Multisymplectic Geometry, Covariant Hamiltonians, and Water Waves

Jerrold E. Marsden*
Control and Dynamical Systems
California Institute of Technology 116-81
Pasadena, CA 91125

Steve Shkoller †
Center for Nonlinear Studies
Los Alamos National Laboratory MS-B258
Los Alamos, NM 87545 USA

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Abstract

This paper concerns the development and application of the multisymplectic Lagrangian and Hamiltonian formalism for nonlinear partial differential equations. This theory generalizes and unifies the classical Hamiltonian formalism of particle mechanics as well as the many presymplectic 2-forms used by Bridges [1997]. In this theory, solutions of a PDE are sections of a fiber bundle Y over a base manifold X of dimension $n+1$, typically taken to be spacetime. Given a connection on Y, a covariant Hamiltonian density $\mathcal{H}$ is then intrinsically defined on the primary constraint manifold $P_{\mathcal{L}}$, the image of the multisymplectic version of the Legendre transformation. One views $P_{\mathcal{L}}$ as a subbundle of $J^1(Y)^\star$, the *affine* dual of $J^1(Y)$, the first jet bundle of Y. A canonical multisymplectic $(n+2)$-form $\Omega_{\mathcal{H}}$ is then defined, from which we obtain a multisymplectic Hamiltonian system of differential equations that is equivalent to both the original PDE as well as the Euler-Lagrange equations of the corresponding Lagrangian. Furthermore, we show that the $n+1$ 2-forms $\omega^{(\mu)}$ defined by Bridges [1997] are a particular coordinate representation for a single multisymplectic $(n+2)$-form, and in the presence of symmetries, can be *assembled* into $\Omega_{\mathcal{H}}$. A generalized Hamiltonian Noether theory is then constructed which relates the action of the symmetry groups lifted to $P_{\mathcal{L}}$ with the conservation laws of the system. These conservation laws are defined by our generalized Noether's theorem which recovers the vanishing of the divergence of the vector of $n+1$ distinct momentum mappings defined in Bridges [1997] and, when applied to water waves, recovers Whitham's conservation of wave action. In our view, the multisymplectic structure provides the natural setting for studying dispersive wave propagation problems, particularly the instability of water waves, as discovered by Bridges. After developing the theory, we show its utility in the study of periodic pattern formation and wave instability.

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*Research partially supported by NSF grant DMS 96–33161 and DOE contract DE–FG0395–ER25251. marsden@cds.caltech.edu

†Research partially supported by the Cecil and Ida M. Green Foundation and DOE. email: shkoller@cnls.lanl.gov

1 Introduction

The canonical symplectic structure on the cotangent bundle of a given configuration space provides a natural correspondence between Hamiltonian vector fields that govern the evolution of conservative ordinary differential equations and the Hamiltonian functions which describe them. The setting of tangent and cotangent bundles also provides a natural setting for the Lagrangian description of dynamics and the Legendre transformation that connects the Lagrangian and Hamiltonian points of view.

In either case, the use of vector fields for the description of the dynamics is natural, because for ordinary differential equations there is a single distinguished variable, time. On the contrary, in systems of partial differential equations, solutions depend on multiple variables, usually spatial as well as temporal, so one can make the case that a single vector field is not the appropriate point of view because it would require collapsing all of the spatial structure of a solution to a single point of phase space. This occurs when a choice is made to consider the time coordinate separately, and describe the dynamics in terms of an infinite-dimensional space of fields at a given instant in time. Although this methodology has been very successful, availing itself to the powerful organizing structure of the theory of evolution operators from a point of view of functional analysis, its immediate affect is a break of manifest covariance.

To maintain a covariant description, one can use a generalization of symplectic geometry known as multisymplectic geometry. This subject has a long and distinguished history that we shall not review in this article; rather, we follow the framework established in Gotay [1991], Gotay et al [1992], and Gotay and Marsden [1992], wherein relativistic field theories with Dirac-Bergmann type constraints are considered in a Lagrangian formalism, while the Hamiltonian formalism relies on a "space + time" (or 3+1) split. These references contain citations to much of the important literature and history of the subject.

It is interesting that the structure of connection is not necessary to intrinsically define the Lagrangian formalism (as shown in the preceding references), while for the intrinsic definition of a covariant Hamiltonian the introduction of such a structure is essential. Of course, one can avoid a connection if one is willing to confine ones attention to local coordinates. We give an intrinsic definition of the covariant Hamiltonian so that we may examine the fundamental interplay of the equivariance of Hamiltonian and Lagrangian densities with respect to group actions.

Our objective is to use a variant of the multisymplectic Hamiltonian formalism to generalize and make intrinsic the seminal and extremely important work of Bridges [1996a, 1996b, 1997] on wave propagation, periodic pattern formation, and linear instability. Roughly speaking, our main result states that in the case of n distinct and possibly unbounded spatial directions, the n+1 pre-symplectic 2-forms introduced by Bridges are actually contained in a single higher degree multisymplectic $(n+2)$-form, and that in the presence of symmetries, these many forms can be *assembled* into this single canonical form. Furthermore, the covariant Hamiltonian Noether theory that we construct, generalizes Bridges' clever decomposition of water wave conservation laws, and is an intrinsic restatement of the constrained variational principles which lead to the existence of water wave instabilities and diagonal periodic pattern formation.

We begin in Section 2 by recalling some of the basic constructions and a few key results from the multisymplectic formalism of Gotay et al [1992]. In Section 3, we add the structure of connection and intrinsically define our covariant Hamiltonian density. In Section 4, we show that our multisymplectic formalism generalizes the classical theory of particle mechanics, as well Bridges' theory of nonlinear partial differential equations. Section 5 is devoted to our development of a covariant Hamiltonian Noether theory. In Section 6, we show how this theory recovers the classical conservation laws of particle mechanics as well as the new conservation laws proposed by Bridges [1996b] for studying water waves. Finally, in Section 7 we show how our general theory applies to the study of periodic pattern formation and the instability of waves.

2 Multisymplectic Geometry

A **covariant configuration bundle** is a finite-dimensional fiber bundle $\pi_{XY} : Y \to X$ over an oriented manifold X. In many examples, especially those occurring in relativistic field theories, X is chosen to be spacetime and the fields of interest are sections of this bundle. For nonrelativistic theories, such as nonlinear waves, one typically chooses X to be classical spacetime (i.e., the product of the reals, $\mathbb{R}$, with the spatial variables).

We shall need a little notation. Denote the fiber $\pi_{XY}^{-1}(x)$ of Y over $x \in X$ by Y_x and the tangent space to X at x by T_xX, etc., and denote sections of π_{XY} by $\Gamma(\pi_{XY})$. We also let $VY \subset TY$ be the **vertical subbundle**; this is the bundle over Y whose fibers are given by

$$V_yY = \{v \in T_yY \mid T\pi_{XY} \cdot v = 0\}, \tag{2.1}$$

where $T\pi_{XY} \cdot v$ denotes the derivative of the map π_{XY} in the direction v.

Just as the covariant configuration bundle is the analogue of the configuration space in particle mechanics, the first jet bundle, defined next, is the field theoretic analogue of the tangent bundle.

Definition 2.1 *The* **first jet bundle** $J^1(Y)$ *is the affine bundle over Y whose fiber over $y \in Y_x$ consists of those linear mappings $\gamma : T_xX \to T_yY$ satisfying*

$$T\pi_{XY} \circ \gamma = \textit{Identity on } T_xX. \qquad \diamond \tag{2.2}$$

The vector bundle underlying this affine bundle is the bundle whose fiber over $y \in Y_x$ is the space $L(T_xX, V_yY)$ of linear mappings of T_xX into V_yY. Note that for each $\gamma \in J^1(Y)_y$, we have the splitting

$$T_yY = \text{image } \gamma \oplus V_yY. \tag{2.3}$$

The choice of the *first* jet bundle $J^1(Y)$ is used for the field theoretic tangent bundle for classical field theories whose Lagrangians depend on the point values of the fields and their *first* derivatives.

For higher order field theories, one uses higher order jet bundles; see Gotay et al [1992] and Gotay and Marsden [1992] for references to this literature.

We let dim $X = n+1$ and the fiber dimension of Y be N. Coordinates on X are denoted $x^\mu, \mu = 1, 2, \ldots, n, 0$, and fiber coordinates on Y are denoted by $y^A, A = 1, \ldots, N$. These induce coordinates $v^A{}_\mu$ on the fibers of $J^1(Y)$. If $\phi : X \to Y$ is a section of π_{XY}, its tangent map at $x \in X$, denoted $T_x\phi$, is an element of $J^1(Y)_{\phi(x)}$. Thus, the map $x \mapsto T_x\phi$ defines a section of $J^1(Y)$ regarded as a bundle over X. This section is denoted $j^1(\phi)$ and is called the **first jet** of ϕ. In coordinates, $j^1(\phi)$ is given by

$$x^\mu \mapsto (x^\mu, \phi^A(x^\mu), \partial_\nu \phi^A(x^\mu)), \tag{2.4}$$

where $\partial_\nu = \partial/\partial x^\nu$. A section of the bundle $J^1(Y) \to X$ which is the first jet of a section of $Y \to X$ is said to be **holonomic**.

The field theoretic analogue of the cotangent bundle is defined next.

Definition 2.2 *The* **dual jet bundle** $J^1(Y)^\star$ *is the vector bundle over Y whose fiber at $y \in Y_x$ is the set of affine maps from $J^1(Y)_y$ to $\Lambda^{n+1}(X)_x$, the bundle of $(n+1)$-forms on X.* $\diamond$

A smooth section of $J^1(Y)^\star$ is therefore an affine bundle map of $J^1(Y)$ to $\Lambda^{n+1}(X)$ covering π_{XY}. We choose affine maps since $J^1(Y)$ is an affine bundle, and we map into $\Lambda^{n+1}(X)$ since we are ultimately thinking of integration as providing the pairing on sections.

Fiber coordinates on $J^1(Y)^\star$ are $(p, p_A{}^\mu)$, which correspond to the affine map given in coordinates by

$$v^A{}_\mu \mapsto (p + p_A{}^\mu v^A{}_\mu) d^{n+1}x, \tag{2.5}$$

where $d^{n+1}x = dx^1 \wedge \cdots \wedge dx^n \wedge dx^0$.

Analogous to the canonical one- and two-forms on a cotangent bundle, there are canonical forms on $J^1(Y)^\star$. To define these, another description of $J^1(Y)^\star$ will be convenient. Namely, let $\Lambda := \Lambda^{n+1}(Y)$ denote the bundle of $(n+1)$-forms on Y, with fiber over $y \in Y$ denoted by Λ_y and with projection $\pi_{Y\Lambda} : \Lambda \to Y$. Let $Z \subset \Lambda$ be the subbundle whose fiber is given by

$$Z_y = \{z \in \Lambda_y \,|\, v \lrcorner (w \lrcorner z) = 0 \text{ for all } v, w \in V_y Y\}, \tag{2.6}$$

where $v \lrcorner \cdot$ denotes left interior multiplication by v.

Elements of Z can be be written uniquely as

$$z = p d^{n+1}x + p_A{}^\mu dy^A \wedge d^n x_\mu, \tag{2.7}$$

where

$$d^n x_\mu = \partial_\mu \lrcorner d^{n+1}x \quad \text{ and, as before, } \quad \partial_\mu = \frac{\partial}{\partial x^\mu}.$$

Hence, fiber coordinates for Z are also $(p, p_A{}^\mu)$.

Corresponding to equating the coordinates $(x^\mu, y^A, p, p_A{}^\mu)$ of Z and of $J^1(Y)^\star$, there is a vector bundle isomorphism

$$\Phi : Z \to J^1(Y)^\star. \tag{2.8}$$

Intrinsically, Φ is defined by pull-back:

$$\Phi(z)(\gamma) = \gamma^* z \in \Lambda^{n+1}(X)_x \tag{2.9}$$

where $z \in Z_y$, $\gamma \in J^1(Y)_y$ and $x = \pi_{XY}(y)$. Using fiber coordinates $v^A{}_\mu$ for γ, the preceding equation becomes

$$\gamma^* dx^\mu = dx^\mu \quad \text{and} \quad \gamma^* dy^A = v^A{}_\mu dx^\mu \tag{2.10}$$

and so

$$\gamma^*(p d^{n+1}x + p_A{}^\mu dy^A \wedge d^n x_\mu) = (p + p_A{}^\mu v^A{}_\mu) d^{n+1}x, \tag{2.11}$$

where we have used $dx^\nu \wedge d^n x_\mu = \delta^\nu_\mu d^{n+1}x$.

One shows that the inverse of Φ can also be defined intrinsically, although it is somewhat more complicated, and thus the spaces $J^1(Y)^\star$ and Z are canonically isomorphic as vector bundles over Y.

There are canonical forms on Z and the isomorphism between $J^1(Y)^\star$ and Z can be used to transfer these to $J^1(Y)^\star$. We first define the **canonical** $(n+1)$**-form** Θ_Λ on Λ by

$$\begin{aligned} \Theta_\Lambda(z)(u_1, \ldots, u_{n+1}) &= z(T\pi_{Y\Lambda} \cdot u_1, \ldots, T\pi_{Y\Lambda} \cdot u_{n+1}) \\ &= (\pi^*_{Y\Lambda} z)(u_1, \ldots, u_{n+1}) \end{aligned} \tag{2.12}$$

where $z \in \Lambda$ and $u_1, \ldots, u_{n+1} \in T_z\Lambda$. Define the **canonical** $(n+2)$**-form** Ω_Λ on Λ by

$$\Omega_\Lambda = -d\Theta_\Lambda. \tag{2.13}$$

Note that if $n = 0$ (i.e., X is one-dimensional), then $\Lambda = T^*Y$ and Θ_Λ is the standard canonical one-form. If $i_{\Lambda Z} : Z \to \Lambda$ denotes the inclusion, the **canonical** $(n+1)$**-form** Θ on Z is defined by

$$\Theta = i^*_{\Lambda Z} \Theta_\Lambda \tag{2.14}$$

and the **canonical** $(n+2)$**-form** Ω on Z is defined by

$$\Omega = -d\Theta = i^*_{\Lambda Z} \Omega_\Lambda. \tag{2.15}$$

The pair (Z, Ω) is called **multiphase space** or **covariant phase space**. It is an example of a **multisymplectic manifold**.

Using (2.7), (2.12), (2.13), (2.14) and (2.15), one finds that the coordinate expression for Θ is

$$\Theta = p_A{}^\mu dy^A \wedge d^n x_\mu + p d^{n+1}x, \tag{2.16}$$

and so

$$\Omega = dy^A \wedge dp_A{}^\mu \wedge d^n x_\mu - dp \wedge d^{n+1}x. \tag{2.17}$$

Let the **Lagrangian density** $\mathcal{L} : J^1(Y) \to \Lambda^{n+1}(X)$, be a given smooth bundle map over X. In coordinates, we write

$$\mathcal{L}(\gamma) = L(x^\mu, y^A, v^A{}_\mu) d^{n+1}x. \tag{2.18}$$

The corresponding **covariant Legendre transformation** associated with $\mathcal{L}$ is a fiber preserving map over Y, $\mathbb{F}\mathcal{L} : J^1(Y) \to J^1(Y)^\star \cong Z$, whose intrinsic definition follows.

Definition 2.3 *If $\gamma \in J^1(Y)_y$, we define $\mathbb{F}\mathcal{L}(\gamma) \in J^1(Y)^\star_y$ (where $y \in Y_x$) to be the first order vertical Taylor approximation to $\mathcal{L}$:*

$$\mathbb{F}\mathcal{L}(\gamma) \cdot \gamma' = \mathcal{L}(\gamma) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{L}(\gamma + \varepsilon(\gamma' - \gamma)) \tag{2.19}$$

where $\gamma' \in J^1(Y)_y$. $\diamond$

A straightforward calculation shows that the covariant Legendre transformation is given in coordinates by

$$p_A{}^\mu = \frac{\partial L}{\partial v^A{}_\mu}, \text{ and } \quad p = L - \frac{\partial L}{\partial v^A{}_\mu} v^A{}_\mu. \tag{2.20}$$

Notice that formally, the second of these equations defines the (negative of the) energy while the first one is reminiscent of the usual relation $p_i = \partial L/\partial \dot{q}^i$ from classical mechanics. One of the nice features of the covariant Legendre transformation is how these two basic aspects of the Legendre transformation arise from a single construction.

Definition 2.4 *The* **Cartan form** *is the $(n+1)$-form $\Theta_\mathcal{L}$ on $J^1(Y)$ defined by*

$$\Theta_\mathcal{L} = (\mathbb{F}\mathcal{L})^* \Theta \tag{2.21}$$

where Θ is the canonical $(n+1)$-form on Z. We also define the $(n+2)$-form $\Omega_\mathcal{L}$ by

$$\Omega_\mathcal{L} = -d\Theta_\mathcal{L} = (\mathbb{F}\mathcal{L})^* \Omega \tag{2.22}$$

where $\Omega = -d\Theta$ is the canonical $(n+2)$-form on Z. $\diamond$

3 The Covariant Hamiltonian

In this section we develop an intrinsic covariant (or multisymplectic) Hamiltonian formalism. We begin by noting that the covariant Legendre transformation $\mathbb{F}\mathcal{L} : J^1(Y) \to J^1(Y)^\star$ is never a fiber bundle diffeomorphism since $\dim J^1(Y)^\star = \dim J^1(Y) + 1$; nevertheless, it is appropriate in many examples to require $\mathbb{F}\mathcal{L}$ to be a smooth bundle diffeomorphism over Y onto its image. In fact, from the second equation in (2.20), the image of $\mathbb{F}\mathcal{L}$ defines the primary constraint of the theory.

Definition 3.1 *We say that $\mathcal{L}$ is* **regular** *if the image of the first jet bundle under the covariant Legendre transformation $P_\mathcal{L} := \mathbb{F}\mathcal{L}(J^1(Y))$ is a smooth manifold and if $\mathbb{F}\mathcal{L}$ is a diffeomorphism onto $P_\mathcal{L}$. We call $P_\mathcal{L}$ the* **primary constraint manifold**. $\diamond$

One should note that many field theories, such as the vacuum Maxwell equations and many others, especially relativistic ones, are not regular because of the presence of constraints (such as $\operatorname{div} \mathbf{E} = 0$ for Maxwell's equations). We are assuming regularity only for simplicity and because it is appropriate for the examples we have in mind. Gotay et al [1992] deal with the more general case in a Lagrangian formalism.

At this point, we introduce the additional structure of a connection. While connections are not particularly needed for the Lagrangian side of field theory, they seem to be essential for the development of an intrinsic Hamiltonian formalism. We recall the definition of an (Eheresmann) connection as a vertical-valued one-form.

Definition 3.2 *A* **connection** *on Y is a vector bundle map $\mathfrak{A} : TY \to VY$ such that on each fiber over $y \in Y$, $\mathfrak{A} : T_yY \to V_yY$ satisfies*

$$\mathfrak{A} = \text{ Identity on } VY. \tag{3.1}$$

The **horizontal space** *at each point $y \in Y$ is defined by $\mathrm{hor}_y = \ker \mathfrak{A}_y$, so that we have $T_yY = \mathrm{hor}_y \oplus V_yY$.* $\diamond$

In coordinates, the action of $\mathfrak{A}$ on a tangent vector to Y, namely (v^ν, v^A) is written as $(0, v^A + \mathfrak{A}^A_\mu v^\mu)$. This defines the coordinate expression for the connection. We remark that it is not entirely necessary to a priori explicitly introduce a connection if one wishes to define the Hamiltonian locally in a coordinate chart, and then use coordinate patches to obtain a global characterization; however, the process of producing a coordinate independent global definition is tantamount to producing a connection.

Next, we reexpress the covariant Legendre transformation $\mathbb{F}\mathcal{L}$ in terms of a vertical derivative of functions on $J^1(Y)$.

Definition 3.3 *For any $x \in X$ and $y \in Y_x$, let $U \subset J^1(Y)_y$ be an open subset, and let $\mathcal{S} \in C^1(U, \Lambda^{n+1}(X)_x)$. Then the* **covariant derivative** *of S associated with the connection $\mathfrak{A}$ maps U into $C^0(J^1(Y)_y, \Lambda^{n+1}(X)_x)$ and is defined, for any $\gamma \in U$, and $\gamma' \in J^1(Y)_y$, by*

$$D_{\mathfrak{A}}\mathcal{S}(\gamma) \cdot \gamma' = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{S}(\gamma + \varepsilon \mathfrak{A}(\gamma')). \qquad \diamond \tag{3.2}$$

It is then natural to consider the covariant derivative of the smooth bundle map $\mathcal{L} : \pi_{X,J^1(Y)} \to \pi_{X,\Lambda^{n+1}(X)}$, so that using (3.2), the Legendre transformation can be written as

$$\mathbb{F}\mathcal{L}(\gamma) \cdot \gamma' = [\mathcal{L}(\gamma) - D_{\mathfrak{A}}\mathcal{L}(\gamma) \cdot \gamma] + D_{\mathfrak{A}}\mathcal{L}(\gamma) \cdot \gamma' \tag{3.3}$$

for all $\gamma, \gamma' \in J^1(Y)_y$ and $y \in Y_x$. Note that this expression is affine with the first two terms being the constant terms and the last one being the linear term.

We may now define the covariant Hamiltonian density on the primary constraint manifold $P_{\mathcal{L}}$.

Definition 3.4 *For a regular Lagrangian, the corresponding* **covariant Hamiltonian** $\mathcal{H} : P_{\mathcal{L}} \to \Lambda^{n+1}(X)$ *is defined by*

$$\mathcal{H}(z) = D_{\mathfrak{A}}\mathcal{L}(\gamma) \cdot \gamma - \mathcal{L}(\gamma) \tag{3.4}$$

where $z = \mathbb{F}\mathcal{L}(\gamma)$. $\diamond$

In coordinates, we may write $\mathcal{H} = H d^{n+1}x$ where

$$H = \frac{\partial L}{\partial v^A{}_\mu}(v^A{}_\mu + \mathfrak{A}^A_\mu) - L.$$

Notice that the covariant Hamiltonian is well defined under the assumption of regularity; namely, the map $\gamma \mapsto z = \mathbb{F}\mathcal{L}(\gamma)$ from $J^1(Y)$ to $P_{\mathcal{L}}$ is a diffeomorphism.

We coordinatize the primary constraint manifold by $(x^\mu, y^A, p_A{}^\mu)$ with p now expressed in terms of the other variables by rewriting the preceding expression for H as

$$H = \frac{\partial L}{\partial v^A{}_\mu}\mathfrak{A}^A_\mu - p,$$

regarded as an implicit equation for p.

Let $i_{J^1(Y)^\star,P_\mathcal{L}} : P_\mathcal{L} \to J^1(Y)^\star$ denote the inclusion. We may pull-back the canonical $(n+1)$- and $(n+2)$-forms on $J^1(Y)^\star$ to $P_\mathcal{L}$ and obtain (using a notation to remind us that this takes the Hamiltonian point of view):

$$\begin{aligned} \Theta_\mathcal{H} &= i^*_{J^1(Y)^\star,P_\mathcal{L}}\Theta, \\ \Omega_\mathcal{H} &= i^*_{J^1(Y)^\star,P_\mathcal{L}}\Omega. \end{aligned} \tag{3.5}$$

In canonical coordinates, we have

$$\begin{aligned} \Theta_\mathcal{H} &= p_A{}^\mu dy^A \wedge d^n x_\mu + (p_A{}^\mu \mathfrak{A}^A_\mu - H) \wedge d^{n+1}x \\ \Omega_\mathcal{H} &= dy^A \wedge dp_A{}^\mu \wedge d^n x_\mu + \left[\tfrac{\partial H}{\partial y^A} dy^A + \left(\tfrac{\partial H}{\partial p_A{}^\mu} - \mathfrak{A}^A_\mu\right) dp_A{}^\mu\right] \wedge d^{n+1}x. \end{aligned} \tag{3.6}$$

For many important examples, we will consider X as the classical spacetime manifold with the locally trivial connection which is simply the natural projection whose action in coordinates is $(0, v^A)$, i.e., the components $\mathfrak{A}^A_\mu = 0$.

Definition 3.5 *Let $\phi \in \Gamma(\pi_{XY})$, and $j^1(\phi)$ its first jet. A section z of $\pi_{X,P_\mathcal{L}}$ is called* **conjugate** *to $j^1(\phi)$ if $z = \mathbb{F}\mathcal{L} \circ j^1(\phi)$. In this case, we shall write $\tilde{j}^1(\phi)$ for z and say that z is holonomic.* $\diamond$

Definition 3.6 *A holonomic section z of $P_\mathcal{L}$ is called* **Hamiltonian** *for $\mathcal{H}$ if*

$$z^*(U \lrcorner \Omega_\mathcal{H}) = 0. \tag{3.7}$$

for any $U \in T(P_\mathcal{L})$. We also refer to the system of equations (3.7) regarded as differential equations for z as the **multihamiltonian system** *of equations associated to $\mathcal{H}$.* $\diamond$

Lemma 3.1 *If $\mathbb{F}\mathcal{L} : J^1(Y) \to P_\mathcal{L}$ is a fiber bundle diffeomorphism over Y and $\phi \in \Gamma(\pi_{XY})$, then the following are equivalent:*

- **(i)** *$\tilde{j}^1(\phi)^*(U \lrcorner \Omega_\mathcal{H}) = 0$ for any $U \in T(P_\mathcal{L})$;*
- **(ii)** *$j^1(\phi)^*(W \lrcorner \Omega_\mathcal{L}) = 0$ for any $W \in T(J^1(Y))$.*

Proof. Assume **(i)** holds and let $U \in T(P_\mathcal{L})$. Since $\mathbb{F}\mathcal{L}$ is a fiber bundle diffeomorphism, there exists $W \in T(J^1(Y))$ such that $T\mathbb{F}\mathcal{L} \circ W = U \circ \mathbb{F}\mathcal{L}$. Hence,

$$\begin{aligned} 0 = \tilde{j}^1(\phi)^*(U \lrcorner \Omega_\mathcal{H}) &= j^1(\phi)^*\mathbb{F}\mathcal{L}^*(T\mathbb{F}\mathcal{L} \cdot W \lrcorner \Omega_\mathcal{H}) \\ &= j^1(\phi)^*(\mathbb{F}\mathcal{L}^* T\mathbb{F}\mathcal{L} \cdot W \lrcorner \mathbb{F}\mathcal{L}^*\Omega_\mathcal{H}) \\ &= j^1(\phi)^*(W \lrcorner \Omega_\mathcal{L}). \end{aligned}$$

Using the same argument, the inverse function theorem guarantees that the converse holds as well. ■

We are thus led to the following conclusion.

Theorem 3.1 *If $\mathbb{F}\mathcal{L} : J^1(Y) \to P_\mathcal{L}$ is a fiber bundle diffeomorphism over Y and $\phi \in \Gamma(\pi_{XY})$, then the following are equivalent:*

- **(i)** *ϕ is a stationary point of $\int_X \mathcal{L}(j^1(\phi))$;*
- **(ii)** *$\tilde{j}^1(\phi)$ is a Hamiltonian section for $\mathcal{H}$.*

Before we prove the theorem, we state the following definition and lemma.

Definition 3.7 *A* **(finite) variation** *of ϕ is a curve $\phi_\lambda = \eta_\lambda \circ \phi$, where η_λ is the flow of a vertical vector field V on Y which is compactly supported in X. One says that ϕ is a* **stationary point** *of the action if*

$$\frac{d}{d\lambda}\left[\int_X \mathcal{L}(j^1(\phi_\lambda))\right]\Big|_{\lambda=0} = 0 \tag{3.8}$$

for all variations ϕ_λ of ϕ.

Lemma 3.2 *If $\mathbb{F}\mathcal{L} : J^1(Y) \to P_\mathcal{L}$ is a fiber bundle diffeomorphism, then*

$$\tilde{j}^1(\phi)^*(U \lrcorner \Omega_\mathcal{H}) = 0$$

for any U which is $\pi_{Y,P_\mathcal{L}}$-vertical or is tangent to $\tilde{j}^1(\phi)$. Similarly,

$$j^1(\phi)^*(W \lrcorner \Omega_\mathcal{L}) = 0$$

for any W which is $\pi_{Y,J^1(Y)}$-vertical or is tangent to $j^1(\phi)$.

Proof. Since $\mathbb{F}\mathcal{L}$ is a *fiber-preserving* bundle diffeomorphism, for any $\pi_{Y,P_\mathcal{L}}$-vertical U, there exists a $\pi_{Y,J^1(Y)}$-vertical W such that $T\mathbb{F}\mathcal{L} \circ W = U \circ \mathbb{F}\mathcal{L}$. Using canonical coordinates, let us write U and W as

$$U = U_A{}^\mu \frac{\partial}{\partial p_A{}^\mu}, \text{ and } W = W_\mu{}^A \frac{\partial}{\partial v^A{}_\mu}.$$

A calculation using (3.6) shows that

$$U \lrcorner \Omega_\mathcal{H} = U_A{}^\mu \left(dy^A \wedge d^n x_\mu + \left(\frac{\partial H}{\partial p_A{}^\mu} - \mathfrak{A}^A_\mu \right) d^{n+1}x \right).$$

Hence, using Lemma 3.1, we have that

$$\tilde{j}^1(\phi)^*(U \lrcorner \Omega_\mathcal{H}) = j^1(\phi)^* \left(-W_{v_\nu{}^B} \frac{\partial L}{\partial v^A{}_\mu \partial v^B{}_\nu} (dy^A \wedge d^n x_\mu - v^A{}_\mu d^{n+1}x) \right), \tag{3.9}$$

which vanishes using (2.10). On the other hand, if U is tangent to the graph of $\tilde{j}^1(\phi)$, then $U = T\tilde{j}^1(\phi) \cdot v$ for some $v \in TX$ so that

$$\tilde{j}^1(\phi)^*(U \lrcorner \Omega_\mathcal{H}) = \tilde{j}^1(\phi)^*((T\tilde{j}^1(\phi) \cdot v) \lrcorner \Omega_\mathcal{H}) = v \lrcorner (\tilde{j}^1(\phi)^* \Omega_\mathcal{H}),$$

which vanishes since $\tilde{j}^1(\phi)^*\Omega_\mathcal{L}$ is an $(n+2)$-form on the $(n+1)$-manifold X. The identical argument works for W. ■

The proof of Lemma 3.2 shows that in canonical coordinates

$$\frac{\partial H}{\partial p_A{}^\mu} = \nu^A{}_\mu + \mathfrak{A}^A_\mu, \tag{3.10}$$

and in the case that $U = U_{y^A} \frac{\partial}{\partial y^A}$ and that $\tilde{j}^1(\phi)^*(U \lrcorner \Omega_\mathcal{H}) = 0$, we obtain that

$$\frac{\partial H}{\partial y^A} = -\frac{\partial p_A{}^\mu}{\partial x^\mu}. \tag{3.11}$$

Thus, equations (3.10) and (3.11) are the coordinate expressions for a multihamiltonian system.

Proof of Theorem 3.1. Let $\phi_\lambda = \eta_\lambda \circ \phi$ of ϕ be a variation corresponding to a π_{XY}-vertical vector field V on Y with compact support in X. Using (2.7) we find that $\mathcal{L}(j^1(\phi)) = j^1(\phi)^*\Theta_\mathcal{L}$ and hence

$$\begin{aligned}\frac{d}{d\lambda}\left[\int_X \mathcal{L}(j^1(\phi_\lambda))\right]\Big|_{\lambda=0} &= \frac{d}{d\lambda}\int_X j^1(\phi_\lambda)^*\Theta_\mathcal{L}|_{\lambda=0} \\ &= \frac{d}{d\lambda}\left[\int_X j^1(\phi)^* j^1(\eta_\lambda)^*\Theta_\mathcal{L})\right]\Big|_{\lambda=0} \\ &= \int_X j^1(\phi)^*\mathfrak{L}_{j^1(V)}\Theta_\mathcal{L}\end{aligned} \tag{3.12}$$

where

$$j^1(V) = \frac{d}{d\lambda} j^1(\eta_\lambda)\Big|_{\lambda=0}$$

is the jet prolongation of V to $J^1(Y)$ (see Definition 5.4, if necessary). Using Cartan's magic formula, we get $\mathfrak{L}_W\Theta_\mathcal{L} = -W \lrcorner \Omega_\mathcal{L} + d(W \lrcorner \Theta_\mathcal{L})$, which, together with (3.12), gives

$$\begin{aligned}\frac{d}{d\lambda}\left[\int_X \mathcal{L}(j^1(\phi_\lambda))\right]\Big|_{\lambda=0} &= -\int_X j^1(\phi)^*(j^1(V) \lrcorner \Omega_\mathcal{L}) \\ &\qquad + \int_X dj^1(\phi)^*(j^1(V) \lrcorner \Theta_\mathcal{L}) \\ &= -\int_X j^1(\phi)^*(j^1(V) \lrcorner \Omega_\mathcal{L})\end{aligned} \tag{3.13}$$

by Stokes' theorem and the fact that V, and hence $j^1(V)$ is compactly supported in X. Lemma 3.1 together with (3.13) shows that **(ii)** implies **(i)**.

The converse follows from the fact that any $\pi_{X,J^1(Y)}$-vertical vector field W may be decomposed as

$$W = j^1(V) + W_1,$$

where V is π_{XY}-vertical and W_1 is $\pi_{Y,J^1(Y)}$-vertical. Then if **(i)** holds, Lemma 3.1 and Lemma 3.2 together with (3.13) show that

$$\int_X \tilde{j}^1(\phi)^*(U \lrcorner \Omega_\mathcal{H}) = 0$$

for all vector fields U on $P_\mathcal{L}$ with compact support in X. Since the space of smooth vector fields on $J^1(Y)$ is a module over the ring of smooth functions on X, an argument like that in the Fundamental Lemma of the Calculus of Variations shows that the integrand must vanish for all vector fields $U \in T(P_\mathcal{L})$ with compact support in X. A partition of unity argument then shows that **(ii)** implies **(i)**. ■

In the next section, we shall demonstrate the machinery of our intrinsic development on two examples: classical mechanics and nonlinear PDEs. We note that the essence of both of the following examples are the equations (3.10) and (3.11). For a development of a generalized Hamiltonian structure based on Hamiltonian vector fields that seems well-suited for ordinary differential equations, we refer the reader to Cantrijn et al [1997].

4 Particle Mechanics and Nonlinear PDEs

In this section, we show that our multisymplectic Hamiltonian formalism generalizes classical particle mechanics and is a natural setting for nonlinear Hamiltonian partial differential equations.

4.1 Particle mechanics

For non-relativistic classical mechanics with a configuration manifold Q (of dimension N), we choose $X = \mathbb{R}$ (so that $n = 0$) and $Y = \mathbb{R} \times Q$. In this case, $J^1(Y) = \mathbb{R} \times TQ$, and the cross-product induces a (flat) connection $\mathfrak{A} : \mathbb{R} \times TQ \to TQ$. The dual jet bundle is given by $J^1(Y)^\star = T^*\mathbb{R} \times T^*Q$ and has canonical coordinates $(t, p, q^1, ..., q^N, p_1, ..., p_N)$.

Given a Lagrangian in the usual sense $L : TQ \to \mathbb{R}$, we define $\mathcal{L} : J^1(Y) \to \Lambda^1(Y)$ by

$$\mathcal{L}(t, q^A, \dot{q}^A) = L(t, q^A, \dot{q}^A)dt.$$

The covariant Legendre transformation is the map $\mathbb{F}\mathcal{L} : J^1(Y) \to J^1(Y)^*$; i.e., $\mathbb{F}\mathcal{L} : \mathbb{R} \times TQ \to T^*\mathbb{R} \times T^*Q$ given by

$$\mathbb{F}\mathcal{L}(t, q^i, \dot{q}^i) = (t, L(t, q^A, \dot{q}^A) - p_A\dot{q}^A, q^A, p_A)$$

where $p_A = \partial L/\partial \dot{q}^A$. In this case, it is clear that $P_\mathcal{L} = \mathbb{R} \times T^*Q$ (as a subbunble of $T^*\mathbb{R} \times T^*Q$) with coordinates (t, q^A, p_A). Assume that the Legendre transformation is nondegenerate in the usual sense so that $\mathbb{F}\mathcal{L} : \mathbb{R} \times TQ \to P_\mathcal{L}$ is a vector bundle diffeomorphism over $\mathbb{R}$ and the corresponding Hamiltonian $H : T^*Q \to \mathbb{R}$ is well defined. The function H corresponds to the density $\mathcal{H} : P_\mathcal{L} \to \Lambda^1(X)$ as

$$\mathcal{H}(t, q^A, p_A) = H(t, q^A, p_A)dt,$$

where

$$\mathcal{H}(z) = D_\mathfrak{A}\mathcal{L}(\gamma) \cdot \gamma - \mathcal{L}(\gamma), \quad z = \mathbb{F}\mathcal{L}(\gamma) \in P_\mathcal{L}$$

has the coordinate expression

$$\mathcal{H}(t, q^A, p_A) = (p_A q^A - L(t, q^A, \dot{q}^A))dt.$$

In this case, we obtain the usual symplectic 2-form on extended phase space

$$\Omega_\mathcal{H} = dq^A \wedge dp_A + dH \wedge dt.$$

Proposition 4.1 *Let $\bar{V}$ be a vector field on $P_\mathcal{L}$ with integral curve $\tilde{j}^1(\phi)$. Then $\bar{V}$ is a Hamiltonian vector field for H if and only if $\tilde{j}^1(\phi)$ is a Hamiltonian section for $\mathcal{H}$.*

Proof. In coordinates, let $\bar{v} = (1, V_q^A, V_p^B)$ and let $(U_t, U) \equiv (U_t, U_q^A, U_p^B)$ be the coordinates for an arbitrary vector field $\bar{U} \in T(P_\mathcal{L})$. Then

$$i_{\bar{U}}\Omega_\mathcal{H} = U_p^B dq^A - U_q^A dp^B + (dH \cdot U)dt - U_t dH, \tag{4.1}$$

and the pull-back of (4.1) under $\tilde{j}^1(\phi)$ vanishes if and only if

$$\dot{q}^A = -\frac{\partial H}{\partial p_A}, \qquad \dot{p}_B = \frac{\partial H}{\partial q^B}, \tag{4.2}$$

and

$$dH \cdot U = 0. \tag{4.3}$$

■

4.2 Nonlinear partial differential equations

To motivate the exposition, consider the nonlinear wave equation given by

$$\frac{\partial^2\phi}{\partial {x^0}^2} - \triangle\phi - V'(\phi) = 0, \quad \phi \in \Gamma(\pi_{XY}), \tag{4.4}$$

where $\triangle$ is the Laplace-Beltrami operator and V is a real-valued C^∞ function of one variable.

We will show that equation (3.7) along the holonomic section $\tilde{j^1}(\phi)$ is equivalent to the nonlinear PDE (4.4) as well as the Bridges [1997] coordinate representation. For clarity of presentation, we will consider only one spatial dimension. In terms of our general notation, we set $X = \mathbb{R}^2$ $(n = 1)$ and $Y = \mathbb{R}^2 \times M$, so that sections of Y have the coordinate expressions (x^0, x^1, ϕ), and $TY = \mathbb{R}^2 \times TM$. The cross-product once again induces a flat connection defined by the natural projection $\mathfrak{A} : \mathbb{R}^2 \times TM \to TM$.

The first jet bundle $J^1(Y)$ is a five-dimensional manifold and sections of $J^1(Y)$ have local coordinates $(x^0, x^1, \phi, \frac{\partial\phi}{\partial x^0}, \frac{\partial\phi}{\partial x^1})$. The affine dual $J^1(Y)^\star$ is six-dimensional with its sections having the local coordinates $(x^0, x^1, \phi, p, p^0, p^1)$. The Lagrangian density $\mathcal{L} : J^1(Y) \to \Lambda^2(X)$ is expressed as

$$\mathcal{L}\left(x^0, x^1, \phi, \frac{\partial\phi}{\partial x^0}, \frac{\partial\phi}{\partial x^1}\right) = L\left(x^0, x^1, \phi\frac{\partial\phi}{\partial x^0}, \frac{\partial\phi}{\partial x^1}\right) dx^1 \wedge dx^0$$

which, for the case of the nonlinear wave equation is

$$\mathcal{L}\left(x^0, x^1, \phi, \frac{\partial\phi}{\partial x^0}, \frac{\partial\phi}{\partial x^1}\right) = \left[\frac{1}{2}(\{\partial_0\phi\}^2 - \{\partial_1\phi\}^2) + V(\phi)\right] dx^1 \wedge dx^0.$$

In this setting, the covariant Legendre transformation $\mathbb{F}\mathcal{L} : J^1(Y) \to J^1(Y)^\star$ is given by

$$\mathbb{F}\mathcal{L}\left(x^0, x^1, \phi, \frac{\partial\phi}{\partial x^0}, \frac{\partial\phi}{\partial x^1}\right) = \left[x^0, x^1, p^0\frac{\partial\phi}{\partial x^0} + p^1\frac{\partial\phi}{\partial x^1}L\left(x^0, x^1, \phi, \frac{\partial\phi}{\partial x^0}, \frac{\partial\phi}{\partial x^1}\right), \phi, \frac{\partial\phi}{\partial x^0}, \frac{\partial\phi}{\partial x^1}\right]$$

where $p^\mu = \partial L/\partial\phi_\mu$ and $\phi_\mu = \partial\phi/\partial x^\mu$. When L is regular (it is for the nonlinear wave equation), we have the primary constraint subbundle $P_\mathcal{L} := \pi_{\mathbb{R}^2,\mathbb{R}^2\times\mathbb{R}^3} \subset J^1(Y)^\star$ with coordinates $(x^0, x^1, \phi, p^0, p^1)$, and the Hamiltonian density on $P_\mathcal{L}$ is written in coordinates as

$$\mathcal{H}(x^0, x^1, \phi, p, p^0, p^1) = \left[p^0\frac{\partial\phi}{\partial x^0} + p^1\frac{\partial\phi}{\partial x^1} - L\left(x^0, x^1, \phi, \frac{\partial\phi}{\partial x^0}, \frac{\partial\phi}{\partial x^1}\right)\right] dx^1 \wedge x^0,$$

while the canonical 3-form on $J^1(Y)^\star$ is given by

$$\Omega_\mathcal{H} = -d\phi \wedge dp^0 \wedge dx^1 + d\phi \wedge dp^1 \wedge dx^0 + dH \wedge dx^1 \wedge dx^0. \tag{4.5}$$

We note that in this case, by global triviality, we may identify $P_\mathcal{L}$ with $\pi_{\mathbb{R}^2,\mathbb{R}^3}$.

Bridges [1997] considers this scalar field theory with the manifold $M = \mathbb{R}$ and a Lagrangian L that has no explicit dependence on time or space. (In particular, all of the fibers of both $J^1(Y)$ and $P_\mathcal{L}$ are identical over X and identified with $\mathbb{R}^3$.) He obtains the following partial differential equation for $\mathcal{Z} \equiv (\phi, p^0, p^1)$:

$$M\frac{\partial\mathcal{Z}}{\partial x^0} + K\frac{\partial\mathcal{Z}}{\partial x^1} = -dH(\mathcal{Z}), \tag{4.6}$$

where the 3×3 matrices M and K are defined to be

$$M = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{4.7}$$

M and K may be identified with a pair of degenerate 2-forms $\omega^{(1)}$ and $\omega^{(2)}$ on $P_{\mathcal{L}}$ which define Bridges' multisymplectic structure, and although it may appear that these two 2-forms provide a distinct structure from that of the 3-form in (4.5), in fact it is just a particular coordinate representation of the intrinsic structure which we have defined.

Proposition 4.2 *If $\mathbb{F}\mathcal{L} : J^1(Y) \to P_{\mathcal{L}}$ is a fiber bundle diffeomorphism and $\phi \in \Gamma(\pi_{XY})$, then $\tilde{j}^1(\phi)$ is a Hamiltonian system for $\mathcal{H}$ if and only if*

$$\frac{\partial}{\partial x^0} \lrcorner \tilde{j}^1(\phi)^*(dp^0 \wedge d\phi) + \frac{\partial}{\partial x^1} \lrcorner \tilde{j}^1(\phi)^*(dp^1 \wedge d\phi) = -dH(\tilde{j}^1(\phi)). \tag{4.8}$$

where (4.8) is equivalent to Bridges' equation (4.6).

Proof. Let $U \in T(P_{\mathcal{L}})$ be an arbitrary vector field which in coordinates is

$$U = U_{x^0}\frac{\partial}{\partial x^0} + U_{x^1}\frac{\partial}{\partial x^1} + U_{\phi}\frac{\partial}{\partial \phi} + U_{p^0}\frac{\partial}{\partial p^0} + U_{p^1}\frac{\partial}{\partial p^1},$$

so that

$$\begin{aligned} U \lrcorner \Omega_{\mathcal{H}} \;=\; & -U_{\phi} dp^0 \wedge dx^1 + U_{p^0} d\phi \wedge dx^1 - U_{x^1} d\phi \wedge dp^0 \\ & + U_{\phi} dp^1 \wedge dx^0 - U_{p^1} d\phi \wedge dx^0 + U_{x^0} d\phi \wedge dp^1 \\ & + \left(\frac{\partial H}{\partial \phi} U_{\phi} + \frac{\partial H}{\partial p^0} U_{p^0} + \frac{\partial H}{\partial p^1} U_{p^1} \right) dx^1 \wedge dx^0 \\ & - \frac{\partial H}{\partial \phi} U_{x^1} d\phi \wedge dx^0 - \frac{\partial H}{\partial p^0} U_{x^1} dp^0 \wedge dx^0 - \frac{\partial H}{\partial p^1} U_{x^1} dp^1 \wedge dx^0 \\ & + \frac{\partial H}{\partial \phi} U_{x^0} d\phi \wedge dx^1 + \frac{\partial H}{\partial p^0} U_{x^0} dp^0 \wedge dx^1 + \frac{\partial H}{\partial p^1} U_{x^0} dp^1 \wedge dx^1. \end{aligned}$$

Then

$$\begin{aligned} \tilde{j}^1(\phi)^*(U \lrcorner \Omega_{\mathcal{H}}) \;=\; & \left[U_{\phi} \left(\frac{\partial p^0}{\partial x^0} + \frac{\partial p^1}{\partial x^1} + \frac{\partial H}{\partial \phi} \right) + U_{p^0} \left(-\frac{\partial \phi}{\partial x^0} + \frac{\partial H}{\partial p^0} \right) + U_{p^1} \left(-\frac{\partial \phi}{\partial x^1} + \frac{\partial H}{\partial p^1} \right) \right. \\ & + U_{x^0} \left(-\frac{\partial \phi}{\partial x^0}\frac{\partial p^1}{\partial x^1} + \frac{\partial \phi}{\partial x^1}\frac{\partial p^1}{\partial x^0} - \frac{\partial H}{\partial \phi}\frac{\partial \phi}{\partial x^0} - \frac{\partial H}{\partial p^0}\frac{\partial p^0}{\partial x^0} - \frac{\partial H}{\partial p^1}\frac{\partial p^1}{\partial x^0} \right) \\ & \left. + U_{x^1} \left(\frac{\partial \phi}{\partial x^0}\frac{\partial p^0}{\partial x^1} - \frac{\partial \phi}{\partial x^1}\frac{\partial p^0}{\partial x^0} - \frac{\partial H}{\partial \phi}\frac{\partial \phi}{\partial x^1} - \frac{\partial H}{\partial p^0}\frac{\partial p^0}{\partial x^1} - \frac{\partial H}{\partial p^1}\frac{\partial p^1}{\partial x^1} \right) \right] dx^1 \wedge dx^0, \end{aligned}$$

which vanishes if and only if

$$dH = -\left(\frac{\partial p^0}{\partial x^0} + \frac{\partial p^1}{\partial x^1} \right) d\phi + \frac{\partial \phi}{\partial x^0} dp^0 + \frac{\partial \phi}{\partial x^1} dp^1,$$

and this is precisely a restatement of (4.8). To see that (4.8) is equivalent to (4.6), simply notice that $\partial_\mu \tilde{j^1}(\phi) = T\tilde{j^1}(\phi) \cdot \partial_\mu$ and that $\partial_\mu \mathcal{Z}$ is the $\pi_{X,P_{\mathcal{L}}}$-vertical component of $\partial_\mu \tilde{j}^1(\phi)$. ■

Thus, we have shown that Bridges' formulation is equivalent to our intrinsically defined multihamiltonian system for the nonlinear wave equation defined over one spatial dimension ($n = 1$). The argument, however, is entirely independent of the number of spatial directions, and obviously holds when $X = \mathbb{R}^{n+1}$ and $Y = \mathbb{R}^{n+1} \times \mathbb{R}$, in which case our multihamiltonian system may be expressed as

$$\frac{\partial}{\partial x^\mu} \lrcorner \tilde{j^1}(\phi)^*(dp^\mu \wedge d\phi) = -dH,$$

or, in terms of Bridges' n+1 2-forms $\omega^{(\mu)}$, as

$$\omega^{(\mu)}\left(\frac{\partial \tilde{j^1}(\phi)}{\partial x^\mu}, U\right) = -dH(\tilde{j^1}(\phi)) \cdot U \text{ for all } U \in T(P_{\mathcal{L}}).$$

More importantly, as we shall show in Section 6, in the presence of symmetry, we can assemble these n+1 distinct 2-forms $\omega^{(\mu)}$ into our single (n+2)-form $\Omega_{\mathcal{H}}$.

5 Covariant Noether Theory

Definition 5.1 *A* **covariant canonical transformation** *is a π_{XZ}-bundle map $\eta_Z : Z \to Z$ covering a diffeomorphism $\eta_X : X \to X$ such that $\eta_Z^* \Omega = \Omega$.* $\diamond$

Definition 5.2 *If $\eta_Y : Y \to Y$ is a π_{XY}-bundle automorphism (also covering a diffeomorphism $\eta_X : X \to X$), its* **canonical lift** *$\eta_Z : Z \to Z$ is defined by*

$$\eta_Z(z) = (\eta_Y^{-1})^*(z). \qquad \diamond \tag{5.1}$$

We may now define the covariant analogue of momentum maps in symplectic geometry.

Let $\mathcal{G}$ denote a Lie group (perhaps infinite-dimensional) with Lie algebra $\mathfrak{g}$ that acts on X by diffeomorphisms and acts on Z (or Y) as π_{XZ} (or π_{XY})-bundle automorphisms. For $\eta \in \mathcal{G}$, let η_X, η_Y and η_Z denote the corresponding transformations of X, Y and Z (the map $\eta_Z : Z \to Z$ is the prolongation of η_Y) and for $\xi \in \mathfrak{g}$, let ξ_X, ξ_Y and ξ_Z denote the corresponding infinitesimal generators. If $\mathcal{G}$ acts on Z by covariant canonical transformations, then the Lie derivative of Ω along ξ_Z is zero:

$$\mathfrak{L}_{\xi_Z} \Omega = 0, \tag{5.2}$$

so that the left Lie algebra action is canonical. In the case that

$$\mathfrak{L}_{\xi_Z} \Theta = 0, \tag{5.3}$$

then $\mathcal{G}$ acts by *special* covariant transformations.

Definition 5.3 *Let a Lie algebra $\mathfrak{g}$ have a canonical left action on Z, and suppose there exists $J \in L(\mathfrak{g}, \Lambda^n(Z))$ covering the identity on Z such that for each $\xi \in \mathfrak{g}$,*

$$\xi_Z \lrcorner \Omega = dJ(\xi).$$

The map $\mathbf{J} : Z \to \mathfrak{g}^* \otimes \Lambda^n(Z)$ defined by

$$\mathbf{J}(z)(\xi) = J(\xi)(z) \tag{5.4}$$

for all $\xi \in \mathfrak{g}$ and $z \in Z$ is called a **covariant momentum mapping** *(or a* **multimomentum mapping***) of the action.* $\diamond$

The covariant momentum map is said to be Ad*-equivariant if the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\mathbf{J}} & \mathfrak{g}^* \otimes \Lambda^n(Z) \\ \Big\downarrow \eta_Z & & \Big\downarrow \mathrm{Ad}^* \otimes \mathrm{Id} \\ Z & \xrightarrow{\mathbf{J}} & \mathfrak{g}^* \otimes \Lambda^n(Z) \end{array}$$

commutes, or equivalently if

$$J(\mathrm{Ad}_\eta^{-1}\xi) = \eta_Z^*[J(\xi)]. \tag{5.5}$$

Lemma 5.1 *If the action on Z is the lifted action η_Z, then $\mathcal{G}$ acts by special covariant transformations, the mapping J defined by*

$$\begin{aligned} J(\xi) &= \xi_Z \lrcorner \Theta \\ &= \pi_{YZ}^*(\xi_Y \lrcorner z), \end{aligned} \tag{5.6}$$

is a multimomentum mapping of the action for the multisymplectic form on Ω, i.e.,

$$\xi_Z \lrcorner \Omega = dJ(\xi), \tag{5.7}$$

and is Ad*-equivariant.

Proof. Differentiating the coordinate expression for (5.1) we find that if $\xi = (\xi^\mu, \xi^A)$, then

$$\xi_Z = (\xi^\mu, \xi^A, -p\xi^\nu{}_{,\nu} - p_B{}^\nu \xi^B{}_{,\nu}, p_A{}^\nu \xi^\mu{}_{,\nu} - p_B{}^\mu \xi^B{}_{,A} - p_A{}^\mu \xi^\nu{}_{,\nu}), \tag{5.8}$$

and hence that $\mathfrak{L}_{\xi_Z}\Theta = 0$. Then,

$$dJ(\xi) = d(\xi_Z \lrcorner \Theta) = \mathfrak{L}_{\xi_Z}\Theta - \xi_Z \lrcorner d\Theta = \xi_Z \lrcorner \Omega.$$

Since $\xi_Y = T\pi_{YZ} \circ \xi_Z$, (5.6) immediately follows, and the last assertion holds because special covariant momentum maps are Ad^*-equivariant (the argument is analogous to that for the cotangent bundle case which is proven in Abraham and Marsden [1978], Theorem 4.2.10). ■

In coordinates this **special covariant momentum map** may be expressed as

$$J(\xi)(z) = (p_A{}^\mu \xi^A + p\xi^\mu) d^n x_\mu - p_A{}^\mu \xi^\nu dy^A \wedge \partial_\nu \lrcorner (\partial_\mu \lrcorner d^{n+1}x). \tag{5.9}$$

Next we describe the prolonged action of the group $\mathcal{G}$ on Y to $J^1(Y)$ and $P_\mathcal{L}$.

Definition 5.4 *Let $\eta_Y : Y \to Y$ be a π_{XY}-bundle automorphism covering a diffeomorphism $\eta_X : X \to X$. Then*

$$\eta_{J^1(Y)}(\gamma) = T\eta_Y \circ \gamma \circ T\eta_X^{-1} \text{ for all } \gamma \in J^1(Y), \tag{5.10}$$

and

$$\eta_{P_\mathcal{L}}(z) = i^*_{J^1(Y)^\star, P_\mathcal{L}} \mathbb{F}\mathcal{L}(\eta_{J^1(Y)} \circ (\mathbb{F}\mathcal{L}|_{P_\mathcal{L}})^{-1} z) \text{ for all } z \in P_\mathcal{L}. \diamond \tag{5.11}$$

Definition 5.5 *We say that the Lagrangian density $\mathcal{L}$ and the Hamiltonian density $\mathcal{H}$ are equivariant with respect to $\mathcal{G}$ if for all $\eta \in \mathcal{G}$, $\gamma \in J^1(Y)$, and $z \in P_{\mathcal{L}}$,*

$$\mathcal{L}(\eta_{J^1(Y)}(\gamma)) = (\eta_X^{-1})^* \mathcal{L}(\gamma), \tag{5.12}$$

and

$$\mathcal{H}(\eta_{P_{\mathcal{L}}}(z)) = (\eta_X^{-1})^* \mathcal{H}(z), \tag{5.13}$$

where $(\eta_X^{-1})^*\mathcal{L}(\gamma)$ means the $(n+1)$-form $\mathcal{L}(\gamma)$ at $x \in X$ is pushed forward to an $(n+1)$-form at $\eta(x)$. $\diamond$

Analogous to Corollary 4.2.14 of Abraham and Marsden [1978], one may readily verify that both $\Theta_{\mathcal{L}}$ and $\Theta_{\mathcal{H}}$ are invariant under the respective group action prolongations, i.e.,

$$\eta^*_{J^1(Y)}\Theta_{\mathcal{L}} = \Theta_{\mathcal{L}} \text{ and } \eta^*_{P_{\mathcal{L}}}\Theta_{\mathcal{H}} = \Theta_{\mathcal{H}}.$$

Lemma 5.2 *Suppose that $\mathcal{L}$ is regular and that $\mathcal{H} : P_{\mathcal{L}} \to \Lambda^{n+1}(X)$ is equivariant, and is not constant on any $P_{\mathcal{L}}$ neighborhood. Then $\mathcal{L} : J^1(Y) \to \Lambda^{n+1}(X)$ is equivariant.*

Proof. Let $\mathcal{G}$ be a group acting on Y by bundle automorphisms. From (5.11), we have by definition that $\mathbb{F}\mathcal{L} : J^1(Y) \to P_{\mathcal{L}}$ is equivariant with respect to $\mathcal{G}$. Hence, for all $\eta \in \mathcal{G}$ and $\gamma \in J^(Y)$,

$$\mathcal{H}(\eta_{P_{\mathcal{L}}}(\mathbb{F}\mathcal{L}(\gamma))) = (\eta_X^{-1})^* \mathcal{H}(\mathbb{F}\mathcal{L}(\gamma)). \tag{5.14}$$

Assume that the Lagrangian density $\mathcal{L}$ is not equivariant with respect to $\mathcal{G}$. Then, there exists $\eta \in \mathcal{G}$ and $\gamma \in J^1(Y)$ for which

$$F(\gamma) \equiv \left[\mathcal{L} \circ \eta_{J^1(Y)}(\gamma) - (\eta_X^{-1})^* \mathcal{L}\right](\gamma) \neq 0.$$

Hence, by continuity of F, there is some neighborhood U in $J^1(Y)$ about γ for which $F(U)$ does not intersect $\{0\}$ in $\Lambda^{n+1}(X)$. We will assume that $\gamma \in J^1(Y)_y$ for some fixed $y \in Y$ and take U to be a fiber neighborhood of γ. By choosing $\varepsilon > 0$ sufficiently small, we see that for all $\gamma' \in J^1(Y)_y$ satisfying $\mathfrak{A} \circ \gamma' = \mathfrak{A}$,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{L}(\eta_{J^1(Y)}(\gamma) + \varepsilon(\eta_{J^1(Y)}(\gamma') - \eta_{J^1(Y)}(\gamma)) \neq (\eta_X^{-1})^* \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{L}(\gamma + \varepsilon(\gamma' - \gamma)),$$

in which case,

$$D\mathfrak{A}(\eta_{J^1(Y)}(\gamma)) \cdot \eta_{J^1(Y)}(\gamma) \neq (\eta_X^{-1})^* D\mathfrak{A}(\gamma) \cdot \gamma$$

for all γ in some $J^1(Y)$-neighborhood U. Since $\mathcal{H}$ cannot be locally constant, (5.14) cannot be true for all $\gamma \in U$, and so $\mathcal{H}$ cannot be equivariant. ■

Theorem 5.1 *Let $\mathcal{L}$ be a regular Lagrangian density, and suppose that $\mathcal{H} : P_{\mathcal{L}} \to \Lambda^{n+1}(X)$ is equivariant and is not constant on any $P_{\mathcal{L}}$-neighborhood. Then the map*

$$J^{\mathcal{L}}(\xi) := \mathbb{F}\mathcal{L}^* J(\xi)$$

is a momentum map for the lifted action of $\mathcal{G}$ on $J^1(Y)$ relative to $\Omega_{\mathcal{L}}$, and the map

$$J^{\mathcal{H}}(\xi) := ((\mathbb{F}\mathcal{L}|_{P_{\mathcal{L}}})^{-1})^* \circ \mathbb{F}\mathcal{L}^* J(\xi) = (\mathbb{F}\mathcal{L} \circ \mathbb{F}\mathcal{L}|^{-1}_{P_{\mathcal{L}}})^* J(\xi) = i^*_{J^1(Y)^\star, P_{\mathcal{L}}} J(\xi)$$

is a momentum map for the lifted action of $\mathcal{G}$ on $P_{\mathcal{L}}$ relative to $\Omega_{\mathcal{H}}$; i.e., for all $\xi \in \mathfrak{g}$,

$$\xi_{J^1(Y)} \lrcorner \Omega_{\mathcal{L}} = dJ^{\mathcal{L}}(\xi), \tag{5.15}$$

and

$$\xi_{P_{\mathcal{L}}} \lrcorner \Omega_{\mathcal{H}} = dJ^{\mathcal{H}}(\xi), \tag{5.16}$$

where $\xi_{J^1(Y)}$ and where $\xi_{P_{\mathcal{L}}}$ are the infinitesimal generators corresponding to ξ.

Proof. Lemma 5.2 asserts that $\mathcal{L}$ is equivariant, from which we may conclude that $\mathbb{F}\mathcal{L} : J^1(Y) \to Z$ is equivariant so that

$$\eta_Z \circ \mathbb{F}\mathcal{L} = \mathbb{F}\mathcal{L} \circ \eta_{J^1(Y)}. \tag{5.17}$$

Indeed, we see that

$$\begin{aligned} \{\eta_{J^1(Y)^\star}[\mathbb{F}\mathcal{L}(\gamma)]\} \cdot \gamma' &= (\eta_X^{-1})^* \{\mathbb{F}\mathcal{L}(\gamma)[\eta_{J^1(Y)}^{-1}(\gamma')]\} \\ &= (\eta_X^{-1})^* \left\{ \mathcal{L}(\gamma) + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{L}(\gamma + \varepsilon[\eta_{J^1(Y)}^{-1}(\gamma') - \gamma]) \right\}. \end{aligned}$$

and that

$$\{\mathbb{F}\mathcal{L}[\eta_{J^1(Y)}(\gamma)]\} \cdot \gamma' = \mathcal{L}(\eta_{J^1(Y)}(\gamma)) + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{L}(\eta_{J^1(Y)}(\gamma)) + \varepsilon[\gamma' - \eta_{J^1(Y)}(\gamma)]), \tag{5.18}$$

which are equal by the equivariance of $\mathcal{L}$. The infinitesimal version of (5.17) yields

$$\xi_Z \circ \mathbb{F}\mathcal{L} = T\mathbb{F}\mathcal{L} \cdot \xi_{J^1(Y)}, \tag{5.19}$$

which is a statement that ξ_Z is $\mathbb{F}\mathcal{L}$-related to $\xi_{J^1(Y)}$. Hence, the pull-back of (5.7) along $\mathbb{F}\mathcal{L}$ gives us (5.15), while a second pull-back of (5.15) along the diffeomorphism $\mathbb{F}\mathcal{L}|_{P_{\mathcal{L}}}^{-1}$ verifies (5.16). ■

In multisymplectic coordinates, the multimomentum mapping $J^{\mathcal{H}}$ is written as

$$J^{\mathcal{H}}(\xi) = (p_A{}^\mu \xi^A + (p_A{}^\nu \mathfrak{A}_\nu^A - H)\xi^\mu) d^n x_\mu - p_A{}^\mu \xi^\nu dy^A \wedge \partial_\nu \lrcorner (\partial_\mu \lrcorner d^{n+1}x), \tag{5.20}$$

where $H : P_{\mathcal{L}} \to \mathbb{R}$ is the Hamiltonian function associated with the Hamiltonian density $\mathcal{H}$ by $\mathcal{H} = H d^{n+1}x$.

Theorem 5.2 (Covariant Hamiltonian Version of Noether's Theorem) *Let $\mathcal{L}$ be regular, and suppose that a group $\mathcal{G}$ acts on Y by π_{XY}-bundle automorphisms, and that the Hamiltonian density $\mathcal{H}$ is equivariant with respect to $\mathcal{G}$, and is not locally constant on any $P_{\mathcal{L}}$-neighborhood. Then for each $\xi \in \mathcal{G}$,*

$$d[\tilde{j}^1(\phi)^* J^{\mathcal{H}}(\xi)] = 0, \tag{5.21}$$

for any $\phi \in \Gamma(\pi_{XY})$ for which $\tilde{j}^1(\phi)$ is a covariant Hamiltonian system.

Proof. Since $\tilde{j}^1(\phi)$ is a Hamiltonian system, $\tilde{j}^1(\phi)^* i_U \Omega_{\mathcal{H}} = 0$ for any vector $U \in T(P_{\mathcal{L}})$ so set $U = \xi_{P_{\mathcal{L}}}$. ■

Under the same hypotheses, Lemma 3.1 gives us the following equivalent statement.

Corollary 5.1 *For each $\xi \in \mathcal{G}$, $d[j^1(\phi)^* J^{\mathcal{L}}(\xi)] = 0$.*

The quantity $\tilde{j}^1(\phi)^* J^{\mathcal{H}}(\xi)$ is called the **Hamiltonian Noether current**, and as we shall show, leads to very useful decompositions of the classical water wave conservations laws. In coordinates, it has the form

$$\tilde{j}^1(\phi)^* J^{\mathcal{H}}(\xi) = \left[(p_A{}^\mu \xi^A + (p_A{}^\nu \mathfrak{A}_\nu^A - H)\xi^\mu) - p_A{}^\mu \phi^A{}_{,\nu} \xi^\nu + p_A{}^\nu \phi^A{}_{,\nu} \xi^\mu \right] d^n x_\mu. \tag{5.22}$$

6 Symmetry and generalized conservation laws

Just as we have shown in Section 4 that the multihamiltonian system generalizes the classical Hamiltonian description of particle mechanics as well as the structures defined in Bridges [1997], we can do the same for the multisymplectic Hamiltonian Noether theory.

6.1 Particle mechanics

We let the groups Diff($\mathbb{R}$) act on $\mathbb{R}$ and G on Q and consider the action of the prolongation of $\mathcal{G} =$ Diff($\mathbb{R}$) $\times G$. With elements of $\mathfrak{g}$ written as (f, ξ), the Hamiltonian Noether current has the simple coordinate form

$$\tilde{j}^1(\phi)^* J^{\mathcal{H}}(\xi) = p_A \xi^A - Hf := J_\xi^H - Hf,$$

where $J_\xi^H = p_A \xi^A$ is the usual momentum map for G acting on Q in Hamiltonian mechanics. Then equation (5.21) asserts that along trajectories of the Hamiltonian vector field,

$$\frac{d}{dt}(J_\xi^H - Hf) = 0,$$

which is equivalent to conservation of *both* J_ξ^H and H.

6.2 Bridges' decomposition of Noether theory

In this section we fix $X = \mathbb{R}^{n+1}$, $Y = \mathbb{R}^{n+1} \times \mathbb{R}$, and $P_{\mathcal{L}} = \pi_{\mathbb{R}^{n+1}, \mathbb{R}^{n+2}}$. We examined this particular geometry in Section 4.2, wherein we showed that Bridges' multisymplectic structure consisting of $n+1$ degenerate 2-forms $\omega^{(\mu)}$ on $P_{\mathcal{L}}$ is an equivalent and particular coordinate representation of our intrinsically defined multihamiltonian system on $P_{\mathcal{L}}$. We now show that in the presence of a general symmetry group action, Bridges' $n+1$ pre-symplectic 2-forms $\omega^{(\mu)}$ can actually be assembled into our single multisymplectic $(n+2)$-form $\Omega^{\mathcal{H}}$.

Bridges forms an $(n+1)$-vector of momenum mappings such that each vector component is associated with a distinct spacetime direction through the pre-symplectic 2-form $\omega^{(\mu)}$. He then relates the action of a general symmetry group $\mathcal{G}$ along the fiber of $P_{\mathcal{L}}$ with the vanishing of the divergence of the vector of momentum mappings. The following result is Theorem 2.2 in Bridges [1996b].

Proposition 6.1 *Let $H : \mathbb{R}^{n+2} \to \mathbb{R}$ be a covariant Hamiltonian with $n+1$ distinct 2-forms $\omega^{(\mu)}$. If $dH \cdot \xi_{P_\mathcal{L}} = 0$ for all $\xi_{P_\mathcal{L}}$ in the Lie algebra $\mathfrak{g}$ of the group $\mathcal{G}$ acting on $\mathbb{R}^{n+2}$, and if P^μ is the momentum mapping associated to $\omega^{(\mu)}$, i.e., for all $\xi_{P_\mathcal{L}} \in \mathfrak{g}$,*

$$\xi_{P_\mathcal{L}} \lrcorner \omega^{(\mu)} = dP^\mu(\xi_{P_\mathcal{L}}), \tag{6.1}$$

then

$$\frac{\partial P^\mu}{\partial x^\mu} = 0. \tag{6.2}$$

We will first show that (6.2) is a particular example of our conservation law (5.21) in the case of the trivial bundle geometry defined above, for which the fields have no explicit dependence on time or space (geometrically, this means that each fiber of the bundle $\pi_{X,P_\mathcal{L}}$ is identical).

Proposition 6.2 *In the case that the action on $P_\mathcal{L} := \pi_{\mathbb{R}^{n+1},\mathbb{R}^{n+2}}$ is the lifted action $\eta_{P_\mathcal{L}}$, then the momentum mappings P^μ defined in (6.1) are the components of the Hamiltonian Noether current, and hence the conservation law (6.2) is contained in Theorem 5.2.*

Proof. We set the diffeomorphism η_X to be the identity, and identify $P_\mathcal{L}$ with $\mathbb{R}^{n+2}$. Hence, the infinitesimal generators $\xi^\mu = 0$, and using (5.22), we see that the Hamiltonian Noether current is given by

$$\tilde{j^1}(\phi)^* J^\mathcal{H}(\xi) = p^\mu \xi d^n x_\mu := N^\mu d^n x_\mu. \tag{6.3}$$

Using (5.8), we easily deduce that the lifted action $\xi_{P_\mathcal{L}}$ is given in coordinates by $(0, \xi, -p^\mu \frac{\partial \xi}{\partial \phi})$ so that the equivariance of $\mathcal{H}$ is equivalent to $dH \cdot \xi_{P_\mathcal{L}} = 0$.

In accordance with Proposition 6.1, all of the group action is along the fiber of $\pi_{X,P_\mathcal{L}}$, identified with $\mathbb{R}^{n+2}$, so we will restrict the exterior derivative d to the fiber.

We claim that $P^\mu = N^\mu$. To see this, we must show that $dN^\mu = \xi_{P_\mathcal{L}} \lrcorner \omega^{(\mu)}$, but this is precisely the case since in coordinates, for each $\mu = 1, ..., n, 0$,

$$dN^\mu = \left(p^\mu \frac{\partial \xi}{\partial \phi}, 0, ..., \xi, ..., 0 \right), \ \xi \text{ in the } (\mu+1)\text{th coordinate.}$$

Then, using the identity

$$d(N^\mu d^n x_\mu) = \partial_\nu N^\mu dx^\nu \wedge d^n x_\mu = \partial_\mu N^\mu d^{n+1} x$$

we have that $d[\tilde{j^1}(\phi)^* J^\mathcal{H}(\xi)] = 0$ implies that $\frac{\partial N^\mu}{\partial x^\mu} = 0$ so that $\frac{\partial P^\mu}{\partial x^\mu} = 0$ and the result is proved. ■

This proposition indicates how we can assemble the $n+1$ 2-forms $\omega^{(\mu)}$ into the single $(n+2)$-form $\Omega_\mathcal{H}$ when lifted symmetries exist. Namely, to each $\omega^{(\mu)}$, there corresponds a momentum mapping $P^\mu(\xi_{P_\mathcal{L}})$ of the symmetry group given by (6.1). By Proposition 6.2, the maps $P^\mu(\xi_{P_\mathcal{L}})$ are the components of the Hamiltonian Noether current. This then defines the Hamiltonian covariant momentum mapping $J^\mathcal{H}(\xi_{P_\mathcal{L}})$ which in turn, by (5.16), defines the canonical multisymplectic $(n+2)$-from $\Omega_\mathcal{H}$ on $P_\mathcal{L}$. In fact, since lifts are special canonical transformations, the covariant momentum map defines the $(n+1)$-$\Theta_\mathcal{H}$ on $P_\mathcal{L}$ as well. We summarize with the following corollary.

Corollary 6.1 *Assume the group $\mathcal{G}$ acts by special canonical transformations, and let the Hamiltonian Noether current $\tilde{j^1}(\phi)^* J^{\mathcal{H}}(\xi)$ be given in multisymplectic coordinates by $N^\mu d^n x_\mu$. Then the $\omega^{(\mu)}$ satisfy $\xi_{P_\mathcal{L}} \lrcorner \omega^{(\mu)} = dN^\mu$. Furthermore, if $\omega^{(\mu)}$ is exact, such that*

$$\omega^{(\mu)} = d\kappa^{(\mu)} \tag{6.4}$$

for 1-forms $\kappa^{(\mu)}$, then

$$N^\mu = \xi_{P_\mathcal{L}} \lrcorner \kappa^{(\mu)}. \tag{6.5}$$

7 The Geometry of Water Waves

It is interesting to note that the covariant Hamiltonian Noether theorem intrinsically contains the mass conservation law for water waves as well as the conservation of wave action and action flux. In particular, the vanishing of the exterior derivative of the Hamiltonian Noether current is an intrinsic restatement of the mass conservation law, while the projected components of the Hamiltonian Noether current P^μ, as defined by Proposition 6.2, are related to the action and action flux. As an example, for the case of two spatial dimensions, the ensemble (or phase) average of P^0 corresponds to Whitham's definition of wave action and that of P^1 and P^2 correspond to the two-component action-flux (see Whitham [1974]), while in the case of one spatial dimension the Hamiltonian density $\mathcal{H}$ is related to the flow force or in some cases the momentum flux (see Bridges [1997]). These observations were first made by Bridges [1996a] in coordinates and seemed to have been the primary motivating factors for defining additional 2-forms $\omega^{(\mu)}$ for each unbounded spatial direction.

Next, we show that our definition for a multihamiltonian system contains the variational principles which are essential to the study of pattern formation and wave instability. For simplicity, we restrict our attention to symmetries given by the circle and $\mathbb{T}^{n+1}$; however, it is important to note that our procedure is general and may be applied to *any subgroup* of the Euclidean group $\mathrm{SE}(n+1)$ and its products. This is significant if one wishes to study hexagonal pattern formations, for example, in addition to merely the periodic ones.

7.1 Pattern formation, action, index, and the loop space

Let $X = \mathbb{R}^{n+1}$ and let Y be the vector bundle $\mathbb{R}$ over X. Consider the semilinear elliptic scalar partial differential equation

$$\triangle\phi + V'(\phi) = 0, \quad \phi \in \Gamma(\pi_{XY}), \tag{7.1}$$

where as above, $\triangle$ is the Laplace-Beltrami operator and V is a real-valued C^∞-bundle map. For this example, it is appropriate to set $P_\mathcal{L} = \pi_{\mathbb{R}^{n+1},\mathbb{R}^{n+2}}$ and $\mathcal{G} = SO(n+1)$, in which case (7.1) may be equivalently expressed as

$$\tilde{j^1}(\phi)^*(U \lrcorner \Omega_\mathcal{H}) = 0 \tag{7.2}$$

for all $U \in T(P_\mathcal{L})$, where in coordinates,

$$\mathcal{H} = Hd^{n+1}x \quad \text{and} \quad H(\tilde{j^1}(\phi)) = \frac{1}{2}p^\mu \cdot p^\mu + V(\phi). \tag{7.3}$$

We show in this section that our intrinsic multisymplectic structure can be used to generalize the notion of action and index on the loop space of the primary constraint subbundle $P_\mathcal{L}$ as defined in Bridges [1996b].

Let the map $\chi : X \to \mathbb{R}$ be defined in coordinates by $\chi(x^\mu) = k_\mu x^\mu$, and identify $\mathbb{T}^1$ with its universal cover $\mathbb{R}\backslash\mathbb{Z}$, so that a smooth 2π-periodic map $\alpha : \mathbb{R} \to P_\mathcal{L}$ may be identified with the smooth map $\alpha : \mathbb{T}^1 \to P_\mathcal{L}$.

Definition 7.1 *The loop space of $P_\mathcal{L}$ is the subset of $\Gamma(\pi_{X,P_\mathcal{L}})$ defined by*

$$Loop(P_\mathcal{L}) = \{z \in \Gamma(\pi_{X,P_\mathcal{L}}) \mid z = \alpha \circ \chi \text{ is holonomic and } \quad \alpha \in C^\infty(\mathbb{T}^1, P_\mathcal{L})\}.$$

We then set

$$loop(P_\mathcal{L}) = \{z \in Loop(P_\mathcal{L}) \mid \dot\alpha \in \mathcal{P}\},$$

where $\dot\alpha = \frac{d\alpha}{d\chi}$ and $\mathcal{P} \subset T(P_\mathcal{L})$ consists of those vector fields on $P_\mathcal{L}$ which are prolongations of vector fields on Y. $\diamond$

Hence, an element $\alpha \circ \chi$ in loop$(P_\mathcal{L})$ is conjugate to the first jet of a section $f \circ \chi$ in π_{XY}, where $f : \mathbb{T}^1 \to Y$.

The diagonal periodic patterns of (7.1) correspond to the restriction of (7.1) to Loop$(P_\mathcal{L})$. Thus, if $\alpha \circ \chi \in$ Loop$(P_\mathcal{L})$, a diagonal periodic pattern satisfies

$$(\alpha \circ \chi)^*(U \lrcorner \Omega_\mathcal{H}) = 0 \text{ for all } U \in T(P_\mathcal{L}). \tag{7.4}$$

Recently, the existence of periodic pattern solutions to (7.1) has been obtained by expressing such solutions as critical points of a constrained variational principle, and using information provided by sensitivity matrices, sometimes called the index, for classification of the critical point type. As it turns out, when the infinitesimal group action coincides with the vector field $\dot\alpha$, the Hamiltonian Noether current naturally and intrinsically verifies these variational principles.

When $\alpha \circ \chi \in$ loop$(P_\mathcal{L})$, we may associate to it the loop space Hamiltonian Noether current

$$\mathcal{N}(\dot\alpha) := (\alpha \circ \chi)^* J^\mathcal{H}(\dot\alpha). \tag{7.5}$$

As with our previous example in Section 4.2, we set the group action on X to be the identity, and identify $P_\mathcal{L}$ with $\mathbb{R}^{n+2}$. We assume that $\mathcal{H}$ is equivariant with respect to the group action on $P_\mathcal{L}$ and is not locally constant. In this case, $d\mathcal{N}(\dot\alpha) = 0$ implies that $\alpha \circ \chi := j^1(f \circ \chi)$ is a Hamiltonian system, a fact that is readily verified by a tedious computation in local coordinates. The computation, however, proceeds easily on the Lagrangian side. Using the coordinate expression

$$\frac{d}{d\chi} j^1(f \circ \chi) = j^1(\dot f) := \left(0, \dot f, \frac{\partial \dot f}{\partial x^\mu} + \frac{\partial \dot f}{\partial y}\frac{\partial (f \circ \chi)}{\partial x^\mu}\right),$$

which is readily obtained from the definition of the vector field prolongation to $J^1(Y)$ given in (5.10), we appeal to Corollary 5.1 and check that $d[j^1(f \circ \chi)^* J^\mathcal{L}(j^1(\dot f))] = 0$. In coordinates, this yields

$$\begin{aligned}\Bigg\{ &\left[\frac{\partial L}{\partial y}(j^1(f \circ \chi)) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial (f \circ \chi)_\mu}(j^1(f \circ \chi))\right)\right][-\dot f \circ (f \circ \chi)] \\ &+ \frac{\partial L}{\partial y}(j^1(f \circ \chi))(\dot f \circ (f \circ \chi)) \\ &+ \frac{\partial L}{\partial (f \circ \chi)_\mu}(j^1(f \circ \chi))\left[\frac{\partial \dot f}{\partial x^\mu} + \frac{\partial \dot f}{\partial y}(f \circ \chi)_\mu\right](j^1(f \circ \chi))\Bigg\} d^{n+1}x\end{aligned}$$

which vanishes if and only if the Lagrangian is equivariant and if $f \circ \chi$ is a stationary point of $\int_X \mathcal{L}(j^1(f \circ \chi))$. Since $\mathcal{H}$ is equivariant, Lemma 5.2 guarantees that $\mathcal{L}$ is equivariant as well, in which case Theorem 3.1 gives us that $\alpha \circ \chi$ is a Hamiltonian system.

To see that our covariant Noether theory contains the classical constrained variational principle in local coordinates, we make the following observations. Let the 2-forms $\omega^{(\mu)}$ and the 1-forms $\kappa^{(\mu)}$ be as defined in Corollary 6.1. By Proposition 6.1, equation (7.4) is satisfied if and only if

$$\begin{aligned} 0 &= -dH(\alpha \circ \chi) - k_\mu(\dot{\alpha} \lrcorner \omega^{(\mu)}) \\ &= d[-H(\alpha \circ \chi) - k_\mu(\dot{\alpha} \lrcorner \kappa^{(\mu)})] \\ &:= d\mathcal{F}(\dot{\alpha}, \chi). \end{aligned}$$

Thus, as noted in Bridges [1996b], $\alpha \circ \chi$ is a diagonal periodic pattern if it is the critical point of $\int_{\mathbb{T}^1} \mathcal{F}(\dot{\alpha}, \chi) d\chi$ (classically, the phase-averaged quantities are considered). From this, we see that the solutions to (7.4) are the critical points of the phase-averaged Hamiltonian with the additional constraints

$$\int_{\mathbb{T}^1} [\dot{\alpha} \lrcorner \kappa^{(\mu)}] d\chi = I_\mu \tag{7.6}$$

so that the k_μ are the Lagrange multipliers. In the case that H may be viewed as an implicit function of the I_μ, we have that

$$k_\mu = -\frac{\partial H}{\partial I_\mu},$$

so that the Hessian matrix of H with respect to the level sets I_μ has components

$$[\mathrm{Hess}_I(H)]_{\mu\nu} = \frac{\partial^2 H}{\partial I_\mu \partial I_\nu} = \frac{\partial k_\mu}{\partial I_\nu}.$$

¿From the implicit function theorem, a diagonal periodic pattern is non-degenerate if $\det[\mathrm{Hess}_I(H)] \neq 0$. Then, the natural definition for the index for such patterns is given by

$$\mathrm{index}(\alpha) = \#\text{ negative eigenvalues of } \mathrm{Hess}_I(H).$$

We refer the reader to Bridges [1996a] for a detailed account and applications.

7.2 Stability of Water Waves

Conservative partial differential equations are often accurate models for water waves, and in this section we will briefly comment upon the connection between our covariant Noether theory and the constrained toral variational principles which lead to characterizations of the instabilities of the system. Our brevity is due to the fact that the $\mathrm{Hess}_I(H)$-matrix is explicitly connected to the linear stability exponents from which we may deduce the behavior of our solutions, and as we gave a fairly detailed description in the previous section of how this matrix arises from the vanishing of the exterior derivative of the Hamiltonian Noether current, we shall herein only discuss the minor modifications necessary for this theory.

To demonstrate the main ideas, let us consider the the manifolds X, Y, and $P_{\mathcal{L}}$ as given in the previous section and the the partial differential equation defined in (4.4), with corresponding covariant Hamiltonian

$$\mathcal{H} = \left[\frac{1}{2} \left(\sum_{\mu=1}^{n} p_\mu^2 - p_0^2 \right) - V(\phi) \right] d^{n+1}x.$$

Unlike the case of pattern formation for which we considered solutions of

$$\tilde{j^1}(\phi)^*(U \lrcorner \Omega_{\mathcal{H}}) = 0 \text{ for all } U \in T(P_{\mathcal{L}}) \tag{7.7}$$

restricted to loop($P_{\mathcal{L}}$), now we restrict consideration to the periodic sections of $P_{\mathcal{L}}$, so that $\tilde{j^1}(\phi) : \mathbb{T}^{n+1} \to P_{\mathcal{L}}$. If we make the change of variables $w^\mu = k_\mu x^\mu$ (no sum), then by Proposition 6.1, periodic solutions of (7.7) are expressed in coordinates by

$$k_\mu \frac{\partial \tilde{j^1}(\phi)}{\partial w^\mu} \lrcorner \omega^{(\mu)} = -dH(\tilde{j^1}(\phi)), \quad (k_0, ..., k_n) \in \mathbb{T}^{n+1}. \tag{7.8}$$

Arguing exactly as we did in the previous section, we may again deduce that the k_μ are the Lagrange multipliers of the system, and thus the Hessian matrix of H with respect to the level set I_μ is identically obtained as for the case of periodic pattern formation. See Bridges [1997] for a discussion of the relationship between $\mathrm{Hess}_I(H)$ and the classical linear stability exponents.

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