual of $\psi_{B_2} \in \Gamma(\mathrm{Aut}(T^*Q))$.

The system (H_1, B_1, F_1^H, W_1^H) is transformed via the inverse Legendre transformation $(q, \dot{q}) = \mathbb{F}H_1(q, p) = (q, m_1(q)^{-1}p)$ to the CL system (L_1, F_1^L, W_1^L), where

$$
L_1(q, \dot{q}) = \frac{1}{2}\langle \dot{q}, m_1 \dot{q} \rangle - V_1(q), \qquad F_1^L = F_1^H \circ \mathbb{F}H_1^{-1} \quad \text{and} \quad W_1^L = W_1^H \circ \mathbb{F}H_1^{-1}.
$$

$$T^*Q \xrightarrow{\;\;\psi_{B_2}^{-1}\;\;} T^*Q$$

$$m_1^{-1} \searrow \qquad \swarrow \tilde{m}_2^{-1}$$
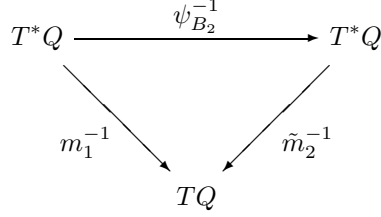
$$TQ$$

FIGURE 3. Diagram of inverse Legendre transformations

The system $(\tilde{H}_2, \tilde{B}_2, \tilde{F}_2^H, \tilde{W}_2^H)$ is transformed via the inverse Legendre transformation $(q, \dot{q}) = \mathbb{F}\tilde{H}_2(q, \tilde{p}) = (q, \tilde{m}_2^{-1}(q)\tilde{p})$ to the CL system (L_2, F_2^L, W_2^L), where

$$L_2(q, \dot{q}) = \frac{1}{2}\langle \dot{q}, \tilde{m}_2\dot{q}\rangle - V_2(q), \qquad F_2^L = \tilde{F}_2^H \circ \mathbb{F}\tilde{H}_2^{-1} \quad \text{and} \quad W_2^L = \tilde{W}_2^H \circ \mathbb{F}\tilde{H}_2^{-1}.$$

The diagram in Figure **??** commutes if and only if $\psi_{B_2} = m_1\tilde{m}_2^{-1}$, which by the definition of $\tilde{m}_2$ in (**??**) is equivalent to

$$\psi_{B_2} = m_2 m_1^{-1} \tag{51}$$

i.e., in a matrix form $K_2(q) = m_2(q)m_1(q)^{-1}$. We will show that

$$(H_1, B_1, F_1^H, W_1^H) \overset{H}{\sim} (H_2, B_2, F_2^H, W_2^H) \iff [(L_1, F_1^L, W_1^L) \overset{L}{\sim} (L_2, F_2^L, W_2^L)] + (\textbf{??}). \tag{52}$$

Under (**??**), one computes

$$\begin{aligned}
m_1\tilde{m}_2^{-1}W_2^L &= \psi_B(\psi_{B_2}^* W_2^H) \circ \mathbb{F}L_2 \\
&= \psi_B(\psi_{B_2}^{-1}W_2^H \circ \psi_{B_2}) \circ \mathbb{F}L_2 \\
&= W_2^H \circ m_1 = W_2^H \circ \mathbb{F}L_1.
\end{aligned}$$

First, under (**??**), **ELM-1** reads

$$W_1^L = m_1\tilde{m}_2^{-1}W_2^L \iff W_1^H = W_2^H$$

which is **HM-1** for (H_1, B_1, F_1, W_1) and (H_2, B_2, F_2, W_2). Second, since $u_1^H, u_2^H \in W_1^H = W_2^H$, **HM-2** can be *equivalently* written in coordinates as (**??**). One can compute

$$\dot{q}_{H_1} - \dot{q}_{H_2} = m_1(q)^{-1}p - K_2(q)^T m_2(q)^{-1}p.$$

Hence, the first half of **HM-2** reads

$$\dot{q}_{H_1} - \dot{q}_{H_2} = 0 \iff K_2(q) = m_2(q)m_1(q)^{-1}$$

which is the commutativity condition (**??**). The remaining half of **HM-2** reads $\dot{p}_{H_1} - \dot{p}_{H_2} \in W_2^H$. By a similar computation carried out in the paragraph entitled **CH-equivalence Proves CL-Equivalence**, one can show that

$$\dot{p}_{H_1} - \dot{p}_{H_2} \in W_2^H \iff \textbf{ELM-2}$$

under (**??**). One can also readily show that condition (**??**) on the inputs u_i^L's is equivalent to the condition (**??**) on the inputs u_i^H's. Therefore we have established (**??**).

The Main Theorem. We summarize the preceding discussion in the following theorem.[2]

Theorem 4.1. *The method of controlled Lagrangians is equivalent to that of controlled Hamiltonians for simple mechanical systems. Namely, the following holds:*

1: *Let (L_i, F_i^L, W_i^L), $i = 1, 2$, be two simple CL systems, and let $(H_i, B_{\mathrm{can}}, F_i^H, W_i^H)$ be the associated simple CH systems Legendre-transformed via $\mathbb{F}L_i$ from the CL system (L_i, F_i^L, W_i^L) for $i = 1, 2$, respectively. Then,*

$$(L_1, F_1^L, W_1^L) \overset{L}{\sim} (L_2, F_2^L, W_2^L) \iff (H_1, B_{\mathrm{can}}, F_1^H, W_1^H) \overset{H}{\sim} (\mathbb{F}L_2 \circ \mathbb{F}L_1^{-1})^*(H_2, B_{\mathrm{can}}, F_2^H, W_2^H),$$

where B_{can} is the canonical Poisson tensor on T^*Q.

2: *Let (H_i, B_i, F_i^H, W_i^H), $i = 1, 2$, be two simple CH systems. Decompose B_i into its regular part $B_{\mathrm{r},i}$ and its gyroscopic part $B_{\mathrm{gr},i}$ so that*

$$(H_i, B_i, F_i^H, W_i^H) \overset{H}{\sim} (H_i, B_{\mathrm{r},i}, F_i^H + F_{\mathrm{gr},i}^H, W_i^H)$$

*where the gyroscopic force $F_{\mathrm{gr},i}^H$ is defined by $\mathrm{vlift}(F_{\mathrm{gr},i}) = B_{\mathrm{gr},i}^\sharp \mathbf{d}H_i$ (see Proposition **??**). Then there exist $\psi_1, \psi_2 \in \Gamma(\mathrm{Aut}(T^*Q))$ and two simple CH systems $(\hat{H}_i, B_{\mathrm{can}}, \hat{F}_i^H, \hat{W}_i^H)$, $i = 1, 2$ such that*

$$\psi_i^*(H_i, B_{\mathrm{r},i}, F_i^H + F_{\mathrm{gr},i}^H, W_i^H) = (\hat{H}_i, B_{\mathrm{can}}, \hat{F}_i^H, \hat{W}_i^H) \qquad i = 1, 2,$$

and finally

$$(H_1, B_1, F_1^H, W_1^H) \overset{H}{\sim} (H_2, B_2, F_2^H, W_2^H) \iff (L_1, F_1^L, W_1^L) \overset{L}{\sim} (L_2, F_2^L, W_2^L)$$

where (L_i, F_i^L, W_i^L) is the simple CL system inverse-Legendre-transformed via $\mathbb{F}\hat{H}_i$ from the CH system $(\hat{H}_i, B_{\mathrm{can}}, \hat{F}_i^H, \hat{W}_i^H)$ for $i = 1, 2$, respectively.

Here is another way to state this result which is shorter and perhaps easier to digest, but gives a little less information.

Corollary 4.2. *The method of controlled Lagrangians is equivalent to that of controlled Hamiltonians for simple mechanical systems in the following sense. For any two simple CL systems (L_i, F_i^L, W_i^L), $i = 1, 2$, there exist two associated simple CH systems (H_i, B_i, F_i^H, W_i^H), $i = 1, 2$, such that*

$$(L_1, F_1^L, W_1^L) \overset{L}{\sim} (L_2, F_2^L, W_2^L) \iff (H_1, B_1, F_1^H, W_1^H) \overset{H}{\sim} (H_2, B_2, F_2^H, W_2^H),$$

and vice-versa.

---

[2]The notation and terminology used in this theorem can be found in the body of the paper as follows. Definition **??** and Definition **??** give the definitions of CL-equivalence, $\overset{L}{\sim}$, and the CH-equivalence relation, $\overset{H}{\sim}$. Definition **??** and Definition **??** give the definitions of simple CL systems and simple CH systems. The beginning of Section **??** provides the definition of the Legendre transformation and the construction of ψ_1 and ψ_2. Proposition **??** and the remarks before it discuss the definition of pull-back systems.

5. The Method of Collocated/Non-Collocated Partial Feedback Linearization

Even though the controlled Lagrangian method was originally developed for stabilization of equilibria using energy shaping and dissipation, here we show that one can put the collocated/non-collocated partial feedback linearization property of underactuated mechanical systems into the controlled Lagrangian framework (see Spong's work [?] and references therein for more about the collocated/non-collocated partial feedback linearization).

For simplicity, take $Q = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ to be the configuration space and use $q = (q_1, q_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ as coordinates. Let $(L, F = 0, W)$ be a given CL system (with no external forces) and with $W(q) = 0 \times \mathbb{R}^{n_2}$ for all $q \in Q$ and

$$L(q, \dot{q}) = \frac{1}{2} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \begin{bmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22}(q) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - V(q).$$

Then the Euler-Lagrange equation of this system can be written in the following form (as in [?]):

$$
\begin{aligned}
m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + h_1(q, \dot{q}) + \phi_1(q) &= 0, \\
m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + h_2(q, \dot{q}) + \phi_2(q) &= \tau
\end{aligned}
$$

with control $\tau : TQ \to W$ where h_i includes all $\dot{q}$-dependent terms and ϕ_i contains the terms from the potential energy.

Collocated Linearization. Define the new CL system (L_c, F_c, W_c) as follows:

$$
\begin{aligned}
L_c(q, \dot{q}) &= \frac{1}{2}\dot{q}_1^T m_{11}\dot{q}_1 + \frac{1}{2}\dot{q}_2^T\dot{q}_2 \\
F_c(q, \dot{q}) &= \begin{bmatrix} -h_1 - \phi_1 + (\mathbf{d}m_{11}[\dot{q}])\dot{q}_1 - \frac{\partial L_c}{\partial q_1} \\ -\frac{\partial L_c}{\partial q_2} \end{bmatrix} \\
W_c &= \langle \mathbf{d}q_2^i - (m_{12})_{\alpha i}\mathbf{d}q_1^\alpha \mid i = 1, \ldots, n_2 \rangle \\
&= \text{the subbundle spanned by the columns of } \begin{bmatrix} -m_{12}(q) \\ I_{n_2} \end{bmatrix}
\end{aligned}
$$

where I_{n_2} is the $n_2 \times n_2$ identity matrix. If a part of F_c is a potential force, i.e., the differential of a function, one can combine it into the Lagrangian L_c as a potential function. A control to the system (L_c, F_c, W_c) can be written via a map $u_c : TQ \to \mathbb{R}^{n_2}$ as follows:

$$\begin{bmatrix} -m_{12}(q) \\ I_{n_2} \end{bmatrix} u_c(q, \dot{q}) \in W_c(q). \tag{53}$$

One can check that the given system $(L, 0, W)$ is CL-equivalent to the system (L_c, F_c, W_c) and the Euler-Lagrange equation of (L_c, F_c, W_c) is given by

$$
\begin{aligned}
m_{11}\ddot{q}_1 + h_1 + \phi_1 &= -m_{12}u_c \tag{54} \\
\ddot{q}_2 &= u_c \tag{55}
\end{aligned}
$$

with $u_c : TQ \to \mathbb{R}^{n_2}$. If we write the control for (L_c, F_c, W_c) in the form (**??**), then equation (**??**) is written as

$$\tau = h_2 + \phi_2 - m_{21}m_{11}^{-1}(h_1 + \phi_1) + (m_{22} - m_{21}m_{11}^{-1}m_{12})u_c$$

so that the closed-loop system $(L, 0, \tau)$ and (L_c, F_c, u_c) produce the same equations of motion. This coincides with the notion of collocated partial feedback linearization in [?].

As already indicated at the end of Section 2, if the control bundle is integrable, it allows us to find a set of local coordinates for the configuration space Q which is convenient for doing stability analysis (as was the case in [3]). The integrability condition of the control bundle W_c is given by (the curvature condition)

$$\frac{\partial A_\alpha^i}{\partial q_2^j} A_\beta^j - \frac{\partial A_\beta^i}{\partial q_2^j} A_\alpha^j + \frac{\partial A_\alpha^i}{\partial q_1^\beta} - \frac{\partial A_\beta^i}{\partial q_1^\alpha} = 0$$

for $1 \le \alpha, \beta \le n_1$ and $1 \le i, j \le n_2$ where A_α^i is the (i,α)-th element of the matrix $m_{21} = m_{12}^T$ and $(q_1, q_2) = (q_1^\alpha, q_2^i)$; this is seen from the fact that W_c is spanned by the set $\{dq_2^i - A_\alpha^i dq_1^\alpha \mid 1 \le \alpha \le n_1, 1 \le i \le n_2\}$.

Non-Collocated Linearization. For non-collocated linearization, [?] makes the assumption that the submatrix $m_{12}(q)$ is onto, i.e., $\text{rank}\,(m_{12}(q)) = l$ for all $q \in Q$. Then, there is a pseudo-inverse $m_{12}^\dagger := m_{12}^T(m_{12}m_{12}^T)^{-1}$ such that $m_{12}m_{12}^\dagger = I_{n_1}$ with I_{n_1} the $n_1 \times n_1$ identity matrix. For this assumption to hold, it is necessary that $n_2 \ge n_1$, i.e., the number of actuation degrees of freedom should be at least as big as the number of unactuated degrees of freedom. The same property was used in [?].

Define the system (L_n, F_n, W_n) as follows:

$$\begin{aligned}
L_n(q, \dot{q}) &= \frac{1}{2}\dot{q}_1^T \dot{q}_1 + \frac{1}{2}\dot{q}_2^T \dot{q}_2, \\
F_n(q, \dot{q}) &= \begin{bmatrix} 0 \\ -m_{12}(q)^\dagger(h_1(q, \dot{q}) + \phi_1(q)) \end{bmatrix} \\
W_n &= \langle \mathbf{d}q_1^i - (m_{12}^\dagger m_{11})_{\alpha i} \mathbf{d}q_2^\alpha \mid i = 1, \dots, n_1 \rangle \\
&= \text{subbundle spanned by the columns of } \begin{bmatrix} I_{n_1} \\ -m_{12}(q)^\dagger m_{11}(q) \end{bmatrix}
\end{aligned}$$

where, as above, one can move any potential force parts of F_n into the Lagrangian L_n as a potential function. A control for the system (L_n, F_n, W_n) can be written via a map $u_n : TQ \to \mathbb{R}^{n_1}$ as follows:

$$\begin{bmatrix} I_{n_1} \\ -m_{12}(q)^\dagger m_{11}(q) \end{bmatrix} u_n(q, \dot{q}) \in W_n(q). \tag{56}$$

Notice that $\dim W_n(q) \le \dim W(q)$. Hence, it is appropriate to use the concept of CL-inclusion rather than CL-equivalence. Indeed, one can easily check that the given system $(L, 0, W)$ *includes* the system (L_n, F_n, W_n) and the Euler-Lagrange equations of (L_n, F_n, W_n) are written as

$$\begin{aligned}
\ddot{q}_1 &= u_n \\
\ddot{q}_2 &= -m_{12}^\dagger m_{11}(u_n + h_1 + \phi_1)
\end{aligned}$$

with $u_n : TQ \to \mathbb{R}^{n_2}$. If we write the control for (L_n, F_n, W_n) in the form (**??**), then condition (**??**) is written as

$$\tau = h_2 + \phi_2 - m_{22}m_{12}^\dagger(h_1 + \phi_1) + (m_{21} - m_{22}m_{12}^\dagger m_{11})u_n$$

such that the two closed-loop systems $(L, 0, \tau)$ and (L_n, F_n, u_n) produce the same equations of motion. This coincides with the non-collocated partial feedback linearization in [?].

Future Directions.

1. It reasonable to expect that one can apply the same techniques to nonholonomic systems, following the work of [?] and [?, ?].

2. It would be interesting to consider the case of degenerate almost Poisson structures. This would take one into the realm of Dirac structures; see [**?**, **?**, **?**, **?**, **?**].

3. Reduction theory for CL and CH systems with symmetry will be the subject of a companion paper.

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