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Stability of Relative Equilibria of Point Vortices on a Sphere and Symplectic Integrators

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Summary. — This paper analyzes the dynamics of N point vortices moving on a sphere from the point of view of geometric mechanics. The formalism is developed for the general case of N vortices, and the details are provided for the (integrable) case $N = 3$. Stability of relative equilibria is analyzed by the energy-momentum method. Explicit criteria for stability of different configurations with generic and non-generic momenta are obtained. In each case, a group of transformations is specified, such that motion in the original (unreduced) phase space is stable modulo this group. Finally, we outline the construction of a symplectic-momentum integrator for vortex dynamics on a sphere.

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1. – Introduction

Helmholtz [1] introduced the model which is today referred to as the point vortex model. Helmholtz' contemporaries developed these models, such as Kirchhoff [2] and his student Gröbli. An account of some of the history of this problem can be found in Aref, Rott and Thomann [3] and Kidambi and Newton [4].

The dynamics of N-vortices on a sphere is a Hamiltonian system (see, Kidambi and Newton [4] and references therein). The Hamiltonian structure can be obtained using reduction techniques starting with the description of ideal hydrodynamics in terms of diffeomorphism groups; see Marsden and Weinstein [5] and Arnold and Khesin [6].

We begin with a description of the dynamics of N point vortices on a sphere using geometric mechanics. For $N = 3$ vortices, we classify relative equilibria and determine their stability by the energy-momentum method (see Marsden [7] and references therein). We refer the reader to Pekarsky and Marsden [8] for the detailed description of the procedure and the associated reduction of the dynamics. The use of the energy-momentum method for the stability of vortices was studied for certain planar cases by Lewis and Ratiu [9].

Numerical simulations of the dynamics of 3 point vortices on a sphere are presented, which confirm the analytical stability results. Finally, the construction of a symplectic-momentum algorithm for the numerical analysis of the vortex dynamics on a sphere is outlined. The structure preserving properties of such an algorithm are believed to be important for the reliability of long time simulations of such problems.

The Phase Space and its Poisson Structure. The phase space for N vortices moving on the two sphere S^2 consists of N copies of the sphere. Namely, we let $P = S^2 \times \cdots \times S^2$ be N copies of the standard sphere with radius R in $\mathbb{R}^3$. We let the nth vortex position ($n = 1, \dots, N$) on the sphere be denoted $\mathbf{x}_n$ so that we have the constraint $\|\mathbf{x}_n\| = R$. We also let $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}^{3N}$. Each vortex has a nonzero vortex strength denoted Γ_n.

The Poisson structure on P is given by

$$\{F, H\}(\mathbf{x}) = \sum_{n=1}^{N} \frac{R}{\Gamma_n} \{F, H\}_n(\mathbf{x}_n) = -\sum_{n=1}^{N} \frac{R}{\Gamma_n} \mathbf{x}_n \cdot (\nabla_n F \times \nabla_n H), \tag{1}$$

where $\{\ ,\ \}_n$ is the Poisson structure on the nth copy of S^2, and $\times$ is the cross product.

The Symmetry Group and Momentum Map. Consider the diagonal action of the group SO(3) on P defined by rotations in each $\mathbb{R}^3$. This action is canonical with respect to the Poisson structure (1). The corresponding Lie algebra is naturally identified with $\mathbb{R}^3$ (having the vector cross product as its Lie bracket operation) and we write $\boldsymbol{\xi}$ for the vector in $\mathbb{R}^3$ corresponding to the element $\xi \in \mathfrak{so}(3)$. We regard ξ as a skew symmetric 3×3 matrix; it is related to $\boldsymbol{\xi} \in \mathbb{R}^3$ in the standard way, namely, $\xi \cdot \mathbf{u} = \boldsymbol{\xi} \times \mathbf{u}$ for each vector $\mathbf{u} \in \mathbb{R}^3$.

The vector field of infinitesimal transformations corresponding to an element ξ in the Lie algebra is given by:

$$\xi_P(\mathbf{x}) := \frac{d}{dt} \exp(\xi t) \cdot \mathbf{x} \Big|_{t=0} = (\boldsymbol{\xi} \times \mathbf{x}_1, \dots, \boldsymbol{\xi} \times \mathbf{x}_N). \tag{2}$$

Recall that a *momentum map* $\mathbf{J} : P \to \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ for this action is defined by requiring the Hamiltonian vector field corresponding to $\langle \mathbf{J}(x), \xi \rangle$ to be equal to the vector field of infinitesimal transformations: $X_{\langle \mathbf{J}(x), \xi \rangle} = \xi_P$, where $\langle \cdot, \cdot \rangle$ is the natural paring between the Lie algebra and its dual. It is readily checked that the momentum map is proportional to the moment of vorticity and is given by: $\mathbf{J}(\mathbf{x}) = -\frac{1}{R} \sum_{n=1}^{N} \Gamma_n \mathbf{x}_n$.

The momentum map is *equivariant*, that is $\mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(\mathbf{x})) = \mathbf{J}(g(\mathbf{x}))$, for all $g \in \mathrm{SO}(3)$. Here, the map $\mathrm{Ad}^*_{k^{-1}} : \mathfrak{so}(3)^* \to \mathfrak{so}(3)^*$, defined for each $k \in \mathrm{SO}(3)$, denotes the coadjoint action of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)^*$. In our case, this can be seen directly from the form of $\mathbf{J}$; the coadjoint action corresponds simply to rotations in the dual space $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$.

It follows from the equivariance of $\mathbf{J}$ that $\|\mathbf{J}\|^2$ is invariant under the coadjoint action. Hence, smooth functions of $\|\mathbf{J}\|^2$ are also invariant. Thus, if $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ are coordinates in the dual $\mathfrak{so}(3)^*$, then any smooth function of $\|\mathbf{b}\|^2$ is a Casimir function. Correspondingly, the generic symplectic leaves of $\mathfrak{so}(3)^*$ are spheres defined by the level sets $\|\mathbf{J}\|^2 = \text{const} \neq 0$. Note that since $\mathrm{SO}(3)$ is compact, its action on both P and $\mathfrak{so}(3)^*$ is proper.

The Hamiltonian. The Hamiltonian describing the motion of N vortices on the surface of a sphere of radius R is given by (see, e.g. Kidambi and Newton [4])

$$H = \frac{1}{4\pi R^2} \sum_{m<n} \Gamma_m \Gamma_n \ln(l^2_{mn}), \tag{3}$$

where $l^2_{mn} = 2(R^2 - \mathbf{x}_m \cdot \mathbf{x}_n)$ is the square of the chord distance between two vortices with positions $\mathbf{x}_m$ and $\mathbf{x}_n$. Keep in mind that the constraints $\|\mathbf{x}_n\| = R$ are assumed. The volume of the parallelepiped formed by the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is denoted V and of course it is given by the triple product, namely, $V = \mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_3)$. Notice that the Hamiltonian (3) is invariant with respect to the diagonal action of $\mathrm{SO}(3)$ on P described above. Hence, the momentum map $\mathbf{J}$ is constant along the flow of this Hamiltonian.

2. – Stability of Relative Equilibria

The Energy-Momentum Method. We shall now utilize the energy-momentum method (see Marsden [7] for a summary and references) for the analysis of the stability of *relative equilibria*. Relative equilibria are dynamical orbits with initial conditions $\mathbf{x}_e$ such that $\mathbf{x}(t) = \exp(\xi_e t)\, \mathbf{x}_e$ for some Lie algebra element ξ_e and any time t. As is well known for relative equilibria, the *augmented energy* function $H_{\xi_e} := H - \langle \mathbf{J} - \mu_e, \xi_e \rangle$ has a critical point at $\mathbf{x}_e$, where $\mu_e = \mathbf{J}(\mathbf{x}_e)$ is the value of the momentum at the relative equilibrium. For notational convenience we will occasionally omit the subscript e.

The orbital stability of a relative equilibrium is equivalent to the stability of the corresponding equilibrium of the reduced system that is induced on the quotient manifold $P/\mathrm{SO}(3)$. The energy momentum method is designed to enable one to test for orbital stability directly on the unreduced manifold P by constructing a special subspace $\mathcal{G} \subset T_{\mathbf{x}_e} P$. This is done by considering a tangent space to the level set of constant momentum $\mathbf{J}^{-1}(\mu_e)$ and eliminating the neutrally stable directions associated to the isotropy subgroup

$$\mathrm{SO}(3)_{\mu_e} := \{ g \in \mathrm{SO}(3) \mid \mathrm{Ad}^*_g \mu_e = \mu_e \}.$$

This subgroup is sometimes called the stabilizer of μ_e since it consists of transformations which leave the momentum value invariant. The energy-momentum method determines stability by examining definiteness of the second variation of H_{ξ_e} restricted to the subspace $\mathcal{G}$. A detailed description of this method can be found in Simo, Lewis and Marsden [10].

If one has a definite second variation, then Patrick's theorem (see Patrick [11]) guarantees stability modulo the isotropy subgroup, provided its action on P is proper, the Lie algebra admits an inner product invariant under the adjoint action of the isotropy subgroup and the momentum map has a regular value. From the expression for the momentum map and the fact that SO(3) is compact, we conclude that the assumptions of Patrick's theorem are automatically satisfied for our applications.

As was mentioned above, relative equilibria are critical points of the augmented Hamiltonian H_ξ. For variational calculations, we extend all functions on P to functions on the ambient space $\mathbb{R}^{3N}$, and then restrict variations to the tangent space to P by requiring $\delta F(\mathbf{x}) \cdot \boldsymbol{\eta} = 0$ for all $\boldsymbol{\eta} \in T_{\mathbf{x}}P$. For the augmented Hamiltonian corresponding to (3), this results in the following conditions on $\mathbf{x}$:

$$\frac{\Gamma_r}{R}\left(\boldsymbol{\xi}(\mathbf{x}) - \frac{1}{2\pi R}\sum_{n\neq r}\Gamma_n \frac{\mathbf{x}_n}{l_{nr}^2}\right) = \kappa_r \frac{\Gamma_r}{R^2}\mathbf{x}_r, \tag{4}$$

where κ_r are constants to be determined.

Equidistant Relative Equilibria. An ***equidistant configuration*** is, by definition, one that satisfies $l_{mn}^2 = l^2$ for all $m \neq n$. Whatever its dynamics, such a configuration is possible only for $N = 2, 3, 4$ (this follows by geometric arguments similar to those used for the study of regular polytopes in three space); we exclude the simple case $N = 2$ from our considerations.

To verify that an equidistant configuration is a relative equilibrium, one checks that indeed the conditions (4) are satisfied. In fact, $\boldsymbol{\xi}(\mathbf{x}) = \sum_n \Gamma_n \mathbf{x}_n / 2\pi R l^2 = -\mathbf{J}(\mathbf{x})/2\pi l^2$ solves (4) with $\kappa_r = \Gamma_r / 2\pi l^2$. Notice that the vectors $\boldsymbol{\xi}$ and $\mathbf{J}$ have opposite directions.

Great Circle Relative Equilibria. For $N = 3$ vortices, we have the following classification of ***great circle equilibria*** (see Kidambi and Newton [4]); we introduce the following notations: $a_1 = l_{23}^2, a_2 = l_{13}^2, a_3 = l_{12}^2$.

1. **Generic momentum**, $\mathbf{J}(\mathbf{x}_e) \neq \mathbf{0}$.
   General relative equilibria correspond to vortices lying on a great circle (and thus satisfying $V = 0$, where V is the volume of the parallelepiped spanned by $\mathbf{x}$), and satisfying the condition

$$2R\left(\frac{a_3 - a_1}{a_2}(\Gamma_1 + \Gamma_3) + \frac{a_1 - a_2}{a_3}(\Gamma_2 + \Gamma_1) + \frac{a_2 - a_3}{a_1}(\Gamma_3 + \Gamma_2)\right) - \frac{1}{R}(a_3(\Gamma_1 - \Gamma_2) + a_2(\Gamma_3 - \Gamma_1) + a_1(\Gamma_2 - \Gamma_3)) = 0\ , \tag{5}$$

   obtained by setting $\dot{V} = 0$. This implicit formula determines another relation (in addition to $V = 0$), between a_1, a_2 and a_3 for each fixed set of Γ's. This is a nonlinear equation and thus can have multiple solutions.

(a) **Isosceles Triangular Great Circle Equilibria.** A particular family of ***isosceles triangular relative equilibria*** for arbitrary values of Γ's is given by the following configuration: $a_1 = a_2 = 2R^2, \quad a_3 = 4R^2$ or, equivalently, $\alpha_1 = \alpha_2 = \pi/2, \quad \alpha_3 = \pi$, as well as configurations obtained from it by cyclic permutations of indices. The whole configuration rotates around the vector $\boldsymbol{\xi}(\mathbf{x}) = -\mathbf{J}(\mathbf{x})/4\pi R^2$.

(b) **Equilateral Triangular Great Circle Equilibria.** A ***great circle equilateral triangle relative equilibrium*** with $l_{mn}^2 = l^2 = 3R^2$ and ξ given by $\boldsymbol{\xi}(\mathbf{x}) = -\mathbf{J}(\mathbf{x})/2\pi l^2$.

Note: When the term *equilateral triangle relative equilibrium* is used, and we do not append "great circle", we will mean that it is a *non-great circle equilateral triangle relative equilibrium*.

2. **Degenerate momentum**, $\mathbf{J}(\mathbf{x}_e) = \mathbf{0}$.
In this case, the vortices again lie on a great circle, and the whole configuration rotates around the vector $\boldsymbol{\xi}(\mathbf{x}) = -\dfrac{1}{2\pi R}\left(\dfrac{\Gamma_1\mathbf{x}_1}{l_{23}^2} + \dfrac{\Gamma_2\mathbf{x}_2}{l_{13}^2} + \dfrac{\Gamma_3\mathbf{x}_3}{l_{12}^2}\right)$.

If we consider the "inverse" problem, namely, given a configuration on a great circle find Γ_n satisfying (5) so that this configuration is a relative equilibrium, then condition (5) becomes a linear equation in Γ_n of the form $\beta_1\Gamma_1 + \beta_2\Gamma_2 + \beta_3\Gamma_3 = 0$, where $\beta_n = \beta_n(a_1, a_2, a_3)$ are functions of a great circle configuration. One would expect this to have a two parameter family of solutions.

Definiteness of the Second Variation. For the calculation of the second variation the Lagrange multiplier method is used. Define the *extended Hamiltonian* $\tilde{H}_\xi := H_\xi + \sum_n \lambda_n(\|\mathbf{x}_n\|^2 - R^2)$, where $\|\mathbf{x}_n\|^2 - R^2 = 0$ constrains the motion of vortices to the sphere S^2. The Lagrange multipliers λ_n are determined by the condition $\delta\tilde{H}_\xi(\mathbf{x}_e) = 0$ and are given by $\lambda_n = -\kappa_n\Gamma_n/2R^2$, where κ_n are determined from (4). Then the second variation at $\mathbf{x}_e$ is well-defined as a bilinear form on $T_{\mathbf{x}_e}P$. It is given by the following expression:

$$\frac{\partial^2\tilde{H}_\xi}{\partial x_s^j \partial x_r^i} = \begin{cases} 2\lambda_r\delta^{ij} - \dfrac{\Gamma_r}{\pi R^2}\displaystyle\sum_{n\neq r}\Gamma_n\frac{x_n^i x_n^j}{l_{nr}^4}, & r = s \\ -\dfrac{\Gamma_r\Gamma_s}{2\pi R^2 l_{rs}^2}\left(\delta^{ij} + 2\dfrac{x_s^i x_r^j}{l_{rs}^2}\right), & r \neq s. \end{cases} \tag{6}$$

We summarize below the stability results in the form of theorems and omit the corresponding calculations of the restriction of the second variation.

Theorem 2.1 (Stability of Non-great Circle Equilateral Triangles).
An equilateral triangle configuration of non-great circle relative equilibria $\mathbf{x}_e$ *is stable modulo* SO(2) *rotations around the vector* $\mathbf{J}(\mathbf{x}_e)$ *if* $\sum_{n<m}\Gamma_n\Gamma_m > 0$ *and is unstable if* $\sum_{n<m}\Gamma_n\Gamma_m < 0$.

This theorem generalizes the known results of Synge [12] for the stability of equilateral relative equilibria of 3 vortices on a plane. Indeed, the stability conditions are independent of the radius R. Thus, in the limit $R \to \infty$ the spherical stability conditions agree with those for the planar case.

Conjecture. *The condition* $\sum_{n<m}\Gamma_n\Gamma_m = 0$ *corresponds to a (degenerate) Hamiltonian bifurcation.*

Theorem 2.2 (Stability of Isosceles Triangle Great Circle Equilibria). *A great circle configuration of relative equilibrium* $\mathbf{x}_e$ *given by* $a_1 = a_2 = 2R^2,\ a_3 = 4R^2$ *is* stable *if* $\Gamma_1^2+\Gamma_2^2 > \sum_{n\neq m}\Gamma_n\Gamma_m$ *and* unstable *if* $\Gamma_1^2+\Gamma_2^2 < \sum_{n\neq m}\Gamma_n\Gamma_m$. *The stability is modulo* SO(2) *rotations around* $\mathbf{J}(\mathbf{x}_e)$.

Stability of Great Circle Equilateral Triangle Relative Equilibria (GCET). The stability analysis of a GCET differs from the non-great circle equilateral triangle case for the following reason. The two-dimensional subspace to which the second variation of the augmented Hamiltonian is restricted in the general case fails to be a transversal subspace to the G_μ orbit (rotations around $\mathbf{J}$) within $\operatorname{Ker} D\mathbf{J}$ but rather degenerates to a one-dimensional subspace. A complimentary direction transversal to the plane of the triangle has to be taken into account. A straightforward computation gives the following expression for the restriction of the second variation:

$$\delta^2\tilde{H}_\xi|_{\mathcal{G}} = \frac{1}{12\pi}\begin{pmatrix} 0 & 0 \\ 0 & 9-(\Gamma_1+\Gamma_2+\Gamma_3)\left(\dfrac{1}{\Gamma_1}+\dfrac{1}{\Gamma_2}+\dfrac{1}{\Gamma_3}\right)\end{pmatrix}.$$

One concludes from this that *these GCET equilibria are at best, neutrally stable.* The reasons for the degeneracy are discussed in [8].

The Degenerate Case $\mathbf{J}(\mathbf{x}_e) = \mathbf{0}$. Stability in this case is a simple task and can be done by a dimension count. This results in the following theorem.

Theorem 2.3 (Stability of Great Circle Equilibria with $\mathbf{J} = \mathbf{0}$). *A relative equilibrium with zero vorticity momentum* $\mathbf{J}(\mathbf{x}_e) = \mathbf{0}$*, which necessarily lies on a great circle, is* stable modulo SO(3).

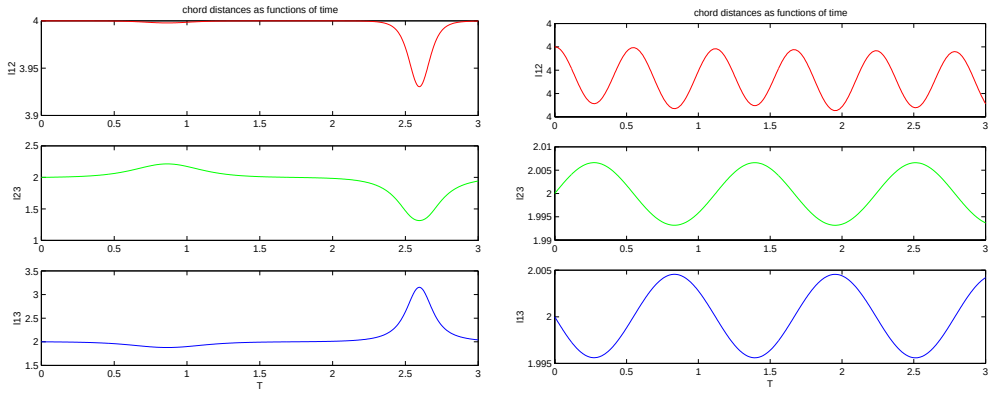
Proof. The isotropy subgroup $\mathrm{SO}(3)_{\mu=0}$ is, in this case, the whole group SO(3) and hence the dimension of $\mathbf{J}^{-1}(\mathbf{0})/\mathrm{SO}(3)_{\mu=0}$ is zero. This implies that

$$\operatorname{Ker} D\mathbf{J}(\mathbf{x}) = T_{\mathbf{x}}(\mathrm{SO}(3)_{\mu=0}\cdot\mathbf{x}).$$

The assumptions of Patrick's theorem are satisfied as SO(3) is compact, and so this proves the theorem. □

3. – Numerical simulations

In this section we outline results of some numerical simulations of the dynamics of three point vortices on a sphere. The numerical integration of the differential equations was performed using the Matlab ODE45 package with the tolerance set to 10^{-10}. Numerical simulations using various values of vorticities Γ have confirmed the stability results of Theorems **2**.1 and **2**.2. Changes in the stability types of equilateral triangle relative equilibria and isosceles triangle great circle relative equilibria have been observed when the following conditions are approximately satisfied: $\sum_{n<m}\Gamma_n\Gamma_m = 0$ and $\Gamma_1^2+\Gamma_2^2 = \sum_{n\neq m}\Gamma_n\Gamma_m$, respectively. The figures demonstrate typical behavior of the

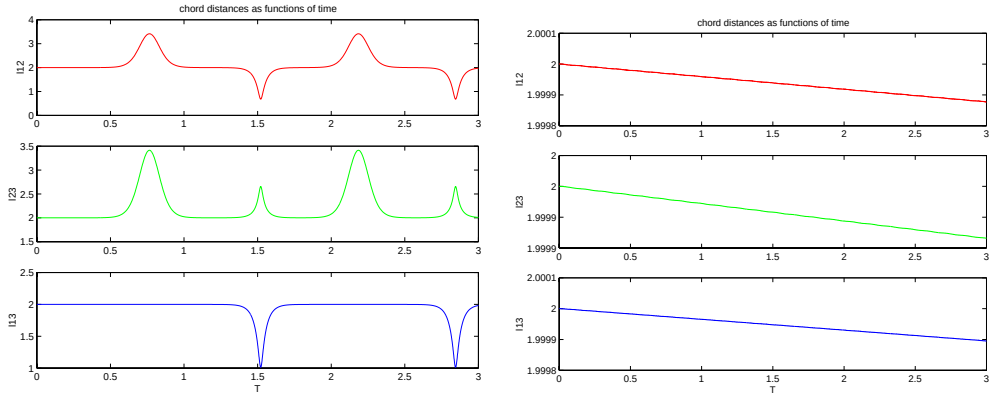


(a) (b)

Fig. 1. – Chord distances as functions of time. Great circle relative equilibrium. (a) Unstable, $\Gamma_1 = 3, \Gamma_2 = 2, \Gamma_3 = 0.2$, $\det \delta^2 H_\xi = -1$; (b) Stable, $\Gamma_1 = 3, \Gamma_2 = 2, \Gamma_3 = 0.02$, $\det \delta^2 H_\xi = 0.8$.

chord distances of an equilibrium as a function of time for stable and unstable types of motion depending on the value of the second variation $\delta^2 H_\xi$.

Notice that while for a stable great circle relative equilibrium the chord distances exhibit small oscillations (Figure 1(b)), one observes a slight drift in the case of a stable



(a) (b)

Fig. 2. – Chord distances as functions of time. Equilateral triangle relative equilibrium. (a) Unstable, $\Gamma_1 = 2, \Gamma_2 = 1, \Gamma_3 = -2$, $\det \delta^2 H_\xi = -4$; (b) Stable, $\Gamma_1 = 2, \Gamma_2 = 1, \Gamma_3 = 4$, $\det \delta^2 H_\xi = 14$.

equilateral triangle relative equilibrium (Figure 2(b)). One possible explanation for this numerical drift is that ODE45 is not a structure preserving algorithm and so this could be a numerical difficulty.

Furthermore, numerical (stability) bifurcation analysis, as $\delta^2 H_\xi$ passes through zero, requires long time simulations, and standard algorithms, such as ODE45, cannot be reliably used. Hence, there is a great need for structure preserving numerical algorithms for the vortex dynamics problem on a sphere, and we shall address this issue in the next section.

4. – Structure preserving algorithms for vortex dynamics on a sphere

Historically, there have been many approaches devised for constructing symplectic integrators, beginning with the original derivations based on generating functions (see de Vogelaere [13]) and proceeding to symplectic Runge-Kutta algorithms, the shake algorithm, etc. A fundamentally new approach to symplectic integrators was that of Veselov [14, 15] who developed a discrete mechanics based on a discretization of Hamilton's principle. For standard mechanical systems defined on TQ of some configuration manifold Q, this method leads in a natural way to symplectic-momentum integrators (see Marsden, Patrick, and Shkoller [16] and references therein). In this section we shall outline the construction of a structure preserving integrator for the vortex dynamics problem on a sphere which does not literally fall into this framework.

As we described, the Hamiltonian description of our problem has phase space $P = S^2 \times \cdots \times S^2$ which is not a cotangent bundle, and the symplectic form Ω on P is not canonical. Consequently, the Hamiltonian of the system is not of the form kinetic plus potential energy, but rather has a logarithmic dependence on conjugate variables. Thus, direct application of the theory in [16] is not appropriate for our setting, and an alternative approach is presented which is founded on Lie-Poisson theory.

We consider vortex dynamics on a sphere as a Hamiltonian Lie-Poisson system for a Lie group G. This setting is described in detail in [17] (for Lie-Poisson systems on Lie groups G see, for example, [18]). The idea is to construct a discrete algorithm which preserves the Lie-Poisson structure, and all of the symmetries associated with it, and this is accomplished by duality with the the Lagrangian side. Namely, the Hamiltonian we consider is reduced from a G-invariant Hamiltonian which can be associated with a G-invariant Lagrangian by the fiber derivative of L, the Legendre transform. Now, on the Lagrangian side, we develop a discrete Euler-Poincaré algorithm whose solution naturally provides an algorithm for time-stepping in $\mathfrak{g}^*$ in such a way that the coadjoint orbits, as well as the orbit structure is manifestly preserved. Details of this general construction may be found in [19, 17]; herein, we shall only give a brief overview.

Consider a Lie group that is a direct product of N copies of SO(3), i.e. $G = \mathrm{SO}(3) \times \cdots \times \mathrm{SO}(3)$. As our phase space, we consider the symplectic manifold (T^*G, Ω), where Ω is the γ_n-weighted canonical symplectic form defined as

$$\Omega(v, w) = \sum_n \gamma_n \Omega_n^{\mathrm{can}}(v_n, w_n),$$

and where γ_n are as yet unspecified constants. Let G act by cotangent lift of *left* multiplication. The coadjoint orbit reduction theorem (see, e.g. [18]) states that the symplectic reduced space $J_G^{-1}(\mu)/G_\mu$ is identified via *left* translation with O_μ, the coadjoint orbit through μ. Moreover, the reduced symplectic form coincides, in our case, with the minus

γ_n-weighted coadjoint symplectic form ω^-, which is induced on each symplectic leaf of $\mathfrak{g}^*$ by the minus (appropriately weighted) Lie-Poisson structure.

Thus, we can conclude that for $\mu = (\mu_1, \ldots, \mu_N) \in \mathbb{R}^{3N}$, ω^- is determined (as a product form) by $\omega_n^- = -\gamma_n dS/\|\mu_n\|$ in each copy of $\mathbb{R}^3$, where dS is the standard area form on a sphere of radius $\|\mu_n\|$ (we abuse notations here as the form dS is obviously not exact). We can fix a particular coadjoint orbit by choosing $\|\mu_n\| = R$ for any n for a fixed R. This orbit corresponds to a tensor product of N spheres of radius R with the following symplectic structure $\omega^- = \sum_n \gamma_n \omega_n^- = -\sum_n \gamma_n \, dS/R$, where dS is the area form on a sphere of radius R.

Notice that this is exactly the phase space corresponding to the dynamics of N point vortices on a sphere , where we set $\gamma_n = \Gamma_n$, the vortex strengths. Thinking of the phase space as a coadjoint orbit, we can extend the Hamiltonian (3) arbitrarily to all of $\mathfrak{g}^*$. Then, the point vortex system may be thought of as a Lie-Poisson system, obtained by reduction from a system on T^*G. This point of view enables us to construct the corresponding Euler-Poincaré system on the Lie algebra $\mathfrak{g}$ by performing Legendre transformations.

Once we obtain the Euler-Poincaré description of the vortex dynamics, its discretization can be performed in the following way [19]. Following Moser and Veselov [20], we start with a Lagrangian L system on TG and discretize TG by $G \times G$. We define the discrete Lagrangian, $\mathbb{L} : G \times G \to \mathbb{R}$, by $\mathbb{L}(g_1, g_2) = L\left((g_1 + g_2)/2, (g_2 - g_1)/h\right)$. The action sum $\mathbb{S} = \sum_{k=0}^{N-1} \mathbb{L}(g_k, g_{k+1})$ is formed and the *discrete Euler-Lagrange (DEL) equations* $D_2\mathbb{L}(g_{k-1}, g_k) + D_1\mathbb{L}(g_k, g_{k+1}) = 0$ as well as the discrete symplectic form $\omega_{\mathbb{L}}$ given in coordinates on $G \times G$ by $\omega_{\mathbb{L}} = (\partial^2\mathbb{L}/\partial g_1^i \partial g_2^j) dg_1^i \wedge dg_2^j$ are obtained by extremizing $\mathbb{S} : G^{N+1} \to \mathbb{R}$ with arbitrary variations. One checks that the discrete Lagrangian $\mathbb{L} : G \times G \to \mathbb{R}$ is left (right) invariant under the diagonal action of G on $G \times G$, whenever $L : TG \to \mathbb{R}$ is left (right) invariant.

The reduction of this system proceeds as follows. Recall that the induced group action is simply the left multiplication in each component: $k : (g_1, g_2) \mapsto (kg_1, kg_2)$ for all $k, g_1, g_2 \in G$. Then the quotient map is given by $\pi : G \times G \to (G \times G)/G \cong G, \quad (g_1, g_2) \mapsto g_2^{-1} g_1$. We note that one may alternatively use $g_1^{-1} g_2$ instead of $g_2^{-1} g_1$; our choice is consistent with other literature (see, for example, [16]). The projection map π defines the *reduced discrete Lagrangian* $\ell : G \to \mathbb{R}$ for any G-invariant $\mathbb{L}$ by $\ell \circ \pi = \mathbb{L}$, so that $\ell(g_2^{-1} g_1) = \mathbb{L}(g_1, g_2)$, and the *reduced action sum* is given by $s = \sum_{k=0}^{N-1} \ell(f_{k+1k})$, where $f_{k+1k} \equiv g_{k+1}^{-1} g_k$ denote points in the quotient space. A reduction of the DEL equations results in the *discrete Euler-Poincaré* (DEP) equations and the corresponding constrained variation principle for the reduced action sum. The resulting algorithm is then formulated in terms of reduced variables only, and it can be readily reconstructed to an integrator on the original unreduced space.

5. – Conclusions

The simple physical system of 3 point vortices on a sphere reveals a surprisingly rich geometrical structure. By applying the energy-momentum method, we have found explicit criteria for the stability of different configurations of relative equilibria with generic and non-generic momenta. In each case we have specified a group of transformations modulo which stability in the unreduced space is understood. Numerical simulations of the vortex dynamics have been performed which confirmed the results of stability Theorems **2**.2 and **2**.1.

We presented an outline of a construction of a symplectic-momentum algorithm for the vortex dynamics on a sphere. We refer the reader to [19, 17] for a detailed description of the vortex dynamics on a sphere as a Lie-Poisson system as well as a discrete counterpart of the Euler-Poincaré reduction and the resulting discrete Euler-Poincaré (DEP) equations. We also note that Patrick has constructed some efficient symplectic integrators for the N vortex problem using splitting methods.

Finally, we mention that it would be of interest to extend the results here to the case of rotating spheres.

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