The Hamiltonian structure of a two-dimensional rigid circular cylinder interacting dynamically with *N* point vortices

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This paper studies the *dynamical* fluid plus rigid-body system consisting of a two-dimensional rigid cylinder of general cross-sectional shape interacting with *N* point vortices. We derive the equations of motion for this system and show that, in particular, if the vortex strengths sum to zero and the rigid-body has a circular shape, the equations are Hamiltonian with respect to a Poisson bracket structure that is the sum of the rigid body Lie–Poisson bracket on $\mathfrak{se}(2)^*$, the dual of the Lie algebra of the Euclidean group on the plane, and the canonical Poisson bracket for the dynamics of *N* point vortices in an *unbounded* plane. We then use this Hamiltonian structure to study the linear and nonlinear stability of the moving Föppl equilibrium solutions using the energy-Casimir method.
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I. INTRODUCTION

The interaction of rigid and deformable bodies with incompressible, vortical fluid flow fields at high Reynolds numbers and, in particular, the interaction with the vortices shed by the bodies themselves due to the dynamics of their thin boundary layers, has been a subject of long-standing interest in fluid mechanics. The classical von Karman vortex street behind a circular cylinder may be considered the archetypical problem in this subject. Viscous effects are confined to the thin boundary layers and it is a reasonable approximation to model the interactions on an inviscid basis as long as the mechanism of vortex shedding itself can be considered unimportant to the dynamics of the interaction.

The subject has, of course, been extensively explored in the traditional aeronautics, mechanical, and civil engineering areas along with numerous applications. Indeed, in areas like aeronautics, strong (though not spatially extensive) vorticity fields are almost always in the vicinity of aircraft wings and bodies. The highly nonlinear nature of these interactions has for a long time, however, ruled out the possibility of sophisticated mathematical modeling. Typical and popular models have usually followed semi-empirical approaches where most of the nonlinear behavior is accounted for by the force coefficients whose values are obtained from experimental data assembled in look-up tables. Moreover, the effect of the moving body on the vorticity field is usually ignored except in acoustical and aeroelastic studies and here too one typically looks only at small oscillations of the body or body surface.

Mathematical advances in nonlinear dynamics in the past three decades, especially in the area of geometrical mechanics, and emerging engineering applications like the design of remotely piloted underwater vehicles,[1] have motivated the authors to look at this subject from the point of view of geometric mechanics and develop, at least on an ideal fluid level, sophisticated nonlinear models to study the dynamics *and* control of these systems. In particular, one overall goal is to study the role of vorticity and model the dynamics of the system as a whole; that is, we want to allow the body and vorticity field to interact freely or in a controlled manner and develop coupled PDEs or ODEs for the simultaneous evolution of the body variables and the vorticity field. Our specific goal in the present paper is to carry this out for a two-dimensional (2D) rigid body interacting dynamically with *N* point vortices. Interacting fluid–solid systems in such a framework have not been well-studied. Indeed, the authors are not aware of the existence in literature of even the equations of motion of the simple system we are considering in this paper. We use these equations to study Hamiltonian structures and stability in the case of rigid bodies with a circular cross section.

Our long-term goal is to understand the geometry, dynamics, and control of a three-dimensional (3D) rigid or deformable body moving in the vortical field of an incompressible, inviscid fluid. Apart from the design of underwater

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vehicles, we expect such investigations to have relevance and applications in several other areas in engineering and physics: high angle-of-attack aerodynamics, the locomotion of fish that shed vortices by flapping their tails,[2] and the dynamics of bubbles.[3] Theoretical investigations in these areas are not new. There are several papers that derive integral expressions for forces and moments on moving bodies in vortical fields, for example,[4,5] but these do not consider the dynamics of the interacting fluid–solid system. The work of Galper and Miloh[6,7] has a dynamics perspective, however, they extend Kirchhoff's equations of motion to the case of a nonuniform potential flow field superimposed on the potential field associated with the moving rigid or deformable body. Extension to vortical fields is not considered. Kadtke and Novikov[8] consider a dynamically interacting vortex-cylinder system but with only one point vortex and their focus is on chaotic capture. The works that come closest to addressing our problem are Koiller[9] and Kelly.[10] We hope to subsequently apply to these problems the many ideas in nonlinear stability, relative equilibria and control that have been developed in the general geometric theory of mechanics[11] and also introduce viscous effects such as boundary layers.[12]

We will focus on some first steps toward this goal: to understand the Hamiltonian structure and stability of a 2D rigid cylinder that interacts *dynamically* with N point vortices external to it. This system may be viewed, in the context of geometric mechanics, as the blend of two simpler, classical systems each with a well-known Hamiltonian structure. One is the system of a 2D rigid cylinder moving in a field with *zero* vorticity. The equations of motion of this system, derived by Kirchhoff, can be shown to be Hamiltonian[1] with respect to the Lie–Poisson bracket structure on $\mathfrak{se}(2)^*$, the dual of the Lie algebra of the Euclidean group on the plane. The other system is that of N point vortices moving externally to a closed, rigid, *stationary* boundary. The equations of the vortices were shown by Lin[13] to be Hamiltonian with respect to the same canonical symplectic structure as that of N vortices in an *unbounded* plane.

We present in this paper the equations of motion of the dynamically interacting system for a cylinder of general cross-sectional shape and show that, at least for circular shapes and when the vortex strengths sum to zero, the equations are Hamiltonian with respect to a Poisson bracket structure that is simply the *sum* of the brackets of the two, simpler systems referred to above, i.e., *Lie–Poisson plus canonical point vortex*. The reason we assume that the sum of the point vortex strengths is zero is that, in this case, as we shall show later on, the relevant momenta depend only on the positions of the vortices with respect to the body and this simplifies matters considerably. We do not expect that this is a fundamental restriction.

The equations of motion are derived from a standard momentum balance analysis in the plane. The flow is assumed to be inviscid, incompressible, at rest at infinity and satisfies the zero normal velocity condition on the body. In the last subsection of this paper the Hamiltonian structure is used to investigate the linear and nonlinear stability of the Föppl equilibrium.[14,15] Stability and control issues of this system in particular those with relevance to some of the areas of application mentioned above will be studied in more detail later.

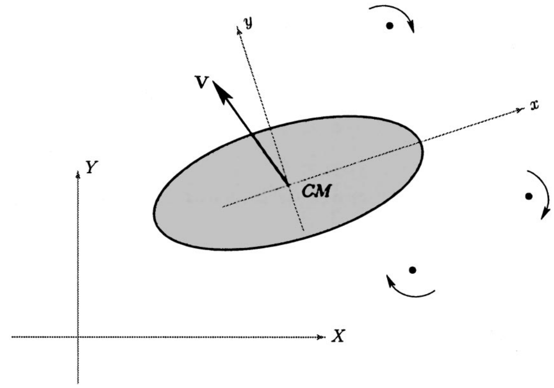

FIG. 1. A 2D rigid cylinder interacting dynamically with point vortices. *XY* is a reference frame fixed in space, while *xy* is a body-fixed frame with origin at the center of mass CM and axes parallel to the principal directions.

II. HAMILTONIAN FORM FOR THE DYNAMICS OF A MOVING CIRCULAR CYLINDER OF RADIUS *R*, AND *N* POINT VORTICES

In this section we present the Hamiltonian equations of a circular cylinder of radius R interacting dynamically with N point vortices in the plane whose strengths sum to zero. A schematic sketch of the configuration when the cylinder has arbitrary shape and the vortex strengths have arbitrary sum is given in Fig. 1. The equations of motion for that more general problem are derived in the Appendix and the equations for the circular case follow directly from them. The general equations (A55), (A56), (A57), and (A58) are reproduced below for convenience:

$$\left(\frac{d}{dt}+\mathbf{\Omega}\times\right)\mathbf{L}=0, \quad \frac{d\mathbf{A}}{dt}+\mathbf{V}\times\mathbf{L}=0,$$

$$\Gamma_k\left(\frac{d\mathbf{l}_k}{dt}+\mathbf{\Omega}\times\mathbf{l}_k+\mathbf{V}\right)=J\left(\frac{\partial W}{\partial \mathbf{l}_k}\right), \quad k=1,\ldots,N,$$

$$\frac{d\mathbf{a}}{dt}=\mathbf{V}+\mathbf{a}\times\mathbf{\Omega},$$

where $\mathbf{V}$ is the velocity of the body center of mass referred to the body-fixed frame, $\mathbf{a}$ is the position vector, referred to the body-fixed frame, of the body center of mass from the origin of the spatially fixed frame, $\mathbf{\Omega}$ is the body rotational velocity, $\mathbf{L}$ and $\mathbf{A}$ are the momenta of the system given by Eq. (A44), $\mathbf{l}_k$ is the position vector of the kth point vortex in the body-fixed frame, and W is the Kirchhoff–Routh function generalized to moving boundaries and given by Eq. (A53).

Let the velocity of the center of mass of the circular cylinder be $\mathbf{V}(t)=[u(t),v(t)]$. Then, with reference to Eqs. (A49) and (A50),

$$\sum \Gamma_k\psi_B(\mathbf{l}_k,\mathbf{V})=R^2\sum \Gamma_k\left\langle\left(-\frac{\sin\theta_k}{l_k},\frac{\cos\theta_k}{l_k}\right),\mathbf{V}\right\rangle, \tag{1}$$

where (l_k,θ_k) are polar coordinates of the kth vortex in the body-fixed frame. Note that ψ_B is independent of $\mathbf{\Omega}$ since the rotation of the circular cylinder has no effect on the fluid. Conversely, the fluid also has no effect on $\mathbf{\Omega}$ since the pressure forces act through the center of the cylinder. Therefore, the equations of motion should give $d\mathbf{\Omega}/dt=0$ and this can indeed be confirmed.

The functions G and g for a circular cylinder can be calculated using the classical circle theorem of Milne-Thomson.[16] This gives a simple representation of the image vorticity in terms of two point vortices—one of the same strength but opposite sign at the inverse point and the other of the same strength and sign at the center of the circle—for each point vortex outside the circle. Thus,

$$g(x,y;x_k,y_k)=\frac{1}{2\pi}\log|(x,y)|-\frac{1}{2\pi}\log|(x,y)-(R^2x_k/l_k^2,R^2y_k/l_k^2)|, \tag{2}$$

where $l_k^2=x_k^2+y_k^2$. Using Eqs. (A48) and (1) the function W can then be easily calculated. For future reference we write

$$W=\sum \Gamma_k\psi_B(\mathbf{l}_k,\mathbf{V})+W_G, \tag{3}$$

where

$$W_G=\sum_{k,j(k>j)}\Gamma_k\Gamma_jG(\mathbf{l}_k;\mathbf{l}_j)+\frac{1}{2}\sum \Gamma_k^2g(\mathbf{l}_k;\mathbf{l}_k).$$

Evaluating the mass matrix M shows that all off-diagonal terms vanish and, further, the first two diagonal terms are the same and are each equal to the *mass plus added mass* of the system. Denoting these terms by c, M simplifies to

$$M=\begin{bmatrix} c & 0 & 0\\ 0 & c & 0\\ 0 & 0 & I\end{bmatrix},$$

where $c=m+\pi R^2$. Therefore,

$$\mathbf{L}=c\mathbf{V}+\mathbf{p},\quad \mathbf{A}=\pi,$$

assuming $\Omega(0)=0$. Next, calculate

$$\mathbf{p}=\sum \Gamma_j\mathbf{l}_j\times\mathbf{k}+\oint_{\partial B}\mathbf{l}\times(\mathbf{n}_b\times(\mathbf{u}_V)_b)ds, \tag{4}$$

$$\pi=-\frac{1}{2}\sum \Gamma_j\langle\mathbf{l}_j,\mathbf{l}_j\rangle\mathbf{k}-\frac{1}{2}\oint_{\partial B}l^2(\mathbf{n}_b\times(\mathbf{u}_V)_b)ds. \tag{5}$$

In this problem $\mathbf{l}=R\mathbf{n}_b$ on the body boundary. The contour integral in $\mathbf{p}$ simplifies as

$$\oint_{\partial B}\mathbf{l}\times(\mathbf{n}_b\times(\mathbf{u}_V)_b)ds=-R\oint_{\partial B}(\mathbf{u}_V)_b\,ds,$$

and that in π vanishes:

$$\oint_{\partial B}l^2(\mathbf{n}_b\times(\mathbf{u}_V)_b)ds=0.$$

It can be checked after performing the necessary integrations that

$$\oint_{\partial B}(\mathbf{u}_V)_b\,ds=\sum \mathbf{k}\times\Gamma_k\left(-\frac{R}{l_k}\cos\theta_k,-\frac{R}{l_k}\sin\theta_k\right).$$

Comparing with Eq. (1) it is seen that the following relation holds in this problem:

$$\sum \Gamma_k\psi_B(\mathbf{l}_k,\mathbf{V})=\oint_{\partial B}\mathbf{l}\times(\mathbf{n}_b\times(\mathbf{u}_V)_b)ds. \tag{6}$$

The general significance of this relation is not yet understood but it plays a simplifying role when constructing the Hamiltonian structure of this system. It is conjectured that an insight into this relation may help understand the Hamiltonian structure for general body shapes.

The equations of motion for the circular cylinder when the vortex strengths sum to zero can now be deduced from Eqs. (A55), (A56), (A57), and (A58). A fairly simple Hamiltonian structure for these equations emerges by inspection. The details are presented below.

For the case when the vortex strengths do not sum to zero this structure does not hold and it is obvious, by looking at the equations, that this is related to the center of mass of the cylinder becoming an additional dynamical variable in the problem. We believe, however, that there exists a Hamiltonian structure for this case too and will be revealed by invoking the same theories mentioned at the end of the Appendix for the problem of general body shapes.

Proposition. *The freely interacting system of a rigid circular cylinder of radius R in an incompressible, inviscid fluid, and N point vortices whose strengths sum to zero and are external to it, is governed by the following system of equations:*

$$\frac{d\mathbf{L}}{dt}=0, \tag{7}$$

$$\frac{d\mathbf{A}}{dt}+\mathbf{V}\times\mathbf{L}=0, \tag{8}$$

$$\Gamma_k\frac{d\mathbf{l}_k}{dt}=-J\frac{\partial H}{\partial\mathbf{l}_k},\quad k=1,\dots,N, \tag{9}$$

where

$$\mathbf{L}(t)=\mathbf{L}(0)=c\mathbf{V}+\sum \Gamma_k\mathbf{l}_k\times\mathbf{k}+R^2\sum \mathbf{k}\times\Gamma_k\left(\frac{x_k}{x_k^2+y_k^2},\ \frac{y_k}{x_k^2+y_k^2}\right)$$

and

$$H(\mathbf{L},\mathbf{l}_k)=-W(\mathbf{L},\mathbf{l}_k)+\frac{1}{2c}\langle\mathbf{L},\mathbf{L}\rangle-\frac{1}{c}\Bigg(\sum\Gamma_k(\mathbf{L}\times\mathbf{l}_k)\cdot\mathbf{k}-\frac{1}{2}\sum\Gamma_k^2\langle\mathbf{l}_k,\mathbf{l}_k\rangle-\sum_k\sum_{j(j>k)}\Gamma_k\Gamma_j\langle\mathbf{l}_k,\mathbf{l}_j\rangle+\frac{R^4}{2}\left\langle\sum\Gamma_k\frac{\mathbf{l}_k}{\langle\mathbf{l}_k,\mathbf{l}_k\rangle},\sum\Gamma_k\frac{\mathbf{l}_k}{\langle\mathbf{l}_k,\mathbf{l}_k\rangle}\right\rangle\Bigg). \tag{10}$$

In the preceding equation, $W(\mathbf{L},\mathbf{l}_k)$ is the Kirchhoff–Routh function for the system and is given by Eq. (3) with $\mathbf{V}$ *re-written in terms of* $\mathbf{L}$ *and* $\mathbf{l}_k$. *This is a Poisson vector field on the space* $P=\mathfrak{se}(2)^*\times(\mathbf{R}^{2N}\setminus(\Delta\cup B))\equiv P_b\times P_v$ *equipped with the following Poisson bracket. For* $F,\ G\in C^\infty(P)$, *define*

$$\{F,G\}=\{F|_{P_b},G|_{P_b}\}_{\text{Lie–Poisson}}+\{F|_{P_v},G|_{P_v}\}_{\text{point vortex}}.$$

Therefore if $p(t)=(\mu(t),\mathbf{l}_k(t))\in P$ *is an integral curve of the system, where* $\mu(t)=(\mathbf{L}(t),A(t))$, *then*

$$\frac{dF}{dt}:=\left\langle\nabla_pF,\frac{dp}{dt}\right\rangle=\langle\nabla_\mu F,ad^*_{\partial H/\partial\mu}\mu\rangle-\sum_{k=1}^{N}\langle\nabla_kF,J^{-1}\nabla_k(H/\Gamma_k)\rangle.$$

Proof. This is a straightforward exercise: one verifies that the right hand side defines a vector field that is obtained from the given Hamiltonian and Poisson bracket.

In verifying the momentum equations, recall that the Lie–Poisson equations on $\mathfrak{se}(2)^*$ are given by

$$\frac{d\mu}{dt}=ad^*_{\delta H/\delta\mu}\mu,\quad \mu\in\mathfrak{g}^*,\quad \delta H/\delta\mu\in\mathfrak{g}$$

for the Hamiltonian H and where

$$ad^*_{(\hat{\delta},w)}(\alpha,s)=(-\langle s,Jw\rangle,-\hat{\delta}s).$$

Making the identification $\mu=(\alpha,s)=(A,\mathbf{L})$:

$$ad^*_{(\partial H/\partial A,\partial H/\partial\mathbf{L})}(A,\mathbf{L})=\left(-\left\langle\mathbf{L},J\frac{\partial H}{\partial\mathbf{L}}\right\rangle,-\frac{\partial H}{\partial A}J\mathbf{L}\right).$$

Now if the momentum equations (7) and (8) are Lie–Poisson, we should have

$$\mathbf{V}\times\mathbf{L}=\left\langle\mathbf{L},J\frac{\partial H}{\partial\mathbf{L}}\right\rangle,\quad 0=\frac{\partial H}{\partial A}J\mathbf{L}.$$

These relations are satisfied if

$$\frac{\partial H}{\partial A}=0,\quad \frac{\partial H}{\partial\mathbf{L}}=\mathbf{V}.\qquad\square$$

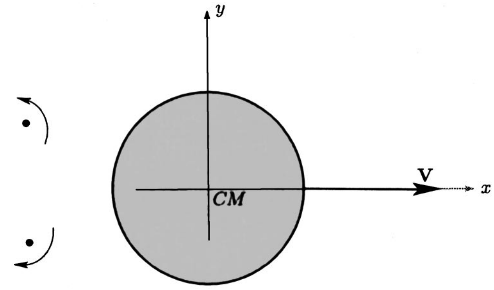

FIG. 2. The Föppl equilibrium when the cylinder moves with constant velocity **V**.

Comments.

(1) Δ in the definition of the phase space P is the set of collision points of vortices and B is the region occupied by the circle.

(2) The system reduces to the correct Hamiltonian system in the two well-known cases: (i) the irrotational case, i.e., no point vortices in the flow and (ii) the stationary body case. In case (i), one obtains the equations of motion for the body as $d\mathbf{V}/dt=0$ and $H=(1/2c)\langle\mathbf{L},\mathbf{L}\rangle=(c/2)\times\langle\mathbf{V},\mathbf{V}\rangle$. Kirchhoff's equations give exactly the same result. In case (ii) $\mathbf{V}=0$, and the terms within the large parentheses in Eq. (10) reduce to $(1/2)\langle\mathbf{L},\mathbf{L}\rangle$ and one obtains $H=-W_G(\mathbf{l}_k)$. The system thus reduces to the one investigated by Lin.[13]

(3) The Hamiltonian can be re-written in terms of $\mathbf{V}$ and $\mathbf{l}_k$ as

$$\begin{aligned}H(\mathbf{V},\mathbf{l}_k)&=-\sum\Gamma_k\psi_B(\mathbf{V},\mathbf{l}_k)-W_G(\mathbf{l}_k)\\&\quad+\frac{1}{2c}\langle c\mathbf{V}+\mathbf{p},c\mathbf{V}+\mathbf{p}\rangle\\&\quad-\frac{1}{c}\left(c\mathbf{V}\times\left(\sum\Gamma_k\mathbf{l}_k\right)\cdot\mathbf{k}+\frac{1}{2}\langle\mathbf{p},\mathbf{p}\rangle\right)\\&=-W_G(\mathbf{l}_k)+\frac{c}{2}\langle\mathbf{V},\mathbf{V}\rangle.\end{aligned}$$

Using Eq. (21) and the L^2-orthogonality of $\mathbf{u}_V$ and $\nabla\Phi_B$ it can be checked that the above is the total kinetic energy of the system (fluid+body) minus the infinite contributions associated with the point vortex velocity field.

III. LINEAR AND NONLINEAR STABILITY OF THE MOVING FÖPPL EQUILIBRIUM

Consider Föppl's results[14,15,17] for equilibria of the system of a circular cylinder in an ambient uniform stream of velocity **V** and two counter-rotating point vortices of equal strength behind the cylinder located symmetrically with respect to the freestream direction. The same results hold in a translating frame if the cylinder moves with velocity V in a fluid at rest at infinity, the point vortices now move with the cylinder at the same velocity and are stationary in the body-fixed frame, as shown in Fig. 2. We call this equilibrium the *moving* Föppl equilibrium to distinguish it from the classical case.

The loci of equilibrium positions are described by the curves

$$l_0^2 - R^2 = \pm 2 l_0 y_0, \tag{11}$$

where

$$l_0^2 = x_0^2 + y_0^2,$$

(x_0, y_0) and $(x_0, -y_0)$ being the positions of the two vortices in the body-fixed frame. At each equilibrium position, there is a linear relation between the vortex strength Γ and V:

$$\Gamma = 4\pi V y_0 \frac{l_0^4 - R^4}{l_0^4}. \tag{12}$$

Linear stability results of the classical Föppl equilibrium[14,15] show that the point vortices are unstable to anti-symmetric infinitesimal perturbations and stable to symmetric ones. Numerical simulations of the perturbed trajectories for finite disturbances have been investigated in de Laat and Coene.[18] An analytic investigation of the nonlinear stability using the second term in the Taylor expansion has been done in Tordella.[19]

A. Linear stability

Analysis of the linear stability of the *moving* Föppl equilibrium differs from the classical one in the following three ways. First, any perturbation of the vortex positions also introduces a perturbation of the cylinder velocity because of the coupled dynamics. The phase space of the system, P, (whose equilibrium we are studying) is larger by two dimensions due to the presence of the additional variable $\mathbf{L}$. The linearized equations for $\mathbf{L}$ are trivial [as can be seen from Eq. (7)], however, the linearized vector field for the point vortex locations has extra terms in it compared to the classical case. Second, because the phase space is larger, the complete set of equilibria defined by Eqs. (11) and (12) defines *a curve* in P. In other words, the equilibria are *not* isolated fixed points in phase space. Third, the eigenvalue behavior of the linearized system under symmetric disturbances is different from that in the classical case, as we shall see. The main details of the linear stability analysis are given below.

The linearized equations about the moving Föppl equilibrium are

$$\begin{pmatrix} \frac{d\,\delta x_1}{dt} \\ \frac{d\,\delta y_1}{dt} \\ \frac{d\,\delta x_2}{dt} \\ \frac{d\,\delta y_2}{dt} \\ \frac{d\,\delta L_x}{dt} \\ \frac{d\,\delta L_y}{dt} \end{pmatrix} = D \cdot \begin{pmatrix} \delta x_1 \\ \delta y_1 \\ \delta x_2 \\ \delta y_2 \\ \delta L_x \\ \delta L_y \end{pmatrix},$$

where D is the 6×6 stability matrix given by

$$D = -\frac{1}{\Gamma}\begin{pmatrix} -\frac{\partial^2 H}{\partial x_1\,\partial y_1} & -\frac{\partial^2 H}{\partial y_1^2} & -\frac{\partial^2 H}{\partial x_2\,\partial y_1} & -\frac{\partial^2 H}{\partial y_2\,\partial y_1} & -\frac{\partial^2 H}{\partial L_x\,\partial y_1} & -\frac{\partial^2 H}{\partial L_y\,\partial y_1} \\ \frac{\partial^2 H}{\partial x_1^2} & \frac{\partial^2 H}{\partial x_1\,\partial y_1} & \frac{\partial^2 H}{\partial x_1\,\partial x_2} & \frac{\partial^2 H}{\partial x_1\,\partial y_2} & \frac{\partial^2 H}{\partial x_1\,\partial L_x} & \frac{\partial^2 H}{\partial x_1\,\partial L_y} \\ \frac{\partial^2 H}{\partial x_1\,\partial y_2} & \frac{\partial^2 H}{\partial y_1\,\partial y_2} & \frac{\partial^2 H}{\partial x_2\,\partial y_2} & \frac{\partial^2 H}{\partial y_2^2} & \frac{\partial^2 H}{\partial L_x\,\partial y_2} & \frac{\partial^2 H}{\partial L_y\,\partial y_2} \\ -\frac{\partial^2 H}{\partial x_1\,\partial x_2} & -\frac{\partial^2 H}{\partial y_1\,\partial x_2} & -\frac{\partial^2 H}{\partial x_2^2} & -\frac{\partial^2 H}{\partial y_2\,\partial x_2} & -\frac{\partial^2 H}{\partial x_2\,\partial L_x} & -\frac{\partial^2 H}{\partial x_2\,\partial L_y} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{|F.e.}.$$

The δ quantities denote infinitesimal perturbations and the D is evaluated at the moving Föppl equilibrium. The Hamiltonian for the case of a circular cylinder translating with velocity $\mathbf{V}$ (variable) and two vortices of equal and opposite strengths, Γ and $-\Gamma$, located at (x_1, y_1) and (x_2, y_2), respectively, is given by

$$H = -W_G(\mathbf{l}_k) + \frac{1}{2c}[\langle \mathbf{L},\mathbf{L}\rangle + \langle \mathbf{p},\mathbf{p}\rangle - 2\langle \mathbf{p},\mathbf{L}\rangle],$$

where

$$W_G(x_1,y_1;x_2,y_2)=\frac{\Gamma^2}{8\pi}\log\left(\frac{l_2^2}{l_1^2}\right)-\frac{\Gamma^2}{8\pi}\log\left[\left(x_1-R^2\frac{x_1}{l_1^2}\right)^2+\left(y_1-R^2\frac{y_1}{l_1^2}\right)^2\right]-\frac{\Gamma^2}{8\pi}\log\left[\left(x_2-R^2\frac{x_2}{l_2^2}\right)^2+\left(y_2-R^2\frac{y_2}{l_2^2}\right)^2\right]$$
$$+\frac{\Gamma^2}{4\pi}\log\left[\left(x_1-R^2\frac{x_2}{l_2^2}\right)^2+\left(y_1-R^2\frac{y_2}{l_2^2}\right)^2\right]-\frac{\Gamma^2}{4\pi}\log[(x_1-x_2)^2+(y_1-y_2)^2]$$
$$=-\frac{\Gamma^2}{8\pi}\log[(l_1^2-R^2)^2]-\frac{\Gamma^2}{8\pi}\log[(l_2^2-R^2)^2]-\frac{\Gamma^2}{4\pi}\log[(x_1-x_2)^2+(y_1-y_2)^2]$$
$$+\frac{\Gamma^2}{4\pi}\log[l_1^2l_2^2+R^4-2R^2(x_1x_2+y_1y_2)],$$

$$\mathbf{L}=c\mathbf{V}+\mathbf{p},$$

and

$$\mathbf{p}=\Gamma\left(y_1-y_2+R^2\left(\frac{y_2}{l_2^2}-\frac{y_1}{l_1^2}\right),\quad x_2-x_1+R^2\left(\frac{x_1}{l_1^2}-\frac{x_2}{l_2^2}\right)\right).$$

At the Föppl equilibrium the vortex positions are related by

$$x_2=x_1=:x_0,\quad y_2=-y_1=:-y_0$$

and one also has

$$L_x=cV+2\Gamma y_0\left(1-\frac{R^2}{l_0^2}\right),\quad L_y=0,\tag{13}$$

where V is the constant translational velocity of the cylinder which can, without loss of generality, be taken in the direction of the positive x-axis (see Fig. 2).

B. Evaluating the stability matrix

To evaluate the eigenvalues of the stability matrix, we follow a procedure similar to the one in the classical Föppl case. At any point $p\in P$, we split the tangent space T_pP as

$$T_pP=\mathbf{F}^s\oplus(\mathbf{F}^s)^c.\tag{14}$$

Here, $\mathbf{F}^s$ is the space of *symmetric* disturbances. It is a three-dimensional subspace of T_pP and is defined by the relations

$$\delta x_1=\delta x_2=:\delta x_s,\quad \delta y_1=-\delta y_2=:\delta y_s,\quad \delta L_y=0.$$

Note that $\mathbf{F}^s$ is an invariant subspace under the vector field of the linearized system. The complementary space $(\mathbf{F}^s)^c$ is the space of *anti-symmetric* disturbances and is defined by the relations

$$\delta x_1=-\delta x_2=:\delta x_a,\quad \delta y_1=\delta y_2=:\delta y_a,\quad \delta L_x=0.$$

It follows from Eq. (14) that $(\mathbf{F}^s)^c$ is also an invariant subspace. The direct sum in Eq. (14) is defined as follows. Write any vector $(\delta x_1,\delta y_1,\delta x_2,\delta y_2,\delta L_x,\delta L_y)\in T_pP$ as

$$\delta x_1=\delta x_s+\delta x_a,\quad \delta y_1=\delta y_s+\delta y_a,\quad \delta x_2=\delta x_s-\delta x_a,$$

$$\delta y_2=-\delta y_s+\delta y_a,\quad \delta L_y=0+\delta L_y,\quad \delta L_x=\delta L_x+0,$$

where $(\delta x_s,\delta y_s,\delta L_x)\in\mathbf{F}^s$ and $(\delta x_a,\delta y_a,\delta L_y)\in(\mathbf{F}^s)^c$.

The above linear change of variables helps identify the eigenvalue behavior of the linearized dynamics in each of $\mathbf{F}^s$ and $(\mathbf{F}^s)^c$ separately. Denote by M the nonsingular matrix that takes the vector $(\delta x_1,\delta y_1,\delta x_2,\delta y_2,\delta L_x,\delta L_y)$ to the vector $(\delta x_s,\delta y_s,\delta L_x,\delta x_a,\delta y_a,\delta L_y)$. The linear stability matrix D assumes the following block form after the transformation of variables:

$$M^{-1}DM=-\frac{1}{\Gamma}\begin{pmatrix}S & U\\ 0 & A\end{pmatrix},$$

where S, A and U are 3×3 matrices.

The matrix A has the following two nontrivial eigenvalues:

$$\lambda_a^2=(a_{21}-a_{23})(a_{12}+a_{14})+(a_{11}-a_{13})^2,$$
$$=-\left(\frac{\partial^2H}{\partial x_1^2}-\frac{\partial^2H}{\partial x_1\,\partial x_2}\right)\left(\frac{\partial^2H}{\partial y_1^2}+\frac{\partial^2H}{\partial y_1\,\partial y_2}\right)$$
$$+\left(\frac{\partial^2H}{\partial x_2\,\partial y_1}-\frac{\partial^2H}{\partial x_1\,\partial y_1}\right)^2,$$

and the matrix S has the following two nontrivial eigenvalues:

$$\lambda_s^2=(a_{21}+a_{23})(a_{12}-a_{14})+(a_{11}+a_{13})^2,$$
$$=\left(\frac{\partial^2H}{\partial x_1^2}+\frac{\partial^2H}{\partial x_1\,\partial x_2}\right)\left(\frac{-\partial^2H}{\partial y_1^2}+\frac{\partial^2H}{\partial y_1\,\partial y_2}\right)$$
$$+\left(\frac{\partial^2H}{\partial x_2\,\partial y_1}+\frac{\partial^2H}{\partial x_1\,\partial y_1}\right)^2.$$

The eigenvalues are functions of the parameter

$$\alpha=\frac{R^2}{l_0^2}.$$

The plots in Figs. 3 and 4 show that $\lambda_s^2>0$ for $0\leqslant\alpha<\alpha_0$ and $\lambda_s^2<0$ for $1>\alpha>\alpha_0$, whereas $\lambda_a^2>0$ everywhere in the domain of α. The plots are of $\lambda^2 f(\alpha)$ vs α, where

$$f(\alpha)=\frac{16\pi^2l_0^4}{\Gamma^4}(1-\alpha)^2(1-\alpha^2)^2.$$

The plots may be interpreted in the following manner. Fix R and, hence, the curve of Föppl equilibrium as per Eq. (11). Then the plots give us the linear stability of the system for different vortex locations on that curve. Alternatively, one could fix l_0 and vary R. The plots then give the linear stability of the system for different vortex locations, all with the same value of l_0 but lying on different Föppl curves.

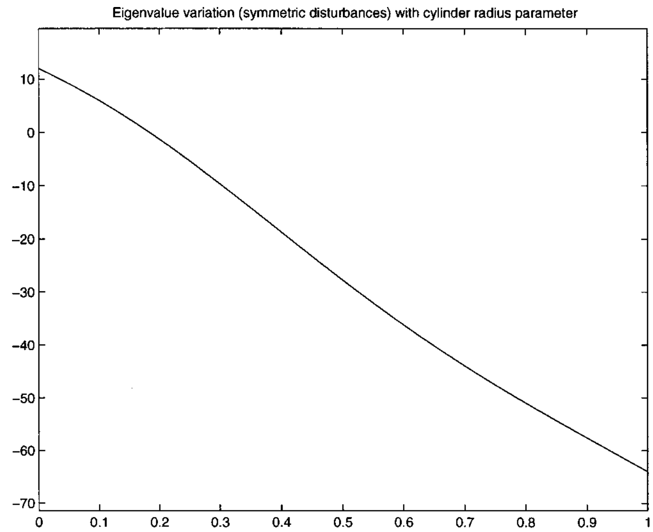

FIG. 3. The plot of $\lambda_s^2 f(\alpha)$ vs $\alpha = R^2/l_0^2$ in the case of symmetric infinitesimal disturbances.

In both interpretations, however, vanishing α corresponds to vanishing effect of the cylinder motion on the system. Indeed, in the limit $\alpha = 0$, the moving Föppl equilibrium becomes the equilibrium of two point vortices of equal but opposite strengths in an unbounded flow translating with velocity $\Gamma/(4\pi y_0)$.

From Fig. 3, it is seen that one gets linear instability for small values of α for symmetric disturbances in contrast to the linear stability for all α of the classical Föppl equilibrium.

To summarize, in the case of infinitesimal *anti-symmetric* disturbances one gets the same results for the moving Föppl equilibria as for the classical equilibria, that is, the equilibria are linearly *unstable* for all α. However, for the case of infinitesimal *symmetric* disturbances there is a difference. There is a range of values $0 \leqslant \alpha < \alpha_0$ for which the moving equilibria on the Föppl curve are linearly *unstable*.

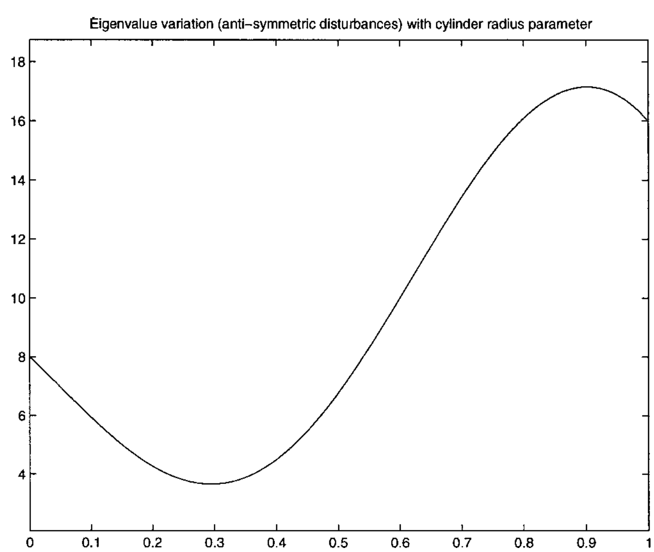

FIG. 4. The plot of $\lambda_a^2 f(\alpha)$ vs $\alpha = R^2/l_0^2$ in the case of anti-symmetric infinitesimal disturbances.

This result is completely missed in the analysis of the classical equilibria where one gets linear stability for all α. The reason for this is the artificial constraint imposed on the dynamics by keeping the free-stream velocity constant which is equivalent to keeping the cylinder velocity fixed in our moving system. It is possible that this instability, when properly understood, can be harnessed for some motion planning goal by a suitable control mechanism.

C. Nonlinear stability and the energy-Casimir method

The study of the stability of the Föppl equilibria to *finite* perturbations does not seem to have been previously undertaken. A weakly nonlinear stability analysis has been done in Tordella.[19] The Hamiltonian structure described in the last section strongly suggests that one can carry out a complete nonlinear stability analysis of the moving equilibria using the *energy-Casimir method*.[11] Nonlinear stability here refers to the Lyapunov definition.

The energy-Casimir method involves showing the existence of $\Phi(C)$, where C is a Casimir function of the system, such that the first variation of the augmented Hamiltonian function

$$H_\Phi = H + \Phi(C) \tag{15}$$

vanishes at the Föppl equilibrium and the second variation quadratic form is positive or negative definite. This is a *sufficient* condition for stability, in the Lyapunov sense, to finite disturbances.

It follows from the linear stability results that one cannot expect nonlinear stability of the moving Föppl equilibria to *arbitrary* finite disturbances. However, stability to *symmetric* finite disturbances can be expected and this is what we show below.

A set of Casimirs for the system of N point vortices and a circular cylinder, as can be easily checked, is

$$C = k_1 \langle \mathbf{L}, \mathbf{L} \rangle + k_2,$$

where k_1 and k_2 are scalar constants. Without loss of generality, one can assume $k_1 = 1$ and $k_2 = 0$. Consider now variations of the function

$$H_\Phi = H + \Phi(C).$$

For the first derivative of this to vanish at the Föppl equilibrium,

$$\frac{\partial(H + \Phi(C))}{\partial \mathbf{L}}_{|F.e.} = 0, \tag{16}$$

which implies

$$\Phi'(C) \frac{\partial C}{\partial \mathbf{L}}_{|F.e.} = -\frac{\partial H}{\partial \mathbf{L}}_{|F.e.} = \frac{1}{c}\left(\Gamma \frac{2y_0(l_0^2 - R^2)}{l_0^2} - L_x(0) \right), \tag{17}$$

and so

$$\Phi'(C)_{|F.e.}=\frac{1}{2cL_x(0)}\left(\Gamma\frac{2y_0(l_0^2-R^2)}{l_0^2}-L_x(0)\right) \tag{18}$$

and since $\partial C/\partial A=\partial C/\partial x_1=.....=\partial C/\partial y_2=0$ the first derivatives of $H+\Phi(C)$ with respect to these variables also vanish.

Now compute the second derivatives. The only nontrivial second derivatives of $\Phi(C)$ at the Föppl equilibria are

$$\left.\frac{\partial^2\Phi(C)}{\partial L_x^2}\right|_{F.e.}=\Phi'(C)_{|F.e.}\frac{\partial^2 C}{\partial L_x^2}+\Phi''(C)_{|F.e.}\left(\frac{\partial C}{\partial L_x}\right)^2$$
$$=2\Phi'(C)_{|F.e.}+\Phi''(C)(2L_x(0))^2,$$

$$\left.\frac{\partial^2\Phi(C)}{\partial L_y^2}\right|_{F.e.}=\Phi'(C)_{|F.e.}\frac{\partial^2 C}{\partial L_y^2}+\Phi''(C)_{|F.e.}\left(\frac{\partial C}{\partial L_y}\right)^2$$
$$=2\Phi'(C)_{|F.e.}.$$

Note that at this point the $\Phi'(C)_{|F.e.}$ has been determined by the vanishing first variation condition Eq. (18). However, $\Phi''(C)_{|F.e.}$ is undetermined and will be used as a handle to make (if possible) the matrix of second variations positive or negative definite.

The matrix of second variations of H_Φ is given by

$$W=\begin{pmatrix}
\frac{\partial^2 H_\Phi}{\partial x_1^2} & \frac{\partial^2 H_\Phi}{\partial x_1\,\partial y_1} & \frac{\partial^2 H_\Phi}{\partial x_1\,\partial x_2} & \frac{\partial^2 H_\Phi}{\partial x_1\,\partial y_2} & \frac{\partial^2 H_\Phi}{\partial L_x\,\partial x_1} & \frac{\partial^2 H_\Phi}{\partial L_y\,\partial x_1}\\
. & \frac{\partial^2 H_\Phi}{\partial y_1^2} & \frac{\partial^2 H_\Phi}{\partial y_1\,\partial x_2} & \frac{\partial^2 H_\Phi}{\partial y_1\partial y_2} & \frac{\partial^2 H_\Phi}{\partial L_x\,\partial y_1} & \frac{\partial^2 H_\Phi}{\partial L_y\partial y_1}\\
. & . & \frac{\partial^2 H_\Phi}{\partial x_2^2} & \frac{\partial^2 H_\Phi}{\partial x_2\,\partial y_2} & \frac{\partial^2 H_\Phi}{\partial L_x\,\partial x_2} & \frac{\partial^2 H_\Phi}{\partial L_y\,\partial x_2}\\
. & . & . & \frac{\partial^2 H_\Phi}{\partial y_2^2} & \frac{\partial^2 H_\Phi}{\partial L_x\,\partial y_2} & \frac{\partial^2 H_\Phi}{\partial L_y\,\partial y_2}\\
. & . & . & . & \frac{\partial^2 H_\Phi}{\partial L_x^2} & \frac{\partial^2 H_\Phi}{\partial L_x\,\partial L_y}\\
. & . & . & . & . & \frac{\partial^2 H_\Phi}{\partial L_y^2}
\end{pmatrix}.$$

Denote the elements of W by w_{ij}. Using the various relations between the second derivatives at the equilibria we get

$$W=\begin{pmatrix}
w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & w_{16}\\
\cdot & w_{22} & -w_{14} & w_{24} & w_{25} & -w_{15}\\
\cdot & \cdot & w_{11} & -w_{12} & w_{15} & w_{16}\\
\cdot & \cdot & \cdot & w_{22} & w_{25} & -w_{15}\\
\cdot & \cdot & \cdot & \cdot & w_{55} & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & w_{66}
\end{pmatrix}.$$

The second variation quadratic form is

$$\delta^2 H_\Phi=(\delta x_1,\delta y_1,\delta x_2,\delta y_2,\delta L_x,\delta L_y)$$
$$\cdot W(\delta x_1,\delta y_1,\delta x_2,\delta y_2,\delta L_x,\delta L_y)^T.$$

Performing the change of variables as in the linear stability analysis gives

$$\delta^2 H_\Phi=(\delta x_s,\delta y_s,\delta L_x,\delta x_a,\delta y_a,\delta L_y)$$
$$\cdot M^T W M(\delta x_s,\delta y_s,\delta L_x,\delta x_a,\delta y_a,\delta L_y)^T,$$

where

$$M^TWM=\begin{pmatrix}
2w_{11}+2w_{13} & 2(-w_{14}+w_{12}) & 2w_{15} & 0 & 0 & 2w_{16}\\
2w_{12}-2w_{14} & 2(-w_{24}+w_{22}) & 0 & 0 & 0 & 0\\
2w_{15} & 0 & w_{55} & 0 & 2w_{25} & 0\\
0 & 0 & 0 & 2(w_{11}-w_{13}) & 2(w_{12}+w_{14}) & 0\\
0 & 0 & 2w_{25} & 2(w_{12}+w_{14}) & 2(w_{22}+w_{24}) & -2w_{15}\\
2w_{16} & 0 & 0 & 0 & -2w_{15} & w_{66}
\end{pmatrix}.$$

We next check this matrix for definiteness. Since

$$\begin{aligned} w_{11}+w_{13} &= \left(\frac{\partial^2 H_\Phi}{\partial x_1^2}+\frac{\partial^2 H_\Phi}{\partial x_1\,\partial x_2}\right) \\ &= \frac{\Gamma^2\alpha^2}{4\pi R^2} \\ &\quad\times\left[\frac{(\alpha-3)(1+2\alpha^2+\alpha)+(1+\alpha)^2(3-\alpha)(1-\alpha)^4}{(1-\alpha)^2(1+\alpha)}\right] \\ &< 0 \end{aligned}$$

(as can be checked by plotting), we check for *negative* definiteness. The second order principal minor is

$$PM_2=-4\lambda_s^2>0, \quad \alpha_0<\alpha<1,$$

where λ_s is an eigenvalue of the linearized system under symmetric disturbances and α_0 is the value at which the plot in Fig. 3 crosses the α-axis. The third order principal minor is

$$PM_3=w_{55}PM_2-8w_{15}^2(w_{22}-w_{24}).$$

Since

$$(w_{22}-w_{24})=\left(\frac{\partial^2 H_\Phi}{\partial y_1^2}-\frac{\partial^2 H_\Phi}{\partial y_1\,\partial y_2}\right)<0, \quad 0<\alpha<1,$$

we get

$$PM_3<0 \Leftrightarrow w_{55}<[8w_{15}^2(w_{22}-w_{24})]/PM_2.$$

Since

$$w_{55}=\frac{\partial^2 H_\Phi}{\partial L_x^2}=\frac{1}{c}+2\Phi'(C)_{|F.e.}+\Phi''(C)_{|F.e.}(2L_x(0))^2,$$

we can make w_{55} as small as possible by a suitable choice of $\Phi''(C)_{|F.e.}$.

Recall that the symmetric subspace is an invariant subspace under the linearized dynamics. It is not difficult to see that *finite* symmetric perturbations of the equilibria also lead to symmetric motions for all time. Hence there exists a symmetric *submanifold* of the phase space P which is invariant under the *full* dynamics. Indeed, using the theory of discrete reduction[11] one can show that this submanifold is the fixed point set under the action of the discrete group $\mathbb{Z}_2$ and is thus a symplectic submanifold of the phase space P. It is invariant under Hamiltonian vector fields on P.

Consequently the upper left 3×3 block of the matrix M^TWM can be viewed as the matrix of second variations of the Hamiltonian subsystem on the symmetric submanifold. The above calculations show that this block is *negative definite* in the range $\alpha_0<\alpha<1$ and this is a sufficient condition for nonlinear stability. Hence, we make the following proposition.

Proposition. *In the range of the radius parameter R^2/l_0^2 where the moving Föppl equilibria are linearly stable to infinitesimal, symmetric disturbances, they are also nonlinearly Lyapunov stable to finite, symmetric disturbances.*

Continuing with the calculations, one finds that the fourth order minor is given by

$$PM_4=2(w_{11}-w_{13})PM_3<0,$$

since

$$w_{11}-w_{13}>0.$$

Hence the matrix M^TWM fails to be positive or negative definite and no sufficient condition for nonlinear stability to arbitrary finite disturbances emerges. This is consistent with the result of linear instability to arbitrary infinitesimal disturbances.

ACKNOWLEDGMENTS

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APPENDIX: EQUATIONS OF MOTION OF A 2D RIGID CYLINDER OF ARBITRARY SHAPE IN AN INVISCID, INCOMPRESSIBLE, VORTICAL FIELD

In this Appendix we derive the equations for the *dynamical* interaction of a 2D rigid body of arbitrary shape moving in a fluid with vorticity.

1. Smooth vorticity fields

The equations are first derived for a smooth vorticity field and then specialized to a field of point vortices. A schematic sketch in the case of point vortices is shown in Fig. 1.

a. Linear momentum

We start by deriving an expression for the linear momentum of the fluid. We make use of the following vector identity[17] (p. 65 in cited reference):

$$\int \mathbf{a}\,dA=\int(\mathbf{r}\times\mathrm{curl}\,\mathbf{a})dA+\oint \mathbf{r}\times(\mathbf{n}\times\mathbf{a})ds, \tag{A1}$$

where $\mathbf{a}$ is a divergence-free vector field on some bounded domain $A\subset\mathbb{R}^2$, $\mathbf{r}$ is the position vector with respect to some fixed reference frame, $\mathbf{n}$ is the unit *inward* normal vector on the boundary. Now let $\mathbf{a}=\mathbf{u}=$ the velocity field of the flow. Let C_R denote a fixed circular boundary of radius R centered on some arbitrary point in the domain. C_R encloses the body and *all* of the vorticity (for all time). Let ∂B denote the moving boundary of the body. Then the momentum of the fluid (of constant, unit density) in the domain A_R between these two boundaries is

$$\begin{aligned} \int_{A_R}\mathbf{u}\,dA &= \int_{A_R}(\omega\mathbf{r}\times\mathbf{k})dA+\oint_{\partial B}\mathbf{r}\times(\mathbf{n}\times\mathbf{u})ds \\ &\quad+\oint_{C_R}\mathbf{r}\times(\mathbf{n}\times\mathbf{u})ds, \end{aligned} \tag{A2}$$

where $\omega\mathbf{k}=\mathrm{curl}\,\mathbf{u}$ is the vorticity field, $\mathbf{k}$ being the unit vector normal to the plane. Note that the normal in the body contour integral points *away* from the body and the normal in the C_R contour integral points radially *inward*. Counterclockwise circulation is considered positive with the associated vorticity vector pointing out of the plane.

Write

$$\mathbf{u}=\nabla\Phi_B+\mathbf{u}_V, \tag{A3}$$

where the notation will now be explained. First of all, $\nabla\Phi_B$ denotes the curl-free velocity field in $\mathbb{R}^2\backslash B$ (here, $B\subset\mathbb{R}^2$ is the region occupied by body) determined uniquely by the motion of the body satisfying the boundary conditions:

$$\nabla\Phi_B\cdot\mathbf{n}=\mathbf{q}\cdot\mathbf{n}\ \text{ on }\ \partial B, \tag{A4}$$

$$\nabla\Phi_B\rightarrow 0,\ \ R\rightarrow\infty, \tag{A5}$$

where $\mathbf{q}$ is the velocity of the body boundary point. Also, $\mathbf{u}_V$ denotes the velocity field due to the vorticity satisfying the boundary conditions

$$\mathbf{u}_V\cdot\mathbf{n}=0\ \text{ on }\ \partial B, \tag{A6}$$

$$\mathbf{u}_V\rightarrow 0,\ \ R\rightarrow\infty. \tag{A7}$$

It should be noted that $\mathbf{u}_V=\mathbf{u}_0+\mathbf{u}_I$, where $\mathbf{u}_0$ is the velocity field due to the vorticity in the absence of boundaries and is naturally defined on all of $\mathbb{R}^2$ ($\mathbf{u}_0\rightarrow 0$ as $R\rightarrow\infty$, and $\nabla\times\mathbf{u}_0=0$ in B). $\mathbf{u}_I$ is the velocity field that is curl-free in $\mathbb{R}^2\backslash B$ and is hence *uniquely determined in* $\mathbb{R}^2\backslash B$ by the boundary conditions

$$\mathbf{u}_I\cdot\mathbf{n}=-\mathbf{u}_0\cdot\mathbf{n}\ \text{ on }\ \partial B, \tag{A8}$$

$$\mathbf{u}_I\rightarrow 0\ \text{ as }\ R\rightarrow\infty. \tag{A9}$$

Now apply Newton's second law to the fluid in A_R. The following assumptions are made during the derivation: the force of gravity on the fluid is balanced by the hydrostatic pressure, there is no other external force on the fluid, the total vorticity in the fluid is constant in time, there is no circulation around the body and the weight of the body is balanced by the force of buoyancy. We further make the simplifying assumption that the fluid and body have constant, uniform density equal to unity. Hence,

$$\mathbf{F}_S+\oint_{C_R}p_R\mathbf{n}\,ds=\frac{d}{dt}\int_{A_R}\mathbf{u}\,dA-\oint_{C_R}\mathbf{u}(\mathbf{u}\cdot\mathbf{n})ds,$$

where $\mathbf{F}_S$ is the force (per unit span) exerted by the solid on the fluid at the boundary ∂B and is equal and opposite to that exerted by the fluid on the solid (denoted by $-\mathbf{F}_S$), $\oint_{C_R}p_R\mathbf{n}\,ds$ is the total contribution of the pressure forces acting on C_R, and $\oint_{C_R}\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\ ds$ is the net flux of momentum across C_R. Since $-\mathbf{F}_S=A_b(d\mathbf{U}/dt)$, where A_b is the cross-sectional area of the cylinder, we get the following vector equation for the system comprising of a rigid body and an incompressible, inviscid fluid in the domain A_R:

$$A_b\frac{d\mathbf{U}}{dt}+\frac{d}{dt}\oint_{\partial B}\mathbf{r}\times(\mathbf{n}\times\nabla\Phi_B)ds+\frac{d}{dt}\int_{A_R}(\omega\mathbf{r}\times\mathbf{k})dA+\frac{d}{dt}\oint_{\partial B}\mathbf{r}\times(\mathbf{n}\times\mathbf{u}_V)ds+\mathbf{P}_R=0, \tag{A10}$$

where

$$\mathbf{P}_R=\frac{d}{dt}\oint_{C_R}\mathbf{r}\times(\mathbf{n}\times\mathbf{u})ds-\oint_{C_R}\mathbf{u}(\mathbf{u}\cdot\mathbf{n})ds-\oint_{C_R}p_R\mathbf{n}\,ds.$$

b. Angular momentum

We use the elementary vector identity[17] (p. 55 and the comment above Eq. (18), p. 65 in cited reference):

$$\int\mathbf{r}\times\mathbf{a}\,dA=-\frac{1}{2}\int(r^2\,\mathrm{curl}\,\mathbf{a})dA-\frac{1}{2}\oint r^2(\mathbf{n}\times\mathbf{a})ds, \tag{A11}$$

where $r=\|\mathbf{r}\|$. Here again $\mathbf{n}$ is the *inward* pointing unit normal. Hence the angular momentum of the fluid in the domain A_R is

$$\int_{A_R}\mathbf{r}\times\mathbf{u}\,dA=-\frac{1}{2}\int_{A_R}\omega r^2\,dA-\frac{1}{2}\oint_{\partial B}r^2(\mathbf{n}\times\mathbf{u})ds-\frac{1}{2}\oint_{C_R}r^2(\mathbf{n}\times\mathbf{u})ds. \tag{A12}$$

Applying Newton's second law for angular momentum for the fluid in A_R, we get

$$\mathbf{M}_S+\oint_{C_R}p_R\mathbf{r}\times\mathbf{n}\,ds=\frac{d}{dt}\int_{A_R}\mathbf{r}\times\mathbf{u}\,dA-\oint_{C_R}\mathbf{r}\times\mathbf{u}(\mathbf{u}\cdot\mathbf{n})ds, \tag{A13}$$

where $\mathbf{M}_S$ is the torque exerted by the solid on the fluid and is equal and opposite to that exerted by the fluid on the solid. The other terms are analogous to those in the force equation. Since $-\mathbf{M}_S=d(A_b\mathbf{b}\times\mathbf{U}+I\mathbf{\Omega})/dt$, where $\mathbf{b}(t)$ is the position vector in the inertial frame of the center of mass of the body, we thus get the following scalar equation from the conservation of angular momentum for the system comprising of a rigid body and an incompressible, inviscid fluid in the domain A_R:

$$\frac{d}{dt}(A_b\mathbf{b}\times\mathbf{U}+I\mathbf{\Omega})-\frac{1}{2}\frac{d}{dt}\oint_{\partial B}r^2(\mathbf{n}\times\nabla\Phi_B)ds-\frac{1}{2}\frac{d}{dt}\int_{A_R}\omega r^2\mathbf{k}\,dA-\frac{1}{2}\frac{d}{dt}\oint_{\partial B}r^2(\mathbf{n}\times\mathbf{u}_V)ds+M_R=0. \tag{A14}$$

Here I is the principal moment of inertia tensor and $\mathbf{\Omega}=\Omega\mathbf{k}$ is the angular velocity of the body (which can be identified as a scalar in this 2D case). The first two terms represent the total angular momentum of the body with respect to the origin of the fixed reference frame, and

$$M_R=-\frac{1}{2}\frac{d}{dt}\oint_{C_R}r^2(\mathbf{n}\times\mathbf{u})ds-\oint_{C_R}p_R(\mathbf{r}\times\mathbf{n})ds-\oint_{C_R}\mathbf{r}\times\mathbf{u}(\mathbf{u}\cdot\mathbf{n})ds. \tag{A15}$$

The contribution of the $\mathbf{P}_R$ *and* M_R *terms*. A straightforward computation shows that the terms $\mathbf{P}_R$ and M_R go to zero in the limit $R\rightarrow\infty$, that is,

$$\mathbf{P}_R=O\left(\frac{1}{R}\right), \tag{A16}$$

$$M_R=O\left(\frac{1}{R}\right). \tag{A17}$$

The details of this simplification are given below, the uninterested reader may directly skip to the next subsection.

The far field behavior of the velocity field is given, at any time t, by

$$\mathbf{u}=u_c\mathbf{s}+\mathbf{u}^{(2)}+\mathbf{u}^{(3)}+O\left(\frac{1}{R^4}\right), \tag{A18}$$

where $\mathbf{s}$ is the unit tangent vector on C_R. The field $u_c\mathbf{s}$ is time-invariant and given by

$$u_c\mathbf{s}=\left(\frac{\int_{A_R}\omega\, dA}{2\pi R}\right)\mathbf{s}. \tag{A19}$$

Here we have made use of the assumption that there is no net circulation about the body and hence $\int_B\omega\, dA=0$. The fields $\mathbf{u}^{(2)}$ and $\mathbf{u}^{(3)}$, which contain the first and second moments of the vorticity distribution, respectively, may be time-varying[20] and their far field behavior is given by

$$\mathbf{u}^{(2)}=O\left(\frac{1}{R^2}\right), \tag{A20}$$

$$\mathbf{u}^{(3)}=O\left(\frac{1}{R^3}\right). \tag{A21}$$

It follows, from the decomposition Eq. (A18), that

$$\oint_{C_R}\mathbf{u}^{(2)}\cdot ds=-\mathbf{k}\cdot\oint_{C_R}\mathbf{n}\times\mathbf{u}^{(2)}\, ds=0, \tag{A22}$$

$$\oint_{C_R}\mathbf{u}^{(3)}\cdot ds=-\mathbf{k}\cdot\oint_{C_R}\mathbf{n}\times\mathbf{u}^{(3)}\, ds=0. \tag{A23}$$

Using $\mathbf{s}\times\mathbf{n}=\mathbf{k}$ and $\mathbf{r}=-R\mathbf{n}$, the integral in the first term in $\mathbf{P}_R$ may be evaluated:

$$\oint_{C_R}\mathbf{r}\times(\mathbf{n}\times\mathbf{u})ds=-\oint_{C_R}R\mathbf{n}\times(\mathbf{n}\times\mathbf{u}^{(2)})ds+O\left(\frac{1}{R}\right). \tag{A24}$$

In the irrotational region traversed by C_R, Eq. (18) can also be written as

$$\mathbf{u}=\nabla\Phi=\nabla\Phi_V+\nabla\Phi^{(2)}+\nabla\Phi^{(3)}+O\left(\frac{1}{R^4}\right),$$

where Φ_V is the *multiple-valued* velocity potential, invariant in time, due to a single vortex of strength $\int_{A_R}\omega\, dA$, and where $\Phi^{(2)}$ is single-valued because of Eq. (A22). We use the identity Eq. (A1) and the divergence theorem to get

$$\oint_{C_R}R\mathbf{n}\times(\mathbf{n}\times\mathbf{u}^{(2)})ds=\oint_{C_R}R\mathbf{n}\times(\mathbf{n}\times\nabla\Phi^{(2)})ds$$
$$=-\oint_{C_R}\Phi^{(2)}\mathbf{n}\, ds. \tag{A25}$$

The leading order term in the second integral in $\mathbf{P}_R$ is easily seen to be $O(1/R^2)$. Using Bernoulli's theorem in the irrotational region traversed by C_R, the pressure integral in $\mathbf{P}_R$ is written as

$$\oint_{C_R}p_R\mathbf{n}\, ds=\oint_{C_R}\left(\frac{\partial\Phi}{\partial t}-\frac{|\mathbf{u}|^2}{2}+f(t)\right)\mathbf{n}\, ds.$$

The only $O(1)$ contribution to the pressure integral comes from the first term on the right. It follows that

$$\frac{d}{dt}\oint_{C_R}R\mathbf{n}\times(\mathbf{n}\times\mathbf{u}^{(2)})ds+\oint_{C_R}\frac{\partial\Phi}{\partial t}\mathbf{n}\, ds=0, \tag{A26}$$

and one obtains Eq. (A16).

The evaluation of M_R proceeds on similar lines. Rewrite the first integral in M_R as

$$\frac{1}{2}\oint_{C_R}\langle\mathbf{r},\mathbf{r}\rangle(\mathbf{n}\times\mathbf{u})ds$$
$$=-\frac{1}{2}R^2\mathbf{k}\int_{A_R}\omega\, dA+\frac{1}{2}R^2\oint_{C_R}(\mathbf{n}\times\mathbf{u}^{(2)})ds$$
$$+\frac{1}{2}R^2\oint_{C_R}(\mathbf{n}\times\mathbf{u}^{(3)})ds+O\left(\frac{1}{R}\right). \tag{A27}$$

On the right hand side, the first term is invariant in time and it follows from Eqs. (A22) and (A23) that the second and third terms vanish. Hence,

$$\frac{d}{dt}\frac{1}{2}\oint_{C_R}\langle\mathbf{r},\mathbf{r}\rangle(\mathbf{n}\times\mathbf{u})ds=0+O\left(\frac{1}{R}\right). \tag{A28}$$

The other terms in M_R give

$$\oint_{C_R}\mathbf{r}\times\mathbf{u}(\mathbf{u}\cdot\mathbf{n})ds=O\left(\frac{1}{R}\right),$$

$$\oint_{C_R}p_R(\mathbf{r}\times\mathbf{n})ds=-\oint_{C_R}p_R(R\mathbf{n}\times\mathbf{n})ds=0 \tag{A29}$$

and one obtains Eq. (A17).

2. Point vortices

Now assume that the given vorticity field is a singular distribution of N point vortices, as shown in Fig. 1.

The vector identities Eqs. (A1) and (A11) are not directly applicable to the given fluid domain but to a *modified* domain in which one removes small circles centered around each point vortex. It can then be shown that the same vector identities hold with the vorticity written as a delta distribution, $\omega(\mathbf{r}_j)=\Sigma\Gamma_j\delta(\mathbf{r}-\mathbf{r}_j)$.

Substituting Eqs. (A16) and (A17) into (A10) and (A14), the following equations in the limit $R\to\infty$ are then obtained:

$$A_b\frac{d\mathbf{U}}{dt}+\frac{d}{dt}\oint_{\partial B}\mathbf{r}\times(\mathbf{n}\times\nabla\Phi_B)ds+\frac{d}{dt}\sum\Gamma_j\mathbf{r}_j\times\mathbf{k}$$
$$+\frac{d}{dt}\oint_{\partial B}\mathbf{r}\times(\mathbf{n}\times\mathbf{u}_V)ds=0, \tag{A30}$$

and

$$\frac{d}{dt}(A_b\mathbf{b}\times\mathbf{U}+I\boldsymbol{\Omega})-\frac{1}{2}\frac{d}{dt}\oint_{\partial B}r^2(\mathbf{n}\times\boldsymbol{\nabla}\Phi_B)ds$$
$$-\frac{1}{2}\frac{d}{dt}\sum\Gamma_j r_j^2\mathbf{k}-\frac{1}{2}\frac{d}{dt}\oint_{\partial B}r^2(\mathbf{n}\times\mathbf{u}_V)ds=0. \quad \text{(A31)}$$

3. Body-fixed frame

We now transfer Eqs. (A30) and (A31), which were derived in a *spatially fixed* or *inertial* frame, XY in Fig. 1, to equations in a *body-fixed* frame. We choose a principal axis frame with origin at the body center of mass, shown as xy in Fig. 1. For a given point in the domain the position vector $\mathbf{r}$ in the inertial frame is related to the position vector $\mathbf{l}$ in the body-fixed frame by

$$\mathbf{r}=R(t)\mathbf{l}+\mathbf{b}(t),$$

where $R(t)\in \mathrm{SO}(2)$ gives the orientation of the body-fixed frame with respect to the inertial frame, and $\mathbf{b}(t)\in\mathbb{R}^2$ is the position vector of the origin of the body-fixed frame measured in the inertial frame. Putting $\mathbf{b}(t)=0$ in the above gives the law for transforming vectors of the same norm. Time derivatives in the inertial frame are related to time derivatives in the body-fixed frame as follows:

$$\frac{d\mathbf{w}}{dt}=R(t)\frac{d\mathbf{v}}{dt}+R(t)(\boldsymbol{\Omega}\times\mathbf{v}),$$

where $\mathbf{w}=R(t)\mathbf{v}$ and $\boldsymbol{\Omega}$ is the angular velocity of the body referred to the body-fixed frame. The following relation is used often:

$$\dot{R}(t)\mathbf{v}=R(t)(\boldsymbol{\Omega}\times\mathbf{v}).$$

We also make repeated use of the following vector identity:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}. \quad \text{(A32)}$$

Transferring Eqs. (A30) and (A31) term by term using the above relations one finally obtains the linear and angular momentum equations as

$$\left(\frac{d}{dt}+\boldsymbol{\Omega}\times\right)\mathbf{L}=0, \quad \text{(A33)}$$

$$\frac{d\mathbf{A}}{dt}+\mathbf{V}\times\mathbf{L}=0, \quad \text{(A34)}$$

where

$$\mathbf{L}=A_b\mathbf{V}+\oint_{\partial B}\mathbf{l}\times(\mathbf{n}_b\times\nabla_b\Phi_B)ds+\sum\Gamma_j\mathbf{l}_j\times\mathbf{k}$$
$$+\oint_{\partial B}\mathbf{l}\times(\mathbf{n}_b\times(\mathbf{u}_V)_b)ds+\left(\sum\Gamma_j\right)\mathbf{a}\times\mathbf{k}, \quad \text{(A35)}$$

$$\mathbf{A}=I\boldsymbol{\Omega}-\frac{1}{2}\oint_{\partial B}l^2(\mathbf{n}_b\times\boldsymbol{\nabla}_b\Phi_B)ds-\frac{1}{2}\sum\Gamma_j\langle\mathbf{l}_j,\mathbf{l}_j\rangle\mathbf{k}$$
$$-\frac{1}{2}\oint_{\partial B}l^2(\mathbf{n}_b\times(\mathbf{u}_V)_b)ds-\frac{1}{2}\left(\sum\Gamma_j\right)\mathbf{a}\times(\mathbf{a}\times\mathbf{k}), \quad \text{(A36)}$$

$\mathbf{U}=R(t)\mathbf{V}$, $\mathbf{b}=R(t)\mathbf{a}$, and the subscript b denotes reference to the body-fixed frame. Since $\mathbf{U}=d\mathbf{b}/dt$, vectors $\mathbf{a}$ and $\mathbf{V}$ are related as

$$\mathbf{V}=\frac{d\mathbf{a}}{dt}+\boldsymbol{\Omega}\times\mathbf{a}. \quad \text{(A37)}$$

The expressions for $\mathbf{L}$ and $\mathbf{A}$ can be written more elegantly as follows. Recall[16] that Φ_B can be linearly decomposed using the *Kirchhoff potentials* as

$$\Phi_B(\mathbf{l},\mathbf{V}(t),\boldsymbol{\Omega}(t))=\mathbf{V}(t)\cdot\boldsymbol{\phi}(\mathbf{l})+\Omega(t)\xi, \quad \text{(A38)}$$

$$=V_x\phi_x+V_y\phi_y+\Omega\xi, \quad \text{(A39)}$$

where the functions ϕ_x, ϕ_y, and ξ are unit potential functions harmonic in the fluid domain, have vanishing gradients at infinity and satisfy the following body boundary conditions:

$$\frac{\partial\phi_x}{\partial n}=n_x \quad \frac{\partial\phi_y}{\partial n}=n_y \quad \frac{\partial\xi}{\partial n}=n_y x-n_x y. \quad \text{(A40)}$$

Making use of Eqs. (A1) and (A11) and the divergence theorem one sees that

$$\oint_{\partial B}\mathbf{l}\times(\mathbf{n}_b\times\nabla_b\Phi_B)ds=\oint_{\partial B}\Phi_B\mathbf{n}_b\,ds, \quad \text{(A41)}$$

and that

$$\frac{1}{2}\oint_{\partial B}l^2(\mathbf{n}_b\times\nabla_b\Phi_B)ds=\oint_{\partial B}\Phi_B(\mathbf{n}_b\times\mathbf{l})ds. \quad \text{(A42)}$$

Using these relations, and the following one obtainable from Green's theorem:

$$\oint_C\left(f\frac{dg}{dn}-g\frac{df}{dn}\right)ds=0, \quad \text{(A43)}$$

for f, g harmonic in a bounded domain, the momentum variables can be re-written as

$$\begin{pmatrix}\mathbf{L}\\ \mathbf{A}\end{pmatrix}=M\begin{pmatrix}\mathbf{V}\\ \boldsymbol{\Omega}\end{pmatrix}+\begin{pmatrix}\mathbf{p}\\ \pi\end{pmatrix}, \quad \text{(A44)}$$

where

$$\mathbf{p}=\sum\Gamma_j\mathbf{l}_j\times\mathbf{k}+\oint_{\partial B}\mathbf{l}\times(\mathbf{n}_b\times(\mathbf{u}_V)_b)ds+\left(\sum\Gamma_j\right)\mathbf{a}\times\mathbf{k}$$

and

$$\pi=-\frac{1}{2}\sum\Gamma_j\langle\mathbf{l}_j,\mathbf{l}_j\rangle\mathbf{k}-\frac{1}{2}\oint_{\partial B}l^2(\mathbf{n}_b\times(\mathbf{u}_V)_b)ds$$
$$-\frac{1}{2}\left(\sum\Gamma_j\right)\mathbf{a}\times(\mathbf{a}\times\mathbf{k}),$$

and M is a 3×3 *symmetric mass tensor* that depends only on the body shape and body mass. Note that if the sum of the point vortex strengths is zero, then $\mathbf{p}$ and π depend only on the positions of the vortices with respect to the body. The contour integrals are uniquely determined from the boundary conditions Eqs. (A8) and (A9).

4. The Kirchhoff–Routh function and the symplectic phase space of the point vortices

The phase space of N point vortices in bounded domains was shown to have a symplectic structure appropriate for the dynamics by Lin.[13] The symplectic form is the same as in unbounded flow, $\Omega_{\mathrm{symp}}=\Sigma\Gamma_k\, dx_k\wedge dy_k$, and the Hamiltonian vector field is

$$\Gamma_k\frac{dx_k}{dt}=-\frac{\partial W}{\partial y_k},\tag{A45}$$

$$\Gamma_k\frac{dy_k}{dt}=\frac{\partial W}{\partial x_k},\tag{A46}$$

where W is the *Kirchhoff–Routh* function given by

$$W=\sum\ \Gamma_k\psi_B(x_k,y_k)+\sum_{k,j(k>j)}\Gamma_k\Gamma_jG(x_k,y_k;x_j,y_j)+\frac{1}{2}\sum\ \Gamma_k^2g(x_k,y_k;x_j,y_j),\tag{A47}$$

with G being a Green's function satisfying appropriate boundary conditions and of the form

$$G(x,y;x_0,y_0)=g(x,y;x_0,y_0)+\frac{1}{4\pi}\log[(x-x_0)^2+(y-y_0)^2],\tag{A48}$$

and ψ_B is the stream function due to agencies other than the point vortices. The function g is harmonic everywhere in the fluid domain and is the stream function of the irrotational velocity field $\mathbf{u}_I$, see Eq. (A8), which annuls the nonzero normal velocities on the body due to the external vortices. All three functions G, g, and ψ_B depend on the body shape.

Lin[13] derived these equations for fixed boundaries. W is an invariant of the motion if ψ_B has no explicit time dependency. The theory remains valid for moving boundaries but W in general will no longer be an invariant. Denote it as follows:

$$W'(\mathbf{r}_k,t)=\sum\ \Gamma_k\psi_B'(\mathbf{r}_k,t)+\sum_{k,j(k>j)}\Gamma_k\Gamma_jG'(\mathbf{r}_k;\mathbf{r}_j;t)+\frac{1}{2}\sum\ \Gamma_k^2g'(\mathbf{r}_k;\mathbf{r}_k;t),\tag{A49}$$

where for any given t the functions G' and g' satisfy the same properties as G and g. To write W' in terms of $\mathbf{l}_k$, $\mathbf{V}(t)$, and $\mathbf{\Omega}(t)$, note that the term $\psi_B'(\mathbf{r}_k,t)$, which in this problem is solely due to the motion of the body, can be written in body-fixed coordinates as

$$\begin{aligned}\psi_B'(\mathbf{r}_k,t)&=\psi_B[\mathbf{l}_k,\mathbf{V}(t),\mathbf{\Omega}(t)],\\&=\mathbf{V}(t)\cdot\eta(\mathbf{l})+\Omega(t)\kappa(\mathbf{l}).\end{aligned}\tag{A50}$$

The fields $\eta(\mathbf{l})$ (of 2-vectors) and $\kappa(\mathbf{l})$ (of 1-vector) depend only on the shape of the body. Their components are the harmonic conjugates of the Kirchhoff potentials that appear in the analogous linear decomposition of the potential function of the irrotational flow associated with the motion of the body Eq. (A38).

We make the following claim for G' and g'. Let $\mathbf{r}$ denote the position vector in the fixed frame and $\mathbf{l}$ the position vector in the body-fixed frame as before. Then we make the following proposition.

Proposition. The following holds:

$$G'(\mathbf{r};\mathbf{r}_0;t)=G(\mathbf{l};\mathbf{l}_0),\tag{A51}$$

$$\begin{aligned}g'(\mathbf{r};\mathbf{r}_0;t)&:=G'(\mathbf{r};\mathbf{r}_0;t)-1/(2\pi)\log\|\mathbf{r}-\mathbf{r}_0\|\\&=G(\mathbf{l};\mathbf{l}_0)-1/(2\pi)\log\|\mathbf{l}-\mathbf{l}_0\|\\&=g(\mathbf{l};\mathbf{l}_0).\end{aligned}\tag{A52}$$

Proof. We check that G' satisfies all the properties outlined by Lin[13] for all t. Note that

$$\nabla G'=R(t)\nabla_bG,\quad \nabla g'=R(t)\nabla_bg,$$

$$\begin{aligned}\nabla\psi_B'(\mathbf{r},t)&=R(t)\nabla_b\psi_B'(R(t)\mathbf{l}+\mathbf{b}(t),t)\\&=R(t)\nabla_b\psi_B(\mathbf{l},t),\end{aligned}$$

$$\nabla^2g'=\nabla_b^2g.$$

(i) $\nabla^2g'=\nabla_b^2g=0$. Hence $g'(\mathbf{r};\mathbf{r}_0;t)$ is harmonic in the domain.

(ii) The condition of zero circulation around the body is

$$\oint_{\partial B}\frac{\partial G'}{\partial n}ds=\oint_{\partial B}\nabla G'\cdot\mathbf{n}\,ds=0.$$

This is satisfied since

$$\oint_{\partial B}\nabla G'\cdot\mathbf{n}\,ds=\oint_{\partial B}R(t)\nabla_bG\cdot R(t)\mathbf{n}_b\,ds=\oint_{\partial B}\nabla_bG\cdot\mathbf{n}_b\,ds=0.$$

(iii) The far-field behavior of G' should be

$$G'(\mathbf{r};\mathbf{r}_0;t)=\frac{1}{2\pi}\log\|\mathbf{r}-\mathbf{r}_0\|+O\left(\frac{1}{\|\mathbf{r}-\mathbf{r}_0\|}\right),$$

$$\frac{\partial G'}{\partial s}=O\left(\frac{1}{\|\mathbf{r}-\mathbf{r}_0\|^2}\right),$$

$$\frac{\partial G'}{\partial n}=\frac{1}{2\pi\|\mathbf{r}-\mathbf{r}_0\|}+O\left(\frac{1}{\|\mathbf{r}-\mathbf{r}_0\|^2}\right).$$

Since $\|\mathbf{r}-\mathbf{r}_0\|=\|\mathbf{l}-\mathbf{l}_0\|$, and using the relations between gradients and vectors in the two frames, one sees that G' does possess the above behavior. □

Thus,

$$\begin{aligned}W'(\mathbf{r}_k,t)&=W(\mathbf{l}_k,\mathbf{V}(t),\mathbf{\Omega}(t))\\&=\sum\ \Gamma_k\psi_B[\mathbf{l}_k,\mathbf{V}(t),\mathbf{\Omega}(t)]\\&\quad+\sum_{k,j(k>j)}\Gamma_k\Gamma_jG(\mathbf{l}_k;\mathbf{l}_j)+\frac{1}{2}\sum\ \Gamma_k^2g(\mathbf{l}_k;\mathbf{l}_k),\end{aligned}\tag{A53}$$

and

$$\frac{\partial W'}{\partial \mathbf{r}_k} = R(t)\,\frac{\partial W}{\partial \mathbf{l}_k}. \tag{A54}$$

The equations of motion of the vortices in the body-fixed frame can then be derived from Eqs. (A45) and (A46) using the above results. For $k=1,.....,N$, this gives

$$\Gamma_k R(t)\left(\frac{d}{dt}\mathbf{l}_k + \boldsymbol{\Omega}\times\mathbf{l}_k + \mathbf{V}\right) = JR(t)\left(\frac{\partial W}{\partial \mathbf{l}_k}\right),$$

$$\Gamma_k\left(\frac{d}{dt}\mathbf{l}_k + \boldsymbol{\Omega}\times\mathbf{l}_k + \mathbf{V}\right) = J\left(\frac{\partial W}{\partial \mathbf{l}_k}\right).$$

Here J is the matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, to summarize, the equations of motion of the dynamically interacting system of a 2D rigid cylinder and N point vortices external to it are

$$\left(\frac{d}{dt} + \boldsymbol{\Omega}\times\right)\mathbf{L} = 0, \tag{A55}$$

$$\frac{d\mathbf{A}}{dt} + \mathbf{V}\times\mathbf{L} = 0, \tag{A56}$$

$$\Gamma_k\left(\frac{d\mathbf{l}_k}{dt} + \boldsymbol{\Omega}\times\mathbf{l}_k + \mathbf{V}\right) = J\left(\frac{\partial W}{\partial \mathbf{l}_k}\right), \quad k=1,...,N, \tag{A57}$$

$$\frac{d\mathbf{a}}{dt} = \mathbf{V} + \mathbf{a}\times\boldsymbol{\Omega}, \tag{A58}$$

where $\mathbf{V}$ is the velocity of the body center of mass referred to the body-fixed frame, $\mathbf{a}$ is the position vector, referred to the body-fixed frame, of the body center of mass from the origin of the spatially fixed frame, $\boldsymbol{\Omega}$ is the body rotational velocity, $\mathbf{L}$ and $\mathbf{A}$ are the momenta of the system given by Eq. (A44), $\mathbf{l}_k$ is the position vector of the kth point vortex in the body-fixed frame, and W is the Kirchhoff–Routh function generalized to moving boundaries and given by Eq. (A53).

This is a $(2N+5)$-dimensional system in the variables $\mathbf{L}$, $\mathbf{A}$, $\mathbf{a}$, and $\mathbf{l}_k$ $(k=1,...,N)$. We note important special cases of these equations: when the vortices are absent, they reduce to Kirchhoff's equations of motion, when the body is absent or stationary, they reduce to the canonical N-point vortex equations and when the fluid is absent, they reduce to the Lie–Poisson equations for a free rigid body.

The Hamiltonian structure of Eqs. (A55), (A56), (A57), and (A58) for general body shapes is, as yet, unknown. We expect, however, a Hamiltonian structure to exist (irrespective of the shape of the body) due to the conservation of the kinetic energy and the fact that the equations form a finite dimensional system. To find this structure and also the associated Lagrangian formulation one has to invoke the full power of reduction theories for systems with symmetry.[21,22] Such a project has already been embarked upon by the authors and Jim Radford (Caltech).

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