DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **2**, Number **1**, February **2002**

Website: http://AIMsciences.org

pp. **57–67**

LYAPUNOV-BASED TRANSFER BETWEEN ELLIPTIC KEPLERIAN ORBITS

Dong Eui Chang
Control and Dynamical Systems 107-81
California Institute of Technology
Pasadena, CA 91125

David F. Chichka
Control and Dynamical Systems 107-81
California Institute of Technology
Pasadena, CA 91125

Jerrold E. Marsden
Control and Dynamical Systems 107-81
California Institute of Technology
Pasadena, CA 91125

Abstract. We present a study of the transfer of satellites between elliptic Keplerian orbits using Lyapunov stability theory specific to this problem. The construction of Lyapunov functions is based on the fact that a non-degenerate Keplerian orbit is uniquely described by its angular momentum and Laplace (-Runge-Lenz) vectors. We suggest a Lyapunov function, which gives a feedback controller such that the target elliptic orbit becomes a locally asymptotically stable periodic orbit in the closed-loop dynamics. We show how to perform a global transfer between two arbitrary elliptic orbits based on the local transfer result. Finally, a second Lyapunov function is presented that works only for circular target orbits.

1. **Introduction.** Low- and moderate-thrust transfer between satellite orbits in an inverse-square gravity field has been a topic of interest for decades. Some of the earliest work in this field is reviewed and extended by Edelbaum [9, 10] where low thrust transfer between elliptic Keplerian orbits was considered. Using variational calculus and considering the effects of thrust to be perturbations about an orbit, Edelbaum derives the optimal thrust histories to effect small changes in orbital elements. His later work extends this to achieve general transfers. More recent work, such as that surveyed in [7], has concentrated on finding optimal trajectories for fixed-time orbit transfer problems between general Keplerian orbits. Generally, the departure and injection points on the respective orbits are defined, as well as the elements of the orbits themselves. Optimal control theory then provides a two-point boundary-value problem, which may be solved to achieve the optimal thrust profile. The resulting calculations are lengthy, and do not lend themselves to closed-form solution or on-line implementation. For the special case of constant acceleration magnitude and fixed transfer time, some simplified results can be obtained.

1991 *Mathematics Subject Classification.* 70F05, 93D15, 93D20.
Key words and phrases. satellite dynamics, feedback stabilization, orbit transfer.

Here, we present a study of the transfer between elliptic orbits about a spherical Earth, in which the final time is not specified and the injection point is free. We define the orbit at all times through the natural quantities of the angular momentum vector and the Laplace, or eccentricity vector. It is shown that every non-degenerate Keplerian orbit can be uniquely described by these two vectors, and conversely, that every such pair defines a unique orbit. We use the difference between current and desired final values of these vectors to define a Lyapunov function. This Lyapunov function gives an asymptotically stabilizing feedback controller such that the target elliptic Keplerian orbit becomes a locally asymptotically stable periodic orbit. We suggest another Lyapunov function for the transfer to circular orbits using the fact that a circular orbit is uniquely determined by its angular momentum (and energy).

A brief exposition of orbit transfer using a Lyapunov function was presented in [12], where the control is based on a function made up of the squares of the errors between the current and final orbital elements. That paper, however, does not provide a full analysis of the method, and convergence is not shown. Our work does provide a different Lyapunov function as well as a rigorous proof of the validity and convergence for the method presented.

The general method of using Lyapunov functions that are mechanically motivated has appeared in literature before, such as in [2], [4], and [5]. However, we believe that this paper is the first to apply such a general methodology to the problem of Keplerian orbit transfer.

2. **Review of the Two Body Problem.** We give a review of some necessary concepts on the two body problem (see [1], [8], [11] among many others for more on orbital mechanics).

The configuration space is $\mathbb{R}^3_0 := \mathbb{R}^3 - \{0\}$, i.e., $\mathbb{R}^3$ minus the origin. Let $T\mathbb{R}^3_0 = (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ be the tangent space of $\mathbb{R}^3_0$. We use $(\mathbf{r}, \dot{\mathbf{r}})$ as coordinates for $T\mathbb{R}^3_0$, and the over-dot as the derivative with respect to time t. The Keplerian equation of motion is given by

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{\|\mathbf{r}\|^3} \tag{1}$$

where μ is the gravitational parameter. We refer to the solutions of (1) as *Keplerian flows* or *Keplerian orbits*. The energy $E : T\mathbb{R}^3_0 \to \mathbb{R}$ is defined by

$$E(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}\|\dot{\mathbf{r}}\|^2 - \frac{\mu}{\|\mathbf{r}\|}. \tag{2}$$

Define $\pi = (\mathbf{L}, \mathbf{A}) : T\mathbb{R}^3_0 \to \mathbb{R}^3 \times \mathbb{R}^3$ by

$$\mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}) = \mathbf{r} \times \dot{\mathbf{r}}, \tag{3}$$

$$\mathbf{A}(\mathbf{r}, \dot{\mathbf{r}}) = \dot{\mathbf{r}} \times (\mathbf{r} \times \dot{\mathbf{r}}) - \mu \frac{\mathbf{r}}{\|\mathbf{r}\|} \tag{4}$$

where $\mathbf{L}$ is the angular momentum and $\mathbf{A}$ is the Laplace vector. The Laplace vector is occasionally referred to as the eccentricity vector (see [3]) because the two are identical, other than a scaling by μ. The three quantities E, $\mathbf{L}$, and $\mathbf{A}$ are constants of the motion of (1) and satisfy the following relations:

$$\mathbf{L} \cdot \mathbf{A} = 0, \tag{5}$$

$$\|\mathbf{A}\|^2 = \mu^2 + 2E\|\mathbf{L}\|^2 \tag{6}$$

where $\| \cdot \|$ is the usual Euclidean norm on $\mathbb{R}^3$.

Let $\mathbf{L}$ be the angular momentum of a Keplerian orbit $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$. If $\mathbf{L} = 0$, then $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ is a degenerate orbit, i.e., $\mathbf{r}(t)$ moves in a straight line. If $\mathbf{L} \neq 0$, then $\mathbf{r}(t)$ traces an ellipse, a parabola, or a hyperbola, depending upon its energy E being negative, zero, or positive, respectively. We will exclude degenerate orbits from consideration. Hence the set

$$\Sigma_e = \{(\mathbf{r}, \dot{\mathbf{r}}) \in T\mathbb{R}^3_0 \mid E(\mathbf{r}, \dot{\mathbf{r}}) < 0, \, \mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}) \neq 0\} \tag{7}$$

becomes the union of all elliptic Keplerian orbits. Define the set

$$D = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{y} = 0, \mathbf{x} \neq 0, \|\mathbf{y}\| < \mu\}. \tag{8}$$

By (5) – (8), it follows that

$$\pi(\Sigma_e) \subset D \text{ and } \pi(T\mathbb{R}^3_0 - \Sigma_e) \cap D = \varnothing \tag{9}$$

which implies

$$\pi^{-1}(D) = \Sigma_e. \tag{10}$$

For any $(\mathbf{x}, \mathbf{y}) \in D$, take

$$(\mathbf{r}, \dot{\mathbf{r}}) = \begin{cases} (\frac{-1}{2H}\frac{1-e}{e}\mathbf{y}, \frac{-2H}{\mu^2}\frac{1}{e(1-e)}\mathbf{x} \times \mathbf{y}) & \text{if } \mathbf{y} \neq 0 \\ (\frac{1}{-2H}(\mathbf{p} \times \mathbf{x}), \mathbf{p}) & \text{if } \mathbf{y} = 0 \end{cases}$$

where $H = (||\mathbf{y}||^2 - \mu^2)/(2||\mathbf{x}||^2)$, $e = ||\mathbf{y}||/\mu$, and $\mathbf{p}$ is a vector satisfying $\mathbf{p} \cdot \mathbf{x} = 0$ with $||\mathbf{p}|| = \sqrt{-2H}$. It is simple to show that $(\mathbf{r}, \dot{\mathbf{r}}) \in \Sigma_e$ and $\pi(\mathbf{r}, \dot{\mathbf{r}}) = (\mathbf{x}, \mathbf{y})$. This implies $D \subset \pi(\Sigma_e)$, which with (9) implies

$$\pi(\Sigma_e) = D. \tag{11}$$

Since $\mathbf{L}$ and $\mathbf{A}$ are constants of the motion of (1), equations (10) and (11) imply that $\pi^{-1}(\mathbf{x}, \mathbf{y})$ consists of a union of elliptic Keplerian orbits for each $(\mathbf{x}, \mathbf{y}) \in D$. Let $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ be any elliptic Keplerian orbit contained in $\pi^{-1}(\mathbf{L}, \mathbf{A}) \subset T\mathbb{R}^3_0$. Since $\mathbf{L}$ is normal to both $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$, the orbit $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ is contained in the set $\Pi \times \Pi$, where $\Pi \subset \mathbb{R}^3$ is the plane through the origin normal, to $\mathbf{L}$. The polar equation (r, θ) of the ellipse traced by $\mathbf{r}(t)$ on the plane Π is given by

$$r = \frac{\|\mathbf{L}\|^2}{\mu + \|\mathbf{A}\| \cos(\theta - \theta_0)} \tag{12}$$

where θ_0 is the polar angle of the periapsis when the orbit is a non-circular ellipse, i.e, when $\mathbf{A} \neq 0$. The tangent vector $\dot{\mathbf{r}}$ at $\mathbf{r}$ is derived from (3) and (4) as follows:

$$\dot{\mathbf{r}} = \frac{\mathbf{L}}{\|\mathbf{L}\|^2} \times \left(\mathbf{A} + \frac{\mu \mathbf{r}}{\|\mathbf{r}\|}\right).$$

It follows that $\pi^{-1}(\mathbf{L}, \mathbf{A})$ consists of a unique (oriented) elliptic Keplerian orbit for $(\mathbf{L}, \mathbf{A}) \in D$. Thus, we have proved the following Proposition.

Proposition 2.1. *The following holds:*

1. Σ_e *is the union of all elliptic Keplerian orbits.*
2. $\pi(\Sigma_e) = D$ *and* $\Sigma_e = \pi^{-1}(D)$.
3. *The fiber* $\pi^{-1}(\mathbf{x}, \mathbf{y})$ *consists of a unique (oriented) elliptic Keplerian orbit for each* $(\mathbf{x}, \mathbf{y}) \in D$.

The following result follows directly from this.

Corollary 2.1. D *is the space of elliptic Keplerian orbits.*

Another important consequence is the following.

COROLLARY 2.2. *The set* $\pi^{-1}(K)$ *is a compact subset of* Σ_e *for any compact subset* K *of* D.

Proof. Take any compact set $K \subset D$. By Proposition 2.1, $\pi^{-1}(K) \subset \Sigma_e$. Choose any sequence $\{a_k\} \subset \pi^{-1}(K)$. Let $b_k = \pi(a_k)$. Since K is compact, $\{b_k\}$ has a convergent subsequence. By passing to the subindex, we assume that $\{b_k\}$ is convergent to some $b \in K$. Then $\pi^{-1}(b)$ is compact since it is homeomorphic to the unit circle by Proposition 2.1. By the continuity of π, the sequence $\{a_k\}$ converges to $\pi^{-1}(b)$. Choose a metric on Σ_e. Let $c_k \in \pi^{-1}(b)$ be a closest point from a_k to $\pi^{-1}(b)$ for each k. Since $\pi^{-1}(b)$ is compact and a distance function is continuous, the sequence $\{c_k\}$ is well-defined. Since $\pi^{-1}(b)$ is compact $\{c_k\}$ has a convergent subsequence $\{c_{k_j}\}$ with a limit $c \in \pi^{-1}(b)$. One can see that $\{a_{k_j}\}$ converges to $c \in \pi^{-1}(b) \subset \pi^{-1}(K)$. Thus, $\pi^{-1}(K)$ is compact. □

REMARK 1. *For notational simplicity, we will sometimes identify a point* $(\mathbf{x}, \mathbf{y}) \in D$ *with the set* $\pi^{-1}(\mathbf{x}, \mathbf{y}) \subset \Sigma_e$.

3. Main Results.

Based on the results in the last section, we design a controller for orbital transfer between two arbitrary elliptic Keplerian orbits by constructing a suitable Lyapunov function. We consider first the case of local transfer, where the initial orbit is within a neighborhood of the target orbit. We then extend the results to transfer between two arbitrary elliptic orbits. Finally, we suggest another Lyapunov function for circular target orbits.

3.1. Local Orbit Transfer.

We design here a Lyapunov-based controller to achieve asymptotically stable local orbit transfer. The equation of motion with a control force $\mathbf{F}$ is given by

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{\|\mathbf{r}\|^3} + \mathbf{F}. \tag{13}$$

Define a metric d_k on $\mathbb{R}^3 \times \mathbb{R}^3$ by

$$\mathrm{d}_k((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)) = \sqrt{\frac{1}{2}k\|\mathbf{x}_1 - \mathbf{x}_2\|^2 + \frac{1}{2}\|\mathbf{y}_1 - \mathbf{y}_2\|^2}$$

with $k > 0$ a parameter we can choose, and $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in \mathbb{R}^3 \times \mathbb{R}^3$. Let $B_{\mathrm{d}_k}((\mathbf{x}, \mathbf{y}), r) \subset \mathbb{R}^3 \times \mathbb{R}^3$ be the open ball of radius r centered at $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3$ in d_k-metric and $\bar{B}_{\mathrm{d}_k}((\mathbf{x}, \mathbf{y}), r)$ its closure.

Let $(\mathbf{L}_T, \mathbf{A}_T) \in D$ be the pair of the angular momentum and the Laplace vector of the target elliptic orbit. Define a (Lyapunov) function V on $T\mathbb{R}^3_0$ by

$$V(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}k\|\mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}) - \mathbf{L}_T\|^2 + \frac{1}{2}\|\mathbf{A}(\mathbf{r}, \dot{\mathbf{r}}) - \mathbf{A}_T\|^2. \tag{14}$$

Notice that $V(\mathbf{r}, \dot{\mathbf{r}})$ is the square of the distance between $(\mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}), \mathbf{A}(\mathbf{r}, \dot{\mathbf{r}}))$ and $(\mathbf{L}_T, \mathbf{A}_T)$ in the metric d_k, i.e.,

$$V(\mathbf{r}, \dot{\mathbf{r}}) = \Big[\mathrm{d}_k\big((\mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}), \mathbf{A}(\mathbf{r}, \dot{\mathbf{r}})),\ (\mathbf{L}_T, \mathbf{A}_T)\big)\Big]^2. \tag{15}$$

We will find a controller $\mathbf{F}$ whose direction maximally reduces this distance at each moment. Along the trajectories of (13),

$$\frac{d}{dt}\mathbf{L}(\mathbf{r},\dot{\mathbf{r}}) = \mathbf{r}\times\mathbf{F}$$
$$\frac{d}{dt}\mathbf{A}(\mathbf{r},\dot{\mathbf{r}}) = \mathbf{F}\times\mathbf{L}(\mathbf{r},\dot{\mathbf{r}}) + \dot{\mathbf{r}}\times(\mathbf{r}\times\mathbf{F}).$$

Hence,

$$\frac{d}{dt}V(\mathbf{r},\dot{\mathbf{r}}) = \mathbf{F}\cdot\Big(k\Delta\mathbf{L}\times\mathbf{r} + \mathbf{L}(\mathbf{r},\dot{\mathbf{r}})\times\Delta\mathbf{A} + (\Delta\mathbf{A}\times\dot{\mathbf{r}})\times\mathbf{r}\Big)$$

where

$$\Delta\mathbf{L} = \mathbf{L}(\mathbf{r},\dot{\mathbf{r}}) - \mathbf{L}_T;\quad \Delta\mathbf{A} = \mathbf{A}(\mathbf{r},\dot{\mathbf{r}}) - \mathbf{A}_T. \tag{16}$$

Take the controller $\mathbf{F}$ as follows:

$$\mathbf{F}(\mathbf{r},\dot{\mathbf{r}};\mathbf{L}_T,\mathbf{A}_T) = -f(\mathbf{r},\dot{\mathbf{r}})\,(k\Delta\mathbf{L}\times\mathbf{r} + \mathbf{L}(\mathbf{r},\dot{\mathbf{r}})\times\Delta\mathbf{A} + (\Delta\mathbf{A}\times\dot{\mathbf{r}})\times\mathbf{r}) \tag{17}$$

with $f(\mathbf{r},\dot{\mathbf{r}}) > 0$ arbitrary. This choice is such that

$$\frac{dV}{dt}(\mathbf{r},\dot{\mathbf{r}}) = -f(\mathbf{r},\dot{\mathbf{r}})||k\Delta\mathbf{L}\times\mathbf{r} + \mathbf{L}(\mathbf{r},\dot{\mathbf{r}})\times\Delta\mathbf{A} + (\Delta\mathbf{A}\times\dot{\mathbf{r}})\times\mathbf{r}||^2 \leq 0. \tag{18}$$

We now use LaSalle's invariance principle to prove asymptotically stable convergence to the target orbit (see [13] for an exposition of LaSalle's invariant principle). For notational simplicity, we will suppress the dependence of $\mathbf{L}$ and $\mathbf{A}$ on $(\mathbf{r},\dot{\mathbf{r}})$ from now on. Let

$$J = \{(\mathbf{x},\mathbf{y})\in\mathbb{R}^3\times\mathbb{R}^3 \mid \mathbf{x}\neq 0, \|\mathbf{y}\| < \mu\} \tag{19}$$

which is open in $\mathbb{R}^3\times\mathbb{R}^3$. There is an $l > 0$ such that

$$\bar{B}_{\mathrm{d}_k}((\mathbf{L}_T,\mathbf{A}_T),l)\subset J.$$

Let

$$\Omega_l = \pi^{-1}(\bar{B}_{\mathrm{d}_k}((\mathbf{L}_T,\mathbf{A}_T),l)).$$

By (15),

$$\Omega_l = \{(\mathbf{r},\dot{\mathbf{r}})\in T\mathbb{R}^3_0 \mid V(\mathbf{r},\dot{\mathbf{r}})\leq l^2\}. \tag{20}$$

Notice that (5) implies $\pi(T\mathbb{R}^3_0)\subset I$, where

$$I = \{(\mathbf{x},\mathbf{y})\in\mathbb{R}^3\times\mathbb{R}^3 \mid \mathbf{x}\cdot\mathbf{y} = 0\}.$$

Then $\Omega_l = \pi^{-1}(\bar{B}_{\mathrm{d}_k}((\mathbf{L}_T,\mathbf{A}_T),l)\cap I)$. Notice that the set $\bar{B}_{\mathrm{d}_k}((\mathbf{L}_T,\mathbf{A}_T),l)\cap I$ is a compact subset of D. Hence, Ω_l is a compact subset of Σ_e by Corollary 2.2. By (18) and (20), the set Ω_l is a positively invariant compact set. We will show that every trajectory of the closed-loop system starting from Ω_l asymptotically converges to the Keplerian orbit $\pi^{-1}(\mathbf{L}_T,\mathbf{A}_T)$. Define

$$\mathcal{E} = \Big\{(\mathbf{r},\dot{\mathbf{r}})\in\Omega_l\Big|\frac{dV}{dt}(\mathbf{r},\dot{\mathbf{r}}) = 0\Big\} = \{(\mathbf{r},\dot{\mathbf{r}})\in\Omega_l \mid \mathbf{F}(\mathbf{r},\dot{\mathbf{r}};\mathbf{L}_T,\mathbf{A}_T) = 0\}$$
$$\mathcal{M} = \text{ the largest invariant subset of } \mathcal{E}.$$

Let $(\mathbf{r}(t),\dot{\mathbf{r}}(t))$ be an arbitrary trajectory contained in $\mathcal{M}$. Since $\mathcal{M}\subset\mathcal{E}$, there is no control force acting on it. Hence, $(\mathbf{r}(t),\dot{\mathbf{r}}(t))$ is an elliptic Keplerian flow. Let E, $\mathbf{L}$, and $\mathbf{A}$ be the respective energy, angular momentum, and Laplace vector of the Keplerian orbit $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$. They are all constant in time t. By the definition of $\mathcal{M}$, $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ satisfies

$$k\Delta\mathbf{L} \times \mathbf{r}(t) + \mathbf{L} \times \Delta\mathbf{A} + (\Delta\mathbf{A} \times \dot{\mathbf{r}}(t)) \times \mathbf{r}(t) = 0. \tag{21}$$

Let Π be the plane through the origin in $\mathbb{R}^3$ which is normal to $\mathbf{L}$, i.e., the plane where the ellipse swept out by $\mathbf{r}(t)$ lies. The inner product of $\mathbf{r}(t)$ and (21) gives

$$0 = \mathbf{r}(t) \cdot (\mathbf{L} \times \Delta\mathbf{A}) = \Delta\mathbf{A} \cdot (\mathbf{r}(t) \times \mathbf{L}). \tag{22}$$

Notice that

$$\Pi = \operatorname{span}\{\mathbf{r}(t) \times \mathbf{L} \mid t \in \mathbb{R}\}, \tag{23}$$

since $\mathbf{r}(t)$ traces an ellipse in Π. By (22) and (23)

$$\Delta\mathbf{A} = c\mathbf{L} \tag{24}$$

for some $c \in \mathbb{R}$. Note that c is constant since both $\Delta\mathbf{A}$ and $\mathbf{L}$ are constant. Substitution of (24) into (21) gives

$$(k\Delta\mathbf{L} - c(\dot{\mathbf{r}}(t) \times \mathbf{L})) \times \mathbf{r}(t) = 0$$

which by (4) gives

$$(k\Delta\mathbf{L} - c\mathbf{A}) \times \mathbf{r}(t) = 0.$$

This implies that the constant vector $(k\Delta\mathbf{L} - c\mathbf{A})$ is parallel to the nonzero vector $\mathbf{r}(t)$ which changes its direction in time since it sweeps an ellipse. It follows that

$$\Delta\mathbf{L} = \frac{c}{k}\mathbf{A}. \tag{25}$$

By (16), (24), and (25),

$$\mathbf{L}_T = \mathbf{L} - \frac{c}{k}\mathbf{A}, \qquad \mathbf{A}_T = \mathbf{A} - c\mathbf{L}. \tag{26}$$

Since $(\mathbf{L}_T, \mathbf{A}_T)$ and $(\mathbf{L}, \mathbf{A})$ are contained in D, (26) implies

$$0 = \mathbf{L}_T \cdot \mathbf{A}_T = -c\left(\|\mathbf{L}\|^2 + \frac{1}{k}\|\mathbf{A}\|^2\right).$$

Since $\|\mathbf{L}\| > 0$ and $k > 0$, it follows that $c = 0$. Substituting $c = 0$ to (26) gives

$$\mathbf{L} = \mathbf{L}_T, \qquad \mathbf{A} = \mathbf{A}_T.$$

By Proposition 2.1, the Keplerian orbit $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ is the same as the target orbit $\pi^{-1}(\mathbf{L}_T, \mathbf{A}_T)$. Thus, the only trajectory lying in $\mathcal{M}$ is the Keplerian orbit $\pi^{-1}(\mathbf{L}_T, \mathbf{A}_T)$. By LaSalle's invariance principle, the following holds:

Proposition 3.1. *Let $(\mathbf{L}_T, \mathbf{A}_T) \in D$ be the pair of the angular momentum and the Laplace vector of the target elliptic orbit. Take any closed ball $\bar{B}_{\mathrm{d}_k}((\mathbf{L}_T, \mathbf{A}_T), l)$ of a radius $l > 0$ centered at $(\mathbf{L}_T, \mathbf{A}_T)$ contained in the following open set J*

$$J = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \mathbf{x} \neq 0, \|\mathbf{y}\| < \mu\}.$$

Then, every trajectory starting in the subset $\pi^{-1}(\bar{B}_{\mathrm{d}_k}((\mathbf{L}_T, \mathbf{A}_T), l))$ of $T\mathbb{R}^3_0$ remains in that subset and asymptotically converges to the target elliptic orbit $\pi^{-1}(\mathbf{L}_T, \mathbf{A}_T)$ in the closed-loop system (13) with the control law in (17).

REMARK 2. *Proposition 3.1 holds for any positive k in the definition of the metric d_k. There are two interpretations of k. One is that k determines the relative weighting between the two quadratic terms in the function V in (14). The other is that k determines the shape of the region of attraction since k determines the shape of the ball B_{d_k} with the metric d_k.*

REMARK 3. *We explain some advantages of using $(\mathbf{L}, \mathbf{A})$ instead of other quantities, such as orbital elements $(a, e, i, \Omega, \omega)$ or equinoctial elements (a, h, k, p, q) (see [3] for definitions of those elements). First, $(\mathbf{L}, \mathbf{A})$ is globally well-defined whereas orbital elements become singular on circular or equatorial orbits. Second, $\mathbf{L}$ and $\mathbf{A}$ are $\mathbb{R}^3$-valued and $\mathbb{R}^3$ has a nice (Lie-)algebraic structure, namely the cross product $\times$ as well as the dot product $\cdot$, and the property*

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) \tag{27}$$

for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. (It is not accidental that $\mathbf{L}$ and $\mathbf{A}$ are $\mathbb{R}^3$-valued. See [8] for more details). Notice that we have exclusively used the usual Euclidean norm $\|\cdot\|$ on $\mathbb{R}^3$ in the definition of the metric d_k and the Lyapunov function V in order to make use of the algebraic structure of $\mathbb{R}^3$. In particular, the property (27) was very useful in the analysis of the set where $dV/dt = 0$ in the application of LaSalle's invariance principle. It will be difficult to analyze $dV/dt = 0$ if one uses orbital elements or equinoctial elements to define a Lyapunov function as a sum of squares of differences of elements, because the elements do not have useful algebraic structures.

3.2. **Global Orbit Transfer.** The basic idea of the global orbit transfer is to use a finite number of intermediate (target) orbits to transfer between two arbitrary elliptic orbits. We will show a way of choosing intermediate target orbits to achieve the global orbit transfer. By proper choice of intermediate orbits we can also avoid undesirable orbits. The essence of the following argument lies in the combination of Proposition 3.1 and the path-connectivity of the set D defined in (8). We first show that D is path-connected. Any two points $(\mathbf{L}_0, \mathbf{A}_0)$ and $(\mathbf{L}_1, \mathbf{A}_1)$ in D can be joined by a path $\mathrm{c} : [0, 1] \to D \subset \mathbb{R}^3 \times \mathbb{R}^3$, for example,

$$\mathrm{c}(t) = \begin{cases} (\mathbf{L}_0, (1-3t)\mathbf{A}_0) & 0 \le t \le 1/3 \\ (\mathrm{d}(3t-1), 0) & 1/3 \le t \le 2/3 \\ (\mathbf{L}_1, (3t-2)\mathbf{A}_1) & 2/3 \le t \le 1 \end{cases}$$

where $\mathrm{d} : [0, 1] \to \mathbb{R}^3 - \{0\}$ is a path connecting $\mathbf{L}_0$ and $\mathbf{L}_1$. The existence of $\mathrm{d}(t)$ is guaranteed by the path-connectivity of $\mathbb{R}^3 - \{0\}$. Hence, D is path-connected.

Choose two arbitrary elliptic Keplerian orbits $(\mathbf{L}_0, \mathbf{A}_0)$ and $(\mathbf{L}_1, \mathbf{A}_1)$ from D where we want to transfer from $(\mathbf{L}_0, \mathbf{A}_0)$ to $(\mathbf{L}_1, \mathbf{A}_1)$. By the path-connectivity of D, one can choose a path $c : [0, 1] \to D \subset \mathbb{R}^3 \times \mathbb{R}^3$ connecting $(\mathbf{L}_0, \mathbf{A}_0)$ and $(\mathbf{L}_1, \mathbf{A}_1)$. Recall that J in (19) is open and $D \subset J$. There is $\tilde{l} > 0$ such that $B_{\mathrm{d}_k}(c(s), \tilde{l}) \subset J$ for all $s \in [0, 1]$ (for example, take any number less than the distance between the compact set $c([0, 1])$ and the boundary of J or just apply the Lebesgue number lemma to $c([0, 1])$ and J. see [14] for the Lebesgue number lemma). Take any positive number l less than $\tilde{l}$. By the uniform continuity of c, we can find a subdivision of $[0, 1]$, say $s_0, \ldots, s_N$ with $s_0 = 0$ and $s_N = 1$ such that for $i = 0, \ldots, N-1$ the set $c([s_i, s_{i+1}])$ is contained in $B_{\mathrm{d}_k}(c(s_{i+1}), l) \cap D$. In particular, $c(s_i) \in B_{\mathrm{d}_k}(c(s_{i+1}), l) \cap D$. Notice that $\bar{B}_{\mathrm{d}_k}(c(s_{i+1}), l) \cap D$ is a region of attraction of $c(s_{i+1})$ with the controller $\mathbf{F}(\,\cdot\,; c(s_{i+1}))$; this follows from Proposition 3.1 since $\bar{B}_{\mathrm{d}_k}(c(s_i), l) \subset B_{\mathrm{d}_k}(c(s_i), \tilde{l}\,) \subset J$ for each i. Hence, we can drive the trajectory

$(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ from the orbit $(\mathbf{L}_0, \mathbf{A}_0)$ to the orbit $(\mathbf{L}_1, \mathbf{A}_1)$ through the intermediate target orbits $\{c(s_i) \mid i = 0, \ldots, N\}$ by using the controllers $\{\mathbf{F}(\ \cdot\ ; c(s_i)) \mid i = 1, \ldots, N\}$ of the form (17) sequentially. The trajectory lies in $\pi^{-1}(K)$ where

$$K = \left(\bigcup_{i=1}^{N} \bar{B}_{\mathrm{d}_k}(c(s_i), l) \right) \cap D.$$

A lower bound of $\|\mathbf{r}(t)\|$ of the total trajectory $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ is given by

$$\min \left\{ \frac{\|\mathbf{L}\|^2}{\mu + \|\mathbf{A}\|} \;\middle|\; (\mathbf{L}, \mathbf{A}) \in K \right\} \tag{28}$$

and an upper bound is given by

$$\max \left\{ \frac{\|\mathbf{L}\|^2}{\mu - \|\mathbf{A}\|} \;\middle|\; (\mathbf{L}, \mathbf{A}) \in K \right\} \tag{29}$$

Remark 4. *Above, we just showed the possibility of global orbit transfer. There can be several ways to achieve global transfer. For example, one can use different radii for each region of attraction, B_{d_k}. Also, one can use different k's for each region of attraction. A discussion on k was given in a remark following Proposition 3.1.*

3.3. **Special Transfer : Transfer to Circular Orbits.** The Lyapunov function suggested in § 3.1 is not the only available Lyapunov function for local orbit transfer. We here suggest another Lyapunov function for the transfer to *circular* orbits.

Notice that a circular Keplerian orbit is uniquely determined by its angular momentum $\mathbf{L}$ because the Laplace vector $\mathbf{A}$ is zero for circular orbits. The corresponding energy E is determined by $\mathbf{L}$ since $\mu^2 + 2E||\mathbf{L}||^2 = 0$ by (6). Let $\mathbf{L}_T$ and E_T be the angular momentum and the energy of a given target circular orbit. Define a function V on $T\mathbb{R}^3_0$ by

$$V(\dot{\mathbf{r}}, \dot{\mathbf{r}}) = \frac{1}{2}k||\mathbf{L}(\dot{\mathbf{r}}, \dot{\mathbf{r}}) - \mathbf{L}_T||^2 + \frac{1}{2}(E(\dot{\mathbf{r}}, \dot{\mathbf{r}}) - E_T)^2 \tag{30}$$

with $k > 0$. Then one can compute

$$\frac{dV}{dt}(\mathbf{r}, \dot{\mathbf{r}}) = \mathbf{F} \cdot (k\Delta\mathbf{L} \times \mathbf{r} + \Delta E \dot{\mathbf{r}})$$

where $\Delta\mathbf{L} := \mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}) - \mathbf{L}_T$ and $\Delta E := E(\mathbf{r}, \dot{\mathbf{r}}) - E_T$. Take the following form of controller

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}) = -f(\mathbf{r}, \dot{\mathbf{r}})(k\Delta\mathbf{L} \times \mathbf{r} + \Delta E \dot{\mathbf{r}}) \tag{31}$$

with $f(\mathbf{r}, \dot{\mathbf{r}}) > 0$ an arbitrary positive function. This choice is such that

$$\frac{dV}{dt}(\mathbf{r}, \dot{\mathbf{r}}) = -f(\mathbf{r}, \dot{\mathbf{r}})||k\Delta\mathbf{L} \times \mathbf{r} + \Delta E \dot{\mathbf{r}}||^2 \leq 0. \tag{32}$$

One can find $l > 0$ with $l < \frac{k}{2}||\mathbf{L}_T||^2$ such that $\Omega_l := V^{-1}([0, l])$ is a compact subset of Σ_e by (6) and Corollary 2.2. Notice that Ω_l is positively invariant by (32). Let $\mathcal{M}$ be the largest invariant subset of the set $\{(\mathbf{r}, \dot{\mathbf{r}}) \in \Omega_l \mid dV/dt = 0\} = \{(\mathbf{r}, \dot{\mathbf{r}}) \in \Omega_l \mid \mathbf{F} = 0\}$. Let $(\mathbf{r}, \dot{\mathbf{r}})$ be an arbitrary trajectory in $\mathcal{M}$. Then it is an elliptic orbit because $\mathbf{F} = 0$. Let $\mathbf{L}$ and E be the angular momentum and the energy, respectively, of the orbit $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$, which are of course, constant in time t. By definition of $\mathcal{M}$, the trajectory $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ satisfies

$$k\Delta\mathbf{L} \times \mathbf{r}(t) + \Delta E \dot{\mathbf{r}}(t) = 0. \tag{33}$$

The constant value ΔE is either zero or nonzero. If $\Delta E = 0$, then $\Delta \mathbf{L} \times \mathbf{r}(t) = 0$ by (33), which implies $\Delta \mathbf{L} = 0$ since the constant vector $\Delta \mathbf{L}$ is parallel to the vector $\mathbf{r}(t)$ which sweeps an ellipse. Hence, the trajectory $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ is the target orbit if $\Delta E = 0$. We now suppose $\Delta E \neq 0$. The inner product of (33) with $\mathbf{r}(t)$ gives $\dot{\mathbf{r}}(t) \cdot \mathbf{r}(t) = 0$, which implies that $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ is a circular orbit. Since $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$ are perpendicular to each other and $\mathbf{r}(t)$ sweeps a circle, it follows from (33) that $\Delta \mathbf{L}$ is parallel to $\mathbf{L}$, which implies that $\mathbf{L}$ is parallel to $\mathbf{L}_T$. Since we chose l less than $\frac{k}{2}||\mathbf{L}_T||^2$, the vector $\mathbf{L}$ cannot be in the opposite direction of $\mathbf{L}_T$ by definition of Ω_l. Hence, $\mathbf{L}$ and $\mathbf{L}_T$ have the same directions. Let $\mathbf{e}_L := \mathbf{L}/||\mathbf{L}|| = \mathbf{L}_T/||\mathbf{L}_T||$. Recall the general formulas for energy and the magnitude of the angular momentum for a circular orbit of radius r as follows:

$$E = -\frac{\mu}{2r}, \quad ||\mathbf{L}|| = \sqrt{(\mu r)} \tag{34}$$

where the second formula is derived from (12). Let r be the radius of the circular orbit $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ and r_T be that of the target circular orbit. By (34), the equation (33) can be written as

$$\left(\sqrt{r} - \sqrt{r_T}\right)\left(k\sqrt{\mu}(\mathbf{e}_L \times \mathbf{r}) + \frac{\mu(\sqrt{r} + \sqrt{r_T})}{2rr_T}\dot{\mathbf{r}}\right) = 0. \tag{35}$$

Notice that $(\mathbf{e}_L \times \mathbf{r}(t))$ is in the same direction as $\dot{\mathbf{r}}(t)$ and that $r \neq r_T$ since we assumed $\Delta E \neq 0$. The left hand side of (35) is not zero, which gives a contradiction. Therefore, the trajectory $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ is the target circular orbit. We have shown $\mathcal{M}$ consists of the target orbit only. By LaSalle's invariance principle, any trajectory starting in Ω_l remains in Ω_l and asymptotically converges to the target orbit with the control law (31). As a remark, we note that the control law (31) can be used in the global transfer too.

4. **Example.** For illustrative purposes, we give an example of a transfer from low-Earth orbit (LEO) to geosynchronous orbit (GEO). The initial LEO is a circular orbit with radius 7000 km and inclination 28.5 deg. The target GEO is also circular with radius $42,000$ km and inclination 0 deg. The maximum thrust level is 9.8×10^{-5}km/sec^2. These data are from pp. 362–374 in [7]. We use canonical units in simulations; 806.812 sec = 1 canonical time unit, 6378.140 km = 1 canonical distance unit, 9.8×10^{-3}km/sec^2 = 1 canonical acceleration unit, and the gravitational parameter $\mu = 1$. In the following, all units are canonical unless otherwise indicated. The initial point is given by

$$x_0 = (-0.70545852988580, -0.73885031681775, -0.40116299069586),$$
$$v_0 = (0.73122658145185, -0.53921753373056, -0.29277123328399)$$

which corresponds to the initial point in the time-optimal case of [7]. The angular momentum and Laplace vector of the target orbit are given by

$$\mathbf{L}_T = (0, 0, 2.56612389857378); \quad \mathbf{A}_T = (0, 0, 0).$$

We use the Lyapunov function in (14) with $k = 2$. To meet the constraint on the magnitude of the thrust, we choose f in (17) such that the control law $\mathbf{F}$ becomes

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}) = \begin{cases} \frac{1}{\epsilon}G(\mathbf{r}, \dot{\mathbf{r}}) & \text{if } \|G(\mathbf{r}, \dot{\mathbf{r}})\| < \epsilon F_{\max} \\ F_{\max}\frac{G(\mathbf{r},\dot{\mathbf{r}})}{\|G(\mathbf{r},\dot{\mathbf{r}})\|} & \text{if } \|G(\mathbf{r}, \dot{\mathbf{r}})\| \geq \epsilon F_{\max} \end{cases}$$

	time-opt. transfer	Lyap. transfer
sim. time	16.14 hr	18.87 hr
a_f	42,000.001 km	41,974.952 km
e_f	0.00097	0.00462
i_f	0.999359 deg	0.202893 deg

Table 1. Comparison of the time-optimal transfer and the Lyapunov-based transfer.

where $F_{\max} = 0.01$, $\epsilon = 0.00001$ and

$$G(\mathbf{r}, \dot{\mathbf{r}}) = -\left(k\Delta\mathbf{L} \times \mathbf{r} + \mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}) \times \Delta\mathbf{A} + (\Delta\mathbf{A} \times \dot{\mathbf{r}}) \times \mathbf{r}\right).$$

One can easily check that $\|\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})\| \leq F_{\max}$. Figure 1 shows a plot of the simulation results for time $13.4 \times 2\pi$. For comparison of the time-optimal transfer in [7] and our Lyapunov-based transfer, we list the final simulation results in Table 1 - semimajor axis a_f, eccentricity e_f, and inclination i_f - where all the data are in real units, and the data of the time-optimal transfer are from [7] in which the time-optimal controller has constant magnitude $F_{\max}$ during the entire transfer. When comparing these two results, one should take into account that our controller is in a simple and analytic form, whereas the time-optimal controller is numerical and computationally challenging. Also, we can improve the simulation result by choosing different values of k or inserting intermediate target orbits. Hence, these results are sufficient to show that this simple scheme produces a transfer comparable to those generated by much more complex and numerically intensive approaches.

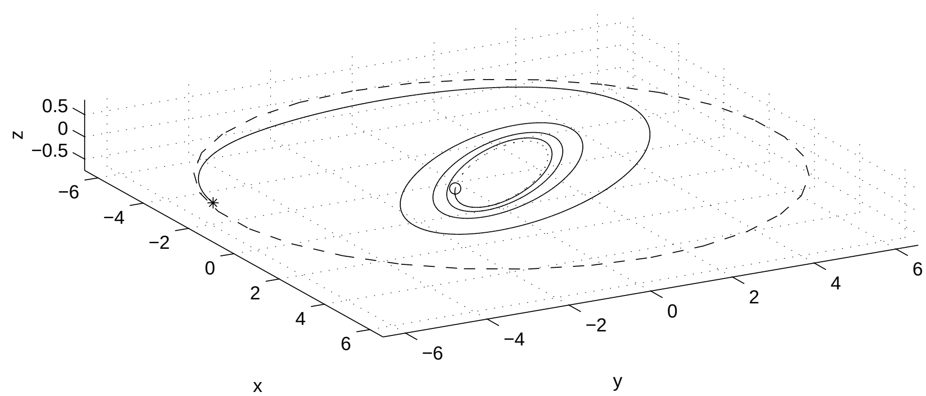

Figure 1. Lyapunov-based LEO-to-GEO transfer in canonical units. The initial and target orbits are dotted $\cdots$ and dashed $--$, respectively. The initial and final points are marked with o and $*$, respectively.

5. **Conclusions and Future Work.** In this work, we have rigorously shown that mechanically motivated Lyapunov function techniques can be used to systematically produce easily implementable, asymptotically stable controllers for orbit transfers between elliptic Kepler orbits.

For long duration, low-thrust transfers, it may be necessary to take into account the J_2 effect, that is, the effect of the bulge of the Earth. We believe that our techniques can be extended to that case, at least in the context of the most important correction terms. This would rely on results on the geometry of the perturbed Kepler problem, given in [6].

A second direction for future research would be to optimize our method. Although we made no attempt at systematic time or fuel optimization in this paper, it would be interesting to pursue this by exploiting, for example, the freedom in the constant k that appears in the Lyapunov function or the freedom in the choice of the function $f(\mathbf{r}, \dot{\mathbf{r}})$ that appears in the control law.

Acknowledgments. D. E. Chang is partially supported by the California Institute of Technology and AFOSR grant F49620-99-1-0190. D. F. Chichka is partially supported by AFOSR grant F49620-99-1-0190. J. E. Marsden is partially supported by the California Institute of Technology and NSF-KDI grant ATM-9873133.

REFERENCES

[1] R. Abraham and J. E. Marsden, " Foundations of Mechanics," 2nd ed., Addison-Wesley, Reading, Massachusetts, 1978, 619–662.
[2] K. J. Åström and K. Furuta, Swinging Up a Pendulum by Energy Control, IFAC Proceedings, **13** (1996).
[3] R. H. Battin, " An Introduction to the Mathematics and Methods of Astrodynamics," in the AIAA Education Series, American Institute of Aeronautics and Astronautics, Inc., New York, New York, 1987.
[4] A. M. Bloch, N. Leonard and J. E. Marsden, Controlled Lagrangians and the Stabilization of Mechanical Systems I: The First Matching Theorem, IEEE Trans. Automat. Control, **45** (2000), 2252–2270.
[5] A. M. Bloch, D. E. Chang, N. Leonard and J. E. Marsden, Controlled Lagrangians and the Stabilization of Mechanical Systems II:Potential Shaping, IEEE Trans. Automat. Control, **46** (2000), 1556–1571.
[6] D. E. Chang and J. E. Marsden, The Geometry of the J_2 Problem, in preparation.
[7] V. A. Chobotov, ed.,"Orbital Mechanics," 2nd ed., in the AIAA Education Series, American Institute of Aeronautics and Astronautics, Inc., New York, New York, 1996, 327–402.
[8] R. H. Cushman and L. M. Bates, "Global Aspects of Classical Integrable Systems," Birkhauser, Boston, 1997, 37–82.
[9] T. N. Edelbaum, Optimum Low-Thrust Rendezvous and Station-keeping, **2** (1964), 1196-1201.
[10] T. N. Edelbaum, Optimum Power-Limited Orbit Transfer in Strong Gravity Fields, *AIAA Journal*, **3** (1965), 921–925.
[11] H. Goldstein, "Classical Mechanics," 2nd ed., Addison-Wesley, Reading, Mass., 1980, 102–104.
[12] M. R. Ilgen, Low Thrust OTV Guidance Using Lyapunov Optimal Feedback Control Techniques, Advances in the Astronautical Sciences, **85** (1980), 1527–1545
[13] H. K. Khalil, "Nonlinear Systems," 2nd ed., Prentice Hall, Inc., New Jersey, 1995, 113–120.
[14] J. R. Munkres, "Topology: a First Course," Prentice-Hall, Inc., New Jersey, 1975, 178–180.

Received August 2001; revised September 2001.

E-mail address: `dchang@cds.caltech.edu`
E-mail address: `chichka@cds.caltech.edu`
E-mail address: `marsden@cds.caltech.edu`