Symmetries in Motion: Geometric Foundations of Motion Control

Jerrold E. Marsden
Control and Dynamical Systems 107-81
California Institute of Technology
Pasadena, CA 91125
marsden@cds.caltech.edu

Jim Ostrowski
School of Engineering and Applied Science
University of Pennsylvania
Philadelphia, PA 19104-6315
jpo@grip.cis.upenn.edu

Abstract

Some interesting aspects of motion and control for systems such as those found in biological and robotic locomotion, attitude control of spacecraft and underwater vehicles, and steering of cars and trailers, involve geometric concepts. When an animal or a robot moves its joints in a periodic fashion, it can move forward or rotate in place. When the amplitude of the motion increases, the resulting net displacements normally increase as well. These observations lead to the general idea that when certain variables in a system move in a periodic fashion, motion of the whole object can result. This property can be used for control purposes; the position and attitude of a satellite, for example, are often controlled by periodic motions of parts of the satellite, such as spinning rotors. Geometric tools that have been useful to describe this phenomenon are "connections", mathematical objects that are extensively used in general relativity and other parts of theoretical physics. The theory of connections, which is now part of the general subject of geometric mechanics, has also been helpful in the study of the stability or instability of a system and in its bifurcations under parameter variations. This approach, currently in a period of rapid evolution, has been used, for example, to design stabilizing feedback control systems in the attitude dynamics of spacecraft and underwater vehicles. The same theory also describes the behavior of systems with constraints, such as those found in a simple, non-slipping rolling wheel or for more complex systems like a car pulling many trailers or a snake sliding across a floor. The presence of symmetries in these systems, often exhibited as position and orientation invariance, leads to a general theory of reduction. In this theory, the salient features of the motion are highlighted in a manner that is also conducive to formulating control inputs. This article explains in a reasonably nontechnical way why some of these tools of geometric mechanics are useful in the study of motion control and locomotion generation.

1 Introduction and Motivation

At the heart of this communication is the description of a geometric framework that has revealed some new insights into locomotion generation and motion control for mechanical systems. This point of view, which has been successfully applied to a wide variety of systems, is primarily based on a single, simple principle: that the generation of locomotion is fundamentally a question of relating internal *shape changes* to net changes in position via a coupling mechanism, most often either interaction with the environment or via some type of conservation law. This basic approach leads one to a powerful geometric structure implicit in a wide variety of systems. The mathematical tools that accompany this theoretical foundation enhance the natural physical intuition one brings to these problems and has provided surprising new insights into locomotion. The purpose of this first section is to provide some basic examples that will serve to motivate and set the stage for further development of this geometric framework and its applications.

Perhaps the most well-known example of the generation of rotational motion using internal shape changes is that of the falling cat. Released from rest with its feet above its head, the cat is able to execute a 180° reorientation and land safely on its feet. One observes that the cat achieves this net change in orientation by wriggling to create changes in its internal shape or configuration. On the surface, this provides a seeming contradiction—since the cat is dropped from rest, it has zero angular momentum at the beginning of the fall and hence, by conservation of angular momentum, throughout the duration of the fall. The cat has effectively changed its angular position while at the same time has zero angular momentum!

The exact process by which this occurs is subtle, and intuitive reasoning can lead one astray. Here is a commonly encountered, but fallacious, proof that a cat with zero angular momentum *cannot turn itself over* — what is wrong with it?

> Since angular momentum is moment of inertia times angular velocity, and the angular momentum of the cat is zero, the angular velocity must also be zero. Since angular velocity is the rate of change of the angular position, the angular position is constant. Thus, the cat cannot turn itself over.

The fallacy in this argument can be seen by trying to make sense of its assertions by thinking about the reasoning more carefully. For example, how does one really define the angular velocity of the cat? One possibility is to fix a rigid frame to the torso of the cat and then define its angular velocity as the rate of change of the rotation matrix taking the initial frame to the current one. If one does this, then the statement that angular momentum is moment of inertia times angular velocity is simply not true! This is because the angular momentum also includes terms from the internal motions of the cat's body parts. Also, keep in mind that the moment of inertia of the cat is not fixed. In fact, the instantaneous moment of inertia, sometimes called the ***locked inertia tensor***, is dependent on the shape of the cat.

To understand how the reorientation works, one has to go back to basics and remember a few fundamental facts from mechanics. First of all, it is true for rigid bodies that the angular momentum is the moment of inertia times its angular velocity (just as linear momentum is mass times linear velocity). Secondly, for a rotating articulated structure, the angular momentum is the sum of the angular momenta of its rigid parts. Each part has its own angular velocity. When one realizes even these basic facts, one sees that the quoted argument is hopelessly naive and incomplete.

Pursuing the mechanical reasoning further shows that something deeper is going on and it has to do with the changing locked inertial tensor due to shape changes. In fact, further analysis shows that these ideas are actually defining what a geometer calls a *connection*. While the study of this problem has a long history, e.g., Kane and Sher [1969], new and interesting insights have recently been discovered using these geometric methods; see Enos [1993], Montgomery [1990] and references therein.

For experienced geometers anxious to know what is really going on here, one can say that this effect is an example of *the holonomy of a natural connection on the principal bundle associated with symplectic reduction*. To grasp the meaning of such a loaded statement, most mechanicians and geometers would have to slow down, take a deep breath, and invest a little time to really understand the geometric and mechanical relationships inherent in this statement. We will be explaining some aspects of this and related topics in the following sections. The bibliography at the end of the paper gives the relevant technical references where the mechanical and mathematical details may be found.

Another example to help visualize this effect is to consider astronauts who wish to reorient themselves in a free space environment. This motion can again be achieved using internal gyrations, or shape changes. For example, consider a motion of the arms similar to the stirring of a large kettle, where the arms are held out forward to lie in a horizontal plane that goes through the shoulders, parallel to the floor. The hands are clasped together, and remain in this horizontal plane during the circular stirring motion. At the point of maximum extension of the arms, the inertia of the body about a vertical axis is also at a maximum. Conservation of angular momentum requires that the body must rotate in an opposite and proportional manner to the motion of the arms. As the arms rotate around and are brought in, however, the inertia of the body is reduced. The motion of the body in reaction is therefore also reduced. Thus, in one complete cycle of arm movement, the body undergoes a net rotation in the opposite direction of arm motion. When the desired orientation is achieved, the astronaut need merely stop the arm motion in order to come to rest. One often refers to the extra motion that is achieved by the name ***geometric phase***.

1.1 An historical perspective on geometric phases

The history of geometric phases and its applications is a long and complex story. We shall only mention a few highlights. The shift in the plane of the swing in the Foucault pendulum (commonly seen in Science Centers) as the earth rotates around its axis is certainly one of the earliest examples of this phenomenon. Anomalous spectral shifts in rotating molecules is another. Phase formulas for special problems such as rigid body motion, elastic rods and polarized light in helical fibers were understood already by the early 1950's although the geometric roots to these problems go back to MacCullagh [1840] and Thomson and Tait [1879, §123–126]. See Berry [1990] and Marsden and Ratiu [1994] for additional historical information.

More recently this subject became better understood, through the work of Berry [1984, 1985] and Simon [1983], whose papers first brought into clear focus the ubiquity of, and the geometry behind, all these phenomena. It was quickly realized that the phenomenon occurs in essentially the same way in both classical and quantum mechanics (see Hannay [1985]). It was also realized by Shapere and Wilczek [1987] that these ideas for classical systems were of great importance in the understanding of locomotion of micro-organisms.

That geometric phases can be linked in a fundamental way with the reduction theory for mechanical systems with symmetry was realized by Gozzi and Thacker [1987] and Montgomery [1988], and developed extensively in terms of reconstruction theory for mechanical systems with symmetry (not necessarily Abelian) by Marsden, Montgomery and Ratiu [1990]. This relation with reduction has played an important role in developing an understanding of the geometric nature of many general forms of locomotion. Other relations with symplectic geometry were found by Weinstein [1990].

The theory of geometric phases has an interesting link with non-Euclidean geometry, a subject first invented for its own sake, without regard to applications. A simple way to explain this link is as follows. Hold your hand at arms length, but allow rotation in your shoulder joint. Move your hand along three great circles, forming a triangle on the sphere and during the motion, keep your thumb "parallel"; that is, forming a *fixed* angle with the direction of motion. After completing the circuit around the triangle, your thumb will return rotated through an angle relative to its starting position. See Figure 1.1. In fact, this angle (in radians) is given by $\Theta = \Delta - \pi$ where Δ is the sum of the angles of the triangle. The fact that $\Theta \neq 0$ is of course one of the basic facts of non-Euclidean geometry — in curved spaces, the sum of the angles of a triangle is not necessarily π. This angle is also related to the *area* A enclosed by the triangle through the relation $\Theta = A/r^2$, where r is the radius of the sphere.

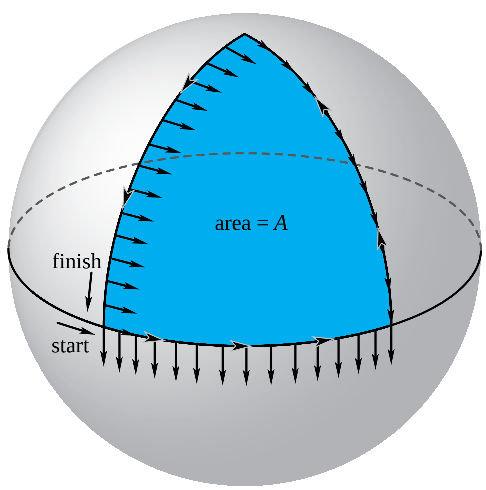

Figure 1.1: A parallel movement of your thumb around a spherical triangle produces a phase shift.

1.2 The role of geometry in control theory

Taking a control theoretic perspective, the motion generated by this sequence of arm movements has some important and useful consequences. Notice first that the directions used to follow the triangle (which can be thought of as control *inputs*), have only two relative components: motion perpendicular to the direction of the thumb and motion along the direction of the thumb. The final state, however, observes a net change in orientation of the thumb— not one of the original allowable input motions! Thus, it is possible to combine a cyclic motion in two of the input states to generate a net motion in a third, "uncontrolled" direction. Furthermore, one can calculate the amount of rotation the thumb experiences simply by determining the area enclosed by the path. As we will see below, these effects are common to a large class of systems which are characterized by what are commonly referred to as ***nonholonomic*** (nonintegrable) constraints.

While the above systems all share a similar geometric structure, another form of motion occurs in a class of seemingly unrelated systems which possess external constraints, such as rolling and sliding constraints, and fluid interaction forces. The issue of parking a car provides a commonplace example of how these constraints arise. The assumption that the wheels roll without slipping places restrictions on the possible motions of the car, i.e., the car can only move along the direction of the wheels and at a rate proportional to the angular velocity of the wheels. However, it is a well-known (and often experienced) fact that the car can move such that a net displacement perpendicular to the direction of the wheels is achieved— this is simply "parallel parking!" The motion is generated by coupling the changes in steering direction with changes in forward motion. In effect, a cyclic "shape" change has resulted in a net motion that could not directly be achieved using either of the inputs alone.

Notice the similarities that the above classes of systems possess, regardless of the means by which the motions are effected. In each case, cyclic motion in one set of variables (often called the *internal, base*, or *shape* variables) produces motion in another set of variables (often called the *group* or *fiber* variables). This idea is central to the basic geometric framework described in the ensuing sections.

In fact, the generation of net translational motion via the mechanism of internal shape changes is common to a wide range of biological and robotic systems. This process of *undulatory locomotion* is exhibited by a number of different biological organisms, including worms, snakes, paramecia, and fish. For example, snakes and paramecia each generate translations by a cyclic manipulation of their internal shape variables (Shapere and Wilczek [1987] and Ostrowski and Burdick [1996]). Just as the net reorientation of the thumb has a geometric interpretation involving the area enclosed by the motion of the hand, so also can the translation of a locomotive system be understood as an enclosed area with respect to internal shape changes.

The existence of oscillatory patterns of inputs is fairly ubiquitous in the generation of motion, both translational and rotational. Even at the most basic levels of intelligence and control, these patterns can be found in nature generating robust forms of locomotion. Similar observations have motivated studies of man-made systems. Examples of this include the links between vibratory motion and translational motion in micromotors studied by Brockett [1989] and the large body of literature devoted to controlling the reorientation of satellites using only internal rotors (Crouch [1984], Nakamura and Mukherjee [1993], and Walsh and Sastry [1995]).

Central to investigations in this area is the following question: how should one control motions of the internal variables so that the desired group (usually translational and rotational) motions are produced? To make progress on this question, one needs to combine experience with simple systems and strategies (such as steering with sinusoids, as in Murray and Sastry [1993] and Tilbury, Walsh, and Sastry [1995]) with a full understanding of the underlying mathematical structure of these systems, both analytical and geometrical. In some cases, careful observations of biological systems may provide useful keys to understanding how to produce these motions, e.g., Hirose's work (Hirose [1993]) with snakes and the biological literature on nonlinear coupled oscillators (Rand, Cohen, and Holmes [1988]). In other cases, tools from optimal control theory may be extremely useful. These tools can aid us in answering basic questions about the optimality of biological gait patterns that have developed through evolution. More practically, they have already seen use in choosing appropriate control inputs to guide a desired motion (for example, Brockett [1981]). As we proceed, we will describe some of the ways in which an understanding of the geometric nature of the problem can provide insights into why cyclic control motions are so tightly coupled with locomotion generation.

2 Connections and Bundles

As we have indicated, one of the fruitful ideas from geometry that has been used in the investigation of mechanical systems is that of a connection. While the notion of a connection is quite precise, connections have many personalities. On the one hand, one thinks of them as describing how curved a space is; in fact, in the classical Riemannian setting used by Einstein in his theory of general relativity, the curvature of the space is constructed out of the connection (in that case, described by the Christoffel symbols). In a related setting of classical spacetime developed by Cartan, the connection is what is responsible for a corrected measure of acceleration; for example if one is on a rotating merry-go-round, one has to correct any measurement of acceleration to take into account the acceleration of the merry-go-round, and this correction can be described by a connection.

In the general theory, connections are associated with mappings, called bundle mappings, that project larger spaces onto smaller ones, as in Figure 2.1. The larger space is called the ***bundle*** and the smaller space is called the ***base***. Directions in the larger space that project to zero are called ***vertical*** directions. The general definition of a ***connection*** is this: it is a specification of a set of directions, called ***horizontal*** directions, at each point, which complements the space of vertical directions.

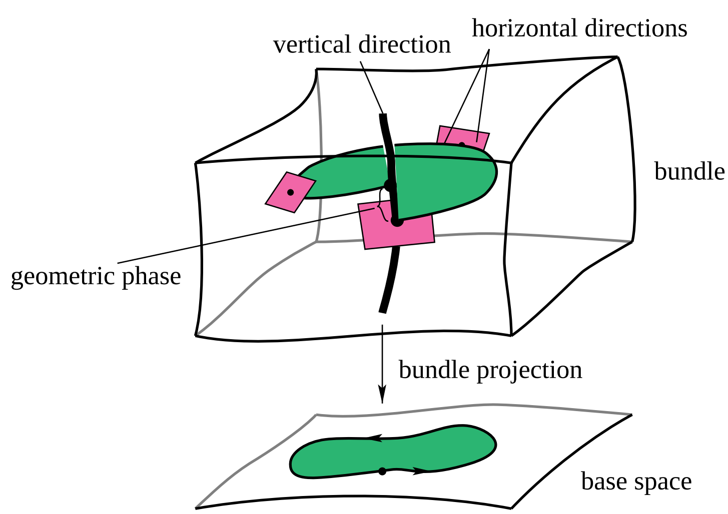

Figure 2.1: A connection divides the space into vertical and horizontal directions.

In the example of moving one's thumb around the sphere, the larger space is the space of all tangent vectors to the sphere, and this space maps down to the sphere itself by projecting a vector to its point of attachment on the sphere. The horizontal directions are the directions with zero acceleration within the intrinsic geometry of the sphere; that is, the directions determined by great circles. In this example, we saw that going around the triangle produces a change in the orientation of the thumb on return. When the thumb moves along a great circle at a fixed angle to its path, it is said to be ***parallel transported***. Equivalently, this motion corresponds to moving in horizontal directions with respect to the connection. The rotational shift that the thumb undergoes during the course of its journey is directly related to the curvature of the sphere (and hence the curvature of the connection) and to the area enclosed by the path that is traced out.

In general, we can expect that for a horizontal motion in the bundle corresponding to a cyclic motion in the base, the horizontal motion will undergo a shift, called a ***phase shift***, between the beginning and the end of its path. The magnitude of the shift will depend on the curvature of the connection and the area that is enclosed by the path in the base space. This shift in the vertical element is often given by an element of a group, such as a rotation or translation group, and is called the *geometric phase.* In many of the examples discussed so far, the base space is the *control space* in the sense that the path in the base space can be chosen by suitable control inputs to the system, e.g., changes in internal shape.

Connections are ubiquitous in geometry and physics. For example, connections are one of the main ingredients in the modern theory of elementary particles, and are the primary fields in Yang-Mills theory, a generalization of Maxwell's electromagnetic theory. In fact, in electromagnetism, the equation $B = \nabla \times A$ for the magnetic field may be thought of as an expression for the curvature B of the connection (or magnetic potential) A.

The use of connections has also proved quite valuable in the study of locomotion for many types of animals, insects, and robots. In this setting, the base space describes the internal shape of the object, and cyclic paths in the shape space correspond to the movements which lead to translational and rotational motion of the body. For example, the cat wriggles in order to effect a net reorientation, much like an inchworm or a fish sends waves down the length of its body to propel itself forward. The vertical elements that are generated as a geometric phase are exactly the overall translations and rotations that the body undergoes. As mentioned above, these can be represented by group elements, most often a subgroup of $SE(3)$, the group of spatial rigid body rotations and translations. The connection in this case does exactly what its name implies—it connects the internal shape motions to the resultant net effects on the position and orientation of the body. In many examples, horizontal motions are then defined to be those motions which correctly encode this connection. As we will see below, however, there are systems for which the geometric phase provides only one component of the total generation of locomotion.

This setting of connections provides a framework in which one can understand the phrase "when one variable in a system moves in a periodic fashion, motion of the whole object can result". Here, the periodic motion is that in the base space (that is, the control space), and the "motion of the whole object" is represented by the geometric phase. The insight provided by this notion is accompanied by many lovely theorems and calculational tools; for example, one of these, derived from Stokes' theorem, shows how to calculate the geometric phase in terms of the integral of the curvature of the connection over an area enclosed by the closed curve on the base. This is one reason that areas so commonly appear in geometric phase formulas.

In some problems, such as that of riding a bicycle, the identification of the shape space with the control space is not so clean (see Getz and Marsden [1995] for example). The shape space that would be analogous to that in the snakeboard is not "fully actuated" in the sense that one does not have direct control over all the degrees of freedom (this is called a *nonminimum phase system*); for the bike, one does not have direct control over the tilt of the bike (as one does with the forward propulsion and the steering).

In mechanics, the basic connection is called the ***mechanical connection***. The mechanical connection arises when the kinetic energy metric is invariant with respect to a Lie group (or symmetry group). The fibers of the bundle in this case are just the group orbits, and so the vertical space is given by those vectors tangent to the group orbits. The mechanical connection is then the unique principal connection whose horizontal space is orthogonal (via the metric) to these fibers. This construction is closely related to that encountered in Kaluza-Klein theory. A construction more familiar to mechanicians would be to compute the effective angular velocity of a system by multiplying the angular momentum by the inverse of the total system moment of inertia tensor. It turns out that this provides a method for describing the mechanical connection by identifying as horizontal those vectors that have zero momentum under this computation. See Marsden, Montgomery and Ratiu [1990] and Marsden [1992] and references therein for the proof of this equivalence and for additional details about these connections.

3 Connections from Constraints: Momentum and Rolling

In many mechanical systems, there are conditions on the allowable velocities of the system, called "constraints." For our purposes these are of two fundamentally different sorts: those given by momentum conservation laws and those generated by friction-type interactions with the environment. The first is typified by the constraint of zero angular momentum for the falling cat. The cat, once released, and before it reaches the ground, cannot change the fact that its angular momentum is zero, no matter how it moves its body parts. We choose the cat's base space to be its shape space, which does indeed literally mean what it says— the space of shapes of its body, which can be specified by giving the angles between its body parts. The bundle in this case consists of these shapes together with a rotation and translation to specify the overall position and orientation of the cat in space. Since the cat is free to manipulate its shape using its muscles, it can perform maneuvers, some of them cyclic, in shape space. Meanwhile, how it turns in space is governed by the law of conservation of angular momentum. It turns out that this law exactly defines the horizontal space of a connection! The connection in this case is called the ***mechanical connection***, which arises due to the symmetries of rigid body rotation. The horizontal subspace is then the vectors orthogonal, with respect to the kinetic energy metric, to pure rigid body rotations (that is, rotations in which the internal shape is fixed). That this corresponds to a connection was discovered through the combined efforts of Smale [1970], Abraham and Marsden [1978], and Kummer [1981]. It was developed by many people such as Guichardet [1984] and Iwai [1987]. Thus, when one puts together the theory of connections with this observation, and throws in control theory, one has the beginnings of the "gauge theory of mechanics".

Observing the motions of a mechanical system in its shape space is related to the theory of *reduction.* This theory seeks to make the phase space of a mechanical system smaller by taking advantage of symmetries to eliminate some of the variables. This method has led to many interesting and unexpected discoveries about mechanics, including, for example, the explanation of the integrability of the Kowaleskaya top in terms of symmetry by Bobenko, Reyman, and Semenov-Tian-Shansky [1989]. Symmetries for mechanical systems often manifest themselves as invariances of the system dynamics with respect to translational or rotational inertial position. For example, net motions that result from the wriggling of the falling cat or the gyrations of the astronaut are independent of the inertial reference frame from which they are observed. As a direct result of Noether's theorem, these invariances imply momentum conservation laws for mechanical systems. The goal of reduction theory is to factor out the invariances in order to provide a simplified analysis in terms of the base (shape) space. Observing motion in shape space alone is similar to watching the shapes change relative to an observer riding in a frame attached to some part of the object. In such a frame, once sees what seem to be extra forces, namely Coriolis and centrifugal forces. In fact, these forces can be understood in terms of the curvature of the mechanical connection. Then the problem of finding the original complete path is one of finding a horizontal path above the given one. This is sometimes called the "reconstruction problem". We note that the generation of geometric phases is closely linked with the reconstruction problem.

One of the simplest systems in which one can see these phenomena is called the ***planar skater***. This device consists of three coupled rigid bodies lying in the plane. They are free to rotate and translate in the plane, somewhat like three linked ice hockey pucks. This has been a useful model example for a number of investigations, and was studied fairly extensively in Oh, Sreenath, Krishnaprasad and Marsden [1989], Krishnaprasad [1989], and references therein. The only forces acting on the three bodies are the forces they exert on each other as they move. Because of their translational and rotational invariance, the total linear and angular momentum remains constant as the bodies move. This holds true even if one applies controls to the joints of the device; this is because the conservation of momentum depends only on externally applied forces and torques. See Figure 3.1.

The planar skater illustrates well some of the basic ideas of geometric phases. If the device starts

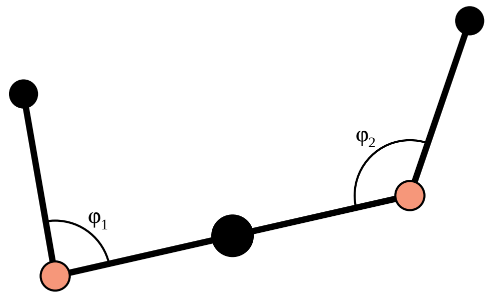

Figure 3.1: The planar skater consists of three interconnected bodies that are free to rotate about their joints.

with zero angular momentum and it moves its arms in a periodic fashion, then the whole assemblage can rotate, keeping, of course, zero angular momentum. This is analogous to the astronauts in free space who rotate their arms or legs in a coordinated fashion and find that they rotate. One can understand this simple example directly by using conservation of angular momentum. In fact, the definition of angular momentum allows one to reconstruct the overall attitude of the device in terms of the motion of the joints using simply freshman calculus. Doing so, one gets a motion generated in the overall attitude of the skater. This is indeed a geometric phase, dependent only on the path followed and not on the speed at which it is traversed or the overall energy of the system. This example is sufficiently simple that one does not need the geometry of connections to understand it, but nonetheless it provides a simple situation in which one can test the ideas. For more complex examples, such as the falling cat, the geometric setting of connections has indeed proven useful.

In the case of the planar skater (and in fact, for all mechanical systems with Lie group symmetries), we can write down the information encoded by the connection in a very simple form. We denote by g the group position (the vertical direction in the fiber bundle) and by r the internal shape configuration. In many mechanical examples one can pick out a distinguished g from the current configuration by putting a body fixed frame on the object and comparing this frame with a reference frame. Then any horizontal motion, that is, any motion compatible with the given connection, must satisfy an equation of the form

$$g^{-1}\dot{g} = -A(r)\dot{r}.$$

The quantity A is often called the local form of the connection. The reader familiar with Lie groups will recognize the left hand side of this equation as an element of the Lie algebra, reflecting the invariance of the system with respect to the group symmetry.

In the event that the angular momentum of the planar skater is not zero, the system experiences a steady drift in addition to the motions caused by the internal shape changes. If we were to fix the shape variables (ϕ_1, ϕ_2), this drift would manifest itself as a steady of angular rotation of the body with speed proportional to the momentum. More generally, the reorientation of the planar skater can always be decomposed into two components: the *geometric phase*, determined by the shape of the path and the area enclosed by it, and the *dynamic phase*, driven by the internal kinetic energy of the system characterized by the momentum. Figure 3.2 shows a schematic representation of this decomposition for general rigid body motion, which also serves to illustrate the motion of the planar skater. In this figure, the sphere represents the reduced space, with a loop in the shape space shown as a circular path on the sphere. The closed circle above the sphere represents the fiber of this bundle attached to the indicated point. Given any path in the reduced (shape) space, there is an associated path, called the *horizontal lift*, that is independent of the time parameterization of the path and of the initial vertical position of the system. Following the lifted path along a loop in the shape space leads to a net change in vertical position along the fiber. This net change is just the geometric phase. On top of that, but decoupled from it, there is the motion of the system driven by the momentum, which leads to the dynamic phase. Combining these two provides the actual trajectory of the system.

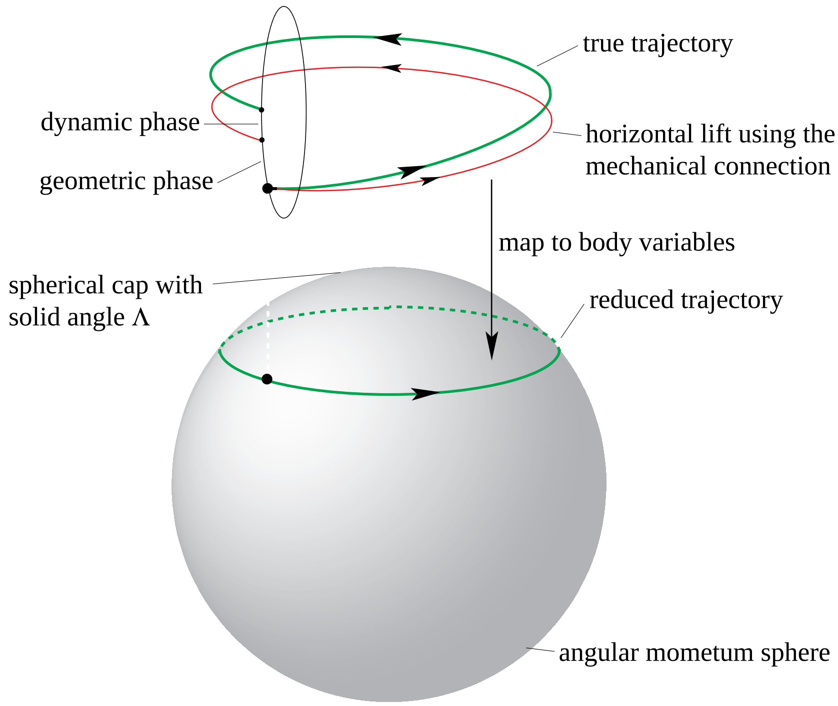

Figure 3.2: The geometric phase formula for rigid body motion; motion in the body angular momentum sphere can be periodic (lower portion of the figure) while the corresponding motion in the space of attitudes and their conjugate momenta, which carries the extra attitude information, is non-periodic (upper portion of the figure). The vertical arrow represents the map from the material representation to body representation.

One can treat systems with nonzero momentum by a simple modification of the above equation—if we denote the momentum by p, then the true trajectory satisfies

$$g^{-1}\dot{g} = -A(r)\dot{r} + (I(r,g))^{-1}p, \tag{3.1}$$

where I is a term related to the inertia of the system (the locked inertia tensor), and p is the (constant) total angular momentum. This equation highlights the clean separation between dynamic and geometric phases, and is again valid for general mechanical systems with symmetries.

To illustrate the significance of the geometry for this problem, consider the simulation of the planar skater shown in Figure 3.3. The cyclic inputs to this system are shown as a base input curve, while the actual trajectory of the motion is shown lifted above the input curve. After completing one cycle of internal shape changes, the skater has undergone a net rotation. The area enclosed by the base inputs is directly proportional to overall rotation of the planar skater. This is exactly the geometric phase, or *holonomy*, associated with the cyclic shape inputs. We will see below that this same sort of relationship occurs for other types of system that involve external constraints.

The nature of this problem is similar to the geometric experiment described in the opening section in which the thumb undergoes a net change in orientation. Several observations concerning the reorientation of the planar skater are in order here. First, the magnitude of the reorientation is again proportional to the amount of area enclosed by the path followed in shape space. This parallels the rotation of the thumb, which is proportional to the solid angle enclosed by the path along the sphere. Additionally, we recognize that the reorientation of the thumb is intrinsically linked to the *curvature* of the sphere. In the case of the planar skater, the connection describing the motion has an associated curvature (exterior derivative) which can analogously be thought of as the key to understanding the geometry of the generated motion.

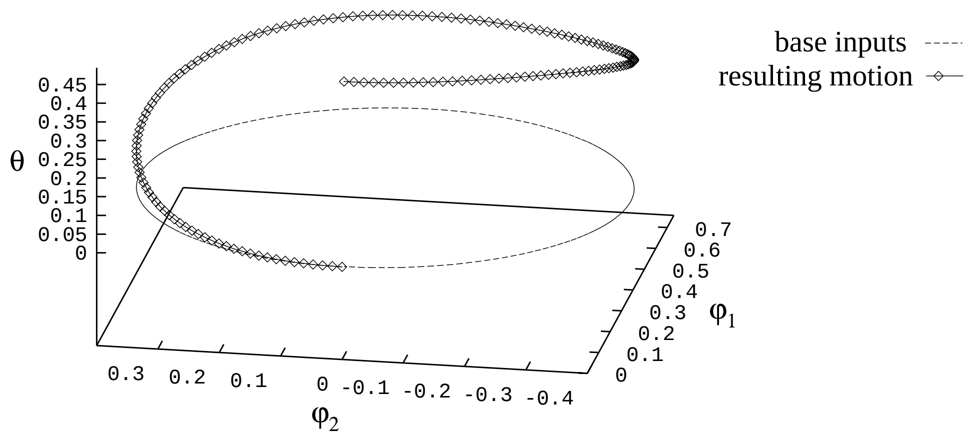

Figure 3.3: Input and output motions for the planar skater.

This gauge theory of mechanics has been successful in simplifying the analysis for a number of important problems in mechanical systems with symmetries (e.g., those governed by momentum conservation laws). Nevertheless, there is another important class of problems that it does not apply to as stated, namely mechanical systems with *rolling constraints*. A prototypical example from this class of problems arises under the constraint assumption that a wheel or ball rolls without slipping on a plane. One very simple idea ties this type of problem to the zero angular momentum constraint problem that was just described. This idea is that of realizing the constraint as the horizontal space of a connection. In fact, the constraint itself defines a connection by declaring the constraint space to be the horizontal space. This, in effect, defines the connection. In the case of rolling constraints, we call this connection the *kinematic* connection to avoid confusion with the mechanical connection described earlier. This point of view for systems with rolling (and rolling-type) constraints was developed by Koiller [1992] and by Bloch, Krishnaprasad, Marsden, and Murray [1996] and Ostrowski [1996]. For locomotion systems the constraints themselves very often possess group symmetries. This additional geometric structure leads to further simplifications of the analysis.

To illustrate the power of these ideas, we note that most car and trailer problems can be formulated in terms of a kinematic connection. Having done so, direct parallels can be drawn between these kinematic systems and the unconstrained mechanical systems described above. In fact, for a car moving under no-slip assumptions, one finds that the motions shown in Figure 3.3 for the planar skater yield strikingly similar output behaviors when considering lateral ("parallel parking") motion of the car. As above, the output motion is directly linked to the curvature of the (kinematic) connection and to the area enclosed by the path in shape space. This again shows that there exist strong connections with geometry at a very basic level. Further evidence of the importance of geometry for locomotion systems has been explored for paramecia (Shapere and Wilczek [1987] and Kelly and Murray [1996], Koiller, Ehlers and Montgomery [1996]), inchworms (Kelly and Murray [1995]), and snakes (Ostrowski and Burdick [1996]).

Things get even more interesting when the system has both rolling constraints and symmetry. Then we have the kinematic connection as well as the symmetry group to deal with, but now they are interlinked. One of the things that makes systems with rolling constraints and symmetries different from free systems is that the law of conservation of angular momentum no longer applies to them. A nice illustration of this can be seen in the motion a famous toy called the rattleback or wobblestone, a canoe shaped piece of wood or plastic that one can purchase in science or game stores (see Figure 3.4).

This toy has the feature that when it rocks on a flat surface like a table, the rocking motion induces a rotational motion. Thus, it can go from zero to nonzero angular momentum about the vertical axis as a result of the interaction between the rocking motion and the rolling constraint with the table. The rattleback also exhibits a preferred spin direction. In one direction, the rattleback spins stably with a constant angular momentum. In the opposite direction, however, it begins to rock (shown simulated

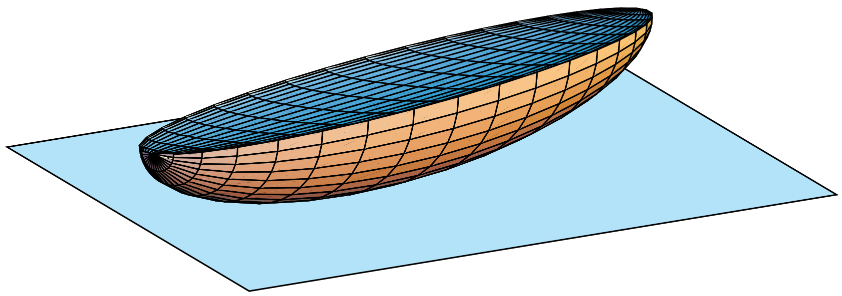

Figure 3.4: The wobblestone or rattleback.

in Figure 3.5 as an oscillation in the angular velocity about a horizontal axis), whereby the angular momentum is altered until the preferred spin direction is achieved (the momentum is also shown in Figure 3.5). A stability analysis of the wobblestone is done in the classical paper of Walker [1896].

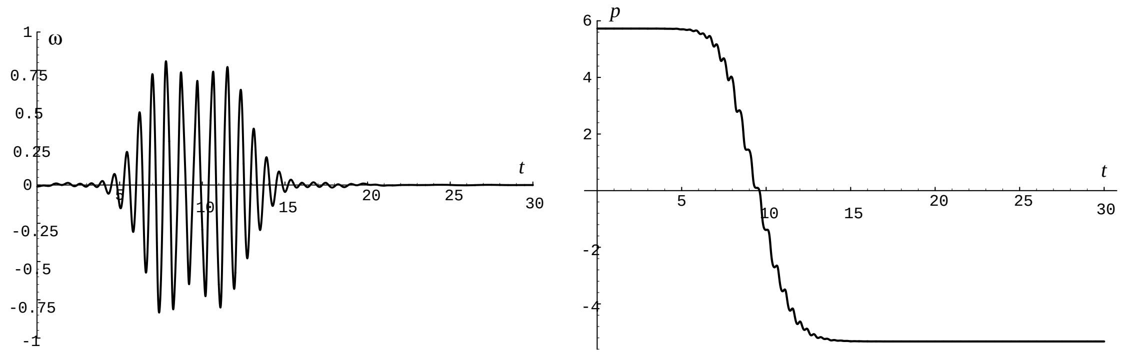

Figure 3.5: Angular velocity, ω, about a principal horizontal axis (rolling) and momentum, p, about the vertical axis (spinning)

Thus, any initial velocities given to this toy will always result in a spin in one particular direction! One can say that it is the forces of constraint that enforce the condition of rolling and upset the balance of angular momentum. This is also the case for the roller racer, a tricycle like toy for children that generates motion with only a periodic movement of the handlebars and no peddling; see Tsakiris and Krishnaprasad [1995] for an analysis. There is a similar non-conservation of linear and angular momentum that occurs for the snakeboard discussed below, and this turns out to be a key point in understanding locomotion generation for this system. One of the key relations to understanding this behavior is that, as shown by Bloch, Krishnaprasad, Marsden, and Murray [1996], there is an attractive equation satisfied by a particular combination of the linear and angular momentum that is called the (nonholonomic) ***momentum equation***. We describe this equation below using the snakeboard to illustrate these ideas.

4 The Snakeboard

The snakeboard is an interesting example of a system in which there is a nontrivial interaction between the forces of constraint and the momentum laws that arise due to symmetries. As such, it has played an important role in understanding the many subtleties that exist when examining the spectrum of mechanical systems with symmetries, ranging from unconstrained mechanical systems to purely kinematic systems.

The *Snakeboard* is a commercially available product marketed by Snakeboard USA, Inc. of Indian

Wells, CA. It is quite similar to a traditional skateboard, with one exception: the front and back pairs of wheels can be rotated independently about their vertical axes. Thus, a rider stands with one foot above each of the wheel bases, and can couple twisting motions of the torso with turning of the feet. The most interesting aspect of this motion is that the rider can begin to move forward without ever having to kick off the ground or crank a pedal, only needing to coordinate the twisting gyrations with the rotation of the wheel axles. The resultant path that is traced out is similar to the serpentine motion of a snake, thus the name *Snakeboard*.

In Figure 4.1 we show a model of the *Snakeboard*, which we will also refer to as the snakeboard, in which the twisting of the torso has been replaced by a rotating inertia wheel.

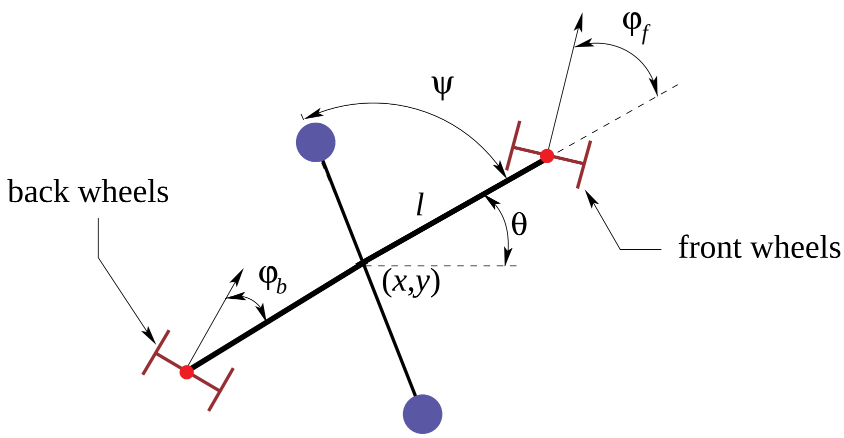

Figure 4.1: A simplified model of the *Snakeboard*

This model has been studied in detail by Ostrowski and Burdick [1995] and Bloch, Krishnaprasad, Marsden and Murray [1996]. From the theoretical point of view, the feature of the snakeboard that sets it truly apart from examples like the planar skater and the falling cat is that even though it has the symmetry group of rotations and translations of the plane, the linear and angular momentum is not conserved. Recall that for the planar skater, no matter what motions the arms of the device make, the values of the linear and angular momentum cannot be altered. Thus, while it is possible to change the orientation of the planar skater, once the internal shape motions stop, the orientation changes also stop. This is not true for the snakeboard—the ability to build up momentum can be traced to the presence of the forces of constraint, just as in the rattleback mentioned earlier. Thus, one might suspect that one should abandon the ideas of linear and angular momentum for the snakeboard. However, a deeper inspection shows that this is not the case. In fact, one finds that there is a particular choice of momentum, roughly corresponding to the angular momentum about the instantaneous center of rotation (the point O in Figure 4.2), that preserves the geometric structure of the problem.

If we call this component the *nonholonomic momentum*, p, one finds that due to the translational and rotational invariance of the whole system, there is a *nonholonomic momentum equation* governing the evolution of p of the form

$$\frac{d}{dt}p = f(r, \dot{r}, p), \tag{4.1}$$

where r represents the internal variables of the system (the three angles shown in the preceding figure). This is one of the very interesting results of the analysis of Bloch, Krishnaprasad, Marsden and Murray [1996], where it was also shown that equation (3.1) for the connection remains valid using the nonholonomic momentum. An important point to recognize is that equation (4.1) does not depend on the rotational and translational position of the system, i.e., there is no explicit g dependence. Thus, if one has a given internal motion, this equation can be solved for p and from it, the attitude and position of the snakeboard calculated by means of another integration using equation (3.1). This strategy thus

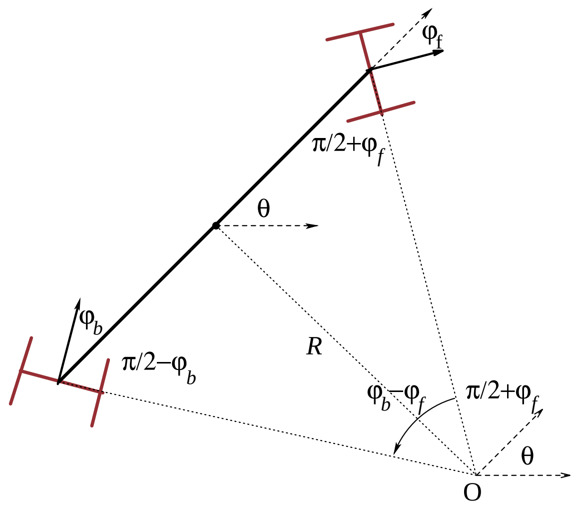

Figure 4.2: The angular momentum about the point O plays an important role in the analysis of the snakeboard.

parallels that used to study the falling cat and the planar skater.

The snakeboard moves by coupling periodic motions of the rotor and wheel axles. Similarly, the kinematic snake, kinematic car, and planar skater each move using coupled internal motions. These are examples of *gaits*, which more generally can be thought of as cyclic patterns of internal shape changes which result in a net displacement. Different gaits correspond to different cyclic input patterns. For example, the snakeboard possesses at least three primary gait patterns, shown in Figures 4.3–4.5. We shall discuss these gaits in more detail in the following paragraph (see for more details, Ostrowski and Burdick [1995]). Research is currently being conducted into the effects of energy dissipation mechanisms on the character of locomotion gaits. It is hoped that this will reveal additional insight into the steady-state nature of cyclic gait patterns.

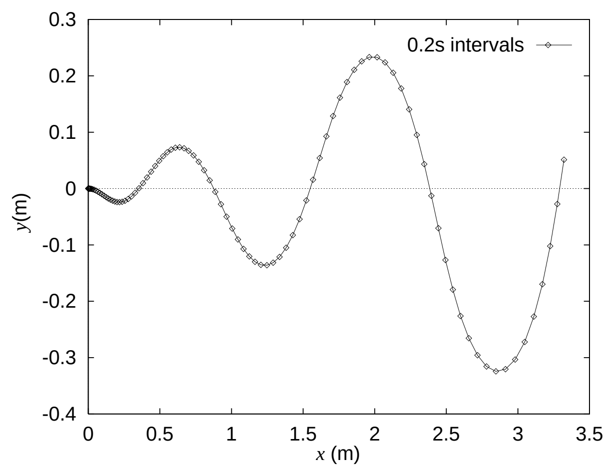

Figure 4.3: Center of mass position (drive gait)

Fig. 4.3 shows the position of the snakeboard's center of mass versus time for the case in which the rotor and wheel axles oscillate with the same frequency (which we term the "drive gait"). This gait closely resembles the motion followed by riders of the *Snakeboard* when they begin moving from a resting position. Fig. 4.4 shows a second possible gait, not yet discovered by *Snakeboard* riders, but which is easily demonstrated by a robotic version that has been built. In this case, the rotor oscillates

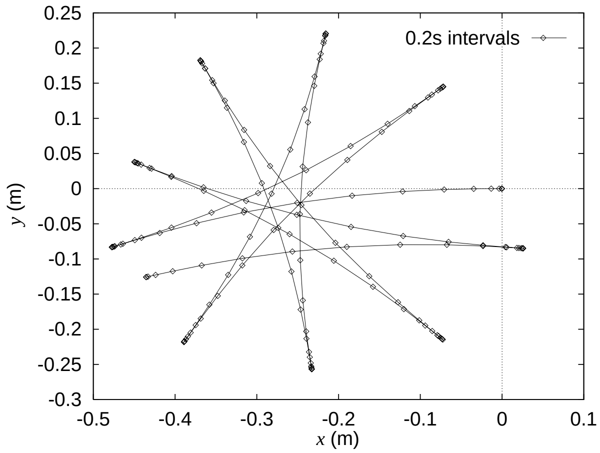

Figure 4.4: Position of the center of mass for the “rotate gait”

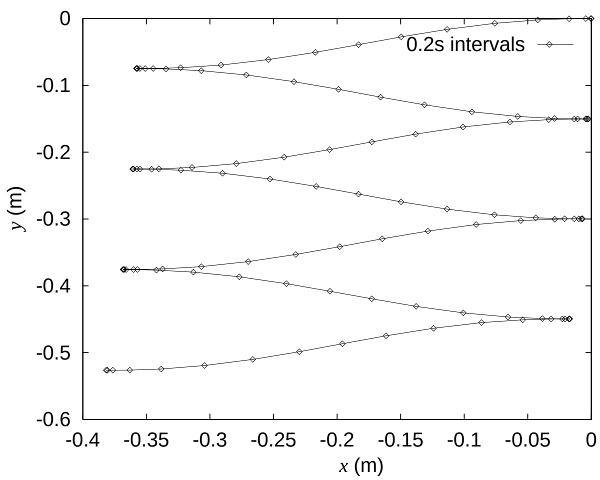

Figure 4.5: Position of the center of mass for the “parallel parking gait”

at twice the frequency of the axles (the “rotate gait,” as the robot essentially rotates in place). Finally, in Fig. 4.5 we show the path followed when generating the third direction of motion (having already generated forward translation and rotation). This gait is the most complicated, requiring the rotor to oscillate three times for every two oscillations of the axles (the “parallel parking gait”).

We do not yet have a complete geometric understanding of the notion of gaits. In general, the net displacement of the mechanism that arises from periodic inputs is the *geometric phase*, or holonomy, associated with the connection. The geometric phase is that part of the motion described by the local form of the connection, A, in equation (3.1). In the case of the snakeboard, however, the net displacement is a non-trivial combination of the geometric phase and the dynamic phase. Understanding the increased complexity of the relationship between geometric and dynamic phases for this class of systems is the subject of current research.

5 Stability, Controllability, and Optimal Control

Control theory adds to the study of dynamical systems the idea that in many instances, one can directly intervene in the dynamics rather than passively watching. For example, while Newton’s equations govern the dynamics of a satellite, we can intervene in these dynamics by controlling onboard gyroscopes, thrusters, or rotors. Quite often, control engineers are tempted to overwhelm the intrinsic dynamics of a system with the controls. However, in many circumstances (for example, fluid control in Bloch and Marsden [1989]) one needs to work with the intrinsic dynamics and make use of its structure.

Two of the basic notions in control theory involve steering and stabilizability. Steering has, as its objective, the design of control inputs that guide the system from an initial position to a desired final position, perhaps following a predefined path. One imagines manipulating the control to achieve this, much the way one steers a car so that the desired final state is achieved. Already this type of question has received much attention and many important and basic questions have been solved. For example, two of the main themes that have developed are the Lie algebraic techniques based on brackets of vector fields (in driving a car, you can repeatedly make two alternating steering motions to produce a motion in a third direction) and the second based on the application of exterior differential systems (a subject invented by Elie Cartan in the mid 1920s whose power is only now being significantly tapped in control theory). The work of Tilbury, Murray, and Sastry [1993], and Walsh and Bushnell [1993], and Leonard and Krishnaprasad [1994] typify some of the modern applications of these ideas.

The analysis of these systems using connections has provided new geometric insight into the generation of control, particularly for locomotion systems. The fact that the same type of input-output relationships characterized by an area rule exist for both the planar skater and the kinematic car is no coincidence! The underlying geometric structure of both these examples is significantly influenced by the presence of group symmetries, which leads to their formulation in terms of a connection. As the snakeboard example further illustrates, this geometric structure provides a great deal of insight into the types of controls that should be used to generate desired motions. Already, tests exist for determining whether or not a given system is controllable, based purely on the information encoded in the local form of the connection and within the nonholonomic momentum equation. Current work focuses on continuing to exploit this structure in order to generate more constructive controllability tests which can be used to do motion planning.

The problem of stabilizability has also received much attention. Here the goal is to take a dynamic motion that might be unstable if left to itself, but which can be made stable through intervention. For example, certain fighter planes can fly in an unstable (forward-swept wing) mode, but which, through delicate control are stabilized. Flying in this mode has the advantage that one can execute tight turns with rather little effort—just appropriately remove the controls! The situation is really not much different from what people do everyday when they ride a bicycle. Interesting systems of this sort are often *balance systems*. Connections can be used to search for stabilizing controls for balance systems, for example, those that would control the onboard gyroscopes in a spacecraft to stabilize the otherwise unstable steady motion about its middle axis; see Bloch, Krishnaprasad, Marsden, and Sanchez [1992] and Kammer and Gray [1993]. Recent work (see, for example, Bloch, Marsden and Sanchez [1996] and Leonard [1996]) is developing a deeper understanding of the problem of stabilization of balance systems. On one level, the approach is to use our knowledge about the source of the instabilities, such as having a saddle point of the effective energy function and knowing this, to design controllers that, in effect, create an energy extremum to achieve the stabilization. Technically, one designs *controlled Lagrangians* to achieve this and makes use of the powerful stability theory available for mechanical systems. In the presence of symmetries this becomes a very interesting endeavor because then one must also deal with geometric phases that can can cause desired or undesired phase drifts.

As might be expected, the geometric and dynamic formulations described above, including the use of connections, yield valuable tools for analyzing both stability and controllability. For example, the instability of a rigid body, e.g., a satellite, about its intermediate axis of inertia is a well-known physical principle. When one takes into account small dissipative effects such as a vibrating antenna, then the rotational motion about the long axis becomes unstable as well, but this effect is more delicate. Corresponding statements for systems like rigid bodies with flexible appendages, or interconnected rigid bodies is more subtle than the dynamics of a single rigid body. There is a powerful method for determining the stability of such solutions called the energy-momentum method. This method is an outgrowth of basic work of Riemann, Poincaré, and others in the last century and more recently by Arnold; further recent developments were made by Simo, Lewis, and Marsden [1991] and Bloch, Krishnaprasad, Marsden, and Ratiu [1994, 1996], Zenkov [1995] and references therein. Key to this method is the ability to reduce the analysis to studying internal shape changes and their effect on rotational and translational motions. The mechanical connection plays a key role in the solution of this problem and it makes many otherwise intractable problems soluble. Recent investigations of Bloch, Marsden, and Zenkov are pointing towards a very interesting generalization of the energy-momentum method to nonholonomic systems.

Another issue of importance in control theory is that of optimal control. Here one has a cost function (literally think of how much you have to pay to have a motion occur in a certain way). The question is not just if one can achieve a given motion but how to achieve it with the least cost. There are many well developed tools to attack this question, the best known of these being the *Pontryagin Maximum Principle*. For mechanical systems with symmetry there are alternative direct approaches using Lagrangian methods that yield the same results. In the context of problems like the falling cat, there is a remarkable consequence of optimality, namely that relative to an appropriate cost function, the optimal trajectory in the base space is the trajectory of a Yang-Mills particle. The equations for a Yang-Mills particle are a generalization of the classical Lorenz equations for a particle with charge e in a magnetic field B:

$$\frac{d}{dt}v = \frac{e}{c}v \times B,$$

where v is the velocity of the particle and where c is the velocity of light. The mechanical connection comes into play though the general formula for the curvature of a connection, generalizing the formula $B = \nabla \times A$ expressing the magnetic field as the curl of the magnetic potential. This remarkable link between optimal control and the motion of a Yang-Mills particle is due to Montgomery [1990, 1991a].

Reduction methods, especially Lagrangian ones, have proven to be quite important in making explicit calculations of optimal trajectories. These methods often lead to a significant reduction in the total number of variables needed to represent the system. This can help make the problem computationally feasible by removing the redundant information encoded in the connection to describe the invariances and symmetries of the system, thereby reducing the number of required calculations.

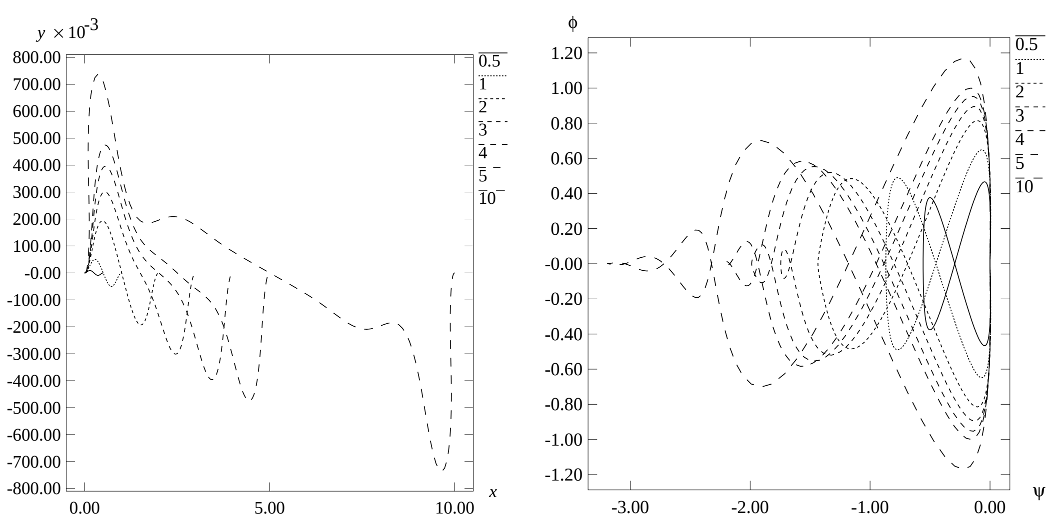

Figure 5.1: Optimal trajectories for the snakeboard "drive" gait. Shown are (left) the center of mass positions (x, y) and (right) the motion of the shape variables, ϕ and ψ.

Investigations of optimal control for systems such as the snakeboard are currently underway (Koon and Marsden [1996], and Ostrowski, Desai, and Kumar [1996]), and are already providing new insight into these systems. To illustrate how optimal controls can be used to determine appropriate input motions, we include here a small sample of two optimal trajectories generated for the snakeboard. In simulating these motions, we have used a Lagrange multiplier technique to turn this constrained optimization problem into an unconstrained variational problem, and have used as our cost function the total energy input into the system. Figure 5.1 shows the optimal trajectories for moving forward a variable distance in the x direction. In each trial, the snakeboard starts and ends with zero velocity. Also shown in this figure are the optimal inputs used to generate this motion. It is interesting to note the apparent gait transition that occurs as the desired final position is increased. In particular, notice the change in the geometry of the input loops, as the number of self-intersections gradually changes from one to four.

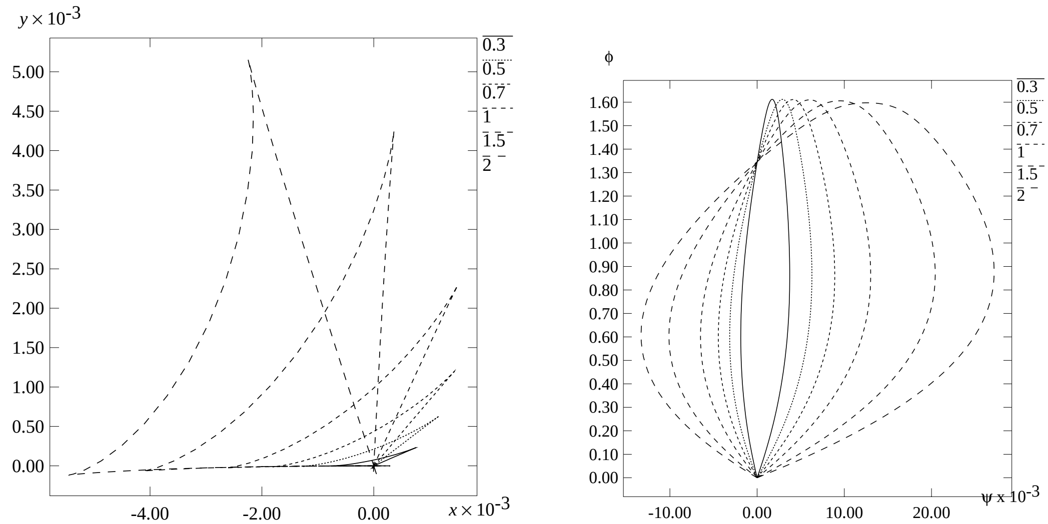

Figure 5.2: Optimal trajectories for the snakeboard "rotate" gait. Shown are (left) the center of mass positions (x, y) and (right) the motion of the shape variables, ϕ and ψ.

Figure 5.2 contains the optimal trajectories used to generate a purely rotational motion (in θ), with zero initial and final momentum. In this motion, the geometry of the inputs remains relatively unchanged. Notice, however, the scales of the motions, as the robot essentially moves in place. This motion directly parallels the motion described above by which an astronaut can rotate in space. Unlike the astronaut, however, the momentum of the snakeboard is not fixed and must be taken into account when planning any motions. Current work includes extending these methods to other systems, performing a dimensional analysis to try to expose the parameters most pertinent to gait transitions, and investigating the effect of constraining the set of allowable inputs, e.g., using only a sinusoidal input and several of its harmonics. It is hoped that this research will expose basic relationships between periodic inputs and optimal trajectories.

Conclusions

The tools of differential geometry, especially the theory of connections and principal fiber bundles have proven to be extremely valuable in studying a wide spectrum of problems in locomotion, control and stabilization. From satellites in space to mobile robots and bicycles moving overland, to the microfluidic undulations of paramecia, these tools have provided new insights into motion generation by providing a powerful geometric framework in which to understand the essential mechanics, dynamics and control theory. This theory has a rich history that provides a significant bridge between the mathematical subjects of geometry and dynamical systems with mechanics and control theory. At the same time, this is an active topic of research that continues to lead to new and useful, and sometimes surprising, discoveries. While the necessity of having to learn topics from more than one area is a challenge, the rewards are well worth it.

Further Information

The list of references provides a large sample of related work in this area. For additional material related to this paper, including MPEG movies of the snakeboard, a postscript copy of the paper by Bloch, Krishnaprasad, Marsden and Murray [1996], and current work in optimal control, the reader is referred to

http://www.cis.upenn.edu/~jpo/papers.html

and

http://cds.caltech.edu/~marsden/

Acknowledgments

Thanks are extended to John Tucker, Tony Bloch, Roger Brockett, Joel Burdick, Jaydev Desai, Bill Goodwine, Scott Kelly, P.S. Krishnaprasad, Vijay Kumar, Naomi Leonard, Andrew Lewis, Richard Montgomery, Richard Murray, Tudor Ratiu, Shankar Sastry, Greg Walsh, and Jeff Wentlandt for their kind advice and help. Thanks to Joel Burdick and Bill Goodwine for help in the preparation of Figure 3.2. Also, thanks to Andy Ruina and an anonymous reviewer for providing suggestions that helped improve this paper.

References

Abraham, R. and J.E. Marsden [1978] *Foundations of Mechanics.* Second Edition, Addison-Wesley.

Berry, M. [1984] Quantal phase factors accompanying adiabatic changes, *Proc. Roy. Soc. London A* **392**, 45–57.

Berry, M. [1985] Classical adiabatic angles and quantal adiabatic phase, *J. Phys. A. Math. Gen.* **18**, 15–27.

Berry, M. [1990] Anticipations of the geometric phase. *Physics Today*, December, 1990, 34–40.

Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden, and R. Murray [1996] Nonholonomic mechanical systems with symmetry. *Arch. Rat. Mech. An.*, **136**, 21–99.

Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden and T.S. Ratiu [1994] Dissipation Induced Instabilities, *Ann. Inst. H. Poincaré, Analyse Nonlineare* **11**, 37–90.

Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden, and T.S. Ratiu [1996] The Euler-Poincaré equations and double bracket dissipation. *Comm. Math. Phys.* **175**, 1–42.

Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden and G. Sánchez de Alvarez [1992] Stabilization of rigid body dynamics by internal and external torques, *Automatica* **28**, 745–756.

Bloch, A.M., N. Leonard and J.E. Marsden [1997] Stabilization of Mechanical Systems Using Controlled Lagrangians, *Proc CDC* **36**, 2356–2361.

Bloch, A.M. and J.E. Marsden [1989] Controlling homoclinic orbits, *Theoretical and Computational Fluid Mechanics* **1**, 179–190.

Bloch, A.M., J.E. Marsden and G. Sánchez [1997] Stabilization of relative equilibria of mechanical systems with symmetry, *Current and Future Directions in Applied Mathematics*, Edited by M. Alber, B. Hu, and J. Rosenthal, Birkhäuser, 43–64.

Bobenko, A.I., A.G. Reyman and M.A. Semenov-Tian-Shansky [1989] The Kowalewski Top 99 years later: A Lax pair, generalizations and explicit solutions, *Comm. Math. Phys.* **122**, 321–354.

Brockett, R.W. [1981] Control theory and singular Riemannian geometry, in *New Directions in Applied Mathematics*, P.J. Hilton and G.S. Young (eds.), Springer.

Brockett, R.W. [1989] On the rectification of vibratory motion, *Sensors and Actuators* **20**, 91–96.

Crouch, P. E. [1984] Spacecraft attitude control and stabilization: Application of geometric control theory to rigid body models. *IEEE Transactions on Automatic Control*, **AC-29**, 321–331.

Enos, M. (Ed.) [1993] *Dynamics and Control of Mechanical Systems: The Falling Cat and Related Problems.* Fields Institute Communications, Vol. 1, American Mathematical Society.

Getz, N.H. and J. E. Marsden [1995] Control for an autonomous bicycle, *International Conference on Robotics and Automation, IEEE*, Nagoya, Japan, May, 1995.

Gozzi, E. and W.D. Thacker [1987] Classical adiabatic holonomy in a Grassmannian system. *Phys. Rev. D* **35**, 2388–2396.

Guichardet, A. [1984] On rotation and vibration motions of molecules, *Ann. Inst. H. Poincaré* **40**, 329–342.

Hannay, J. [1985] Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian, *J. Phys. A: Math. Gen.* **18**, 221–230.

Hirose, S. [1993] *Biologically Inspired Robots: Snake-like Locomotors and Manipulators*, Oxford University Press, Oxford. Translated by Peter Cave and Charles Goulden.

Iwai, T. [1987] A geometric setting for classical molecular dynamics, *Ann. Inst. Henri Poincaré, Phys. Th.* **47**, 199–219.

Kane, T.R. and M. Scher [1969] A dynamical explanation of the falling cat phenomenon, *Int. J. Solids Structures* **5**, 663–670.

Koiller, J. [1992] Reduction of some classical nonholonomic systems with symmetry, *Arch. Rat. Mech. An.* **118**, 113–148.

Krishnaprasad, P.S. [1989] Eulerian many-body problems, *Cont. Math. AMS* **97**, 187–208.

Kammer, D. C., and G.L. Gray [1993] A nonlinear control design for energy sink simulation in the Euler-Poinsot problem. *J. Astron. Sci.* **41**, 53–72.

Koiller, J., K. Ehlers, and R. Montgomery [1996] Problems and progress in microswimming, *J. Nonlin. Sci.*. **6**, 507–541.

Koon, W.S. and J.E. Marsden [1997] Optimal control for holonomic and nonholonomic mechanical systems with symmetry and Lagrangian reduction. *SIAM J. Control and Optim.* **35**, 901–929.

Kummer, M. [1981] On the construction of the reduced phase space of a Hamiltonian system with symmetry, *Indiana Univ. Math. J.* **30**, 281–291.

Leonard, N.E. [1995a] Control synthesis and adaptation for an underactuated autonomous underwater vehicle, *IEEE J. of Oceanic Eng.* **20**, 211–220.

Leonard, N.E. [1995b] Periodic forcing, dynamics and control of underactuated spacecraft and underwater vehicles, *Proc. 34th IEEE Conf. Decision Ctrl.*, 3980–3985.

Leonard, N.E. [1996] Stabilization of underwater vehicle dynamics with symmetry breaking potentials, *in preparation.* (See also Leonard, N.E., Stabilization of steady motions of an underwater vehicle, *Proc. 35th IEEE Conference on Decision and Control*, Kobe, Japan, 1996.)

Leonard, N.E. [1997] Stability of a bottom-heavy underwater vehicle, *Automatica J. IFAC* **33**, 331–346.

Leonard, N.E. and P.S. Krishnaprasad [1995] Motion control of drift-free, left-invariant systems on Lie groups, *IEEE Trans. Automat. Control* **40**, 1539–1554.

Leonard, N.E. and J.E. Marsden [1997] Stability and drift of underwater vehicle dynamics: mechanical systems with rigid motion symmetry, *Physica D* **105**, 130–162.

Lewis, A., R.M. Murray, J.P. Ostrowski and J. Burdick [1994] Nonholonomic mechanics and locomotion: The snakeboard example. *IEEE Intern. Conf. on Robotics and Automation, to appear.*

Marsden, J.E., R. Montgomery and T.S. Ratiu [1990] *Reduction, symmetry, and phases in mechanics.* Memoirs AMS **436**.

Marsden, J.E. and T.S. Ratiu [1994] *An Introduction to Mechanics and Symmetry.* Texts in Applied Mathematics, Springer-Verlag, NY.

MacCullagh, J. [1840] On the rotation of a solid body, *Proc. Roy. Irish Acad. Dublin*, **2**, 520–525, 542–545, **3**, 370–371.

Montgomery, R. [1988] The connection whose holonomy is the classical adiabatic angles of Hannay and Berry and its generalization to the non-integrable case. *Comm. Math. Phys.* **120**, 269–294.

Montgomery, R. [1990] Isoholonomic problems and some applications. *Comm. Math Phys.* **128**, 565–592.

Montgomery, R. [1991a] *Optimal Control of Deformable Bodies and Its Relation to Gauge Theory* in The Geometry of Hamiltonian Systems, T. Ratiu ed., Springer-Verlag.

Montgomery, R. [1991b] How much does a rigid body rotate? A Berry's phase from the 18^{th} century. *Am. J. Phys.* **59**, 394–398.

Murray, R.M. and S. S. Sastry [1993] Nonholonomic Motion Planning: Steering Using Sinusoids. *IEEE Trans. on Automatic Control*, **38**, 700–716.

Nakamura, Y. and R. Mukherjee [1993] Exploiting nonholonomic redundancy of free-flying space robots. *IEEE Transactions of Robotics and Automation*, **9(4)**, 499–506.

Oh, Y.G., N. Sreenath, P.S. Krishnaprasad and J.E. Marsden [1989] The dynamics of coupled planar rigid bodies Part 2: Bifurcations, periodic solutions, and chaos, *Dynamics and Diff. Eq'ns.* **1**, 269–298.

Ostrowski, J. [1996] *Geometric Perspectives on the Mechanics and Control of Undulatory Locomotion*, PhD dissertation, California Institute of Technology. Available electronically as ftp://avalon.caltech.edu/pub/cds/techreports/postscript/cds95-027.ps.gz.

Ostrowski and Burdick [1996] Gait kinematics for a serpentine robot, *IEEE Int. Conf. on Robotics and Automation*, 1294-9, Minneapolis, April, 1996.

Ostrowski, J., J. W. Burdick, A. D. Lewis & R. M. Murray [1995] The mechanics of undulatory locomotion: The mixed kinematic and dynamic case. *IEEE Int. Conf. on Robotics and Automation*, 1945-1951, Nagoya, Japan, May, 1995.

Ostrowski, J., J. P. Desai, and V. Kumar. Optimal gait selection for nonholonomic locomotion systems. Submitted to the *IEEE Conf. on Robotics and Automation*, September 1996.

Rand, R., A. H. Cohen, and P. J. Holmes. Systems of coupled oscillators as models of central pattern generators. In *Neural Control of Rhythmic Movements in Vertebrates*, A. H. Cohen, S. Rossignol, and S. Grillner, editors, pages 333–367. Wiley, New York, 1988.

Shapere, A. and F. Wilczek [1987] Self-propulsion at low Reynolds number, *Phys. Rev. Lett.* **58**, 2051-2054 and Geometry of self-propulsion at low Reynolds number, *J. Fluid Mech.* [1989], **198**, 557–585.

Simo, J.C., D.R. Lewis and J.E. Marsden [1991] Stability of relative equilibria I: The reduced energy momentum method. *Arch. Rat. Mech. Anal.* **115**, 15-59.

Simon, B. [1983] Holonomy, the Quantum Adiabatic Theorem, and Berry's Phase. *Phys. Rev. Letters* **51**, 2167–2170.

Smale, S. [1970] Topology and Mechanics, *Inv. Math.* **10**, 305–331, **11**, 45–64.

Sreenath, N., Y.G. Oh, P.S. Krishnaprasad and J.E. Marsden [1988] The dynamics of coupled planar rigid bodies. Part 1: Reduction, equilibria and stability, *Dyn. and Stab. of Systems* **3**, 25–49.

Thomson, W (Lord Kelvin) and P.G. Tait [1879] *Treatise on Natural Philosophy.* Cambridge University Press.

Tilbury, D., R. Murray, and S. Sastry [1993] Trajectory generation for the N-trailer problem using Goursat normal form. *Proc. IEEE Control and Decision Conf, San Antonio, Dec. 1993*.

Walker, G.T. [1896] On a dynamical top. *Quart. J. Pure Appl. Math.* **28**, 175–184.

Walsh, G. and L. Bushnell [1993] Stabilization of multiple input chained form control systems. *Proc. IEEE Control and Decision Conf, San Antonio, Dec. 1993*.

Walsh, G. and S. Sastry [1995] On reorienting linked rigid bodies using internal motions. *IEEE Trans. on Robotics and Automation*, **11(1)**, 139–146.

Weinstein, A. [1990] Connections of Berry and Hannay type for moving Lagrangian submanifolds. *Adv. in Math.* **82**, 133–159.

Zenkov, D.V. [1995] The Geometry of the Routh Problem, *J. Nonlinear Sci.* **5**, 503–519.

Zenkov, D.V., A.M. Bloch, and J.E. Marsden [1997] The Energy Momentum Method for the Stability of Nonholonomic Systems *Dyn. Stab. of Systems.*, (to appear).