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Point vortices on a sphere: Stability of relative equilibria

Sergey Pekarsky[a] and Jerrold E. Marsden[b]
Control and Dynamical Systems 107-81, California Institute of Technology, Pasadena, California 91125

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In this paper we analyze the dynamics of N point vortices moving on a sphere from the point of view of geometric mechanics. The formalism is developed for the general case of N vortices, and the details are worked out for the (integrable) case of three vortices. The system under consideration is SO(3) invariant; the associated momentum map generated by this SO(3) symmetry is equivariant and corresponds to the moment of vorticity. Poisson reduction corresponding to this symmetry is performed; the quotient space is constructed and its Poisson bracket structure and symplectic leaves are found explicitly. The stability of relative equilibria is analyzed by the energy–momentum method. Explicit criteria for stability of different configurations with generic and nongeneric momenta are obtained. In each case a group of transformations is specified, modulo which one has stability in the original (unreduced) phase space. Special attention is given to the distinction between the cases when the relative equilibrium is a nongreat circle equilateral triangle and when the vortices line up on a great circle.
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I. INTRODUCTION

The problem of vortex motion has a long and interesting history. It was Helmholtz[1] who introduced the model that is referred to today as point vortices. Several of Helmholtz' contemporaries immediately seized upon and developed the treasures in his paper, such as Kirchhoff[2] and his student Gröbli. An account of some of the history of this problem can be found in Aref, Rott and Thomann[3] and Kidambi and Newton.[4]

We mention a few more historical facts relevant to the present paper. The problem of configurations of vortices that could move without change of shape, namely *relative equilibria* in the language of Poincaré, was analyzed by Thomson,[5] the later Lord Kelvin, and stability aspects of this motion were studied in his later paper, Thomson.[6] The geometric construction was rediscovered, updated and added to by Novikov[7] a century later for the case of equal strength vortices. Synge[8] developed a qualitative classification of all possible motions of three planar vortices and was the first to introduce trilinear coordinates. Aref[9] also treats the case of three vortices of general strength.

The paper by Bogomolov[10] contains the first systematic and thorough derivation of the equations of motion for point vortices on both rotating and nonrotating spheres. A later paper of Bogomolov[11] contains an analysis of the motion of three identical point vortices on the sphere, generalizing the planar result by Novikov.[7] The paper by Kidambi and Newton[4] treats the case of three vortices on a sphere for general vortex strengths, thus generalizing the planar results of Synge[8] and Aref.[9]

The topology of the problem of vortices moving on a sphere is considered by Kirwan,[12] though this paper mainly deals with the symplectic reduction (in the sense of Marsden and Weinstein[13]) of the problem and the study of the topology of the symplectically reduced phase spaces and the number of equilibria, by computing, in the spirit of Smale,[14] some topological invariants, such as Betti numbers.

The dynamics of N vortices on a sphere is a Hamiltonian system (see, e.g., Kidambi and Newton[4] and references therein). This Hamiltonian structure can be obtained using general reduc-

[a]Electronic mail: sergey@cds.caltech.edu
[b]Electronic mail: marsden@cds.caltech.edu

tion techniques starting with the geometrical description of ideal hydrodynamics in terms of diffeomorphism groups; see Marsden and Weinstein[13] and Arnold and Khesin.[15]

In this paper we explicitly carry out Poisson reduction for the 3-vortex problem on a sphere. We calculate the induced Poisson structure on the Poisson reduced space and analyze the associated symplectic stratification. Furthermore, relative equilibria are classified and their stability is determined by the energy–momentum method (see Marsden[16] and references therein). The use of the energy–momentum method for the stability of vortices was studied for certain planar cases by Lewis and Ratiu.[17] As in the basic example of the rigid body, stability in the reduced space can also be studied by considering intersections of the energy levels with the symplectic leaves.

A. The phase space and its Poisson structure

The phase space of the dynamical system of N-vortices moving on the two sphere consists of N copies of a sphere, namely $P=S^2\times\cdots\times S^2$ regarded as being embedded in N copies of three space $\mathbb{R}^{3N}$ as the set defined by $\|\mathbf{x}_n\|=R$, where R is the radius of the sphere and $n=1,\ldots,N$ labels the location of the nth vortex. There are nonzero vortex strengths Γ_n ascribed to each vortex; i.e., to each S^2. The Poisson structure on P is given by

$$\{\cdot,\cdot\}=\sum_{n=1}^{N}\frac{R}{\Gamma_n}\{\cdot,\cdot\}_n, \tag{1}$$

where $\{\ ,\ \}_n$ is the Poisson structure on the nth copy of S^2 corresponding to the natural area symplectic form on S^2; that is, the Poisson structure in each copy of $\mathbb{R}^3$ is the standard Lie–Poisson structure on $\mathbb{R}^3$ considered as $\mathfrak{so}(3)^*$, the dual of the Lie algebra of the rotation group SO(3). (See, for example, Marsden and Ratiu[18] for the basic definitions used here.) The restriction of the Lie–Poisson bracket operation on $\mathbb{R}^3$ to a sphere [which is a symplectic leaf in $\mathfrak{so}(3)^*$] defines an area form. For two functions F,H on the nth copy of $\mathbb{R}^3$, the Lie–Poisson structure is

$$\{F,H\}_n(\mathbf{x}_n)=-\mathbf{x}_n\cdot(\nabla_nF\times\nabla_nH), \tag{2}$$

where $\times$ denotes the vector cross product.

B. The symmetry group and momentum map

Consider the diagonal action of the group SO(3) on P defined by rotations in each $\mathbb{R}^3$. This action is canonical with respect to the Poisson structure (1). The corresponding Lie algebra is naturally identified with $\mathbb{R}^3$ (having the vector product as its Lie bracket operation) and we write $\boldsymbol{\xi}$ for the vector in $\mathbb{R}^3$ corresponding to the matrix $\xi\in\mathfrak{so}(3)$; thus,

$$\mathfrak{so}(3)\simeq(\mathbb{R}^3,\times),\quad \text{i.e.,}\quad [\xi,\eta]=\boldsymbol{\xi}\times\boldsymbol{\eta},\quad \text{for any}\quad \xi,\eta\in\mathfrak{so}(3). \tag{3}$$

The vector field of infinitesimal transformations corresponding to an element ξ in the Lie algebra is given by

$$\xi_P(\mathbf{x})\coloneqq\frac{d}{dt}\exp(\xi t)\cdot\mathbf{x}\Big|_{t=0}=(\boldsymbol{\xi}\times\mathbf{x}_1,\ldots,\boldsymbol{\xi}\times\mathbf{x}_N). \tag{4}$$

The space $\mathfrak{so}(3)^*$, which, as mentioned above, is the dual of the Lie algebra $\mathfrak{so}(3)$, is equipped with the natural Lie–Poisson structure given by (2) (after identifying the dual of $\mathbb{R}^3$ with itself using the standard dot product on $\mathbb{R}^3$).

Recall that a *momentum map* $\mathbf{J}:P\to\mathfrak{so}(3)^*\simeq\mathbb{R}^3$ for this action is defined in terms of the Poisson bracket of an arbitrary function F on P by

$$\{F,J(\xi)\}=\xi_P[F],$$

where $J:\mathfrak{so}(3)\to C^\infty(P)$ is related to $\mathbf{J}$ by

$$J(\xi)(z)=\langle\mathbf{J}(z),\xi\rangle,$$

for all $\mathbf{x}\in\mathbf{P}$, $\xi\in\mathfrak{so}(3)$, and where $\langle\cdot,\cdot\rangle$ is the natural paring between the Lie algebra and its dual.

It is readily checked that the momentum map is proportional to the moment of vorticity and is given by

$$\mathbf{J}(\mathbf{x})=-\frac{1}{R}\sum_{n=1}^{N}\Gamma_n\mathbf{x}_n. \tag{5}$$

Denote its components by $\mathbf{J}=(\mathscr{Q},\mathscr{P},\mathscr{S})$. The momentum map is *not surjective* since

$$\max_n|\Gamma_n|-\sum_{m\neq n}|\Gamma_m|<\|\mathbf{J}\|<\sum_n|\Gamma_m|. \tag{6}$$

Denote the range of $\mathbf{J}$ by $\mathscr{R}$.

The momentum map is *equivariant*, that is,

$$\mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(\mathbf{x}))=\mathbf{J}(g(\mathbf{x})), \tag{7}$$

for all $g\in\mathrm{SO}(3)$. Here, the map $\mathrm{Ad}^*_k:\mathfrak{so}(3)^*\to\mathfrak{so}(3)^*$, defined for each $k\in\mathrm{SO}(3)$, denotes the coadjoint action of SO(3) on $\mathfrak{so}(3)^*$. That one can find an *equivariant* momentum map is consistent with general theorems for compact or semisimple groups. In our case, this can be seen directly from (5) as the coadjoint action corresponds simply to rotations in the dual space $\mathfrak{so}(3)^*\simeq\mathbb{R}^3$.

It follows from equivariance of $\mathbf{J}$ or directly, that $\|\mathbf{J}\|^2$ is invariant under the coadjoint action. Hence, smooth functions of $\|\mathbf{J}\|^2$ are also invariant. Thus, if $\mathbf{b}=(b_1,b_2,b_3)\in\mathbb{R}^3$ are coordinates in the dual $\mathfrak{so}(3)^*$, then any smooth function of $\|\mathbf{b}\|^2$ is a Casimir function. Correspondingly, the generic symplectic leaves of $\mathfrak{so}(3)^*$ are spheres defined by the level sets $\|\mathbf{J}\|^2=\mathrm{const}\neq 0$. Note that as SO(3) is compact, the action of it on both P and $\mathfrak{so}(3)^*$ is proper.

C. The Hamiltonian

The Hamiltonian describing the motion of N vortices on the surface of a sphere of radius R is given by (see, e.g., Kidambi and Newton[4])

$$H=\frac{1}{4\pi R^2}\sum_{m<n}\Gamma_m\Gamma_n\ln(l^2_{mn}), \tag{8}$$

where $l^2_{mn}=2(R^2-\mathbf{x}_m\cdot\mathbf{x}_n)$ is the square of the chord distance between two vortices with positions $\mathbf{x}_m$ and $\mathbf{x}_n$. The constraints $\|\mathbf{x}_n\|=R$ are assumed. Notice that the Hamiltonian is invariant with respect to the diagonal action of SO(3) on P described above. Hence, the momentum map $\mathbf{J}$ defined by (5) is constant along the flow of this Hamiltonian.

II. POISSON AND SYMPLECTIC REDUCTION

A. Poisson quotients

In performing Poisson reduction, one normally constructs the quotient space by the symmetry group and calculates its naturally induced Poisson bracket. As is well known, singularities in the quotient space may arise if the group action on the phase space is not free. Strictly speaking, the phase space of the problem is not $S^2\times\cdots\times S^2$ but rather

$$S^2\times\cdots\times S^2\bigg\backslash\sum_k\cup_{i_n\neq i_m}\Delta_{i_1\cdots i_k},$$

where $\Delta_{i_1\cdots i_k}=\{\mathbf{x}|$ any two or more of $\mathbf{x}_{i_n}$ coincide$\}$. This is because the self-interaction and collision of vortices (which lead to infinite energy) have been excluded from consideration. This restriction guarantees that the diagonal action of SO(3) on P is free provided $N\geqslant 3$, i.e., there are 3 or more vortices. The action is also proper, as was mentioned above.

Thus, *the quotient* $T=P/\mathrm{SO}(3)$ *is a smooth* $2N-3$ *dimensional Poisson manifold*. In coordinatizing this quotient we shall use the quantities l_{mn}^2, which are functions on P that are invariant with respect to the SO(3) action, with the conditions $l_{mn}^2\neq 0$. In general, there are $1+2(N-2)=2N-3$ independent functions l_{mn}^2, and other invariant functions can be expressed in terms of them.

To describe a configuration of N vortices on a sphere (up to a global rotation), it is sufficient to specify the chord distance between some two vortices and the chord distances from the remaining $N-2$ to those two (to remove the ambiguity of reflection consider, for example, a stereographic projection and choose two vortices such that all the rest lie to one side of the line connecting those two).

When 3 or more vortices are aligned on a great circle, this coordinate system is degenerate, i.e., there are less than $2N-3$ independent functions l_{mn}^2, and so we shall introduce other coordinates in the neighborhood of such points in the quotient space. Specifically, it is easy to see that the differentials of the three square distances associated to three vortices are linearly dependent when the three vortices lie on a great circle. This analysis, obviously, agrees with the dimension of the Poisson quotient. Also, the variables l_{mn}^2 naturally appear in the Hamiltonian for the N-vortex problem on the sphere, which makes the calculation of the reduced Hamiltonian h easy.

It follows from (5) that the square of the momentum map is given by

$$\mathbf{J}^2=:\|\mathbf{J}\|^2=\left(\sum \Gamma_n\right)^2-\frac{1}{R^2}\sum_{n<m}\Gamma_n\Gamma_m l_{nm}^2, \tag{9}$$

which, as we mentioned, is invariant under the SO(3) action. Other invariants are given by l_{nm}^2. Denote $\Gamma=\Sigma\Gamma_n$ and define a map $\Phi_\mu:T\to\mathbb{R}$ by

$$\Phi_\mu=(\mu-\Gamma)(\mu+\Gamma)+\frac{1}{R^2}\sum_{n<m}\Gamma_n\Gamma_m l_{nm}^2. \tag{10}$$

Notice that the relation (9) between the variables l_{nm}^2 and $\mathbf{J}$ can be expressed as $\Phi_J(l_{nm^2})=0$.

B. Reduction for the 3-vortex problem

Now we consider the 3-vortex problem and the structure of the corresponding Poisson reduced space in more detail. The phase space of the 3-vortex problem is trivial in a sense that it is diffeomorphic to a product of SO(3) with a ''shape-phase space'' U; that is, $P\cong\mathrm{SO}(3)\times U$, where

$$U=\{(a,\alpha_1,\alpha_2)|-R<a<R,0<\alpha_1+\alpha_2<2\pi,\alpha_1<\alpha_2\}\subset\mathbb{R}^3. \tag{11}$$

Here, a can be interpreted as the height of the triangle of the vortices with respect to the sphere and α_n corresponds to an angle opposite the nth vortex. For the computations we will use another atlas which consists of three charts, two of which are nearly identical—they differ only in the orientation and are connected by a $\mathbb{Z}_2$ reflection. That is, for the same chord distances vectors $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3$ can form a right-handed or left-handed coordinate system, corresponding to different orientations and thus defining two different configurations, one in each of these two charts.

Denote the coordinates on these charts by $a_1=l_{23}^2,a_2=l_{13}^2,a_3=l_{12}^2$, so that the a_n are the squares of the sides of the triangle inscribed in a circle of radius $r<R$. Thus, all admissible values of a_n can be parameterized by any two angles α_n,α_m. The chart can be given parametrically by an open set $\mathscr{T}\subset\mathbb{R}^3$ defined as the set of triples (a_1,a_2,a_3) given by

$$a_1=2r^2(1-\cos\,\alpha_1),\qquad a_2=2r^2(1-\cos\,\alpha_2),\qquad a_3=2r^2(1-\cos(\alpha_1+\alpha_2)), \tag{12}$$

where $0<\alpha_1+\alpha_2<2\pi$, $0<r<R$. The third chart contains an open neighborhood of the set of great circles and smoothly connects different orientations. Indeed, for great circles $\mathbf{x}_n$ become linearly dependent, and $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3$ fail to define either the right- or left-handed coordinate system. The chart can be coordinatized by $V=\mathbf{x}_1\cdot(\mathbf{x}_2\times\mathbf{x}_3)$, i.e., the orientable volume of the parallelepiped formed by the vectors $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3$, and any two chord distances a_n,a_m. The sign of V deter-

mines the orientation (by distinguishing between right- and left-handed coordinate systems) and thus specifies one of the above two charts, while $V=0$ corresponds to the great circles. The change of coordinates is checked to be nondegenerate in the open intersections of the charts.

We summarize our results on Poisson reduction for the 3-vortex problem in the following.

Proposition II.1 (Poisson reduction for the 3-vortex problem): The quotient $T=P/\mathrm{SO}(3)$ *is a smooth* 3 *dimensional manifold diffeomorphic to the shape phase space U defined by (11). The natural projection of P to T is a surjective submersion with fibers being the SO(3)-orbits on P.*

The manifold T carries the quotient Poisson structure given as follows in the coordinate charts described above. Let f and h be given functions defined on the set $\mathscr{T}\subset T$ *and let* (a_1,a_2,a_3) *lie in the set* $\Phi_\mu(a_1,a_2,a_3)=0$. *Then*

$$\{f,h\}_T(a_1,a_2,a_3)=\frac{4R^3V}{\Gamma_1\Gamma_2\Gamma_3}\nabla\Phi_\mu\cdot(\nabla f\times\nabla h), \tag{13}$$

where V is regarded as a function of $\mathbf{a}$; *its sign, which corresponds to different orientations, distinguishes between the two charts* $\mathscr{T}$. *The Poisson bracket along the set of great circles is given by the following expression:*

$$\{f,h\}_T(V,a_2,a_3)=B_2\left(\frac{\partial f}{\partial a_2}\frac{\partial h}{\partial V}-\frac{\partial f}{\partial V}\frac{\partial h}{\partial a_2}\right)+B_3\left(\frac{\partial f}{\partial a_3}\frac{\partial h}{\partial V}-\frac{\partial f}{\partial V}\frac{\partial h}{\partial a_3}\right). \tag{14}$$

Here

$$B_2=4R\left(\frac{2(a_1+a_2-a_3)R^2-a_1a_2}{\Gamma_1}-\frac{2(a_2+a_3-a_1)R^2-a_2a_3}{\Gamma_3}\right)$$

and

$$B_3=4R\left(\frac{2(a_2+a_3-a_1)R^2-a_2a_3}{\Gamma_2}-\frac{2(a_1+a_3-a_2)R^2-a_1a_3}{\Gamma_1}\right),$$

in which a_1 *is regarded as a function of* a_2,a_3 *(since they are dependent when* $V=0$*).*

Casimir functions on T are generated by Φ_μ; *that is, any function of* Φ_μ *is a Casimir function. The level sets,* $\Phi_\mu=0$, *determine the symplectic leaves; these leaves are isomorphic to the symplectic-reduced spaces* $P_\mu=\mathbf{J}^{-1}(\mu)/\mathrm{SO}(3)_\mu$. *The generic leaves are those not containing the great circle equilibria with* $\mathbf{J}=\mathbf{0}$ *and are open planes that foliate* $\mathscr{T}$. *For every fixed choice of* Γ_n *they are parallel to each other and none of them contains the central line* $a_1=a_2=a_3$. *If* $0\in Range\ \mathbf{J}$, *then there is a unique nongeneric zero dimensional symplectic leaf that corresponds to a great circle configuration with* $\mathbf{J}=0$.

Proof: Define $F=f\circ\pi$, where $\pi:P\to T$ is the projection. The Poisson bracket on P is given by (1) and (2). One computes, in a straightforward way, $\{F,H\}$ using the chain rule to get (13). Then (14) is obtained upon change of coordinates in the chart intersections and setting $V=0$ afterwards; we omit here the required simple but tedious calculations.

The structure of generic symplectic leaves follows from the linearity of the Casimir function (10). □

A Hamiltonian H on P that is invariant under the diagonal action of SO(3) induces a reduced Hamiltonian h on $T=P/\mathrm{SO}(3)$. The corresponding reduced equations on the leaves $\Phi_\mu=0$ in $\mathscr{T}$ are checked to be given by the following (Euler-like) equations (see Kummer[19] and Kirk and Marsden and Silber[20]):

$$\dot{\mathbf{a}}=\frac{4R^3V}{\Gamma_1\Gamma_2\Gamma_3}\nabla h\times\nabla\Phi_\mu, \tag{15}$$

where $\mathbf{a}=(a_1,a_2,a_3)$.

For the Hamiltonian (8) the reduced equations are

$$\dot{a}_i = \frac{V}{\pi R}\Gamma_i\left(\frac{1}{a_j} - \frac{1}{a_k}\right), \tag{16}$$

where (i,j,k) is a cyclic permutation of (1,2,3). Along the set of great circles the equations are represented in a different way, as is the Poisson bracket; in fact, they are given by

$$\begin{aligned}\dot{V} = \frac{1}{8\pi}\Bigg(&2R\left(\frac{a_3-a_1}{a_2}(\Gamma_1+\Gamma_3) + \frac{a_1-a_2}{a_3}(\Gamma_2+\Gamma_1) + \frac{a_2-a_3}{a_1}(\Gamma_3+\Gamma_2)\right)\\ &-\frac{1}{R}(a_3(\Gamma_1-\Gamma_2)+a_2(\Gamma_3-\Gamma_1)+a_1(\Gamma_2-\Gamma_3))\Bigg),\end{aligned} \tag{17}$$

together with $\dot{a}_2 = 0$ and $\dot{a}_3 = 0$.

These results reproduce, in the spirit of geometric mechanics, some of the results of Kidambi and Newton.[4] For instance, the second invariant in this reference is interpreted as a linear function of the square of the momentum map, $\|\mathbf{J}\|^2$. They differ only in an overall factor and an additive constant. As it was mentioned above, $\|\mathbf{J}\|^2$ determines the symplectic leaves in $\mathfrak{so}(3)^*$ and naturally leads to conserved quantities.

III. STABILITY OF RELATIVE EQUILIBRIA

A. The energy–momentum method

We will now utilize the energy–momentum method (see Marsden[16] for a summary and references) for the analysis of the stability of *relative equilibria*, i.e., dynamical orbits with initial conditions $\mathbf{x}_e$ such that $\mathbf{x}(t) = \exp(\xi_e t)\mathbf{x}_e$ for some Lie algebra element ξ_e and any time t. As is well known for relative equilibria, the *augmented energy* function $H_{\xi_e} := H - \langle \mathbf{J} - \mu_e, \xi_e\rangle$ has a critical point at $\mathbf{x}_e$, where $\mu_e = \mathbf{J}(\mathbf{x}_e)$ is the value of the momentum at the relative equilibrium. For notational convenience we will occasionally omit the subscript e.

The orbital stability of a relative equilibrium is equivalent to the stability of the corresponding equilibrium of the reduced system that is induced on the symplectic leaves P_μ of the quotient manifold $P/\mathrm{SO}(3)$. The energy momentum method is designed to enable one to test for orbital stability directly on the unreduced manifold P by constructing a subspace $\mathcal{S} \subset T_{\mathbf{x}_e}P$ which is isomorphic to $T_{\mathbf{x}_e}P_{\mu_e}$. This is done by considering a tangent space to the level set of constant momentum $\mathbf{J}^{-1}(\mu_e)$ and eliminating the neutrally stable directions associated to the isotropy subgroup,

$$\mathrm{SO}(3)_{\mu_e} := \{g \in \mathrm{SO}(3) \,|\, \mathrm{Ad}_g^* \mu_e = \mu_e\}.$$

The energy–momentum method determines stability by examining definiteness of the second variation of H_{ξ_e} restricted to the subspace $\mathcal{S}$. A detailed description of this method can be found in Simo, Lewis and Marsden.[21]

If one has a definite second variation, then Patrick's theorem (see Patrick[22]) guarantees stability modulo the isotropy subgroup, provided its action on P is proper and the Lie algebra admits an inner product invariant under the adjoint action of the isotropy subgroup. Note that as SO(3) is compact, the assumptions of Patrick's theorem are automatically satisfied for our applications.

As was mentioned above, relative equilibria are critical points of the augmented Hamiltonian H_ξ. For variational calculations, we extend all functions on P to functions on the ambient space $\mathbb{R}^{3N}$, and then restrict variations to the tangent space to P by requiring

$$\delta F(\mathbf{x}) \cdot \boldsymbol{\eta} = 0,$$

for all $\boldsymbol{\eta} \in T_{\mathbf{x}}P$. For the augmented Hamiltonian corresponding to (8), this results in the following conditions on $\mathbf{x}$:

$$\frac{\Gamma_r}{R}\left(\boldsymbol{\xi}(\mathbf{x}) - \frac{1}{2\pi R}\sum_{n\neq r}\Gamma_n \frac{\mathbf{x}_n}{l_{nr}^2}\right) = \kappa_r \frac{\Gamma_r}{R^2}\mathbf{x}_r\,, \tag{18}$$

where κ_r are constants to be determined.

B. Equidistant relative equilibria

An ***equidistant configuration*** is, by definition, one that satisfies $l_{mn}^2 = l^2$ for all $m \neq n$. Whatever its dynamics, such a configuration is possible only for $N = 2,3,4$ (this follows by geometric arguments similar to those used for the study of regular polytopes in three space); we exclude the simple case $N = 2$ from our considerations.

To verify that an equidistant configuration is a relative equilibrium, one need only check that conditions (18) are satisfied. It is easy to see that

$$\boldsymbol{\xi}(\mathbf{x}) = \frac{1}{2\pi R l^2} \sum_n \Gamma_n \mathbf{x}_n = -\frac{1}{2\pi l^2} \mathbf{J}(\mathbf{x}) \tag{19}$$

solves (18) with $\kappa_r = \Gamma_r / 2\pi l^2$. Notice, that in (19) the vectors $\boldsymbol{\xi}$ and $\mathbf{J}$ have opposite directions. These observations prove the following.

Proposition III.1: Equidistant configurations of relative equilibria satisfying $\mathbf{J}(\mathbf{x}_e) \neq 0$ *are possible only for* $N = 3$ *and* 4 *and are given by equilateral triangles and a tetrahedron, respectively; the associated values of the momentum and the Lie algebra element for these relative equilibria satisfy (19).*

Condition (18) together with $\boldsymbol{\xi} = 0$ defines ***static equilibria***. It follows from (19) that equidistant static equilibria are possible only in the degenerate case of zero momentum. This necessarily implies for $N = 3$ that the vortices lie on a great circle, and for both $N = 3$ and $N = 4$ that all Γ_n are equal, i.e., $\Gamma_n = \Gamma$. Moreover, a tetrahedral configuration with zero momentum $\mathbf{J} = 0$ is necessarily a static equilibrium.

C. Great circle relative equilibria

For $N = 3$ vortices, we have the following classification of ***great circle equilibria*** (see Kidambi and Newton[4]); recall the notations $a_1 = l_{23}^2, a_2 = l_{13}^2, a_3 = l_{12}^2$.

1. Generic momentum, $\mathbf{J}(\mathbf{x}_e) \neq \mathbf{0}$

General relative equilibria correspond to vortices lying on a great circle (and thus satisfying $V = 0$) and also satisfying the following condition:

$$\begin{aligned} 2R\left(\frac{a_3 - a_1}{a_2}(\Gamma_1 + \Gamma_3) + \frac{a_1 - a_2}{a_3}(\Gamma_2 + \Gamma_1) + \frac{a_2 - a_3}{a_1}(\Gamma_3 + \Gamma_2) \right) \\ - \frac{1}{R}(a_3(\Gamma_1 - \Gamma_2) + a_2(\Gamma_3 - \Gamma_1) + a_1(\Gamma_2 - \Gamma_3)) = 0, \end{aligned} \tag{20}$$

obtained by setting $\dot{V} = 0$ in (17). This implicit formula determines another relation (in addition to $V = 0$), between a_1, a_2 and a_3 for each fixed set of Γ's. This is a nonlinear equation and thus can have multiple solutions.

(a) Isosceles triangular great circle equilibria. A particular family of ***isosceles triangular relative equilibria*** *for arbitrary values of* Γ*'s is given by the following configuration:*

$$a_1 = a_2 = 2R^2, \quad a_3 = 4R^2, \tag{21}$$

or, equivalently, $\alpha_1 = \alpha_2 = \pi/2$, $\alpha_3 = \pi$, as well as configurations obtained from it by cyclic permutations of indices. The whole configuration rotates around the vector

$$\boldsymbol{\xi}(\mathbf{x}) = -\frac{1}{4\pi R^2} \mathbf{J}(\mathbf{x}) \tag{22}$$

and the constants κ_n in (18) are given by

$$\kappa_1 = \frac{\Gamma_1}{4\pi R^2} - \frac{\Gamma_2}{8\pi R^2}, \quad \kappa_2 = \frac{\Gamma_2}{4\pi R^2} - \frac{\Gamma_1}{8\pi R^2}, \quad \kappa_3 = \frac{\Gamma_3}{4\pi R^2}.$$

(b) Equilateral triangular great circle equilibria. A ***great circle equilateral triangle relative equilibrium*** with $l_{mn}^2 = l^2 = 3R^2$ and ξ given by (19).

Note: When the term *equilateral triangle relative equilibrium* is used, and we do not append ''great circle,'' we will mean that it is a *nongreat circle equilateral triangle relative equilibrium*.

2. Degenerate momentum, $\mathbf{J}(\mathbf{x}_e) = 0$

In this case, the vortices again lie on a great circle, and the whole configuration rotates around the vector

$$\boldsymbol{\xi}(\mathbf{x}) = -\frac{1}{2\pi R}\left(\frac{\Gamma_1 \mathbf{x}_1}{l_{23}^2} + \frac{\Gamma_2 \mathbf{x}_2}{l_{13}^2} + \frac{\Gamma_3 \mathbf{x}_3}{l_{12}^2}\right). \tag{23}$$

Remarks.

(1) Another specific family of great circle solutions can be found in case two of the Γ's are equal; for instance, $\Gamma_1 = \Gamma_2$. In fact, any isosceles triangle with the corresponding sides of the triangle being also equal, that is, $a_1 = a_2$ for $\Gamma_1 = \Gamma_2$, solves to (20) and hence is a relative equilibrium for any value of Γ_3.

(2) If we consider the ''inverse'' problem, namely, given a configuration on a great circle find Γ_n satisfying (20) so that this configuration is a relative equilibrium, then condition (20) becomes a linear equation in Γ_n of the form

$$\beta_1 \Gamma_1 + \beta_2 \Gamma_2 + \beta_3 \Gamma_3 = 0,$$

where $\beta_n = \beta_n(a_1, a_2, a_3)$ are functions of a great circle configuration. One would expect this to have a two parameter family of solutions.

The structure of the symplectic leaves sheds light on the stability of relative equilibria of the system. In particular, generically, great circle configurations satisfying (20) form a family of one dimensional curves in the Poisson manifold T that intersect symplectic leaves in a point. Similarly, equilateral configurations are isolated points within the symplectic leaves, and stability analysis is done by restricting a proper second variation to the tangent space to these leaves.

D. Geometry of the tangent space of phase space

Following the outline in the beginning of this section, consider a generic *regular* relative equilibrium $\mathbf{x}_e$, that is, its symmetry subgroup is finite, i.e., for each nonzero element $\boldsymbol{\xi}$ of the Lie algebra, the corresponding infinitesimal generator evaluated at $\mathbf{x}_e$, denoted $\boldsymbol{\xi}_P(\mathbf{x}_e)$, is nonzero. Then, the isotropy subgroup of the corresponding nonzero momentum value $\mu_e = \mathbf{J}(\mathbf{x}_e)$ is the group SO(2) of rotations around the vector $\mathbf{J}$. For $\mu_e = 0$ the isotropy subgroup is SO(3) itself; (the stability in this is case is simple and will be considered in the end of the section). The isotropy Lie subalgebra is defined by

$$\mathfrak{so}(3)_{\mu_e} = \left\{ \boldsymbol{\xi} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{\xi} = \varrho\, \mathbf{J}(\mathbf{x}_e) = -\frac{\varrho}{R} \sum_n \Gamma_n \mathbf{x}_{e,n}, \ \ \varrho \ \text{ a constant} \right\}. \tag{24}$$

Hence, the tangent space to the $\mathrm{SO}(3)_\mu$ orbit at $\mathbf{x}_e$, which corresponds to the neutrally stable direction, is given by

$$T_{\mathbf{x}_e}(\mathrm{SO}(3)_{\mu_e} \cdot \mathbf{x}_e) = \{ \boldsymbol{\xi} \times \mathbf{x}_e \,|\, \boldsymbol{\xi} = \varrho\, \mathbf{J}(\mathbf{x}_e) \}, \tag{25}$$

where again ϱ is a constant. For regular relative equilibria, $\mathrm{Ker} D\mathbf{J}(\mathbf{x}_e) = T_{\mathbf{x}_e} \mathbf{J}^{-1}(\mu_e)$. The derivative of the momentum map $D\mathbf{J}$ as a mapping from TP to $T\mathbb{R}^3$ can be easily computed from (5) to produce

$$D\mathbf{J}(\mathbf{x})\cdot\mathbf{y}=-\frac{1}{R}\sum_n \Gamma_n\mathbf{y}_n,$$

where $\mathbf{y}:=(\mathbf{y}_1,\ldots,\mathbf{y}_N)\in T_{\mathbf{x}}P$ and $\mathbf{y}_n\in T_{\mathbf{x}_n}S^2$ is a tangent vector to the sphere S^2 at the point $\mathbf{x}_n$. Thus, the kernel is determined by the following condition:

$$\mathrm{Ker}D\mathbf{J}(\mathbf{x})=\left\{\mathbf{y}\in T_{\mathbf{x}}P\,\middle|\,\sum_n \Gamma_n\mathbf{y}_n=0\right\}, \tag{26}$$

and is $2N-3$ dimensional.

Using Eqs. (25) and (26) it is easy to see that

$$T_{\mathbf{x}_e}(\mathrm{SO}(3)_\mu\cdot\mathbf{x}_e)\subset\mathrm{Ker}D\mathbf{J}(\mathbf{x}_e).$$

Indeed,

$$\sum_n \Gamma_n\mathbf{y}_n=\sum_n \Gamma_n\varrho\mathbf{J}\times\mathbf{x}_n=\varrho\mathbf{J}\times\mathbf{J}=0.$$

We proceed to find a subspace $\mathcal{G}\subset\mathrm{Ker}D\mathbf{J}(\mathbf{x}_e)$ that is transversal to the tangent space to the $\mathrm{SO}(3)_\mu$ orbit at $\mathbf{x}_e$. It is done in the following way. Chose two arbitrary vectors $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ such that the plane through them contains no vortices. Then, tangent vectors at each of the vortices,

$$\mathbf{y}_n^{(1)}:=\gamma_n^{(1)}\mathbf{D}^{(1)}\times\mathbf{x}_n,\quad \mathbf{y}_n^{(2)}:=\gamma_n^{(2)}\mathbf{D}^{(2)}\times\mathbf{x}_n \tag{27}$$

span $T_{\mathbf{x}}P$. Notice that (27) guarantees that all $\mathbf{y}_n^{(i)}$ lie in a plane perpendicular to $\mathbf{D}^{(i)}$. Thus, for each $\mathbf{D}^{(i)}$ there are $N-2$ independent zero linear combinations of $\mathbf{y}_n$'s. Also, it follows from (26) that if the coefficients $\gamma_n^{(i)}$ are chosen to satisfy

$$\sum_n \Gamma_n\gamma_n^{(i)}\mathbf{x}_n=\mathbf{D}^{(i)}\quad\text{or}\quad\sum_n \Gamma_n\gamma_n^{(i)}\mathbf{x}_n=0,\quad i=1,2, \tag{28}$$

then the corresponding tangent vectors belong to the $\mathrm{Ker}D\mathbf{J}$.

Any of the equalities in (28) has $N-3+1=N-2$ linearly independent solutions for each $\mathbf{D}^{(i)}$, and, hence, a transversal subspace $\mathcal{G}$ is defined by

$$\mathcal{G}=\mathrm{span}\{\mathbf{y}^{(1)}:=(\gamma_n^{(1)}\mathbf{D}^{(1)}\times\mathbf{x}_n),\ \mathbf{y}^{(2)}:=(\gamma_n^{(2)}\mathbf{D}^{(2)}\times\mathbf{x}_n)\}, \tag{29}$$

and $\dim\mathcal{G}=2N-4$. The isotropy subgroup transformations, i.e., rotations around the axis $\mathbf{J}$, is determined by tangent vectors

$$\mathbf{y}_n:=-\frac{1}{R}\mathbf{J}\times\mathbf{x}_n$$

and corresponds to an additional one-dimensional neutrally stable subspace in $\mathrm{Ker}D\mathbf{J}$.

We note that special choice of $\mathbf{D}^{(i)}$ would result in a diagonal structure of the second variation of H_ξ. We shall see an instance of this below.

E. Definiteness of the second variation

For the calculation of the second variation the Lagrange multiplier method is used. Define the *extended Hamiltonian* $\tilde{H}_\xi$,

$$\tilde{H}_\xi:=H_\xi+\sum_n \lambda_n(\mathbf{x}_n^2-R^2),$$

where $(\mathbf{x}_n^2 - R^2) = 0$ constrains the motion of vortices to the sphere S^2. The Lagrange multipliers λ_n are determined by the condition $\delta \tilde{H}_\xi(\mathbf{x}_e) = 0$ and are given by

$$\lambda_n = -\frac{\kappa_n \Gamma_n}{2R^2},$$

where κ_n are determined from (18). Then the second variation at $\mathbf{x}_e$ is well-defined as a bilinear form on $T_{\mathbf{x}_e}P$. It is given by the following expression:

$$\frac{\partial^2 \tilde{H}_\xi}{\partial x_s^j \partial x_r^i} = \begin{cases} 2\lambda_r \delta^{ij} - \dfrac{\Gamma_r}{\pi R^2} \displaystyle\sum_{n \neq r} \Gamma_n \frac{x_n^i x_n^j}{l_{nr}^4}, & r = s, \\ -\dfrac{\Gamma_r \Gamma_s}{2\pi R^2 l_{rs}^2} \left(\delta^{ij} + 2\dfrac{x_s^i x_r^j}{l_{rs}^2} \right), & r \neq s. \end{cases} \tag{30}$$

In the case of an equilateral triangle configuration, when $l_{rs}^2 = l^2$, one can choose

$$\mathbf{D}^{(1)} = \mathbf{x}_1 + \mathbf{x}_2 \quad \text{and} \quad \mathbf{D}^{(2)} = \mathbf{x}_2 + \mathbf{x}_3,$$

as a set of vectors defining a basis of the transversal subspace $\mathscr{S}$ according to (27) with the constants $\gamma_n^{(i)}$ that satisfy conditions (28) being given by $\gamma_1^{(1)} = 1/\Gamma_1, \gamma_2^{(1)} = 1/\Gamma_2, \gamma_3^{(1)} = 0$ and $\gamma_1^{(2)} = 0$, $\gamma_2^{(2)} = 1/\Gamma_2, \gamma_3^{(2)} = 1/\Gamma_3$. Then, the restriction of the second variation to it is given by the following expression:

$$\delta^2 \tilde{H}_\xi|_{\mathscr{S}} = \frac{V^2}{\pi R^2 l^4} \begin{pmatrix} -\dfrac{\Gamma_3}{\Gamma_1} - \dfrac{\Gamma_3}{\Gamma_2} & 1 \\ 1 & -\dfrac{\Gamma_1}{\Gamma_2} - \dfrac{\Gamma_1}{\Gamma_3} \end{pmatrix}. \tag{31}$$

The second variation is definite provided $\det(\delta^2 \tilde{H}_\xi)$ is positive. Hence, the following.

Theorem III.2 (stability of nongreat circle equilateral triangles): *An equilateral triangle configuration of nongreat circle relative equilibria* $\mathbf{x}_e$ *is stable modulo* SO(2) *rotations around the vector* $\mathbf{J}(\mathbf{x}_e)$ *if*

$$\sum_{n<m} \Gamma_n \Gamma_m > 0, \tag{32}$$

and unstable if

$$\sum_{n<m} \Gamma_n \Gamma_m < 0. \tag{33}$$

This theorem generalizes the known results of Synge[8] for the stability of equilateral relative equilibria of 3 vortices on a plane. Indeed, conditions (32) and (33) are independent of the radius R. Thus, in the limit $R \to \infty$ the spherical stability conditions agree with those for the planar case.

Conjecture: The condition $\Sigma_{n<m} \Gamma_n \Gamma_m = 0$ *corresponds to a (degenerate) Hamiltonian bifurcation.*

Next we analyze stability for the family of great circle relative equilibria given by (21). Choose

$$\mathbf{D}^{(1)} = \mathbf{x}_1 + \mathbf{x}_3 \quad \text{and} \quad \mathbf{D}^{(2)}: (\mathbf{D}^{(2)}, \mathbf{x}_n) = 0, \ \|\mathbf{D}^{(2)}\| = R,$$

as a set of vectors defining a basis of the transversal subspace $\mathscr{S}$ according to (27) with the constants $\gamma_n^{(i)}$ satisfying conditions (28) being given by $\gamma_1^{(1)} = 1/\Gamma_1, \gamma_2^{(1)} = 0$, $\gamma_3^{(1)} = 1/\Gamma_3$ and $\gamma_1^{(2)} = 1/\Gamma_1, \gamma_2^{(2)} = 1/\Gamma_2, \gamma_3^{(2)} = 0$. Then we obtain the following expression for the restriction of the second variation:

$$\delta^2 \tilde{H}_\xi|_{\mathscr{G}} = \frac{1}{8\pi}\begin{pmatrix} \dfrac{\Gamma_2}{\Gamma_1} & 0 \\ 0 & \left(\dfrac{\Gamma_2}{\Gamma_1}+\dfrac{\Gamma_1}{\Gamma_2}\right)-2\left(1+\dfrac{\Gamma_3}{\Gamma_1}+\dfrac{\Gamma_3}{\Gamma_2}\right)\end{pmatrix}.$$

Stability then follows from a direct analysis of its definiteness; that is, whether or not the two diagonal entries have the same sign or not. In other words, one has stability if the determinant is positive and instability if it is negative. Carrying out this simple calculation gives the following result.

Theorem III.3 (stability of isosceles triangle great circle equilibria): *A great circle configuration of relative equilibrium* $\mathbf{x}_e$ *given by (21) is* stable *if*

$$\Gamma_1^2+\Gamma_2^2 > \sum_{n\neq m}\Gamma_n\Gamma_m \tag{34}$$

and unstable *if*

$$\Gamma_1^2+\Gamma_2^2 < \sum_{n\neq m}\Gamma_n\Gamma_m . \tag{35}$$

The stability is modulo SO(2) *rotations around* $\mathbf{J}(\mathbf{x}_e)$.

F. Stability of great circle equilateral triangle relative equilibria

The stability analysis of a GCET, a great circle equilateral triangle relative equilibrium is different from the nongreat circle equilateral triangle case. The reason is that the two-dimensional subspace to which the second variation of the augmented Hamiltonian is restricted in the general case fails to be a transversal subspace to the G_μ orbit (rotations around $\mathbf{J}$) within Ker$D\mathbf{J}$ but rather degenerates to a one-dimensional subspace. A complimentary direction transversal to the plane of the triangle has to be taken into account similar to the case of other great circle relative equilibria. Using the notations developed in the section on the geometry of the tangent space, we choose $\mathbf{D}^{(1)} = n\mathbf{x}_1 + m\mathbf{x}_2$ and $\mathbf{D}^{(2)}:(\mathbf{D}^{(2)},\mathbf{x}_n)=0$, $\|\mathbf{D}^{(2)}\|=R$ as a set of vectors defining a basis of the transversal subspace $\mathscr{G}$ according to (27) with the constants $\gamma_n^{(i)}$ satisfying conditions (28) being given by $\gamma_1^{(1)}=n/\Gamma_1, \gamma_2^{(1)}=m/\Gamma_2, \gamma_3^{(1)}=0$ and $\gamma_1^{(2)}=1/\Gamma_1, \gamma_2^{(2)}=1/\Gamma_2, \gamma_3^{(2)}=1/\Gamma_3$.

Using this basis, a straightforward computation gives the following expression for the restriction of the second variation:

$$\delta^2 \tilde{H}_\xi|_{\mathscr{G}} = \frac{1}{12\pi}\begin{pmatrix} 0 & 0 \\ 0 & 9-(\Gamma_1+\Gamma_2+\Gamma_3)\left(\dfrac{1}{\Gamma_1}+\dfrac{1}{\Gamma_2}+\dfrac{1}{\Gamma_3}\right)\end{pmatrix}.$$

One concludes from this that *these GCET equilibria are at best, neutrally stable*.

In the paragraphs below, we explore this in a little more detail and identify the source of the zero eigenvector. Compute the gradient of the Casimir function Φ_μ, given by equation (10), which gives the normal direction to the symplectic leaf:

$$\nabla\Phi_\mu = \left(\frac{\partial\Phi_\mu}{\partial\mathbf{x}_n}\right) = -\frac{2}{R^2}\begin{pmatrix}\Gamma_1\Gamma_2\mathbf{x}_2+\Gamma_1\Gamma_3\mathbf{x}_3 \\ \Gamma_1\Gamma_2\mathbf{x}_1+\Gamma_2\Gamma_3\mathbf{x}_3 \\ \Gamma_1\Gamma_3\mathbf{x}_1+\Gamma_2\Gamma_3\mathbf{x}_2\end{pmatrix}.$$

Evaluate this gradient at the point corresponding to the GCET, and take the gradient in the direction corresponding to the family of equilateral triangles. To determine such a direction, recall that in the coordinates of the trivialization this family is defined by the following curve: $a = a, \alpha_1 = \alpha_2 = 2\pi/3$, where a is the curve parameter. The tangent vector to this curve is (1,0,0) and so in coordinates of the ambient space, the variation of the GCET configuration along the family of equilateral triangles is given by the following expression:

$$w_{\mathrm{gcet}} := \begin{pmatrix} \mathbf{x}_1 \times \mathbf{x}_2 \\ \mathbf{x}_1 \times \mathbf{x}_2 \\ \mathbf{x}_1 \times \mathbf{x}_2 \end{pmatrix} \in T_x P,$$

i.e., the same tangent vector $\mathbf{x}_1 \times \mathbf{x}_2$ is attached at each vortex position.

Intuitively, one can understand this in the following way. Fix a horizontal plane going through the center of the sphere, intersecting it along a great circle. Inscribe an equilateral triangle giving us precisely the GCET configuration. Constrain each vortex to move along a great circle going through its original position and the North Pole. Then, shifting the plane vertically up and down and keeping track on its cross-section with the sphere, defines a family of equilateral triangles. Obviously, the vector of infinitesimal translation at the GCET configuration is given by w_{GCET} above, i.e., at each vortex the vector points strictly vertically.

The gradient $\nabla\Phi_\mu$ evaluated on w_{GCET} at GCET is zero; the volume function V, being the mixed vector product, vanishes at the great circle. This means that such a direction, i.e., the equilateral triangle family of equilibria, is tangential to the leaf at this point. In this sense the GCET is a nonisolated equilibrium within its symplectic leaf. Thus, further analysis of the stability of the GCET equilibrium requires applications of some other, nonstandard techniques.

The preceding considerations are not applicable to a nongreat circle equilateral triangle configuration, for which one shows that in the coordinates given by chord distances l_{nm}, the family of equilateral triangles given by $l_{nm}=l$ for all n,m intersects symplectic leaves, which are planes [see equation (10), Φ_μ is linear], *transversally*.

G. The degenerate case $\mathbf{J}(\mathbf{x}_e)=0$

Stability in this case is a simple task and can be done by a dimension count. This results in the following theorem.

Theorem III.4 (stability of great circle equilibria with $\mathbf{J}=0$): *A relative equilibrium with zero vorticity momentum* $\mathbf{J}(\mathbf{x}_e)=0$, *which necessarily lies on a great circle, is* stable modulo SO(3).

Proof: The isotropy subgroup $\mathrm{SO}(3)_{\mu=0}$ is, in this case, the whole group SO(3) and hence the dimension of $\mathbf{J}^{-1}(0)/\mathrm{SO}(3)_{\mu=0}$ is zero. This implies that

$$\mathrm{Ker} D\mathbf{J}(\mathbf{x}) = T_{\mathbf{x}}(\mathrm{SO}(3)_{\mu=0}\cdot \mathbf{x}).$$

The assumptions of Patrick's theorem are satisfied as SO(3) is compact, and so this proves the theorem. □

H. Stability in the reduced space

One can also study the stability of equidistant configurations of fixed equilibria in the reduced space by analyzing level sets of the integrals of motion. In general, each such integral defines a codimension 1 surface, and trajectories are confined to lie in the intersection of these surfaces. In our case, the flow lines are given by intersecting the 2d energy levels $h=\mathrm{const}$ with the coadjoint orbits which are planes. This is analogous to the rigid body flow on the angular momentum spheres, where the orbits are given by the intersection of the energy ellipsoids $h=\mathrm{const}$ with the coadjoint orbits that are two-spheres (see, e.g., Marsden and Ratiu[18]). Similar to the Energy–Casimir method, this approach, while defining stability conditions, does not specify the transformations in the unreduced space modulo which the stability is understood.

The equidistant fixed equilibria in T are determined by the central line $a_1=a_2=a_3=a$. In the neighborhood of such an equilibrium $a_i=a(1+\epsilon_i)$, where ϵ_i are small, and the energy levels are given by

$$h=\frac{1}{4\pi R^2}\left(\ln a\sum_{n<m}\Gamma_n\Gamma_m+\Gamma_1\Gamma_2\epsilon_3+\Gamma_1\Gamma_3\epsilon_2+\Gamma_2\Gamma_3\epsilon_1\right)$$
$$-\frac{1}{4\pi R^2}\frac{1}{2}(\Gamma_1\Gamma_2\epsilon_3^2+\Gamma_1\Gamma_3\epsilon_2^2+\Gamma_2\Gamma_3\epsilon_1^2)+\ldots. \tag{36}$$

The symplectic leaves are planes; up to a constant they are given by a linear part in (36). Thus, in a small enough neighborhood of an equilibrium, trajectories are determined by the intersections of these planes with the surfaces defined by the quadratic part in (36). Depending on the mutual signs of Γ's these surfaces are either ellipsoids or hyperboloids of one sheet or hyperboloids of two sheets. For instance, if all Γ_n have the same sign, then the quadratic surface is an ellipsoid, and its intersection with any plane is an ellipse. Hence, the fixed point is surrounded by closed planar orbits and is therefore stable. Note that the condition (32) is satisfied. On the contrary, if the signs of Γ_n are different, the quadratic surface is an hyperboloid, and its intersections with a plane are either ellipses or hyperbolas, depending on the position of the plane. This results in either stable or unstable fixed point, respectively, and is determined precisely by the conditions (32) and (33).

IV. CONCLUSIONS

A simple physical system of 3 point vortices on a sphere reveals a surprisingly rich geometrical structure. In this paper we have explicitly constructed the quotient manifold $T = P/\mathrm{SO}(3)$ of the problem and calculated its inherited Poisson bracket.

An analysis of the symplectic structure of the symplectic leaves in this quotient manifold sheds light on the classification of relative equilibria and their stability. By applying the energy–momentum method, we have found explicit criteria for the stability of different configurations of relative equilibria with generic and nongeneric momenta. In each case we have specified a group of transformations modulo which stability in the unreduced space is understood.

In work in progress, we shall explore the link with dual pairs (see Marsden and Weinstein[13] and Weinstein[23]) more thoroughly. Indeed, $P/\mathrm{SO}(3) \overset{\pi}{\leftarrow} P \overset{J}{\rightarrow} \mathscr{R}$ is a full dual pair. This duality is also one way of viewing noncommutative complete integrability of the 3-vortex problem on a sphere.

We also will be exploring the geometric phase (in the sense of Marsden, Montgomery and Ratiu[24]) for the three-vortex problem on a sphere.

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[1] H. Helmholtz, ''On integrals of the hydrodynamical equations which express vortex-motion (trans. p.g. Tait),'' Philos. Mag. **33**(4) (1867).
[2] G. Kirchhoff, *Vorlesungen Über Mathematische Physik*, 3rd ed. (Teubner, Leipzig, 1883).
[3] H. Aref, N. Rott, and H. Thomann, ''Gröbli's solution of the three-vortex problem,'' Annu. Rev. Fluid Mech. **24** (1992).
[4] R. Kidambi and P. Newton, ''Motion of three point vortices on a sphere,'' Physica D **116**, 143–175 (1998).
[5] Sir W. (Lord Kelvin) Thomson, ''On vortex atoms,'' Proc. R. Soc. Edinburgh **6**, 94–105 (1867).
[6] Sir W. (Lord Kelvin) Thomson, ''Floating magnets (illustrating vortex systems),'' Nature (London) **XVIII**, 13–14 (1878).
[7] E. A. Novikov, ''Dynamics and statistics of a system of vortices,'' Sov. Phys. JETP **41**, 937–943 (1975).
[8] J. L. Synge, ''On the motion of three vortices,'' Can. J. Math. **1**, 257–270 (1949).
[9] H. N. Aref, ''Motion of three vortices,'' Phys. Fluids **22**, 393–400 (1979).
[10] V. A. Bogomolov, ''Dynamics of vorticity at a sphere,'' Fluid Dyn. (USSR) **6**, 863–870 (1977).
[11] V. A. Bogomolov, ''Two dimensional fluid dynamics on a sphere,'' Izv. Atmos. Ocean. Phys. **15**, 18–22 (1979).
[12] F. Kirwan, ''The topology of reduced phase spaces of the motion of vortices on a sphere,'' Physica D **30**, 99–123 (1988).
[13] J. Marsden and A. Weinstein, ''Coadjoint orbits, vortices and Clebsch variables for incompressible fluids,'' Physica D **7**, 305–323 (1983).
[14] S. Smale, ''Topology and mechanics,'' Invent. Math. **10**, 305–331 (1970).
[15] V. I. Arnold, and B. A. Khesin, *Topological Methods in Hydrodynamics* (Springer-Verlag, Berlin, 1997).
[16] J. E. Marsden, *Lectures on Mechanics* (Cambridge University Press, 1992).
[17] D. K. Lewis and T. S. Ratiu, ''Rotating n-gon/kn-gon vortex configurations,'' J. Nonlinear Sci. **6**, 385–414 (1996).
[18] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*, Vol. 17 of *TAM* (Springer-Verlag, Berlin, 1994).
[19] M. Kummer, ''On resonant classical Hamiltonians with *n* frequencies,'' J. Diff. Eqns. **83**, 220–243 (1990).
[20] V. Kirk, J. E. Marsden, and M. Silber, ''Branches of stable three-tori using Hamiltonian methods in Hopf bifurcation on a rhombic lattice,'' Dynam. Stability Systems **11**, 267–302 (1996).
[21] J. C. Simo, D. Lewis, and J. E. Marsden, ''Stability of relative equilibria. part i: the reduced energy–momentum method,'' Arch. Ration. Mech. Anal. **115**, 15–59 (1991).

[22] G. Patrick, ''Relative equilibria in Hamiltonian systems: The dynamic interpretation of nonlinear stability on a reduced phase space,'' J. Geom. Phys. **9**, 111–119 (1992).
[23] A. Weinstein, ''The local structure of Poisson manifolds,'' J. Diff. Geom. **18**, 523–557 (1983).
[24] J. E. Marsden, R. Montgomery, and T. S. Ratiu, ''Reduction, symmetry, and phases in mechanics,'' Mem. AMS **436** (1990).