Stabilization of the Pendulum on a Rotor Arm by the Method of Controlled Lagrangians

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Abstract

This paper obtains feedback stabilization of an inverted pendulum on a rotor arm by the "method of controlled Lagrangians". This approach involves modifying the Lagrangian for the uncontrolled system so that the Euler-Lagrange equations derived from the modified or "controlled" Lagrangian describe the closed-loop system. For the closed-loop equations to be consistent with available control inputs, the modifications to the Lagrangian must satisfy "matching" conditions. The pendulum on a rotor arm requires an interesting generalization of our earlier approach which was used for systems such as a pendulum on a cart.

1 Introduction

We present a method for stabilizing an inverted pendulum attached to the end of a rotating robotic arm (a system described in Åström and Furuta [1996]). We use our constructive approach for stabilizing (underactuated) Lagrangian mechanical systems, which we refer to as the *method of controlled Lagrangians*. The idea is to consider a class of control laws that yield closed-loop dynamics which remain in Lagrangian form. The advantage of requiring Lagrangian closed-loop dynamics is that stabilization can be understood in terms of energy, and the associated energy provides a Lyapunov function. Being Lyapunov-based, the method yields large and computable basins of stability, which become asymptotically stable when dissipative controls are added. The Lagrangian for the closed-loop system is called the *controlled Lagrangian*. The conditions which ensure that the Euler-Lagrange equations derived from the controlled Lagrangian are consistent with available control inputs, i.e., they match the controlled Euler-Lagrange equations for the given mechanical system, are called *matching* conditions. The method of controlled Lagrangians is developed in Bloch, Leonard and Marsden [1997], [1998a,b] and has its origins in Bloch, Krishnaprasad, Marsden and Sánchez de Alvarez [1992] and Bloch, Marsden and Sánchez de Alvarez [1997].

Our earlier work discussed systems that fell into two classes depending on the nature of the controlled Lagrangian required. The simplest class includes the pendulum on a cart while the second is designed for Euler–Poincaré systems such as a satellite with momentum wheels. The pendulum on a rotor arm is a nontrivial unification of these two classes of systems. Full details of the general unified approach will be presented in a forthcoming paper.

This paper is restricted to controlled Lagrangians that modify the system's kinetic energy. One can also consider modifications to the potential energy for stabilization and tracking purposes. In a forthcoming paper, we make modifications to both the potential energy and the kinetic energy. Our shaping of potential energy is done in the spirit of van der Schaft [1986] and Leonard [1997]. Other relevant work involving energy methods in control and stabilization includes Wang and Krishnaprasad [1992], Koditschek and Rimon [1990], Baillieul [1993], and Åström and Furuta [1996].

This paper is organized as follows. In §2, we outline the controlled Lagrangian approach to stabilization. In §3 we discuss briefly the pendulum on a cart. In §4 we describe the general matching theorem. In §5 we apply the theory to the pendulum on a rotor arm.

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2 Controlled Lagrangian Approach

The controlled Lagrangian approach begins with a mechanical system with an uncontrolled (free) Lagrangian equal to kinetic energy minus potential energy. We modify the kinetic energy to produce a new controlled Lagrangian which describes the dynamics of the controlled closed-loop system.

Suppose our system has configuration space Q and that a Lie group G acts freely and properly on Q. It is useful to keep in mind the case in which $Q = S \times G$ with G acting only on the second factor by acting on the left by group multiplication.

For example, for the inverted planar pendulum on a cart, $Q = S^1 \times \mathbb{R}$ with $G = \mathbb{R}$, the group of reals under addition (corresponding to translations of the cart), while for a rigid spacecraft with a rotor, $Q = \mathrm{SO}(3) \times S^1$, where now the group is $G = S^1$, corresponding to rotations of the rotor.

Our goal is to control the variables lying in the *shape space* Q/G (in the case in which $Q = S \times G$, then $Q/G = S$) using controls that act directly on the variables lying in G. Assume that the Lagrangian is invariant under the action of G on Q, where the action is on the factor G alone. In many examples the invariance amounts to the Lagrangian being cyclic in the G-variables. Accordingly, this produces a conservation law for the free system. The construction preserves the invariance of the Lagrangian, thus providing a modified or *controlled* conservation law. Throughout this paper we will assume that G is an abelian group.

The essence of the modification of the Lagrangian involves changing the metric tensor $g(\cdot,\cdot)$ that defines the kinetic energy $\frac{1}{2}g(\dot q, \dot q)$. The tangent space to Q can be split into a sum of horizontal and vertical parts defined as follows: for each tangent vector v_q to Q at a point $q \in Q$, we can write a unique decomposition

$$v_q = \mathrm{Hor}\, v_q + \mathrm{Ver}\, v_q, \tag{2.1}$$

such that the vertical part is tangent to the orbits of the G-action and where the horizontal part is the metric orthogonal to the vertical space; that is, it is uniquely defined by requiring the identity

$$g(v_q, w_q) = g(\mathrm{Hor}\, v_q, \mathrm{Hor}\, w_q) + g(\mathrm{Ver}\, v_q, \mathrm{Ver}\, w_q) \tag{2.2}$$

where v_q and w_q are arbitrary tangent vectors to Q at the point $q \in Q$. This choice of horizontal space coincides with that given by the *mechanical connection*; see, for example, Marsden [1992].

For the kinetic energy of our controlled Lagrangian, we use a modified version of the right hand side of equation (2.2). The potential energy remains unchanged. The modification consists of three ingredients:

1. a new choice of horizontal space, denoted Hor_τ,
2. a change $g \to g_\sigma$ of the metric on horizontal vectors and
3. a change $g \to g_\rho$ of the metric on vertical vectors.

Let ξ_Q denote the infinitesimal generator corresponding to a Lie algebra element $\xi \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G (see Marsden [1992] or Marsden and Ratiu [1994]). Thus, for each $\xi \in \mathfrak{g}$, ξ_Q is a vector field on the configuration manifold Q and its value at a point $q \in Q$ is denoted $\xi_Q(q)$.

Definition 2.1 *Let τ be a Lie algebra valued horizontal one form on Q; that is, a one form with values in the Lie algebra $\mathfrak{g}$ of G that annihilates vertical vectors. The **τ-horizontal space** at $q \in Q$ consists of tangent vectors to Q at q of the form $\mathrm{Hor}_\tau v_q = \mathrm{Hor}\, v_q - [\tau(v)]_Q(q)$, which also defines $v_q \mapsto \mathrm{Hor}_\tau(v_q)$, the **$\tau$-horizontal projection**. The **τ-vertical projection operator** is defined by $\mathrm{Ver}_\tau(v_q) := \mathrm{Ver}(v_q) + [\tau(v)]_Q(q)$.*

Definition 2.2 *Given g_σ, g_ρ and τ, the **controlled Lagrangian** is the following Lagrangian, which equals a modified kinetic minus the given potential energy:*

$$\begin{aligned} L_{\tau,\sigma,\rho}(v) = \frac{1}{2}[&g_\sigma(\mathrm{Hor}_\tau v_q, \mathrm{Hor}_\tau v_q) \\ &+ g_\rho(\mathrm{Ver}_\tau v_q, \mathrm{Ver}_\tau v_q)] - V(q). \end{aligned} \tag{2.3}$$

The equations corresponding to this Lagrangian will be our closed-loop equations. The new terms appearing in those equations corresponding to the directly controlled variables are interpreted as control inputs. The modifications to the Lagrangian are chosen so that no new terms appear in the equations corresponding to the variables that are not directly controlled. We refer to this process as *matching*.

Once the control law is derived using the controlled Lagrangian, the closed-loop stability of an equilibrium can be determined by energy methods, using any available freedom in the choice of τ, g_σ and g_ρ.

Under some reasonable assumptions on the metric g_σ, $L_{\tau,\sigma,\rho}(v)$ has the following useful structure.

Theorem 2.3 *Assume that $g = g_\sigma$ on* Hor *and* Hor *and* Ver *are orthogonal for g_σ. Then*

$$L_{\tau,\sigma,\rho}(v) = L(v + \tau(v)_Q) + \frac{1}{2}g_\sigma(\tau(v)_Q, \tau(v)_Q) + \frac{1}{2}\varpi(v)$$

where $v \in T_qQ$ and $\varpi(v) = (g_\rho - g)(\mathrm{Ver}_\tau(v), \mathrm{Ver}_\tau(v))$.

3 The Inverted Pendulum on a Cart

Before giving the general matching result, we will go briefly through the basic example of the inverted pendulum on a cart (see also Bloch, Leonard and Marsden [1997], [1998b]). This example shows the effectiveness of the method for the stabilization of balance systems and is useful for understanding the more complex pendulum on a rotor arm.

First, we set up the Lagrangian for the cart-pendulum system. Let s denote the position of the cart

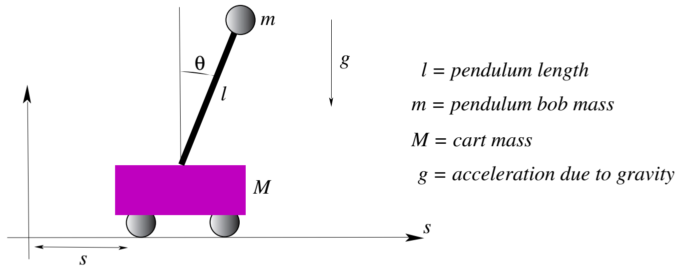

Figure 3.1: The pendulum on a cart.

on the s-axis and let θ denote the angle of the pendulum with the upright vertical, as in Figure 3.1.

The configuration space for this system is $Q = S \times G = S^1 \times \mathbb{R}$, with the first factor being the pendulum angle θ and the second factor being the cart position s. The velocity phase space, TQ has coordinates $(\theta, s, \dot{\theta}, \dot{s})$. The mass of the pendulum is m and that of the cart M.

The symmetry group G of the pendulum-cart system is that of translation in the s variable, so $G = \mathbb{R}$. We do not destroy this symmetry when doing stabilization in θ.

For notational convenience, write the Lagrangian as

$$L(\theta, s, \dot{\theta}, \dot{s}) = \frac{1}{2}(\alpha\dot{\theta}^2 + 2\beta\cos\theta\dot{s}\dot{\theta} + \gamma\dot{s}^2) + D\cos\theta\,, \quad (3.1)$$

where $\alpha = ml^2, \beta = ml, \gamma = M + m$ and $D = -mgl$ are constants. Note that $\alpha\gamma - \beta^2 > 0$. The momentum conjugate to θ is $p_\theta = \partial L/\partial\dot{\theta} = \alpha\dot{\theta} + \beta\cos\theta\dot{s}$ and the momentum conjugate to s is $p_s = \partial L/\partial\dot{s} = \gamma\dot{s} + \beta\cos\theta\dot{\theta}$. The relative equilibrium defined by $\theta = 0, \dot{\theta} = 0$ and $\dot{s} = 0$ is unstable since $D < 0$.

The equations of motion for the cart pendulum system with a control force u acting on the cart are

$$\frac{d}{dt}p_\theta + \beta\sin\theta\dot{s}\dot{\theta} + D\sin\theta = 0,$$

that is,

$$\frac{d}{dt}(\alpha\dot{\theta} + \beta\cos\theta\dot{s}) + \beta\sin\theta\dot{s}\dot{\theta} + D\sin\theta = 0 \qquad (3.2)$$

and

$$\frac{d}{dt}p_s = \frac{d}{dt}(\gamma\dot{s} + \beta\cos\theta\dot{\theta}) = u\,.$$

Next, we form the controlled Lagrangian by modifying only the kinetic energy of the free pendulum-cart system according to the procedure given in the preceding section. This involves a nontrivial choice of τ and g_σ. The parameter ρ in the previous section is not needed it this example, but will be required for the pendulum on a rotor arm.

The most general s-invariant horizontal one form τ is given by $\tau = k(\theta)d\theta$ and we choose g_σ to modify g in the group direction by a constant scalar factor σ (in general, σ need not be a constant).

$$L_{\tau,\sigma} := \frac{1}{2}(\alpha\dot{\theta}^2 + 2\beta\cos\theta(\dot{s} + k\dot{\theta})\dot{\theta} + \gamma(\dot{s} + k\dot{\theta})^2) + \frac{\sigma}{2}\gamma k^2\dot{\theta}^2 + D\cos\theta. \qquad (3.3)$$

Notice that the variable s is still cyclic. We look for the feedback control by looking at the change in the conservation law. Associated to the new Lagrangian $L_{\tau,\sigma}$, we have the conservation law

$$\frac{d}{dt}\left(\frac{\partial L_{\tau,\sigma}}{\partial\dot{s}}\right) = \frac{d}{dt}(\beta\cos\theta\dot{\theta} + \gamma(\dot{s} + k\dot{\theta})) = 0, \qquad (3.4)$$

which we can rewrite in terms of the conjugate momentum p_s for the uncontrolled Lagrangian as

$$\frac{d}{dt}p_s = u := -\frac{d}{dt}(\gamma k(\theta)\dot{\theta}). \qquad (3.5)$$

Thus, we identify the term on the right hand side with the *control force* exerted on the cart.

Using the controlled Lagrangian and equation (3.4), the θ equation is computed to be

$$\left(\alpha - \frac{\beta^2}{\gamma}\cos^2\theta + \sigma\gamma k^2(\theta)\right)\ddot{\theta} \qquad (3.6)$$
$$+\left(\frac{\beta^2}{\gamma}\cos\theta\sin\theta + \sigma\gamma k(\theta)k'(\theta)\right)\dot{\theta}^2 + \ D\sin\theta = 0\,.$$

Next we choose k and σ so that (3.6) using the controlled Lagrangian agrees with the θ equation for the controlled cart (3.2), where the control law is given by (3.5). The θ equation for the controlled cart is

$$\left(\alpha - \frac{\beta^2}{\gamma}\cos^2\theta - \beta k(\theta)\cos\theta\right)\ddot{\theta} \qquad (3.7)$$
$$+\left(\frac{\beta^2}{\gamma}\cos\theta\sin\theta + -\beta\cos\theta k'(\theta)\right)\dot{\theta}^2 + \ D\sin\theta = 0\,.$$

Comparing equations (3.6) and (3.7) we see that we require (twice) the matching condition $\sigma\gamma[k(\theta)]^2 = -\beta k(\theta)\cos\theta$. Since σ was assumed to be a constant we set $k(\theta) = \kappa\beta/\gamma\cos\theta$ where κ is a dimensionless constant (so $\sigma = -1/\kappa$). Substituting for $\ddot{\theta}$ and k in (3.5) we obtain the desired nonlinear control law:

$$u = \frac{\kappa\beta\sin\theta\left(\alpha\dot{\theta}^2 + \cos\theta D\right)}{\alpha - \frac{\beta^2}{\gamma}(1+\kappa)\cos^2\theta} \qquad (3.8)$$

By examining either the energy or the linearization of the closed-loop system, one can see that the equilibrium $\theta = \dot{\theta} = \dot{s} = 0$ is stable if

$$\kappa > \frac{\alpha\gamma - \beta^2}{\beta^2} = \frac{M}{m} > 0\,. \qquad (3.9)$$

In summary, *we get a stabilizing feedback control law for the inverted pendulum provided* κ *satisfies* (3.9).

A simple calculation shows that the denominator of u is nonzero for θ satisfying $\sin^2\theta < E/F$ where $E = \kappa - (\alpha\gamma - \beta^2)/\beta^2$ (which is positive if the stability condition holds) and $F = \kappa + 1$. This range of θ tends to the range $-\pi/2 < \theta < \pi/2$ for large κ.

4 The Master Matching Theorem.

This section gives a general matching theorem for mechanical systems that generalizes the cases discussed in Bloch, Leonard and Marsden [1998a,b]. This matching theorem is constructive and exhibits explicitly how to pick the controlled Lagrangian to achieve the desired matching in a way that generalizes the preceding example of the inverted pendulum.

Firstly, one proves the following coordinate formula for $L_{\tau,\sigma,\rho}$:

$$L_{\tau,\sigma,\rho}(v) = L(x^\alpha, \dot{x}^\beta, \dot{\theta}^a + \tau^a_\alpha \dot{x}^\alpha) + \frac{1}{2}\sigma_{ab}\tau^a_\alpha \tau^b_\beta \dot{x}^\alpha \dot{x}^\beta$$
$$+\frac{1}{2}\varpi_{ab}(\dot{\theta}^a + g^{ac}g_{\alpha c}\dot{x}^\alpha + \tau^a_\alpha \dot{x}^\alpha)(\dot{\theta}^b + g^{bd}g_{\beta d}\dot{x}^\beta + \tau^b_\beta \dot{x}^\beta),$$

where θ^a are coordinates for the abelian symmetry group G and x^α are coordinates on the shape space Q/G; σ_{ab} and ϖ_{ab} are the coefficients for the last two terms, respectively, of the expression for $L_{\tau,\sigma,\rho}$ in Theorem 2.3, and we let $\rho_{ab} = g_{ab} + \varpi_{ab}$. This equation shows that the associated ***controlled conserved quantity*** is given by

$$\begin{aligned}\tilde{J}_a &:= \frac{\partial L_{\tau,\sigma,\rho}}{\partial \dot{\theta}^a} \\ &= \frac{\partial L}{\partial \dot{\theta}^a}(x^\alpha, \dot{x}^\alpha, \dot{\theta}^b + \tau^b_\alpha \dot{x}^\alpha) + \varpi_{ab}(\dot{\theta}^b + g^{bd}g_{\alpha d}\dot{x}^\alpha + \tau^b_\alpha \dot{x}^\alpha) \\ &= g_{\alpha a}\dot{x}^\alpha + g_{ab}(\dot{\theta}^b + \tau^b_\alpha \dot{x}^\alpha) + \varpi_{ab}(\dot{\theta}^b + g^{bd}g_{\alpha d}\dot{x}^\alpha + \tau^b_\alpha \dot{x}^\alpha) \\ &= \rho_{ab}(\dot{\theta}^b + g^{bd}g_{\alpha d}\dot{x}^\alpha + \tau^b_\alpha \dot{x}^\alpha). \qquad (4.1)\end{aligned}$$

It is possible to show that matching is achieved under the following assumptions:

Assumption GM-1. $\tau^b_\alpha = -\sigma^{ab}g_{\alpha a}$.

Assumption GM-2. $\sigma^{bd}(\sigma_{ad,\alpha} + g_{ad,\alpha}) = 2g^{bd}g_{ad,\alpha}$

Assumption GM-3. $\varpi_{ab,\alpha} = 0$.

Assumption GM-4. Letting $\zeta^a_\alpha = g^{ac}g_{\alpha c}$,

$$\tau^b_{\alpha,\delta} - \tau^b_{\delta,\alpha} + \varpi_{ad}\rho^{bd}\left(\zeta^a_{\alpha,\delta} - \zeta^a_{\delta,\alpha}\right)$$
$$- \varpi_{ad}\rho^{dc}g_{ce,\delta}\rho^{eb}\zeta^a_\alpha - \rho^{db}g_{ad,\alpha}\tau^a_\delta = 0,$$

Theorem 4.1 *Under Assumptions GM-1–4 the Euler-Lagrange equations for the controlled Lagrangian $L_{\tau,\sigma,\rho}$ coincide with the controlled Euler-Lagrange equations.*

The proof of this result will be given in a forthcoming publication – for a slightly simpler case see Bloch, Leonard and Marsden [1998b]. Below we shall illustrate how these conditions are satisfied for the pendulum on a rotor arm. They are, of course, also satisfied for the nonlinear pendulum on a cart.

5 The Pendulum on a Rotor Arm

Consider the pendulum shown in Figure 5.1. It is a planar pendulum whose suspension point is attached to another mass M by means of a vertical shaft, as shown. The plane of the pendulum is orthogonal to the radial arm of length R. The shaft is subject to a torque u. We ignore frictional effects here.

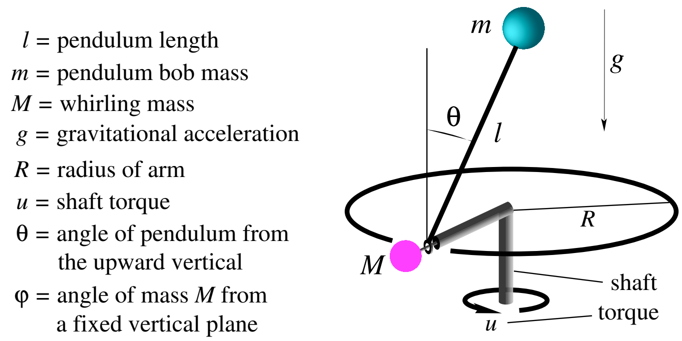

Figure 5.1: A whirling pendulum.

Equations of motion. Erect an xyz-coordinate system, with the z axis vertical along the shaft and the xy-plane in the plane of the horizontal rod. Denote the angle of the horizontal rod with respect to the positive x-axis by ϕ. Refer to Figure 5.2.

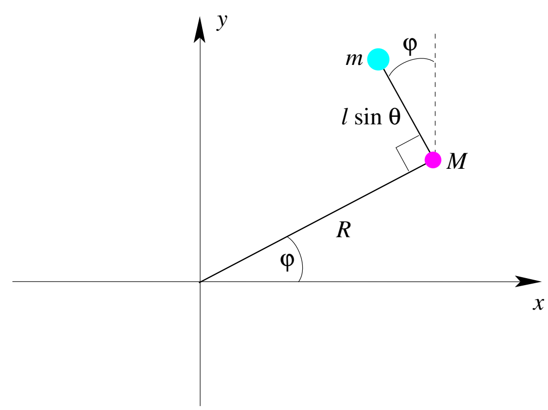

Figure 5.2: Looking down on on the whirling pendulum.

The coordinates of the mass M are $x = R\cos\phi$, $y = R\sin\phi$, and $z = 0$ and so the velocity is

$$\dot{x} = -R\dot{\phi}\sin\phi;\ \dot{y} = R\dot{\phi}\cos\phi;\ \dot{z} = 0.$$

The kinetic energy of the mass M is therefore

$$K_M = \frac{M}{2}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2] = \frac{1}{2}MR^2\dot{\phi}^2$$

The coordinates of the pendulum bob with mass m are $x = R\cos\phi - l\sin\theta\sin\phi$, $y = R\sin\phi + l\sin\theta\cos\phi$ and $z = -l\cos\theta$. The velocity of the bob is the vector with components

$$\dot{x} = -R\dot{\phi}\sin\phi - l\dot{\phi}\sin\theta\cos\phi - l\dot{\theta}\cos\theta\sin\phi$$
$$\dot{y} = R\dot{\phi}\cos\phi - l\dot{\phi}\sin\theta\sin\phi + l\dot{\theta}\cos\theta\cos\phi$$
$$\dot{z} = l\dot{\theta}\sin\theta.$$

The kinetic energy of the bob is thus given by

$$\begin{aligned} K_m &= \frac{1}{2}m[\dot{x}^2 + \dot{y}^2 + \dot{z}^2] \\ &= \frac{1}{2}m[R^2\dot{\phi}^2 + l^2\dot{\phi}^2\sin^2\theta + l^2\dot{\theta}^2 + 2Rl\dot{\phi}\dot{\theta}\cos\theta]. \end{aligned}$$

The potential energy is $V = mgl\cos\theta$ and defining α, β, γ and D precisely as for the pendulum on a cart, the Lagrangian is thus given by

$$\begin{aligned} L &= \frac{\gamma}{2}R^2\dot{\phi}^2 \\ &+ \frac{1}{2}[\alpha\dot{\phi}^2\sin^2\theta + \alpha\dot{\theta}^2 + 2R\beta\dot{\phi}\dot{\theta}\cos\theta] + D\cos\theta. \end{aligned}$$

This Lagrangian is defined on $T(S^1 \times S^1)$, with the variables being ϕ, θ and $\dot{\phi}, \dot{\theta}$. The controlled Euler-Lagrange equations are given by

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} &= u. \end{aligned} \tag{5.1}$$

In our case, the conjugate momenta are

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = \alpha\dot{\theta} + \beta R\dot{\phi}\cos\theta, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = \gamma R^2\dot{\phi} + \alpha\dot{\phi}\sin^2\theta + \beta R\dot{\theta}\cos\theta, \end{aligned}$$

while the derivatives of L with respect to θ and ϕ are: $\partial L/\partial\theta = \alpha\dot{\phi}^2\sin\theta\cos\theta - \beta R\dot{\phi}\dot{\theta}\sin\theta - D\sin\theta$ and $\partial L/\partial\phi = 0$.

Thus, the controlled Euler-Lagrange equations are

$$\frac{d}{dt}[\alpha\dot{\theta} + \beta R\dot{\phi}\cos\theta] \tag{5.2}$$
$$= \alpha\dot{\phi}^2\sin\theta\cos\theta - \beta R\dot{\phi}\dot{\theta}\sin\theta - D\sin\theta;$$
$$\frac{d}{dt}[\gamma R^2\dot{\phi} + \alpha\dot{\phi}\sin^2\theta + \beta R\dot{\theta}\cos\theta] = u. \tag{5.3}$$

We leave the second equation (5.3) as it is and simplify the first equation (5.2):

$$\ddot{\theta} + \frac{R}{l}\ddot{\phi}\cos\theta - \dot{\phi}^2\sin\theta\cos\theta - \frac{g}{l}\sin\theta = 0. \tag{5.4}$$

Relative equilibria. For the unforced ($u = 0$) case, the relative equilibria (relative to the group of rotations about the z-axis) are obtained by putting $\dot{\theta} = 0$ and $\dot{\phi} = \omega$, a constant. In this case, equation (5.4) becomes

$$-\omega^2\sin\theta\cos\theta - \frac{g}{l}\sin\theta = 0.$$

while equation (5.3) is automatically satisfied.

The relative equilibria are given by states of the form $\theta_e, \phi = \omega t, \dot{\theta} = 0, \dot{\phi} = \omega$ where ω is a constant and where θ_e is a root of

$$\omega^2\sin\theta\cos\theta + \frac{g}{l}\sin\theta = 0$$

The roots are given by $\theta = 0, \pi$ (the straight down and the straight up states) and by the roots of

$$\omega^2\cos\theta + \frac{g}{l} = 0$$

i.e., we have no additional roots if $\omega^2 \le g/l$, and two additional roots if $\omega^2 \ge g/l$, namely at $\theta_e = \pm\cos^{-1}\{-g/(\omega^2 l)\}$. Notice that we have a supercritical pitchfork bifurcation of relative equilibria as the value of ω is increased. These new equilibria appear near the straight down state (i.e., near $\theta = \pi$ as it loses stability). The straight up state is always unstable.

Matching. We now apply our general results to this pendulum on a rotor arm problem: here, g_{ab}, σ_{ab} and $\rho_{ab} = g_{ab} + \varpi_{ab}$ are scalars and $g_{ab} = \gamma R^2 + \alpha\sin^2\theta$.

Assumption GM-2 holds with the choice $\sigma_{ab} = cg_{ab}^2 + g_{ab}$, where c is a constant. Assumption GM-1 defines τ_α^b and Assumption GM-3 requires that ϖ_{ab} be a constant or equivalently that $\rho_{ab} = g_{ab} + d$, where d is a constant. Then, we can satisfy Assumption GM-4 by choosing $d = 1/c$. i.e., we take $\rho_{ab} = g_{ab} + 1/c$.

In this problem we have $\tau = -\beta R\cos\theta/\sigma_{ab}$ and

$$\begin{aligned} L_{\tau,\sigma,\rho} &:= \frac{1}{2}\alpha\dot{\theta}^2 + \beta R\cos\theta\left(\dot{\phi} - \frac{\beta R}{\sigma_{ab}}\cos\theta\dot{\theta}\right)\dot{\theta} \\ &+ \frac{1}{2}g_{ab}\left(\dot{\phi} - \frac{\beta R}{\sigma_{ab}}\cos\theta\dot{\theta}\right)^2 + \frac{\beta^2 R^2}{2\sigma_{ab}}\cos^2\theta\dot{\theta}^2 \\ &+ \frac{1}{2c}\left(\dot{\phi} + \beta R\cos\theta\frac{c}{1 + cg_{ab}}\dot{\theta}\right)^2 + D\cos\theta. \end{aligned} \tag{5.5}$$

The controlled conserved quantity $\tilde{J}$ is given by

$$\tilde{J} = \left(g_{ab} + \frac{1}{c}\right)\dot{\phi} + \beta R\cos\theta\dot{\theta}. \tag{5.6}$$

Comparing this with the free conservation law as in the pendulum on a cart we see the control is given by

$$u = -\frac{1}{c}\ddot{\phi}. \tag{5.7}$$

We use the θ equation (5.4) and the conservation law $d\tilde{J}/dt = 0$ to write u as an explicit control law in terms of positions and velocities (as was done for the pendulum on a cart). Defining

$$\kappa = -\frac{1}{1 + c\gamma R^2}$$

to be the dimensionless scalar control gain, we compute the explicit control law as

$$u = \frac{\kappa\beta R\sin\theta\left(\alpha\dot{\theta}^2 - 2\alpha\frac{l}{R}\cos\theta\dot{\theta}\dot{\phi} - \alpha\cos^2\theta\dot{\phi}^2 + D\cos\theta\right)}{\alpha - \frac{\beta^2}{\gamma}(1 + \kappa)\cos^2\theta + \alpha(1 + \kappa)\frac{\alpha}{\gamma R^2}\sin^2\theta}.$$

Stabilization. One can compute that the second variation of the controlled energy evaluated at $\theta = \dot{\theta} = 0$ and $\tilde{J} = \mu$ is

$$\begin{pmatrix} D - \frac{2\alpha\mu^2}{\gamma^2 R^4}(1+\kappa)^2 & 0 \\ 0 & \alpha - \frac{\beta^2}{\gamma}(1+\kappa) \end{pmatrix}.$$

Note that for $\mu = 0$ this is precisely the same as in the case of the planar inverted pendulum on a cart. For stability, therefore, we should choose

$$\kappa > \frac{\alpha\gamma - \beta^2}{\beta^2} = \frac{M}{m},$$

i.e., this makes the second variation negative definite for any value of μ.

The denominator of the control law u is the sum of the denominator of the control u for the planar pendulum plus a term proportional to $\alpha \sin^2\theta$, i.e., the term

$$\alpha(1+\kappa)\frac{\alpha}{\gamma R^2}\sin^2\theta$$

Note that this term disappears in the limit $R/l \to \infty$. However, for finite R/l this additional term affects the possible region of stability as compared to the planar pendulum case. In particular, the denominator of the u above is nonzero (strictly negative) for θ satisfying

$$\sin^2\theta < \frac{\frac{\beta^2}{\gamma}(1+\kappa) - \alpha}{\frac{\beta^2}{\gamma} + \frac{\alpha^2}{\gamma R^2}(1+\kappa)}$$

Note that the numerator is positive when the stability condition holds. For large κ the range of θ tends to the range

$$\sin^2\theta < \frac{R^2}{R^2 + l^2}.$$

This is no longer the whole range of non-downward point states, except in the limit when R/l goes to infinity.

A more general approach to stabilization and asymptotic stabilization in this setting will be given in the sequel to Bloch, Leonard and Marsden [1998b].

6 Final Remarks

The stabilization scheme in this paper is systematic, algorithmic, and makes use of the Euler-Lagrange structure of mechanical systems. The resulting energy expressions provide Lyapunov functions that are used to prove stability and also provide a means to design additional dissipation control terms that will achieve asymptotic stability. Results on asymptotic stability in the context of the method of controlled Lagrangians can be found in Bloch, Leonard and Marsden [1998b]. Results suitable for Euler–Poincaré systems, such as spacecraft and underwater vehicles may be found in Bloch, Leonard and Marsden [1997]. Results on combined kinetic and potential shaping for complete stabilization are the subject of a forthcoming publication.

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