On Billiard Solutions of Nonlinear PDE's *

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Abstract

This letter presents some special features of a class of integrable PDE's admitting billiard-type solutions, which set them apart from equations whose solutions are smooth, such as the KdV equation. These billiard solutions are weak solutions that are piecewise smooth and have first derivative discontinuities at peaks in their profiles. A connection is established between the peak locations and finite dimensional billiard systems moving inside n-dimensional quadrics under the field of Hooke potentials. Points of reflection are described in terms of theta-functions and are shown to correspond to the location of peak discontinuities in the PDE's weak solutions. The dynamics of the peaks is described in the context of the algebraic-geometric approach to integrable systems.

1 Introduction

Camassa and Holm [1993] described classes of n-soliton weak solutions, or "peakons," for an integrable equation arising in the context of shallow water theory. Of particular interest is their description of peakon dynamics in terms of a system of completely integrable Hamiltonian ode's for the locations of the "peaks" of the solution, the points at which its spatial derivative changes sign. In other words, each peakon solution can be associated with a mechanical system of moving particles. Calogero [1995] and Calogero and Francoise [1996] further extended the class of mechanical systems of this type.

The r-matrix approach was applied to the Lax pair formulation of an n-peakon system by Ragnisco and Bruschi [1996], who also pointed out the connection of this system with the classical Toda lattice. A discrete version of the Adler-Kostant-Symes factorization method was used by Suris [1996] to study a discretization of the peakon lattice, realized as a discrete integrable system on a certain Poisson submanifold of $\mathrm{gl}(n)$ equipped with r-matrix Poisson bracket. Generalized peakon systems are obtained for any simple Lie algebra and their complete integrability is demonstrated in Alber *et al.* [1999b].

Antonowicz and Fordy [1987a,b, 1988, 1989] and Antonowicz *et al.* [1991] investigated *energy dependent* Schrödinger operators having potentials with poles in the spectral parameter in connection with certain N-component systems of integrable evolution equations. Using this formalism, they obtained multi Hamiltonian structures for this class of systems of equations. Recently energy dependent Schrödinger operators has been studied in Manas *et al.* [1999] in connection with the flows on the strata of the Grassmanians.

Alber *et al.* [1994, 1995, 1999a] showed that the presence of a pole in the potential is essential in a special limiting procedure that allows for the formation of "billiard solutions" of N-component systems, solutions with discontinuities in first derivative. Using algebraic-geometric methods, one finds that these billiard solutions are related to finite dimensional integrable dynamical systems with reflections. This provides insight into the study of quasi-periodic and solitonic billiard solutions of nonlinear PDE's. This method can be used for a number of equations including the shallow water equation, Dym-type equations, as well as N-component systems with poles and the equations in their hierarchies.

The purpose of this letter is to exhibit some of the special features of integrable PDE's admitting billiard-type solutions that differentiates these equations from those whose solutions are smooth, such as the KdV equation. In particular, quasi-periodic solutions of these equations are integrated on nonlinear subvarieties of Jacobi varieties. This, amongst other things, provides examples of integrable systems of a new type. We make a link between nonsmooth solutions of these equations and billiard dynamical systems. Finally, we establish a connection between Hamiltonian systems describing peakons obtained in Camassa and Holm [1993] and the corresponding systems obtained by tracking the location of discontinuities in the time flows of "μ" variables on Riemann surfaces in the context of the algebraic-geometric approach.

The basic technique of the present letter uses a connection between profiles of wave solutions and geodesic flows with reflections in domains bounded by n-dimensional quadrics. Namely, the time dynamics of peaks in the wave solutions is linked to the points of reflection of the billiard systems. Solutions of the Hamiltonian systems describing the motion of the peaks are obtained by studying certain limits of action-angle representations for quasi-periodic and soliton solutions. The limiting systems of action-angle variables determine Jacobi inversion problems that are solved on *nonlinear* subvarieties of generalized Jacobians.

While the techniques are rather general and can be applied to a large class of N-component integrable evolution equations, we shall illustrate them in detail for two specific integrable PDE's. The dependent variable in these two equations may be interpreted as a horizontal fluid velocity $U(x,t)$. One of these equations is a member of the Dym hierarchy that has been studied by, amongst others, Kruskal [1975], Cewen [1990], Hunter and Zheng [1994] and Alber *et al.* [1995]. Using subscript notation for partial derivatives, this equation is

$$U_{xxt} + 2U_x U_{xx} + U U_{xxx} - 2\kappa U_x = 0\,. \tag{HD}$$

The other equation, derived from the Euler equations of hydrodynamics in Camassa and Holm [1993], is

$$U_t + 3UU_x = U_{xxt} + 2U_x U_{xx} + U U_{xxx} - 2\kappa U_x\,. \tag{SW}$$

In both equations, κ is a real parameter.

Geodesics on Quadrics. We begin by giving an algebraic-geometric description of geodesic motion and motion in the field of a Hooke-type potential on n-dimensional quadrics. As shown by many authors (see, e.g., Rauch-Wojciechowski [1995]), there exists an infinite hierarchy of integrable generalizations of the problem describing a motion on an n-dimensional quadric $\tilde{Q}$:

$$\tilde{Q} := \{(x_1, x_2 \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_1^2/a_1 + \cdots + x_n^2/a_{n+1} = 1\},$$

in the force field of certain polynomial homogeneous potentials $V_p(x_1, \dots, x_{n+1})$, $p \in \mathbf{N}$. The simplest integrable potential is the Hooke potential, that is, the potential of an elastic string joining the center of the ellipsoid $\tilde{Q}$ to the point mass on it:

$$V_1 = \frac{1}{2}\sigma(x_1^2 + \cdots + x_{n+1}^2), \quad \sigma = \text{const.}$$

In this case, in terms of elliptic coordinates $\mu_j' s$ on $\tilde{Q}$, the Lagrangian takes the Stäckel form,

$$L = \frac{1}{8}\sum_{i=1}^{n} \frac{\prod_{j\neq i}(\mu_i - \mu_j)\mu_i}{\Phi(\mu_i)} \left(\frac{d\mu_i}{d\tau}\right)^2 - \frac{\sigma}{2}\sum_{i=1}^{n} \mu_i, \tag{1.1}$$

where $\Phi(\mu) = (\mu - a_1)\cdots(\mu - a_{n+1})$ and the variable τ (called time) parametrizes motion along the trajectory. The equations of motion can be written in Hamiltonian form and constitute a completely integrable system solvable by quadratures, determined in this case by the following Abel–Jacobi equations

$$\sum_{k=1}^{n} \frac{\mu_k^i \; d\mu_k}{2\sqrt{\mathcal{R}(\mu_k)}} = \delta_{in}\; d\tau, \quad i = 1, \dots, n, \tag{1.2}$$

where $\mathcal{R}(\mu) = -L_0\mu\Phi(\mu)[(\mu - c_1)\cdots(\mu - c_{n-1}) - \sigma\mu^n]$ and $c_1, \dots, c_{n-1}$ are constants of motion and δ_{ij} is a Kronecker delta. Notice that for $\sigma = 0$ the order of the polynomial $\mathcal{R}(\mu)$ is odd, whereas for $\sigma \neq 0$ the order is even. The case when $\sigma = 0$ corresponds to free (geodesic) motion on $\tilde{Q}$.

Equations (1.2) contain $(n-1)$ holomorphic differentials and one meromorphic differential defined on the genus n hyperelliptic Riemann surface $\tilde{\mathcal{C}} = \{w^2 = \mathcal{R}(\mu)\}$. Thus the corresponding Abel–Jacobi mapping is inverted on a nonlinear subvariety of the generalized Jacobian of $\tilde{\mathcal{C}}$. On the other hand, under passage to a new variable s defined as

$$\tau = \int_0^s \frac{\mu_1 \cdots \mu_n}{L_0}\, ds\,, \tag{1.3}$$

these equations are transformed to standard form, containing only holomorphic differentials. In this case solution to the inversion problem is well known and follows from the classical results of Jacobi [1884] (see Weierstrass [1878] and Knörrer [1982]). Notice also that the sign of the leading coefficient L_0 should be taken into consideration.

Billiards in Domains Bounded by Quadrics. Applying the limiting procedure in which the quadrics is “flattened” in a certain direction, one obtains descriptions for geodesic billiards and for billiards in the field of a Hooke potential inside lower dimensional quadrics. The sequence of points of reflection gives a solution to completely integrable discrete systems. This result adds to the list of interesting discrete integrable systems considered by Moser and Veselov [1991]. Namely, suppose that one of the semi-axes of the ellipsoid $\tilde{Q}$ tends to zero, say, $a_{n+1} \to 0$. In the limit, $\tilde{Q}$ passes into the interior of the $(N-1)$-dimensional ellipsoid

$$Q := \{(x_1, x_2 \dots, x_n) \in \mathbf{R}^n \mid x_1^2/a_1 + \cdots + x_n^2/a_n = 1\}.$$

The geodesic motion on $\tilde{Q}$ transforms to *billiard* motion inside the ellipsoid Q with Birkhoff reflection conditions. Also, the motion on $\tilde{Q}$ under the Hooke force passes to the motion inside Q under the action of the Hooke force with the potential $V = \sigma(x_1^2 + \cdots + x_n^2)/2$ with elastic reflections along Q. Thus, we have "a generalized ellipsoidal billiard with potential". This system, as well as the billiard limits of the systems with the higher order potentials $V_p(x_1, \dots, x_n, x_{n+1} = 0)$, are completely integrable (see e.g., Kozlov and Treschev [1991]).

Under the limit $a_{n+1} \to 0$ and the change (1.3), the Abel–Jacobi equations can be written in the following integral form

$$\sum_{k=1}^{n} \int_{\mu_0}^{\mu_k} \frac{\mu^{i-1}\, d\mu}{2\sqrt{R(\mu)}} = \phi_i = \text{const}, \quad i = 1, \dots, n-1, \tag{1.4}$$

$$\sum_{k=1}^{n} \int_{\mu_0}^{\mu_k} \frac{d\mu}{2\mu\sqrt{R(\mu)}} = s \tag{1.5}$$

where $R(\mu) = -(\mu - a_1) \cdots (\mu - a_n)[(\mu - c_1) \cdots (\mu - c_{n-1}) - \sigma\mu^n]$ and which contain $(n-1)$ holomorphic differentials on the hyperelliptic Riemann surface $\mathcal{C} = \{w^2 = R(\mu)\}$ of genus $g = (n-1)$ and one differential of third kind Ω having a pair of simple poles $\mathcal{Q}_-$, $\mathcal{Q}_+$ on $\mathcal{C}$ with $\mu(\mathcal{Q}_\pm) = 0$.

The solution of the problem of inversion for (1.4) leads to expressions for the Cartesian coordinates $x_i(s)$ of the point moving inside the ellipsoid Q in terms of quotients of generalized theta-functions. (For details see Fedorov [1999] and Alber *et al.* [1999a].)

Discrete Dynamical Systems. Using these expressions and induction, the coordinates of the whole sequence of impact points are found in the form

$$x_i(N) = \kappa_i \frac{\theta[\Delta + \eta_{(i)}](z_0 + Nq)}{\theta[\Delta](z_0 + Nq)}, \tag{1.6}$$

in case $\sigma = 0$ and

$$x_i(N) = \kappa_i' \frac{\theta[\Delta + \eta_{(i)}](z_0 + Nq)}{\sqrt{\theta[\Delta](z_0 - \hat{q}/2 + Nq)\theta[\Delta](z_0 + \hat{q}/2 + Nq)}}, \tag{1.7}$$

for $\sigma \neq 0$, where $i = 1, \dots, n$, $\kappa_i, \kappa_i' = \text{const}$ and N is the number of impacts encountered so far. Also $q \in \mathbf{C}^g$ is the vector of b-periods of the normalized meromorphic differential Ω and $\hat{q} \in \mathbf{C}^g$ indicates the difference between Abel–Jacobi mappings associated with the two infinite points on $\mathcal{C}$. The vector of constant phases $z_0 = (z_{10}, \dots, z_{g0})^T$ is the same for all the segments of the billiard trajectory. Notice that $\theta[\Delta + \eta_{(i)}](z)$ and $\theta[\Delta](z)$ are the standard theta-functions related to the Riemann surface $\mathcal{C}$ with appropriately chosen half-integer theta-characteristics Δ and $\eta_{(i)}$. The sequence of points (1.6), (1.7) provide solutions of the above discrete dynamical system.

Billiard Solutions of PDE's. Wave solution profiles of the shallow water equation (SW) and the equation of the Dym hierarchy (HD) are, as mentioned before, associated with geodesic motions or motion under the influence of forces determined by a polynomial potential. This results in profiles of billiard weak solutions of these equations being associated with billiard motions inside quadrics and billiards inside quadrics in the presence of a Hooke potential, respectively. This is achieved by using the trace formula at time $t = t_0$; for $\sigma = 0$ we get

$$U(x,t_0) = \sum_{j=1}^{n} \mu_j = \sum_{i=1}^{n} a_i - x^2 - \frac{2x\partial_U \theta[\Delta](z_0) - \partial_U^2 \theta[\Delta](z_0)}{\theta[\Delta](z_0)}, \tag{1.8}$$

while for $\sigma \neq 0$ we get

$$U(x,t_0) = \sum_{j=1}^{n} \mu_j = \sum_{i=1}^{n} a_i - \frac{e^x \theta[\Delta](z_0 + \hat{q}/2) + e^{-x}\theta[\Delta](z_0 - \hat{q}/2)}{\theta[\Delta](z_0)}, \tag{1.9}$$

where $U \in \mathbf{C}^g$ is the vector of b-periods of the normalized differential of the second kind on $\mathcal{C}$ with a double pole at the infinite point and the theta-functions are the same as in (1.7).

Formulae (1.6), (1.7) yield an explicit map for location of the impact points at time t_0,

$$U_N = \sum_{i=1}^{n} x_i^2(N) + \sum_{i=1}^{n} a_i\,,$$

where N is an integer. This, in turn, describes the peaks in the profiles of a weak billiard solution for the nonlinear PDE's under consideration.

The time evolution of the PDE may be viewed as a sequence of snapshots parametrized by t_0. At every fixed time the location of the impact points is recomputed from the initial condition $x_1(N)$. Geometrically, the peaks are moving along the boundary, an ellipsoid of lower dimension. The peak moves in time according to equations similar to those obtained from the Lagrangian (1.1).

The PDE solutions obtained in this way include new quasi-periodic and solitonic billiard solutions, as well as peaked solitons with compact support. (For details see Alber *et al* [1999a].) An example is provided in what follows.

Dynamics of Peaks. For the case of the shallow water equation (SW) we now show how the Hamiltonian structure for the motion of the peaks is obtained by using the algebraic-geometric method. We then show how this can be linked to the Hamiltonian structure found in Camassa and Holm [1993].

The soliton case is obtained from the quasi-periodic case after a transformation of the level sets (first integrals) that shrinks the finite segments of continuous spectrum, therebu creating new points of the discrete spectrum of the associated spectral problem. (For details concerning the inverse scattering transform method, see Ablowitz and Segur [1981] and concerning soliton transition, see Ablowitz and Ma [1981] and Alber and Alber *et al.* [1985].)

Namely, in the case of a 2-peakon solution of the shallow water equation we have the following system:

$$\left.\begin{aligned} \frac{\partial \mu_1}{\partial x} &= \mathrm{Sign}(\mu_1)\frac{(\mu_1-a_1)(\mu_1-a_2)}{(\mu_1-\mu_2)} \\ \frac{\partial \mu_2}{\partial x} &= \mathrm{Sign}(\mu_2)\frac{(\mu_2-a_1)(\mu_2-a_2)}{(\mu_2-\mu_1)} \end{aligned}\right\} \tag{1.10}$$

where μ_1 and μ_2 are evaluated between a_1 and 0 and a_2 and 0, respectively. The corresponding time-flow is

$$\left.\begin{aligned} \frac{\partial \mu_1}{\partial t} &= \mathrm{Sign}(\mu_1)B_1(\mu_1)\frac{(\mu_1-a_1)(\mu_1-a_2)}{(\mu_1-\mu_2)} \\ \frac{\partial \mu_2}{\partial t} &= \mathrm{Sign}(\mu_2)B_1(\mu_2)\frac{(\mu_2-a_1)(\mu_2-a_2)}{(\mu_2-\mu_1)} \end{aligned}\right\} \tag{1.11}$$

and the first order polynomial B_1 for equation (SW) is given by

$$B_1(\mu_1) = -\mu_2 + a_1 + a_2, \qquad B_1(\mu_2) = -\mu_1 + a_1 + a_2 \,.$$

Soliton solutions of the nonlinear evolution equations can be constructed using the trace formula (1.9).

The positions of the peaks can be identified in terms of μ-variables as follows: $\mu_1 = 0$ for the first peak and $\mu_2 = 0$ for the second peak. The dynamics of the peaks is defined by introducing functions $q_1(t)$ and $q_2(t)$ such that

$$\mu_1(q_1(t), t) = 0, \;\; \mu_2(q_2(t), t) = 0 \tag{1.12}$$

and functions $y_1(t)$ and $y_2(t)$ defined as follows:

$$y_1 = \mu_2(q_1(t), t), \;\; y_2 = \mu_1(q_2(t), t). \tag{1.13}$$

Differentiating $\mu_1(q_1(t), t)$ and $\mu_2(q_2(t), t)$ in (1.12) results in

$$\left.\begin{aligned} \frac{d\mu_1}{dt} &= 0 = \frac{\partial \mu_1}{\partial x}\left(\frac{dq_1}{dt} + B_1(\mu_1)\right) = \frac{\partial \mu_1}{\partial x}\left(\frac{dq_1}{dt} - y_1 + a_1 + a_2\right) \\ \frac{d\mu_2}{dt} &= 0 = \frac{\partial \mu_2}{\partial x}\left(\frac{dq_2}{dt} + B_1(\mu_2)\right) = \frac{\partial \mu_2}{\partial x}\left(\frac{dq_2}{dt} - y_2 + a_1 + a_2\right). \end{aligned}\right\} \tag{1.14}$$

This leads to the following system

$$\left.\begin{aligned} \frac{dq_1}{dt} &= U(q_1) = (\mu_1 + \mu_2 - a_1 - a_2)\Big|_{q_1} = \mu_2(q_1) - a_1 - a_2 = y_1 - a_1 - a_2 \\ \frac{dq_2}{dt} &= U(q_2) = (\mu_1 + \mu_2 - a_1 - a_2)\Big|_{q_2} = \mu_1(q_2) - a_1 - a_2 = y_2 - a_1 - a_2 \,, \end{aligned}\right\} \tag{1.15}$$

which coincides with the jump conditions for weak solutions (see Alber *et al.* [1999]). Finally, differentiate y_1 and y_2 to find

$$\left.\begin{aligned} \frac{dy_1}{dt} &= \frac{\partial \mu_2}{\partial x}\frac{dq_1}{dt} + \frac{\partial \mu_2}{\partial t} = \frac{\partial \mu_2}{\partial x}\left(\frac{dq_1}{dt} + B_1(\mu_2)\right) \\ \frac{dy_2}{dt} &= \frac{\partial \mu_1}{\partial x}\frac{dq_2}{dt} + \frac{\partial \mu_1}{\partial t} = \frac{\partial \mu_1}{\partial x}\left(\frac{dq_2}{dt} + B_1(\mu_1)\right) \end{aligned}\right\}$$

which yields

$$\left.\begin{aligned} \frac{dy_1}{dt} &= \mathrm{Sign}(y_1)(y_1 - a_1)(y_1 - a_2) \\ \frac{dy_2}{dt} &= \mathrm{Sign}(y_2)(y_2 - a_1)(y_2 - a_2)\,. \end{aligned}\right\} \tag{1.16}$$

Thus, the equations of evolution for y_1 and y_2 decouple from those for q_1 and q_2. The decoupled equations (1.16) can be solved first and q_1 and q_2 can be subsequently determined from (1.15) by quadratures. The final result, for $a_1 > a_2$, is

$$y_1(t) = \frac{a_1 + a_2 c_1 e^{-(a_1-a_2)t}}{1 + c_1 e^{-(a_1-a_2)t}}, \qquad y_2(t) = \frac{a_1 + a_2 c_2 e^{(a_1-a_2)t}}{1 + c_1 e^{(a_1-a_2)t}}$$

and

$$q_1(t) = -a_1 t + d_1 + \log\left(c_1 + e^{(a_1-a_2)t}\right), \qquad q_2(t) = -a_2 t + d_2 - \log\left(1 + c_2 e^{(a_1-a_2)t}\right),$$

where the constants (c_1, d_1) and (c_2, d_2) are related to the initial values of (y_1, q_1) and (y_2, q_2), respectively, and these parameters are in turn related through the x-system for the μ-functions (1.10) (for details see Alber and Miller [1999]) . Figure 1.1 shows the contour plot of a two-peakon solution $U(x,t)$ for a slower peakon being overtaken by a faster one. The spectrum a_1, a_2 is chosen so that $a_2 = 2a_1$, in which case the slower soliton does not experience a phase shift (see Camassa and Holm [1993]). This is different from the KdV situation, in which interacting solitons always exhibit phase shifts. It is interesting to examine the connection of the set of variables q_i and y_i, $i = 1, 2$, with the q_i and p_i, $i = 1, 2$, introduced by Camassa and Holm [1993]. By definition, the q's are the same in both sets. Notice that the (q, y) system does not have a canonical Hamiltonian form. As to the y's, notice that

$$\left.\begin{aligned} U(q_1) &= y_1 - a_1 - a_2 = p_1 + p_2 e^{-|q_1-q_2|} \\ U(q_2) &= y_2 - a_1 - a_2 = p_2 + p_1 e^{-|q_1-q_2|} \end{aligned}\right\}, \tag{1.17}$$

which provides an expression for the variables y's in terms of p's and q's, and together with (1.15), yields the first set of equations for the Hamiltonian system derived by Camassa and Holm [1993],

$$\left.\begin{aligned} \frac{dq_1}{dt} &= p_1 + p_2 e^{-|q_1-q_2|} \\ \frac{dq_2}{dt} &= p_2 + p_1 e^{-|q_1-q_2|}\,. \end{aligned}\right\}$$

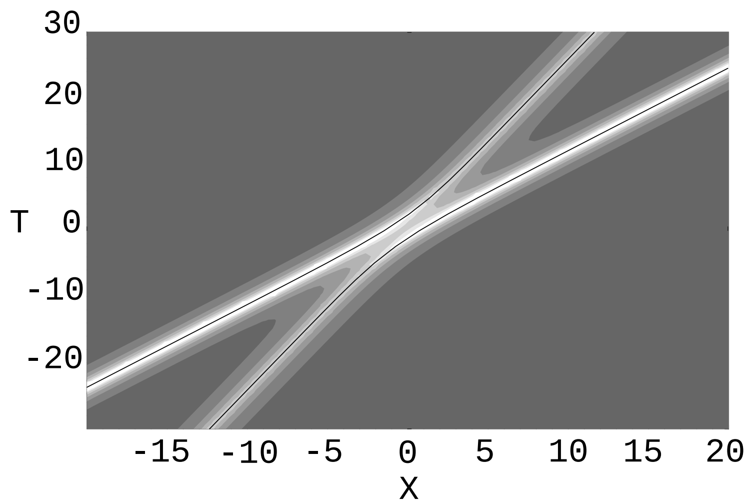

Figure 1.1: Contour plot of a fast peakon catching up with a slower one. Here $a_1 = -0.4$, $a_2 = -0.8$, and the peaks have the same height at time $t = 0$. The thin solid lines represent the peak trajectories obtained by solving systems (1.16) and (1.12) for $q_1(t)$ and $q_2(t)$. For this ratio $(a_2/a_1 = 2)$ of terminal speeds, the slower soliton does not experience any phase shift.

The evolution equation for the p's follows from (1.16), (1.15), and the transformation (1.17):

$$\left.\begin{aligned} \frac{dp_1}{dt} &= \operatorname{sgn}(q_2 - q_1) p_1 p_2 e^{-|q_1 - q_2|} \\ \frac{dp_2}{dt} &= \operatorname{sgn}(q_1 - q_2) p_2 p_1 e^{-|q_1 - q_2|} \,. \end{aligned}\right\}$$

Notice that the constants of motion a_1 and a_2 can be incorporated into first integrals involving p's and q's as follows: $P_{12} = p_1 + p_2 = -(a_1 + a_2)$ and

$$H_{12} = \frac{1}{2}(p_1^2 + p_2^2) + p_1 p_2 e^{-|q_1 - q_2|} = \frac{1}{2}(a_1^2 + a_2^2)\,.$$

These first integrals are the total momentum and Hamiltonian for the (q, p)-flow, respectively.

The case of three or more derivative-shock singularities x_i, y_i, $i = 1, 2, \ldots, n \geq 3$ proceeds in complete analogy with the case $n = 2$ above. Once again, the q-flows decouple from those of the y's. At variance with the $n = 2$ case, the equations forming the system that governs the y-flow are now coupled, and it is not immediately obvious that this system is integrable. A closer inspection however reveals that the y-flow shares the same structure as that of the μ-variables flow and is therefore integrable by a similar argument.

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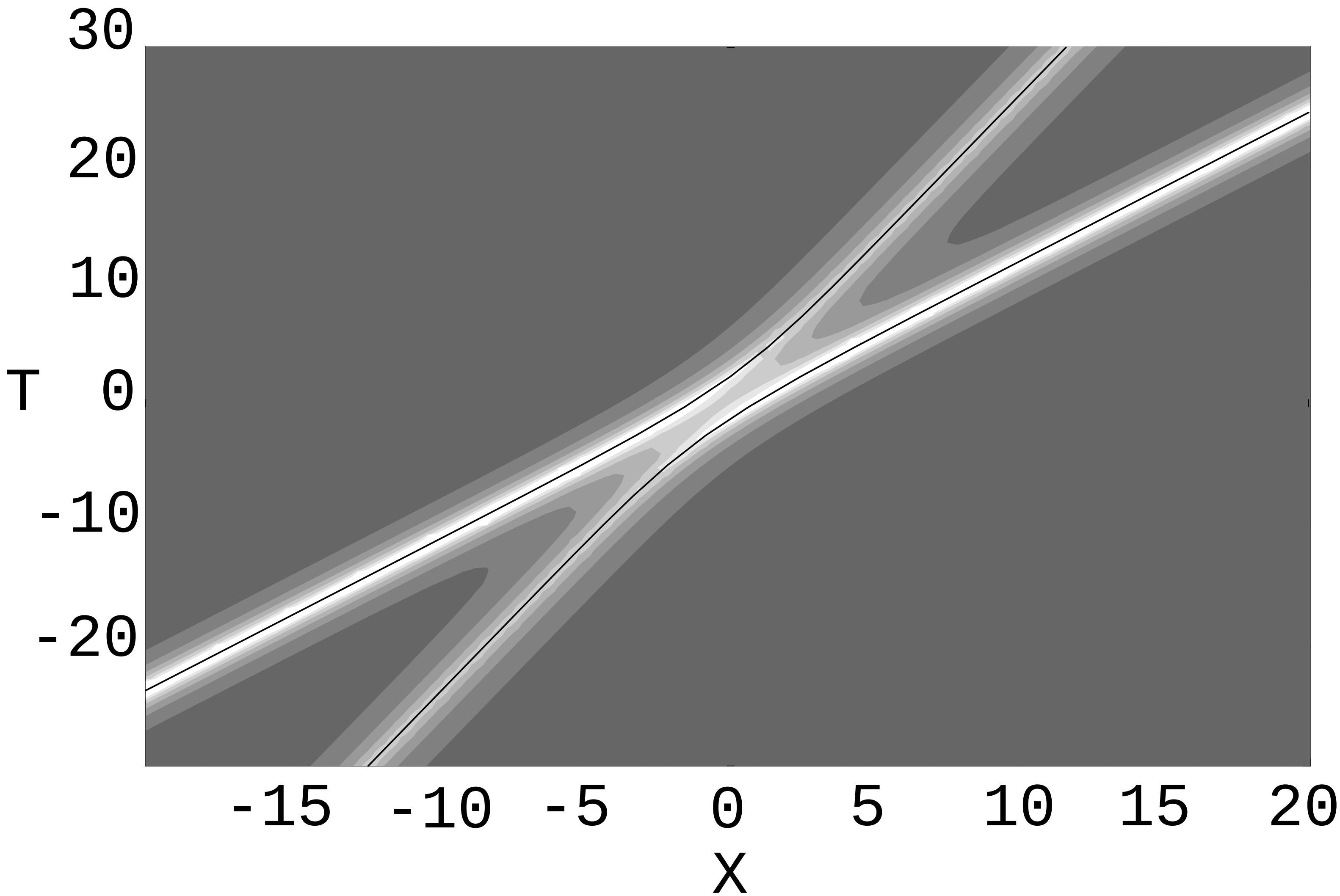
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