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Branches of stable three-tori using Hamiltonian methods in Hopf bifurcation on a rhombic lattice

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Abstract. *This paper uses Hamiltonian methods to find and determine the stability of some new solution branches for an equivariant Hopf bifurcation on* $\mathbb{C}^4$. *The normal form has a symmetry group given by the semi-direct product of* D_2 *with* $T^2 \times S^1$. *The Hamiltonian part of the normal form is completely integrable and may be analyzed using a system of invariants. The idea of the paper is to perturb relative equilibria in this singular Hamiltonian limit to obtain new three-frequency solutions to the full normal form for parameter values near the Hamiltonian limit. The solutions obtained have fully broken symmetry, that is, they do not lie in fixed point subspaces. The methods developed in this paper allow one to determine the stability of this new branch of solutions. An example shows that the branch of three-tori can be stable.*

1 Introduction

A standard approach in the bifurcation analysis of spatially extended systems (such as Rayleigh–Benard convection in an infinite plane) is to restrict to a class of functions that is spatially periodic with respect to a hexagonal, square or rhombic lattice. The restricted equations have symmetry group $D_n \times_s T^2$, where $n = 6$, 4 or 2 respectively, and where $\times_s$ denotes semi-direct product. In the event of a Hopf bifurcation, there is an additional S^1 symmetry in the normal form. In this paper,

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we consider Hopf bifurcation on a rhombic lattice. Center manifold reduction leads to a bifurcation problem on $\mathbb{C}^4$ which in normal form has $(D_2 \times_s T^2) \times S^1$ symmetry. This bifurcation problem arises, for example, in the study of spatially periodic travelling waves in anisotropic systems (Silber *et al.*, 1992), oscillatory magnetoconvection (Clune & Knobloch, 1994) and transverse patterns in lasers (Feng *et al.*, 1994).

In this paper, we develop Hamiltonian methods to prove the existence and stability of a primary branch of three-tori in the $D_2 \times_s T^2$-equivariant Hopf bifurcation problem. Because this bifurcation problem is eight-dimensional and complicated, a complete analysis of the normal form has not been possible. However, the methods of equivariant bifurcation theory (Golubitsky *et al.*, 1988) have yielded a fairly complete analysis of the dynamics on the two- and four-dimensional invariant subspaces (Silber & Knobloch, 1991; Silber *et al.*, 1992). The solutions obtained in this way retain some of the symmetries of the normal form. Solutions that completely break the symmetry of the bifurcation problem do not lie in an invariant subspace. One class of non-symmetric solutions that is known to exist consists of structurally stable heteroclinic cycles (Silber *et al.*, 1992). These solutions are composed of saddle-type periodic orbits and heteroclinic connections between the periodic orbits; each connection lies entirely within a four-dimensional invariant subspace, but the heteroclinic cycle as a whole does not reside in any invariant subspace. Little else is known about non-symmetric solutions of this bifurcation problem, although Knobloch and Silber (1992) conjectured the existence of non-symmetric three-tori, and Clune and Knobloch (1994) presented numerical evidence for the existence of these solutions.

The branch of three-tori that we investigate in this paper are non-symmetric relative equilibria. By relative equilibrium, we mean an orbit $\mathbf{z}(t) \in \mathbb{C}^4$ that has the form $\mathbf{z}(t) = \exp(t\xi)\mathbf{z}(0)$ where $\exp(t\xi)$ is a one-parameter subgroup of the continuous part of the symmetry group. If X is our $T^2 \times S^1$-equivariant vector field on $\mathbb{C}^4$, then X induces a vector field $[X]$ on the quotient (orbit) space $\mathbb{C}^4/(T^2 \times S^1)$, excluding the singular points in the quotient; the relative equilibria of X project to fixed points for the dynamics of $[X]$. It is, in principle, possible to find relative equilibria in the full problem by looking for fixed points of $[X]$. With this direct approach, one would prove the existence of a primary branch of three-tori by checking that the candidate fixed point of $[X]$ is non-symmetric and persists to the bifurcation point. For our problem, this method turns out to be intractable in all its generality.

Here we develop a systematic approach to the problem that exploits the integrability of a Hamiltonian limit, where the integrability is a consequence of the symmetries. The Hamiltonian problem is relatively easy to analyze, because the methods of Hamiltonian reduction (see, for example, Marsden, 1992) effectively allow us to reduce the dynamics to two-dimensions. We adopt a particular realization of this reduction that uses invariants due to Kummer (1986, 1990). This technique is compatible with the orbit space reduction of the original non-Hamiltonian problem.

In the Hamiltonian limit for the problem we study, non-symmetric solutions appear as three- and four-tori, and orbits homoclinic to the three-tori. This limit provides a natural starting point for perturbatively proving existence of three-tori. Such a perturbative approach is delicate because the bifurcation parameter vanishes in the Hamiltonian limit and three-tori are not isolated. We remove the problem with the bifurcation parameter by using ideas of Field and Swift (1994) to decouple the dependence of the dynamics on the bifurcation parameter. This is achieved by means of a rescaling of time and the dynamic variables. Crucial to the success of this is that the decoupling is done before introducing the Hamiltonian limit. The difficulty with non-isolated three-tori is averted by rewriting the defining relative equilibrium equations in such a way that we can apply the implicit function theorem.

Once the branch of three-tori is shown to exist in the normal form, one can use a result of Field (1996) to deduce that the branch persists even when the S^1 symmetry, introduced in the process of reduction to the normal form, is removed. Thus, we conclude that a primary branch of three-tori exists for the $D_2 \times_s T^2$-equivariant Hopf bifurcation problem.

In the general situation, computing the stability of the three-tori is a daunting task because, even for the reduced problem, there are four eigenvalues to be found and no remaining symmetries to simplify the calculation. For our problem, the Hamiltonian limit is again useful, because we can calculate two eigenvalues using an eigenvalue movement formula for nearly Hamiltonian systems. Once the movement of two eigenvalues is known, the remaining two eigenvalues can be calculated directly.

In many systems for which one might try to perform a reduction using invariants, there can be singularities in the orbit space, and these singularities are often very important; see, for example, Haller and Wiggins (1993, 1996) and references therein. Although singularities occur in our problem, we shall not be concerned with them since the relative equilibria we locate occur at non-singular points of the reduced space.

From a broad perspective, the questions raised in this paper concern what information about a dynamical system can be obtained from its 'Hamiltonian part'. Similar questions were addressed in Lewis and Marsden (1989) who studied the decomposition of normal forms into Hamiltonian and non-Hamiltonian parts; in the planar case, they showed that a surprising amount of dynamic information was nascent in the Hamiltonian part. Another paper in this spirit is Knobloch *et al.* (1994), where the role of Hamiltonian normal forms in the problem of system symmetry breaking and symmetry-induced instabilities in certain fluid problems was studied. The approach taken in the present paper is somewhat similar to that used by van Gils and Silber (1995). They proved uniqueness, near a Hamiltonian limit, of a branch of quasi-periodic solutions in the Hopf bifurcation problem with $D_4 \times S^1$ symmetry. The existence of this solution branch had been shown by Swift (1988), who conjectured its uniqueness.

The outline of the paper is as follows. In Section 2, we describe the specific normal form we will study, its invariants and the quotient dynamics. We calculate the five-dimensional dynamical system induced on the space of invariants (there are six invariants with one relation between them). An additional reduction to a four-dimensional system (five variables and one relation) is obtained by a rescaling of the variables and time. We calculate the Hamiltonian limit within this context and give a partial analysis of the corresponding dynamics. Appendix A contains information about the Hamiltonian structure of the problem.

In Section 3, we show that there exist branches of three-tori in certain parameter regimes for the full problem. To do this, we use direct algebraic manipulation to find particular equilibria of the reduced (four-dimensional) dynamics and then apply the implicit function theorem to an associated system to show that these equilibria persist under non-Hamiltonian perturbations. Using the framework developed in Section 2, we deduce that the equilibria correspond to branches of three-tori in the full problem.

In Section 4, we perform a stability analysis of the solutions found. The Hamiltonian case can be analyzed completely. When we perturb from the Hamiltonian limit, we use an eigenvalue movement formula derived in Appendix B to compute the movement of two of the eigenvalues of the system, and show how to calculate the remaining eigenvalues using direct techniques. We illustrate the approach by looking at a specific example. A short summary of our method is given in Section 5.

2 Preliminaries

2.1 *The normal form*

The following action of $T^2 \times S^1$ on $\mathbb{C}^4$ occurs in Hopf bifurcation problems with $D_n \times_s T^2$ symmetry ($n=2, 4$):

$$\begin{aligned}(\theta_1, \theta_2) &: \mathbf{w} \mapsto (e^{i\theta_1}w_1, e^{i\theta_2}w_2, e^{-i\theta_1}w_3, e^{-i\theta_2}w_4)\\ \phi &: \mathbf{w} \mapsto e^{i\phi}\mathbf{w}\end{aligned} \tag{1}$$

where $\mathbf{w}=(w_1, w_2, w_3, w_4) \in \mathbb{C}^4$, $(\theta_1, \theta_2) \in T^2$, and $\phi \in S^1$. Moreover, in the case that $D_n = D_6$, this action applies in an eight-dimensional invariant subspace of the twelve-dimensional equivariant bifurcation problem.

The action (1) arises naturally in the study of Hopf bifurcation to spatially periodic travelling waves in spatially extended systems. For this class of problems, w_1 and w_3 are the amplitudes of spatial Fourier modes

$$\exp(i\mathbf{k}_1 \cdot \mathbf{x}), \qquad \exp(-i\mathbf{k}_1 \cdot \mathbf{x})$$

respectively, and w_2, w_4 are amplitudes of

$$\exp(i\mathbf{k}_2 \cdot \mathbf{x}), \qquad \exp(-i\mathbf{k}_2 \cdot \mathbf{x})$$

respectively, where $\mathbf{k}_1, \mathbf{k}_2 \in \mathbb{R}^2$ are linearly independent, and $|\mathbf{k}_1|=|\mathbf{k}_2|$. The T^2 symmetry arises from spatial translations perpendicular to $\mathbf{k}_1$ and $\mathbf{k}_2$, and S^1 is the normal form symmetry associated with Hopf bifurcation. The discrete group D_n is related to the angle $\phi \in (0, \pi/2]$ between $\mathbf{k}_1$ and $\mathbf{k}_2$; $n=4$ if $\phi=\pi/2$, $n=6$ if $\phi=\pi/3$, and $n=2$ otherwise.

In this paper, we consider the case where the discrete group is D_2. It is generated by the following two reflections (see Silber *et al.*, 1992):

$$\begin{aligned}\kappa_1 &: \mathbf{w} \to (w_4, w_3, w_2, w_1)\\ \kappa_2 &: \mathbf{w} \to (w_2, w_1, w_4, w_3)\end{aligned} \tag{2}$$

The cubic truncation of the normal form of the Hopf bifurcation problem that commutes with the action (1) of $T^2 \times S^1$, and the action (2) of D_2 is (from Silber *et al.*, 1992)

$$\begin{aligned}\dot{w}_1 &= \mu w_1 + (a|w_1|^2 + b|w_2|^2 + c|w_3|^2 + d|w_4|^2)w_1 + f w_2 \bar{w}_3 w_4\\ \dot{w}_2 &= \mu w_2 + (a|w_2|^2 + b|w_1|^2 + c|w_4|^2 + d|w_3|^2)w_2 + f w_1 \bar{w}_4 w_3\\ \dot{w}_3 &= \mu w_3 + (a|w_3|^2 + b|w_4|^2 + c|w_1|^2 + d|w_2|^2)w_3 + f w_2 \bar{w}_1 w_4\\ \dot{w}_4 &= \mu w_4 + (a|w_4|^2 + b|w_3|^2 + c|w_2|^2 + d|w_1|^2)w_4 + f w_1 \bar{w}_2 w_3\end{aligned} \tag{3}$$

Here an overbar denotes complex conjugation. The coefficients a, b, c, d and f are complex; we denote their real and imaginary parts using subscripts r and i, as in

$a = a_r + ia_i$. The coefficient μ is purely imaginary at the Hopf bifurcation point; in a neighborhood of the bifurcation point $\mu \equiv \lambda - i\omega$, where λ is the bifurcation parameter ($|\lambda| \ll 1$). For the normal form of the $D_4 \times_s T^2$-equivariant Hopf bifurcation problem, an additional reflection forces $c = d$ in (3) (see Silber & Knobloch, 1991). In the case of the $D_6 \times_s T^2$-equivariant Hopf bifurcation, the normal form (3) applies in an eight-dimensional fixed-point subspace of the twelve-dimensional bifurcation problem (see Clune & Knobloch, 1994, and references therein).

Solutions that retain some of the symmetry of the normal form (3) have been investigated elsewhere (Silber *et al.*, 1992). In this paper, we are interested in solution branches that have fully broken the symmetry and thus do not lie in an invariant fixed-point subspace of $\mathbb{C}^4$. Our methods give us information about simple non-symmetric solution branches in a particular region of the coefficient space, specifically for $f_r \approx 0$ and $a_r \approx b_r \approx c_r \approx d_r$.

2.2 *Invariants for the action of $T^2 \times S^1$ on $\mathbb{C}^4$*

A natural way in which to study relative equilibria in systems with continuous symmetries is to use the invariants of the symmetry as coordinates. This has been done in a Hamiltonian context by Kummer (1986, 1990), while others have exploited these methods in their study of non-Hamiltonian equivariant bifurcation problems (see, for example, Chossat, 1993). Using the invariants of continuous symmetries as coordinates results in an identification of all points on the orbit of the continuous symmetry operation with a single point in the space of invariants. As we see in this paper, this has the advantage of effectively reducing the dimension of the problem. A trade-off in this reduction is that the reduced state space of the dynamical system may have singularities that prevent it from being a manifold. However, in our system, the singularities occur for symmetric states that are not part of the present investigation. Another potential drawback in using invariants as coordinates is that it can obscure the symmetry of the problem. This is not an issue here as we are explicitly interested in solutions that have fully broken symmetry.

A set of $T^2 \times S^1$ invariants of (1) is

$$\begin{aligned} X + iY &\equiv w_1 \bar{w}_2 w_3 \bar{w}_4 \\ Z &\equiv \tfrac{1}{4}(|w_1|^2 - |w_2|^2 + |w_3|^2 - |w_4|^2) \\ L &\equiv \tfrac{1}{4}(|w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2) \\ J &\equiv \tfrac{1}{2}(|w_1|^2 - |w_3|^2) \\ K &\equiv \tfrac{1}{2}(|w_2|^2 - |w_4|^2) \end{aligned} \tag{4}$$

These are calculated by looking for all monomials invariant under the action (1). In this sense, this is a complete set of invariants. In fact, all other smooth invariant functions can be written in terms of the six real invariants X, Y, Z, J, K and L. (See, for example, Golubitsky *et al.* (1988, Chapter XII, Section 4) for the theory relevant for this discussion, and Rumberger and Scheurle (1996) for the finite differentiability case.) Note that $Z \in [-(L - |J|), L - |K|]$ since

$$\begin{aligned} |w_1|^2 &= L + Z + J \geq 0, \qquad & |w_2|^2 &= L - Z + K \geq 0 \\ |w_3|^2 &= L + Z - J \geq 0, \qquad & |w_4|^2 &= L - Z - K \geq 0 \end{aligned} \tag{5}$$

The action (2) of $\kappa_1, \kappa_2 \in D_2$ on the invariants is

$$\begin{aligned} \kappa_1(X, Y, Z, J, K, L) &= (X, -Y, -Z, -K, -J, L) \\ \kappa_2(X, Y, Z, J, K, L) &= (X, -Y, -Z, K, J, L) \end{aligned} \tag{6}$$

Moreover, the invariants are not independent; they satisfy the following relation

$$X^2 + Y^2 = [(L+Z)^2 - J^2][(L-Z)^2 - K^2] \tag{7}$$

We use the normal form (3) to derive the following equations for the evolution of the invariants

$$\begin{aligned} \dot{X} &= 4[\lambda + (\sigma+\nu)L]X - 4\gamma f_i Z Y + 4 f_r (L^2 - Z^2) L - 2 f_r [L(J^2 + K^2) - Z(J^2 - K^2)] \\ \dot{Y} &= 4[\lambda + (\sigma+\nu)L]Y + 4\gamma f_i Z X - 4 f_i (L^2 - Z^2) Z + 2 f_i [L(J^2 - K^2) - Z(J^2 + K^2)] \\ \dot{Z} &= 2(\lambda + 2\sigma L) Z + \tilde{\alpha}(J^2 - K^2) + 2 f_i Y \\ \dot{J} &= 2[\lambda + (\tilde{\alpha}+\sigma)(L+Z) + \nu(L-Z)]J + 2\tilde{\beta}(L+Z)K \\ \dot{K} &= 2[\lambda + (\tilde{\alpha}+\sigma)(L-Z) + \nu(L+Z)]K + 2\tilde{\beta}(L-Z)J \\ \dot{L} &= 2[\lambda + (\sigma+\nu)L]L + \tilde{\alpha}(J^2 + K^2) + 2\tilde{\beta} J K + 2(\sigma - \nu) Z^2 + 2 f_r X \end{aligned} \tag{8}$$

where

$$\begin{aligned} \gamma &\equiv (a_i - b_i + c_i - d_i)/f_i \\ \tilde{\alpha} &\equiv a_r - c_r \\ \tilde{\beta} &\equiv b_r - d_r \\ \sigma &\equiv a_r + c_r \\ \nu &\equiv b_r + d_r \end{aligned}$$

The algebraic relation (7) can replace one of the six equations in (8) to yield five independent evolution equations. That is, the relation (7) defines an invariant five-dimensional variety (a manifold possibly with singularities) on which the six-dimensional dynamical system evolves. To determine the behavior in the original (unreduced) space $\mathbb{C}^4 \cong \mathbb{R}^8$, we need to supplement (8) with equations for the evolution of three phases. Specifically, let $w_j = r_j e^{i\psi_j}$, $j = 1, \ldots, 4$, where $r_j \geq 0$ is the (real) amplitude and $\psi_j \in [0, 2\pi)$ is the phase. Equation (3) allows us to derive evolution equations for three linearly independent phases $(\psi_1 - \psi_3)/2$, $(\psi_2 - \psi_4)/2$ and $(\psi_1 + \psi_2 + \psi_3 + \psi_4)/4$:

$$\begin{aligned} \frac{\dot{\psi}_1 - \dot{\psi}_3}{2} &= (a_i - c_i)J + (b_i - d_i)K + \frac{J(f_r Y - f_i X)}{(L+Z)^2 - J^2} \\ \frac{\dot{\psi}_2 - \dot{\psi}_4}{2} &= (a_i - c_i)K + (b_i - d_i)J - \frac{K(f_r Y + f_i X)}{(L-Z)^2 - K^2} \\ \frac{\dot{\psi}_1 + \dot{\psi}_2 + \dot{\psi}_3 + \dot{\psi}_4}{4} &= -\omega + (a_i + b_i + c_i + d_i)L + \frac{(f_i X + f_r Y)(L-Z)}{2[(L-Z)^2 - K^2]} \\ &\quad - \frac{(f_r Y - f_i X)(L+Z)}{2[(L+Z)^2 - J^2]} \end{aligned} \tag{9}$$

Note that the phase ψ_j is not defined when $w_j = 0$, and thus the evolution equations (9) are not defined when $Z = L \pm K$, $-L \pm J$. The solutions we will focus on are not at these points.

The three combinations of phases are chosen so that the evolution equations for these combinations, i.e. equations (9), decouple from (8); the decoupling is a consequence of the $T^2 \times S^1$ symmetry. The equation for the evolution of the phase $\psi_1 - \psi_2 + \psi_3 - \psi_4$, which is associated with the complex invariant $X + iY$ in (4), does not decouple. To find relative equilibria in the full eight-dimensional system, we can look for equilibria of the five-dimensional system defined by (7) and (8).

2.3 Reduction to a four-dimensional system

The system (8) is further simplified by scaling (X, Y, Z, J, K) by an appropriate power of L and simultaneously rescaling time. Specifically, noting that J, K, L, Z are all $\mathcal{O}(|\mathbf{w}|^2)$, and X, Y are $\mathcal{O}(|\mathbf{w}|^4)$, we let

$$(x, y, z, j, k) = \left(\frac{X}{L^2}, \frac{Y}{L^2}, \frac{Z}{L}, \frac{J}{L}, \frac{K}{L}\right)$$

and introduce the new time $\tau(t)$, where

$$\dot{\tau} = L(t), \qquad \tau(0) = 0$$

Note that $L \geq 0$ and that $L = 0$ corresponds to the trivial equilibrium solution of (3). It follows from (8) that

$$\begin{aligned}
L' &= 2\lambda + [2\sigma(1+z^2) + 2\nu(1-z^2) + \tilde{\alpha}(j^2+k^2) + 2\tilde{\beta}jk + 2f_r x]\,L \\
x' &= -4\gamma f_i zy + 2f_r(2 - j^2 - k^2 - 2z^2 - 2x^2 + zj^2 - zk^2) - 2\tilde{\alpha}x(j^2+k^2) - 4\tilde{\beta}xjk \\
&\quad - 4(\sigma - \nu)xz^2 \\
y' &= 4\gamma f_i zx - 2f_i(2z - j^2 + k^2 - 2z^3 + zj^2 + zk^2) - 4f_r xy - 2\tilde{\alpha}y(j^2+k^2) - 4\tilde{\beta}jky \\
&\quad - 4(\sigma - \nu)yz^2 \\
z' &= 2f_i y + 2(\sigma - \nu)(1 - z^2)z - 2f_r xz + \tilde{\alpha}(j^2 - k^2 - zj^2 - zk^2) - 2\tilde{\beta}jkz \\
j' &= \tilde{\alpha}(2 + 2z - j^2 - k^2)j - 2f_r xj + 2(\sigma - \nu)(1 - z)jz + 2\tilde{\beta}(1 + z - j^2)k \\
k' &= \tilde{\alpha}(2 - 2z - j^2 - k^2)k - 2f_r xk - 2(\sigma - \nu)(1 + z)kz + 2\tilde{\beta}(1 - z - k^2)j
\end{aligned} \tag{10}$$

where the prime indicates differentiation with respect to the new time τ. The relation (7) now reads

$$x^2 + y^2 = [(1+z)^2 - j^2][(1-z)^2 - k^2] \tag{11}$$

Inspection of (5) shows that $z \in [|j| - 1, 1 - |k|]$ and $j, k \in [-2, 2]$ with $|j| + |k| \leq 2$.

A nice feature of this transformation is that the (x, y, z, j, k) dynamics have decoupled from that of L. Moreover, relation (11) and the evolution equations for (x, y, z, j, k) are independent of the bifurcation parameter λ. We are interested in finding fixed points of the reduced system of equations for (x, y, z, j, k), with $L \propto \lambda$

Table 1. Non-trivial fixed-point subspaces, and the generators of the isotropy subgroups that fix them. $((\theta_1, \theta_2), \phi) \in \mathrm{T}^2 \times \mathrm{S}^1$, where $(\theta_1, \theta_2) \in \mathrm{T}^2$ and $\phi \in \mathrm{S}^1$; $w_1, w_2 \in \mathbb{C}$. The element $((\pi, \pi), \pi)$, which is in every isotropy subgroup, is not listed

Orbit representative	Generators of isotropy subgroup	Relations satisfied by scaled $\mathrm{T}^2 \times \mathrm{S}^1$ invariants in the fixed-point subspace
$\mathbf{w}=(w_1, 0, 0, 0)$	$((-\phi, \theta), \phi), \quad \phi, \theta \in \mathrm{S}^1$	$x=y=k=0,\ z=\frac{1}{2}j=1$
$\mathbf{w}=(w_1, w_1, 0, 0)$	$\kappa_2, ((-\phi, -\phi), \phi), \quad \phi \in \mathrm{S}^1$	$x=y=z=0,\ j=k=1$
$\mathbf{w}=(w_1, 0, 0, w_1)$	$\kappa_1, ((-\phi, \phi), \phi), \quad \phi \in \mathrm{S}^1$	$x=y=z=0,\ j=-k=1$
$\mathbf{w}=(w_1, 0, w_1, 0)$	$\kappa_1\kappa_2, ((0, \theta), 0), \quad \theta \in \mathrm{S}^1$	$x=y=j=k=0,\ z=1$
$\mathbf{w}=(w_1, w_1, w_1, w_1)$	κ_1, κ_2	$y=z=j=k=0,\ x=1$
$\mathbf{w}=(w_1, iw_1, w_1, iw_1)$	$\kappa_1((0, \pi), \frac{\pi}{2}), \quad \kappa_2((0, \pi), \frac{\pi}{2})$	$y=z=j=k=0,\ x=-1$
$\mathbf{w}=(w_1, w_2, 0, 0)$	$((-\phi, -\phi), \phi), \quad \phi \in \mathrm{S}^1$	$x=y=0,\ j=1+z,\ k=1-z$
$\mathbf{w}=(w_1, 0, 0, w_2)$	$((-\phi, \phi), \phi), \quad \phi \in \mathrm{S}^1$	$x=y=0,\ j=1+z,\ k=z-1$
$\mathbf{w}=(w_1, 0, w_2, 0)$	$((0, \theta), 0), \quad \theta \in \mathrm{S}^1$	$x=y=k=0,\ z=1$
$\mathbf{w}=(w_1, w_1, w_2, w_2)$	κ_2	$y=z=0,\ j=k,\ x=1-j^2$
$\mathbf{w}=(w_1, iw_1, w_2, iw_2)$	$\kappa_2((0, \pi), \frac{\pi}{2})$	$y=z=0,\ j=k,\ x=j^2-1$
$\mathbf{w}=(w_1, w_2, w_2, w_1)$	κ_1	$y=z=0,\ j=-k,\ x=j^2-1$
$\mathbf{w}=(w_1, iw_2, w_2, iw_1)$	$\kappa_1((0, \pi), \frac{\pi}{2})$	$y=z=0,\ j=-k,\ x=1-j^2$
$\mathbf{w}=(w_1, w_2, w_1, w_2)$	$\kappa_1\kappa_2$	$j=k=0,\ x^2+y^2=(1-z^2)^2$

then determined from the first of equations (10). In the setting of a class of equivariant Hopf bifurcation problems, Field and Swift (1994) have proved a more general result related to the observation that the equations for (x, y, z, j, k) decouple from those of L. They find conditions under which a similar decoupling results in the existence of a branch of attracting invariant spheres that capture all the local dynamics, not just the equilibria.

The symmetries (1) and (2) ensure that equations (3) possess certain dynamically invariant subspaces. These subspaces are fixed by isotropy subgroups of the full symmetry group $\Gamma \equiv \mathrm{D}_2 \times_S \mathrm{T}^2 \times \mathrm{S}'$. For example, the subspace

$$\{\mathbf{w} \in \mathbb{C}^4 | w_1 \in \mathbb{C},\ w_2 = w_3 = w_4 = 0\}$$

is fixed by spatial translations $(\theta_1, \theta_2) \in \mathrm{T}^2$ followed by the normal form phase shift $\phi = -\theta_1 \in \mathrm{S}^1$, where the action of these group elements is given in (1). The action of Γ on this subspace generates additional invariant subspaces. In Table 1, we present an orbit representative of each fixed-point subspace in terms of the original coordinates on $\mathbb{C}^4$ (see Silber *et al.*, 1992) and also in terms of the scaled invariants (x, y, z, j, k). Note that if a solution satisfies $xz(|j|+|k|) \neq 0$, then it does not possess any symmetry other than the reflection $((\pi, \pi), \pi)$, which is in every isotropy subgroup. Moreover, non-symmetric equilibria of the reduced system of equations (8) correspond to three-tori in $\mathbb{R}^8$, since, generically, the frequencies (9) are non-resonant when $xz(|j|+|k|) \neq 0$.

The relation (11) defines a subset $E^4 \subset \mathbb{R}^5$. Since it is the zero level set of the real-valued function $F(x, y, z, j, k) = x^2 + y^2 - [(1+z)^2 - j^2][(1-z)^2 - k^2]$, E^4 is a smooth four-dimensional manifold at points where the differential of F does not vanish, that is, at points where not all five partial derivatives of F are zero simultaneously. One checks that E^4 is non-smooth only at points in certain of the fixed point subspaces of Table 1.

The system of equations (10) are equivariant under the transformations

$$
\begin{aligned}
&(x, y, z, j, k, L) \to (x, -y, -z, -k, -j, L) \\
&(x, y, z, j, k, L) \to (x, -y, -z, k, j, L) \\
&(x, y, z, j, k, L) \to (x, y, z, -j, -k, L)
\end{aligned}
\tag{12}
$$

due to the discrete symmetries (6). Hence the non-symmetric three-tori we seek will come in sets of four that are symmetry-related.

2.4 Hamiltonian limit

The four-dimensional problem consisting of the equations for (x, y, z, j, k) in (10) and the relation (11) has a Hamiltonian structure when

$$f_r = \tilde{\alpha} = \tilde{\beta} = \sigma - v = 0 \tag{13}$$

In this case, (x, y, z) evolve according to

$$
\begin{aligned}
x' &= -4f_i\gamma zy \\
y' &= 2f_i(2\gamma zx - 2z + j^2 - k^2 + 2z^3 - zj^2 - zk^2) \\
z' &= 2f_i y
\end{aligned}
\tag{14}
$$

where (x, y, z) satisfy the relation

$$\phi_{j,k}(x, y, z) \equiv (x^2 + y^2) - [(1+z)^2 - j^2][(1-z)^2 - k^2] = 0, \quad z \in [|j| - 1, 1 - |k|] \tag{15}$$

and j and k are conserved quantities associated with the action of T^2. That these equations have a Hamiltonian structure becomes clear if we introduce the Hamiltonian

$$H(x, y, z) = f_i(\gamma z^2 + x) \tag{16}$$

and observe that equations (14) can be written

$$V' = \nabla H \times \nabla \phi_{j,k} \tag{17}$$

where $V = (x, y, z)$ and ∇ is the standard gradient operator in Cartesian coordinates. The preceding equations are actually in Hamiltonian form of rigid body type. We explain this in a little more detail in Appendix A.

We refer to the particular choice of coefficient values (13) as 'the Hamiltonian limit'. Note that this does not correspond to a Hamiltonian limit of the original normal form (3), which occurs when, in addition to (13), we require $\lambda = \sigma = 0$. In that case, L is an additional conserved quantity associated with the normal form symmetry, and the bifurcation parameter has been removed from the problem.

Provided $|j| + |k| < 2$, and $jk \neq 0$, the relation (15) defines a compact two-dimensional manifold in $\mathbb{R}^3$ for each choice of j and k. We call this manifold the 'egg'. The condition $|j| + |k| < 2$ follows from the inequality $|j| - 1 \leq z \leq 1 - |k|$, together with the observation that if $|j| + |k| = 2$, then the manifold collapses to the point $x = y = 0$, $z = |j| - 1 = 1 - |k|$. If $j = 0$ (resp. $k = 0$), the surface defined by (15) is not smooth; it has a corner at $x = y = 0$, $z = -1$ (resp. $z = 1$). In any case, for each choice of the conserved quantities j and k, we think of the equations (14) as defining a two-dimensional Hamiltonian system on the surface defined by (15). A solution that satisfies either $|j| + |k| = 2$ or $jk = 0$ may correspond to a symmetric

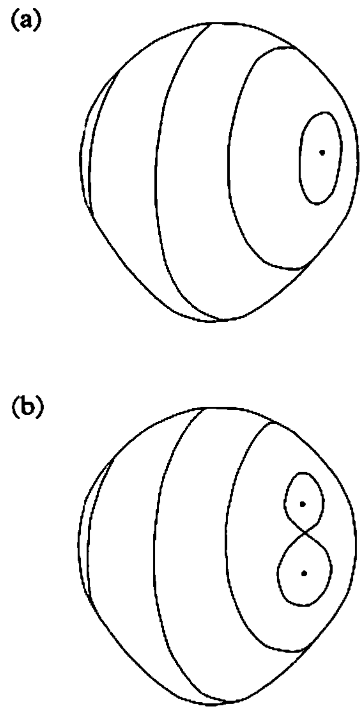

Fig. 1. Typical flows corresponding to equations (14) for some choice of j and k. Solutions lie on the surface of the egg defined by (15). See text for further description of the solutions.

state of the original system (see Table 1); because we are interested in solutions that fully break the symmetry, we avoid these cases from now on.

For a particular choice of j and k, orbits of (14) occur where the level sets of H intersect the corresponding egg. We determine the dynamics by slicing the egg defined by the relation $\phi_{j,k}=0$ with the parabolic surfaces of constant H. Equilibria occur when there is a tangency between some surface of constant H and the egg, with all other intersections corresponding to periodic or homoclinic orbits. Note that equilibria found on the egg correspond to three-tori in the original eight-dimensional phase space, and the periodic solutions correspond to four-tori.

Two typical examples of flows on an egg are presented in Fig. 1. In the first example, there are two elliptic equilibria; the rest of the egg is filled with periodic orbits. In the second example, there are four equilibria, three being elliptic and one hyperbolic. In addition, there are two isolated homoclinic orbits, with the rest of the egg being filled with periodic orbits.

A complete analysis of the dynamics in the Hamiltonian limit is beyond the scope of the present paper; instead, we give the results for the special case $j^2=k^2$ and more generally for j^2 near k^2, and indicate what needs to be done to extend these results to arbitrary j and k. In what follows, it is convenient to work with the scaled Hamiltonian $\tilde{H}=H/f_i$.

It follows immediately from (14) that $y=0$ at an equilibrium solution. We determined the other coordinates at equilibrium by calculating values of x, z and $\tilde{H}$ that satisfy the equations

$$x = \tilde{H} - \gamma z^2 \tag{18}$$

$$2(\gamma^2 - 1)z^3 + (2 + k^2 + j^2 - 2\gamma\tilde{H})z + (k^2 - j^2) = 0 \tag{19}$$

$$(\gamma^2 - 1)z^4 + (j^2 - k^2)z + (1 - j^2)(1 - k^2) - \tilde{H}^2 = 0 \tag{20}$$

Equation (18) comes from rearranging (16), equation (19) expresses the requirement that there be a tangency between the egg and some surface of constant H at an equilibrium, and (20) is obtained by using (18) and (19) to simplify (15). Note that without loss of generality we can consider the case $\gamma \geq 0$; the results for $\gamma < 0$ follow from the observation that equations (18)–(20) are unchanged by the transformation $(\tilde{H}, \gamma, x) \to (-\tilde{H}, -\gamma, -x)$.

Substituting $j^2 = k^2$ into (19) and (20) and solving these equations with (18), we find that if $0 \leq \gamma < \bar{\gamma} \equiv (1 + j^2)/(1 - j^2)$ there are two equilibria on the egg, at

$$\begin{aligned} y &= z = 0 \\ x &= \tilde{H} = \pm(1 - j^2) \end{aligned} \tag{21}$$

(cf. Fig. 1(a)) while if $\gamma > \bar{\gamma}$ there are four equilibria on the egg, two at the values given in (21) and two at

$$\begin{aligned} y &= 0 \\ z &= \pm\sqrt{1 + j^2 - \frac{2|j|\gamma}{\sqrt{\gamma^2 - 1}}} \\ \tilde{H} &= \gamma(1 + j^2) - 2|j|\sqrt{\gamma^2 - 1} \\ x &= \frac{2|j|}{\sqrt{\gamma^2 - 1}} \end{aligned} \tag{22}$$

(cf. Fig. 1(b)). This result is illustrated graphically in Fig. 2, where values of z that solve (19) and (20) are plotted as $\tilde{H}$ is varied; an intersection between the two graphs corresponds to an equilibrium solution on the egg so long as the z coordinate is in the range $z \in (|j| - 1, 1 - |j|)$. We see that there is a Hamiltonian pitchfork bifurcation when $\gamma = \bar{\gamma}$.

The dynamics for $j^2 \neq k^2$ but j^2 near k^2 can be determined qualitatively by sketching the graphs of (19) and (20) again. The situation for $j^2 - k^2$ positive but sufficiently small is shown in Fig. 3. It can be seen that there will still be parameter values for which there are two equilibria and other values for which there are four equilibria but the transition between the two cases now occurs via a Hamiltonian saddle-node bifurcation at some $\gamma = \gamma^\star \approx \bar{\gamma}$.

We have not determined the dynamics for the case of arbitrary j and k. Factors to be considered in the general case include whether there could be six equilibria on the egg (it is straightforward to show that there can be no more than six equilibria on any egg) and whether there can be more than two equilibria when $|\gamma| < 1$.

3 The existence of a branch of three-tori

3.1 Introduction and set-up

In this section, we prove the existence of a branch of three-tori that bifurcates from the trivial solution at $\lambda = 0$. The three-tori are relative equilibria that have fully

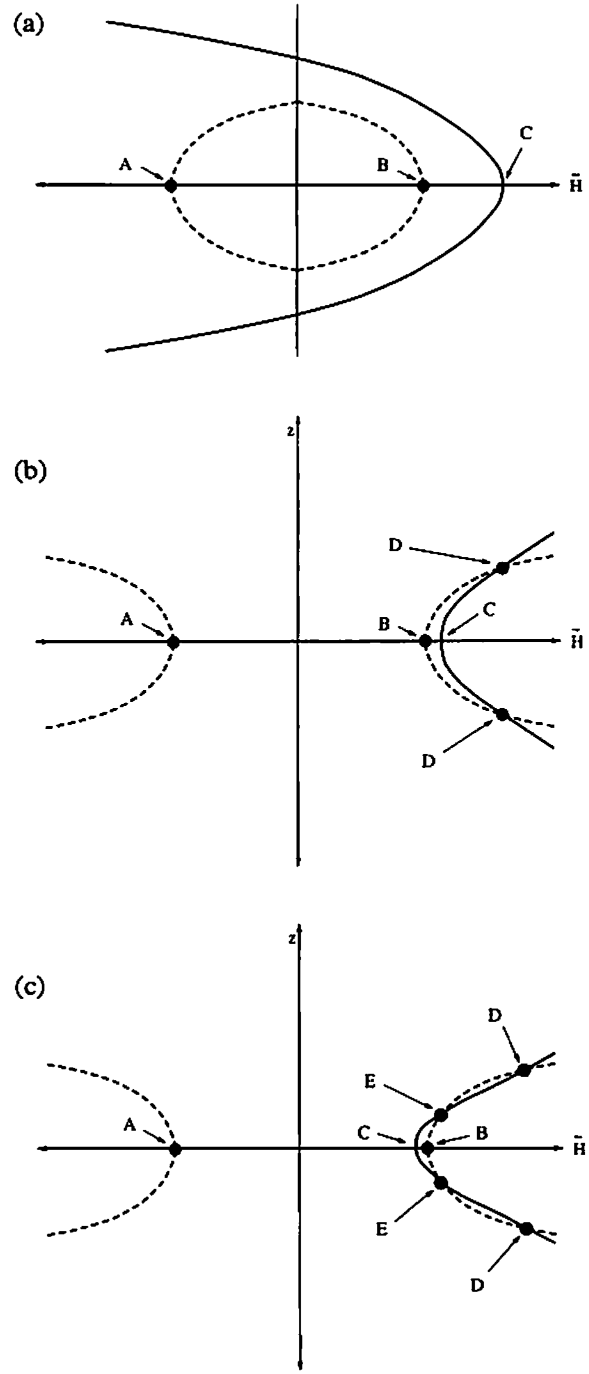

Fig. 2. Solutions of (19) and (20) for the case $j^2=k^2$ when: (a) $0<\gamma<1$; (b) $1<\gamma<\bar{\gamma}$; (c) $\gamma>\bar{\gamma}$; where $\bar{\gamma}=(1+j^2)/(1-j^2)$ is the value of γ at the pitchfork bifurcation. The solid (resp. dashed) curves mark values of $\tilde{H}$ and z that solve (19) (resp. (20)). Note that one of the solid curves lies on the $\tilde{H}$-axis. The large dots mark values of $\tilde{H}$ and z that satisfy both equations. Dots marked A have $\tilde{H}=j^2-1$; B means $\tilde{H}=1-j^2$; C means $\tilde{H}=(1+j^2)/\gamma$; D means $\tilde{H}=(1+j^2)\gamma+2|j|\sqrt{\gamma^2-1}$; E means $\tilde{H}=(1+j^2)\gamma-2|j|\sqrt{\gamma^2-1}$. Note that the dots marked D have values outside the allowable range $z\in(|j|-1,\ 1-|j|)$, and therefore do not correspond to equilibrium solutions on the egg.

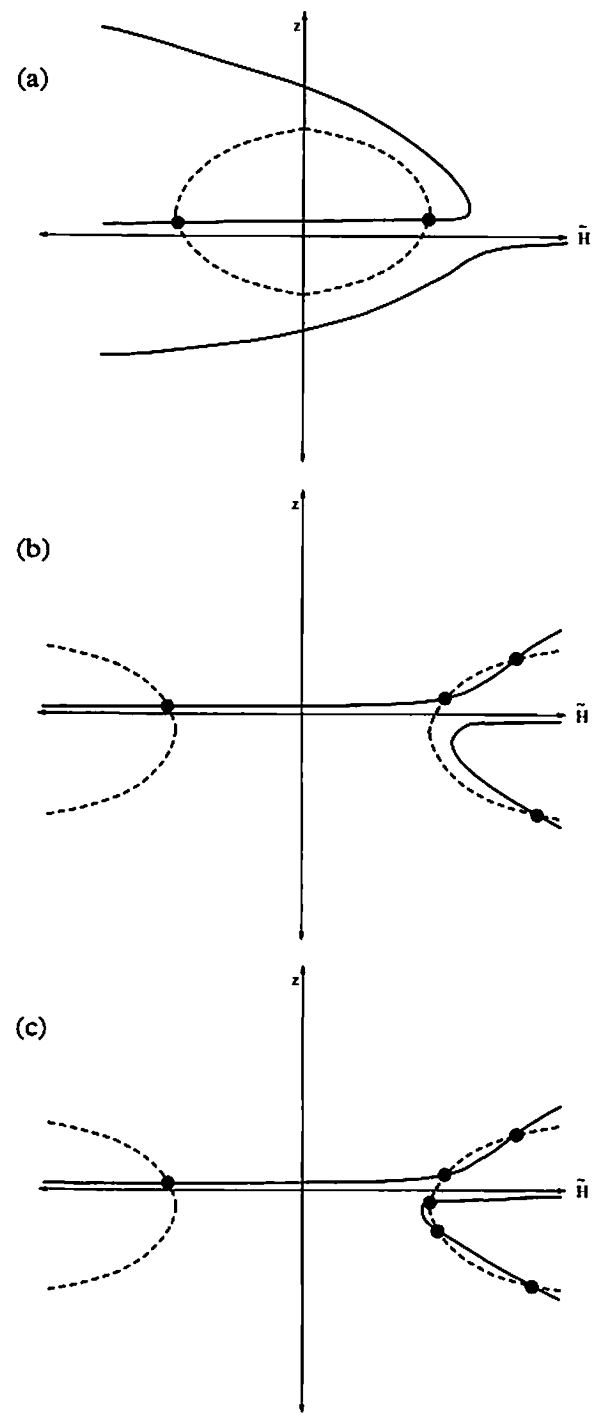

Fig. 3. Qualitative plot of solutions to (19) and (20) for the case j^2-k^2 positive but small when: (a) $0<\gamma<1$; (b) $1<\gamma<\gamma^*$; (c) $\gamma>\gamma^*$; where γ^* is the value of γ at the saddle-node bifurcation. The solid (resp. dashed) curves mark values of $\tilde{H}$ and z that solve (19) (resp. (20)). The large dots mark values of $\tilde{H}$ and z that satisfy both equations. For j^2-k^2 small enough, the dots with the two largest $\tilde{H}$ coordinates will have z values outside the allowable range $z\in(|j|-1,\ 1-|k|)$, and therefore will not correspond to equilibrium solutions on the egg.

broken the symmetry of the bifurcation problem. We proceed by introducing a small parameter ε such that the Hamiltonian limit is $\varepsilon = 0$. Specifically, let

$$f_r = \varepsilon\delta, \qquad \tilde{\alpha} = \varepsilon\alpha, \qquad \tilde{\beta} = \varepsilon\beta \tag{23}$$

We further restrict the problem by choosing

$$\nu = \sigma + \varepsilon x + \varepsilon^2 \rho \tag{24}$$

Because ρ will not enter our calculations at first order in ε, this reduces by one the number of parameters in the problem and significantly simplifies the equations for $(j'k - k'j)/\varepsilon$ and $(jj' + kk')/\varepsilon$ below. Equations (10) now become

$$\begin{aligned}
L' &= 2\lambda + [4\sigma + \mathcal{O}(\varepsilon)]L \\
x' &= -4\gamma f_i zy + 2\varepsilon\delta(2 - j^2 - k^2 - 2z^2 - 2x^2 + zj^2 - zk^2) - 2\varepsilon\alpha x(j^2 + k^2 - 2z^2) \\
&\quad - 4\varepsilon\beta xjk + \mathcal{O}(\varepsilon^2) \\
y' &= 4\gamma f_i zx - 2f_i(2z - j^2 + k^2 - 2z^3 + zj^2 + zk^2) - 4\varepsilon\delta xy - 2\varepsilon\alpha y(j^2 + k^2 - 2z^2) \\
&\quad - 4\varepsilon\beta jky + \mathcal{O}(\varepsilon^2) \\
z' &= 2f_i y - 2\varepsilon\alpha(1 - z^2)z - 2\varepsilon\delta xz + \varepsilon\alpha(j^2 - k^2 - zj^2 - zk^2) - 2\varepsilon\beta jkz + \mathcal{O}(\varepsilon^2) \\
j' &= \varepsilon[\alpha(2 + 2z^2 - j^2 - k^2)j - 2\delta xj + 2\beta(1 + z - j^2)k] + \mathcal{O}(\varepsilon^2) \\
k' &= \varepsilon[\alpha(2 + 2z^2 - j^2 - k^2)k - 2\delta xk + 2\beta(1 - z - k^2)j] + \mathcal{O}(\varepsilon^2)
\end{aligned} \tag{25}$$

Note that the higher-order terms in ε are bounded because the dynamics takes place on a compact manifold.

For the purpose of finding a branch of non-symmetric three-tori bifurcating from the origin in the full problem, we will be interested in the case that the L'-equation in (25) has an attracting non-trivial equilibrium; this is the case for $\sigma < 0$, $\lambda > 0$, at least for ε sufficiently small. For these values, there is an attracting seven-sphere of radius $\sqrt{L} \equiv \sqrt{(|w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2)/4} = \sqrt{\lambda/(-2\sigma)}$ that bifurcates from the origin at $\lambda = 0$ in the full problem. Thus, when $\sigma < 0$, $\lambda > 0$, we can locate a supercritical branch of non-symmetric three-tori in the eight-dimensional problem by looking for non-trivial equilibria of the four-dimensional system defined by the evolution equations for (x, y, z, j, k) and the relation (11).

The system of equations (25) shows that relative equilibria of the perturbed problem with $jk \neq 0$ satisfy the following algebraic system of equations:

$$\begin{aligned}
\phi_{jk} &= x^2 + y^2 - [(1+z)^2 - j^2][(1-z)^2 - k^2] = 0 \\
g_{1,\varepsilon} &= 2f_i(2\gamma zx - 2z + j^2 - k^2 + 2z^3 - zj^2 - zk^2) + \mathcal{O}(\varepsilon) = 0 \\
g_{2,\varepsilon} &= 2f_i y + \mathcal{O}(\varepsilon) = 0 \\
g_{3,\varepsilon} &= 2\beta[z(j^2 + k^2) - (j^2 - k^2)] + \mathcal{O}(\varepsilon) = 0 \\
g_{4,\varepsilon} &= \alpha(j^2 + k^2)(2 - j^2 - k^2 + 2z^2) + 2\beta jk(2 - j^2 - k^2) - 2\delta x(j^2 + k^2) + \mathcal{O}(\varepsilon) = 0
\end{aligned} \tag{26}$$

Here we have replaced the x'-equation in (25) by the relation (11), and the j', k' equations by $(j'k - k'j)/\varepsilon$ and $(j'j + k'k)/\varepsilon$. The reason why we replace equations (25) by equations (26) is as follows. We would like to use the implicit function theorem to prove persistence for $\varepsilon \neq 0$ of equilibria we find at $\varepsilon = 0$. However, in the case of equations (25), these equilibria are not isolated since $j' = k' = 0$ when $\varepsilon = 0$. In equations (26), we retain the $\mathcal{O}(\varepsilon)$ terms from the j' and k' equations. This amounts to dividing the j' and k' equations by ε before taking the limit of $\varepsilon=0$; this works because we are looking for equilibria. As we will see, equilibria in equations (26) are isolated, and we can then apply the implicit function theorem to prove persistence. Note that j' and k' are $\mathcal{O}(\varepsilon^2)$ for the values of j and k that satisfy equations (26) with $\varepsilon=0$.

3.2 *The unperturbed case*

We prove the following theorem.

Theorem 1. Assume that $\beta\neq 0$ and $f_i\neq 0$. Then the equations

$$\phi_{jk}=g_{1,0}=g_{2,0}=g_{3,0}=g_{4,0}=0 \tag{27}$$

have solutions with $0<j^2+k^2<4$ and $jk\neq 0$ if and only if $|\gamma|>1$.

These solutions with the value of $L=-\lambda/(2\sigma)$ give equilibria for the unperturbed system associated with (25). In the next subsection, we will see how to perturb these solutions to the case of non-zero ε.

Proof. Our proof is constructive. We see immediately that any solution has $y=0$. The equation $g_{3,0}=0$ gives us an expression for z in terms of j and k, and squaring this expression allows us to write z^2 as a function of the quantities $S\equiv j^2+k^2$ and $(jk)^2$. Substituting into the equations $\phi_{jk}=0$ and $g_{1,0}=0$ and solving for $x, y, z^2, (jk)^2$ in terms of S yields

$$\begin{aligned} y&=y_0\equiv 0\\ x&=x_0\equiv\frac{4CS(4-S)}{\gamma}\\ (jk)^2&=(j_0k_0)^2\equiv CS^3(4-S)\\ z^2&=z_0^2\equiv\left(\frac{j_0^2-k_0^2}{S}\right)^2=1-4CS(4-S)\end{aligned} \tag{28}$$

where

$$C\equiv\frac{\gamma^2}{16(\gamma^2-1)} \tag{29}$$

When $\alpha\neq 0$ the equation $g_{4,0}=0$ gives

$$j_0k_0=\frac{(4-S)S(8CS(\delta+\alpha\gamma)-\alpha\gamma)}{2\beta\gamma(2-S)} \tag{30}$$

whereas when $\alpha=0$, j_0k_0 satisfies

$$\beta j_0k_0(2-S)=\delta xS \tag{31}$$

Squaring equation (30) and substituting from (28) for $(j_0k_0)^2$ leads to the following cubic equation for S when $\alpha\neq 0$:

$$P_3(S)\equiv c_3S^3+c_2S^2+c_1S+c_0=0 \tag{32}$$

where

$$\begin{aligned} c_3 &= \beta^2(\gamma^2-1)+(\delta+\alpha\gamma)^2 \\ c_2 &= -4((\gamma^2-1)(\beta^2+\alpha^2+\delta\alpha/\gamma)+(\delta+\gamma\alpha)^2) \\ c_1 &= 4(\gamma^2-1)(\beta^2+5\alpha^2+4\delta\alpha/\gamma-\alpha^2/\gamma^2) \\ c_0 &= -16\frac{\alpha^2(\gamma^2-1)^2}{\gamma^2} \end{aligned} \tag{33}$$

If $\alpha=0$, a similar procedure applied to equation (31) leads to a quadratic equation for S:

$$P_2(S) \equiv c_3S^2+c_2S+c_1=0 \tag{34}$$

where c_1, c_2 and c_3 are as in (33) but with α set to zero.

In both cases, we require the value of $S \equiv j_0^2+k_0^2$ to satisfy the inequality $0<S<4$ since $j^2+k^2 \le (|j|+|k|)^2<4$. Given an S in this range, it follows from the requirement $(j_0k_0)^2>0$ that a necessary condition for a solution to exist is

$$\gamma^2>1$$

i.e. $C>0$. Indeed, this condition is both necessary and sufficient: if $\gamma^2>1$ and $\alpha\neq 0$, the observation that

$$\begin{aligned} P_3(0) &= -16\frac{\alpha^2(\gamma^2-1)^2}{\gamma^2}<0 \\ P_3(4) &= 16\beta^2(\gamma^2-1)>0 \end{aligned}$$

shows that there is at least one real solution $S\in(0,4)$ of the cubic (32), while if $\alpha=0$ and $\gamma^2>1$, the quadratic (34) has real roots, $S=2\pm 2|\delta|/\surd(\gamma^2-1)\beta^2+\delta^2$, in $(0,4)$.

Given this solution S, we can solve for the remaining variables using the expressions (28). Note that both roots of the expression for z_0^2 in (28) will give rise to acceptable solutions. Note also that the magnitudes of j_0 and k_0 are determined by (30) or (31) and the equation $j_0^2+k_0^2=S$, and that the sign of j_0k_0 is determined by (30) or (31), but we have two choices for the signs of j_0 and k_0. The choices for the signs of z_0, j_0 and k_0 give us four distinct solutions that are related by the symmetries (12). ■

3.3 *Perturbation analysis*

In this section, we determine conditions under which the implicit function theorem ensures the persistence of the branches of three-tori found in the proof of Theorem 1.

Assume that the conditions of Theorem 1 are satisfied, i.e. $\beta f_i\neq 0$ and $|\gamma|>1$. We use the relation (11) to express x, implicitly, as a function of y, z, j, k, and consider the system of four algebraic equations for y, z, j, k:

$$\mathbf{g}_\varepsilon \equiv (g_{1,\varepsilon}, g_{2,\varepsilon}, g_{3,\varepsilon}, g_{4,\varepsilon})=0 \tag{35}$$

where the functions $g_{i,\varepsilon}$ were defined in equations (26). In the proof of Theorem 1, we found a solution of the unperturbed system $\mathbf{g}_0\equiv(g_{1,0}, g_{2,0}, g_{3,0}, g_{4,0})=\mathbf{0}$. To ensure that this equilibrium persists under the $\mathcal{O}(\varepsilon)$ perturbations, we invoke the implicit function theorem. Specifically, we must show that $\det(\mathbf{Dg}_0) \neq 0$, where the Jacobian matrix,

$$\mathbf{Dg}_0 = \begin{pmatrix} \frac{\partial g_{1,0}}{\partial y} & \frac{\partial g_{1,0}}{\partial z} & \frac{\partial g_{1,0}}{\partial j} & \frac{\partial g_{1,0}}{\partial k} \\ \frac{\partial g_{2,0}}{\partial y} & \frac{\partial g_{2,0}}{\partial z} & \frac{\partial g_{2,0}}{\partial j} & \frac{\partial g_{2,0}}{\partial k} \\ \frac{\partial g_{3,0}}{\partial y} & \frac{\partial g_{3,0}}{\partial z} & \frac{\partial g_{3,0}}{\partial j} & \frac{\partial g_{3,0}}{\partial k} \\ \frac{\partial g_{4,0}}{\partial y} & \frac{\partial g_{4,0}}{\partial z} & \frac{\partial g_{4,0}}{\partial j} & \frac{\partial g_{4,0}}{\partial k} \end{pmatrix} \tag{36}$$

is evaluated on the unperturbed solution branch. We find that the implicit function theorem applies if $\beta f_i \neq 0$ and

$$\begin{aligned} \mathscr{D}(\alpha, \beta, \gamma, \delta) \equiv\; & 4j_0k_0[\alpha(1-S+z_0^2) - \beta j_0k_0 - \delta x_0]\,[S+2(\gamma^2-1)z_0^2] \\ & + 4j_0k_0[(\gamma^2 S+1)(4-S) - 4\gamma^2]\,[\alpha(1-S+z_0^2) - \beta j_0k_0 - \delta x_0] \\ & + \beta S^2(2-S)(\gamma^2 z_0^2 + 1 - z_0^2) + \gamma j_0k_0 S(2-S)(\delta S + 4\alpha\gamma z_0^2 + 4\delta\gamma^2 z_0^2) \\ & + \gamma S(2-S)(\beta x_0 + \delta j_0k_0)\,[(\gamma^2 S+1)(4-S) - 4\gamma^2] \neq 0 \end{aligned} \tag{37}$$

The value of $S \in (0, 4)$ is determined by the cubic (32) or the quadratic (34), whereas z_0^2 and x_0 are given as functions of S in (28), and j_0k_0 is given as a function of S in (30) or (31).

The rather complicated expression in (37) is not manifestly identically zero. However, in order to allay fears of a conspiracy, we evaluate it for fixed values of α, β and δ and a range of values of γ. We thereby show that the inequality is satisfied in an open region of the $(\alpha, \beta, \gamma, \delta, f_i)$-coefficient space. Specifically, we pick $\alpha = 0$, $\beta = 1$ and $\delta = 2$. Equation (34) then yields $S = 2 \pm 4/\sqrt{\gamma^2+3}$. We pick $S = 2 - 4/\sqrt{\gamma^2+3}$, use equation (28) and (31) to evaluate x_0, z_0 and j_0k_0, and subsitute these values into (37). The resulting expression, which depends on γ, is plotted in Fig. 4 for $\gamma \in (1, 50]$. It is clear from the figure that for $f_i \neq 0$ and for the values of α, β, γ and δ considered, $\mathscr{D} \neq 0$, from which we conclude that the equilibrium persists under $\mathscr{O}(\varepsilon)$ perturbations. Evaluation of $\mathscr{D}$ for the choice $S = 2 + 4/\sqrt{\gamma^2+3}$ similarly leads to the conclusion that the corresponding equilibrium persists under perturbation.

The arguments given in this section prove the existence of a branch of three-tori for equations (3), the cubic truncation of the normal form. These arguments can be modified to also give persistence under inclusion of higher-order terms in the normal form. Furthermore, a result of Field (1996) indicates that the branch of three-tori will persist under addition of sufficiently high-order terms that break the S^1 symmetry introduced by the process of reduction to normal form. Thus, we can conclude that a primary branch of three-tori exists for the $D_2 \times_s T^2$-equivariant Hopf bifurcation problem.

We note that the dynamics on a three-torus for the normal form equations is very simple; because a three-torus is a relative equilibrium, the three frequencies are typically independent so that the flow is quasi-periodic. Removing the normal

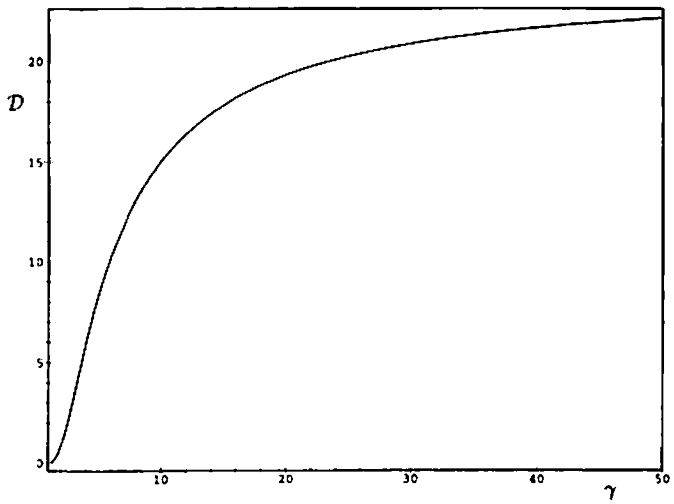

Fig. 4. Plot of $\mathcal{S}(\alpha, \beta, \gamma, \delta)$ for the case $\alpha=0$, $\beta=1$, $\delta=2$, $\gamma \in (1, 50]$, and with the choice $S=2-4/\sqrt{\gamma^2+3}$.

form symmetry at high order will not change this dynamics qualitatively. This can be seen by noting that a three-torus for the eight-dimensional, non-S^1-symmetric equations will correspond to an invariant circle in the quotient space $\mathbb{C}^4/T^2$; because the frequencies associated with a three-torus in the normal form are non-zero, the dynamics on the corresponding invariant circle in $\mathbb{C}^4/T^2$ will be periodic (no stationary solutions), at least for small enough non-S^1-symmetric perturbations.

4 Stability analysis

In this section, we compute the stability of the relative equilibria found in the previous section. The computation of the eigenvalues of the appropriate Jacobian matrix proceeds in three steps. We first observe that each equilibrium solution of the unperturbed problem, obtained by setting $\varepsilon=0$ in (25), also corresponds to an equilibrium in the Hamiltonian limit where (x, y, z) evolve according to equation (14) and $j'=k'=0$. To see this, note that the equations for the evolution of x, y and z are the same in both cases and the j, k dynamics are trivial in the Hamiltonian limit.

In the next subsection, we compute the eigenvalues of the Jacobian matrix associated with the Hamiltonian limit. Of the four relevant eigenvalues, two are zero due to conservation of j and k in the Hamiltonian limit. The remaining two are either a real pair with equal magnitude but opposite signs, in which case the corresponding equilibrium is hyperbolic, or a purely imaginary pair, in which case the equilibrium is elliptic.

The next step in the computation is to calculate how the eigenvalues move under non-Hamiltonian perturbations. We do not need to do any further calculations to determine stability of the perturbed solution in the hyperbolic case; the perturbed equilibrium must have at least one positive real eigenvalue and hence be unstable, at least for small enough perturbations. In the elliptic case, we use an eigenvalue movement formula derived in the second appendix to determine how the pair of purely imaginary eigenvalues moves. Once we know the corrections to the complex eigenvalues, we can determine the movement of the remaining zero eigenvalues. Because the expressions we obtain for movement of the zero eigenvalues are rather ugly, we do not calculate stabilities in the general case but restrict attention to a specific example.

4.1 Eigenvalues in the Hamiltonian limit

In this subsection, we compute the eigenvalues of the Jacobian matrix associated with the Hamiltonian limit (14) evaluated at one of the equilibria in the unperturbed problem. To carry out this computation, we choose coordinates so that the Hamiltonian structure on the egg is canonical. In the following, we abbreviate the notation: we fix some j and k and let the function defining the egg be called simply ϕ. This function, previously called ϕ_{jk}, is given by (15).

A normal vector to the egg at an equilibrium is given by the gradient of ϕ evaluated at that equilibrium; we write (leaving out the evaluation at the equilibrium in the notation)

$$\mathbf{n}=\left(\frac{\partial\phi}{\partial x}, 0, \frac{\partial\phi}{\partial z}\right)$$

The second component of $\mathbf{n}$ is zero since the equilibria we found in the previous section have $y=0$. Tangent vectors to the egg are vectors orthogonal to $\mathbf{n}$; we choose these to be

$$\mathbf{e}_1=(0, 1, 0) \qquad \text{and} \qquad \mathbf{e}_2=\left(-\frac{\partial\phi}{\partial z}, 0, \frac{\partial\phi}{\partial x}\right)$$

Equation (17) yields

$$\begin{aligned} x' &= -\frac{\partial\phi}{\partial y}\frac{\partial H}{\partial z} \\ y' &= \frac{\partial\phi}{\partial x}\frac{\partial H}{\partial z}-\frac{\partial\phi}{\partial z}\frac{\partial H}{\partial x} \\ z' &= \frac{\partial\phi}{\partial y}\frac{\partial H}{\partial x} \end{aligned} \tag{38}$$

Here we have used the fact that H is independent of y (see equation (16)). Let $\mathbf{A}$ denote the linearization of this system at the equilibrium described above. Regarding $\mathbf{A}$ as a 3×3 matrix, a straightforward calculation shows that

$$\mathbf{A}\mathbf{e}_1=a\mathbf{e}_2, \qquad \mathbf{A}\mathbf{e}_2=-b\mathbf{e}_1, \qquad \mathbf{A}\mathbf{n}=c\mathbf{e}_1$$

where

$$a=\frac{\partial^2\phi}{\partial y^2}\left(\frac{\partial\phi}{\partial z}\right)^{-1}\frac{\partial H}{\partial z}$$

$$b=\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial x^2}\frac{\partial H}{\partial z}-\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2 H}{\partial z^2}+\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial z^2}\frac{\partial H}{\partial x}$$

and c is some non-zero coefficient. Note that in calculating a we use the y'-equation in (38) to say that at an equilibrium

$$\frac{\partial H}{\partial x}=\frac{\partial \phi}{\partial x}\frac{\partial H}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)^{-1}$$

The matrix of $\mathbf{A}$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ is thus

$$\mathbf{A}=\begin{bmatrix} 0 & a & c \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From which it is seen that the eigenvalues of $\mathbf{A}$ are 0, $\pm\sqrt{-ab}$. If $ab<0$ the equilibria of the unperturbed problem are saddles. If $ab>0$, we put $\mathbf{A}$ into canonical form by choosing $\mathbf{e}_1=p\mathbf{u}_1$ and $\mathbf{e}_2=p^{-1}\mathbf{u}_2$ where $p=\sqrt{a/b}$; the matrix of $\mathbf{A}$ in the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}\}$ is given by

$$\mathbf{A}=\begin{bmatrix} 0 & \omega_0 & cp \\ -\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\omega_0=\sqrt{ab}$. Substituting the solution (28) into the expressions for a and b given above, we find ω_0^2 in terms of S:

$$\omega_0^2=4f_i^2[\gamma^2S^2+(1-4\gamma^2)S+4(\gamma^2-1)] \tag{39}$$

In summary, we have proved the following theorem.

Theorem 2. If $\omega_0^2>0$, then the solutions constructed in Theorem 1 define stable equilibria (centers) at $y=0$ for the Hamiltonian dynamics induced on the egg by the system (14). If $\omega_0^2<0$, then the solutions correspond to unstable saddles.

Equation (39) shows that if the value of S at one of the solutions constructed in Theorem 1 satisfies

$$S\in\left(\frac{(4\gamma^2-1)-\sqrt{1+8\gamma^2}}{2\gamma^2}, \frac{(4\gamma^2-1)+\sqrt{1+8\gamma^2}}{2\gamma^2}\right) \tag{40}$$

the unperturbed equilibria are unstable saddles, as are the perturbed equilibria, at least for small enough ε. If, on the other hand,

$$S\in\left(0, \frac{(4\gamma^2-1)-\sqrt{1+8\gamma^2}}{2\gamma^2}\right)\cup\left(\frac{(4\gamma^2-1)+\sqrt{1+8\gamma^2}}{2\gamma^2}, 4\right) \tag{41}$$

the unperturbed equilibria are centers. We devote the remainder of the paper to calculating the stability of these elliptic equilibria under small perturbations.

4.2 Movement of imaginary eigenvalues under the addition of non-Hamiltonian terms

Now we use the eigenvalue movement formula from Appendix B to determine whether the purely imaginary pair of eigenvalues $\pm i\omega_0$ associated with an elliptic equilibrium of the unperturbed problem moves to the left or right half of the complex plane with the addition of the non-Hamiltonian terms.

In the $\{\mathbf{u}_1, \mathbf{u}_2\}$ basis of the previous subsection, the linearized equations on the egg are Hamiltonian and they are in canonical form; the symplectic form on the tangent space to the egg at the relative equilibrium is represented by the matrix

$$\mathbf{J}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

If we regard this in the five-dimensional (x, y, z, j, k) space rather than on the tangent space to the egg, then one adds three rows and columns of zeros to the matrix for the symplectic form. The resulting two-form is degenerate, and this is why we allow degeneracy in the eigenvalue movement formula in Appendix B.

The eigenvectors associated with the complex eigenvalues $\pm i\omega_0$ are given in the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ by $\mathbf{v}_1=(1, i)$ and $\mathbf{v}_2=(1, -i)$. If we convert these to five-vectors, the remaining components are zero. In the notation of Appendix B, we take $\mathbf{v}_r=(1, 0, 0, 0, 0)$ and $\mathbf{v}_i=(0, 1, 0, 0, 0)$. We then find that

$$\mathbf{v}_r^T\mathbf{J}\mathbf{v}_i=1$$

so the denominator in the eigenvalue movement formula is simply 2. The numerator is given by

$$N=\mathbf{v}_r^T\mathbf{J}\mathbf{B}_d\mathbf{v}_i-\mathbf{v}_i^T\mathbf{J}\mathbf{B}_d\mathbf{v}_r$$

Here $\varepsilon\mathbf{B}_d$ is a 5×5 Jacobian matrix associated with the non-Hamiltonian terms in the vector field, i.e. the $\mathcal{O}(\varepsilon)$ terms in the (x', y', z', j', k') equations of (25); it is evaluated at the unperturbed equilibrium. In this formula, we must express everything in the basis $\mathscr{B}=(\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}, \mathbf{e}_4, \mathbf{e}_5)$, where $\mathbf{e}_4=(0, 0, 0, 1, 0)$ and $\mathbf{e}_5=(0, 0, 0, 0, 1)$ are the standard basis vectors associated with the variables j and k. A short calculation gives

$$N=B_{11}+B_{22}$$

where B_{ii} is the ii entry of the matrix $\mathbf{B}_d$ with respect to the above basis. Thus, the real part of the perturbation of the eigenvalue $i\omega_0$ is given by

$$Re(\lambda(\varepsilon))=\varepsilon\left(\frac{1}{2}(B_{11}+B_{22})\right)+\mathcal{O}(\varepsilon^2)$$

The following theorem summarizes.

Theorem 3. The perturbation of the eigenvalues $\pm i\omega_0$ is into the left half-plane if $\varepsilon(B_{11}+B_{22})<0$ and into the right half-plane if $\varepsilon(B_{11}+B_{22})>0$.

We now derive an explicit formula for $B_{11}+B_{22}$ in terms of the solution S of (32); recall that this formula only applies to equilibria that are centers for the Hamiltonian problem restricted to the egg, i.e. S must satisfy (41).

Let $\varepsilon\mathbf{f}_d$ be the non-Hamiltonian part of the vector field and write $\mathbf{f}_d=(f_{d,x}, f_{d,y}, f_{d,z}, f_{d,j}, f_{d,k})$, where the subscripts x, y, z, j, k specify the component of the (x', y', z', j', k') equation. After the appropriate change of basis, we find

$$B_{11}=\frac{\partial f_{d,y}}{\partial y} \tag{42}$$

and

$$B_{22}=\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]^{-1}\left[\left(\frac{\partial\phi}{\partial z}\right)^2\frac{\partial f_{d,x}}{\partial x}-\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial x}\left(\frac{\partial f_{d,x}}{\partial z}+\frac{\partial f_{d,z}}{\partial x}\right)+\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial f_{d,z}}{\partial z}\right] \tag{43}$$

We use (28)–(30) to express $B_{11}+B_{22}$ in terms of $S\equiv j^2+k^2$, and find that

$$B_{11}+B_{22}=\frac{d_5S^5+d_4S^4+d_3S^3+d_2S^2+d_1S^1+d_0}{(2-S)(\gamma^2-1)\left[\gamma^4S(4-S)+(1+4\gamma^2)(1-\gamma^2)\right]} \tag{44}$$

where

$$\begin{aligned}
d_5&=\gamma^4(\delta\gamma+\alpha)\\
d_4&=\gamma^3(\delta-15\delta\gamma^2-4\alpha\gamma^3-10\alpha\gamma)\\
d_3&=\gamma^2(79\delta\gamma^3-15\delta\gamma+32\gamma^4\alpha+37\alpha\gamma^2-5\alpha)\\
d_2&=\gamma(83\delta\gamma^2-178\delta\gamma^4-\delta-96\alpha\gamma^5-33\alpha\gamma^3+33\alpha\gamma)\\
d_1&=2(\gamma^2-1)(82\delta\gamma^3-8\delta\gamma+64\alpha\gamma^4+27\alpha\gamma^2-\alpha)\\
d_0&=-8(\gamma^2-1)^2(5\delta\gamma+8\alpha\gamma^2+2\alpha)
\end{aligned} \tag{45}$$

In summary, to compute the movement of the purely imaginary eigenvalues associated with the egg dynamics, one must first find S by solving (32) or (34). Then, provided S lies in the range (41) so that the associated equilibria are centers for the Hamiltonian dynamics, one applies (44) to determine whether the eigenvalues move into the left or right half-plane when the non-Hamiltonian perturbations are introduced. If $\varepsilon(B_{11}+B_{22})>0$, then the eigenvalues move to the right and the equilibria and the associated three-tori are unstable, at least for sufficiently small ε. If, on the other hand, $\varepsilon(B_{11}+B_{22})<0$, then we must complete the stability computation by determining how the zero eigenvalues associated with j and k conservation move with the introduction of the ε perturbations. This problem is tackled in the next section.

4.3 *The other eigenvalues*

We focus on the case where, in the Hamiltonian limit, the eigenvalues of the Jacobian matrix evaluated at an equilibrium are $\pm i\omega_0$, 0, 0, with $\omega_0\in\mathbb{R}$. Then, under the non-Hamiltonian perturbations, the simple eigenvalues will move to $i\omega_0+\varepsilon\lambda_1+\mathcal{O}(\varepsilon^2)$ and $-i\omega_0+\varepsilon\bar{\lambda}_1+\mathcal{O}(\varepsilon^2)$, where $\mathrm{Re}(\lambda_1)$ was calculated in the previous subsection. Meanwhile, the two zero eigenvalues will move an amount given by an expression of the form $\varepsilon^p\mu_1+o(\varepsilon^p)$ and $\varepsilon^q\mu_2+o(\varepsilon^q)$, for some $p\geq q>0$. (This comes from perturbation theory of spectra, as in, for example, Kato (1984, Chapter 2).) In this case, the trace of the Jacobian matrix is $\mathcal{O}(\varepsilon^r)$, where $r=q$ if $q<1$ and $r=1$ otherwise, and the determinant is $\mathcal{O}(\varepsilon^{p+q})$. Thus, $p=q=1$ if and only if the determinant is $\mathcal{O}(\varepsilon^2)$ and the trace is $\mathcal{O}(\varepsilon)$. Later in this subsection we show that we do in fact have $p=q=1$ for our system.

Given that $p=q=1$, we can determine the movement of the zero eigenvalues by computing the trace and determinant of the Jacobian matrix. In particular, the trace is $\varepsilon(2\mathrm{Re}(\lambda_1)+\mu_1+\mu_2)+o(\varepsilon)$ and the determinant is $\varepsilon^2\omega_0^2\mu_1\mu_2+o(\varepsilon^2)$. Since

$2\mathrm{Re}(\lambda_1)=B_{11}+B_{22}$ and ω_0^2 are known (see equations (39) and (44)), we can compute $\mu_1+\mu_2$ and $\mu_1\mu_2$ by computing the trace and determinant. The equilibria are stable under perturbation when $\omega_0^2>0$, $\varepsilon\mathrm{Re}(\lambda_1)<0$, $\varepsilon(\mu_1+\mu_2)<0$ and $\mu_1\mu_2>0$. If any of these inequalities are reversed, then the perturbed equilibria are unstable.

We compute the Jacobian matrix from the (y', z', j', k') equations in (25), remembering that x is determined implicitly in terms of y, z, j, k through the relation (11). In contrast to the calculation in the previous section, here the Jacobian matrix is evaluated at the perturbed equilibrium solution, which we specify as

$$(x_0, y_0, z_0, j_0, k_0)+\varepsilon(x_1, y_1, z_1, j_1, k_1)+\mathcal{O}(\varepsilon^2)$$

The unperturbed solution $(x_0, y_0, z_0, j_0, k_0)$ was given in equations (28)–(34) and the first-order corrections $(x_1, y_1, z_1, j_1, k_1)$ will be calculated as needed.

The Jacobian matrix is computed to have the form

$$\mathbf{M}=\begin{pmatrix} \varepsilon a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & \varepsilon a_{22} & \varepsilon a_{23} & \varepsilon a_{24} \\ \varepsilon^2 a_{31} & \varepsilon a_{32} & \varepsilon a_{33} & \varepsilon a_{34} \\ \varepsilon^2 a_{41} & \varepsilon a_{42} & \varepsilon a_{43} & \varepsilon a_{44} \end{pmatrix} \tag{46}$$

where the a_{ij} are order one in ε. From this, we establish that

$$\begin{aligned} \mathrm{Tr}(\mathbf{M}) &=\varepsilon(a_{11}+a_{22}+a_{33}+a_{44}) \\ \mathrm{Det}(\mathbf{M}) &=\varepsilon^2 a_{21}[a_{12}(a_{34}a_{43}-a_{33}a_{44})+a_{13}(a_{32}a_{44}-a_{34}a_{42}) \\ &\quad +a_{14}(a_{33}a_{42}-a_{32}a_{43})]+\mathcal{O}(\varepsilon^4) \end{aligned} \tag{47}$$

At leading order in ε, we find

$$\begin{aligned} a_{11}&=-4\delta x_0-2\alpha S-4\beta j_0k_0+4\alpha z_0^2-4\gamma f_i z_0 y_1/x_0 \\ a_{12}&=4\gamma f_i x_0-2f_i(2-6z_0^2+S)-8\gamma f_i z_0^2(1-z_0^2)/x_0 \\ a_{13}&=4f_i j_0\{1-z_0-\gamma z_0[(1-z_0)^2-k_0^2]/x_0\} \\ a_{14}&=-4f_i k_0\{1+z_0+\gamma z_0[(1+z_0)^2-j_0^2]/x_0\} \\ a_{21}&=2f_i \\ a_{22}&=-2\alpha(1-3z_0^2)-2\delta x_0-\alpha S-2\beta j_0k_0+4\delta z_0^2(1-z_0^2)/x_0 \\ a_{32}&=4\alpha z_0 j_0+2\beta k_0+4\delta j_0 z_0(1-z_0^2)/x_0 \\ a_{33}&=\alpha[2(1+z_0^2-j_0^2)-S]-2\delta x_0-4\beta j_0k_0+2\delta j_0^2[(1-z_0)^2-k_0^2]/x_0 \\ a_{34}&=-2\alpha j_0k_0+2\beta(1+z_0-j_0^2)+2\delta j_0k_0[(1+z_0)^2-j_0^2]/x_0 \\ a_{42}&=4\alpha z_0k_0-2\beta j_0+4\delta k_0 z_0(1-z_0^2)/x_0 \\ a_{43}&=-2\alpha j_0k_0+2\beta(1-z_0-k_0^2)+2\delta j_0k_0[(1-z_0)^2-k_0^2]/x_0 \\ a_{44}&=\alpha[2(1+z_0^2-k_0^2)-S]-2\delta x_0-4\beta j_0k_0+2\delta k_0^2[(1+z_0)^2-j_0^2]/x_0 \end{aligned} \tag{48}$$

where S is a solution of the cubic (32). Note that the leading order terms in the trace and the determinant depend on the perturbation of the solution through the component y_1 in a_{11} only; from the z'-component of (25) we find

$$f_i y_1 = \alpha(1 - z_0^2)z_0 + \delta x_0 z_0 + \beta j_0 k_0 z_0 \tag{49}$$

Here we use the fact that $j_0^2 - k_0^2 = z_0 S$ at the equilibrium (see equation (28)).

A fairly lengthy computation determines that the trace can be expressed in terms of the solution S of (32) as follows:

$$\mathrm{Tr}(\mathbf{M}) = \varepsilon \frac{(e_4 S^4 + e_3 S^3 + e_2 S^2 + e_1 S + e_0)}{\gamma\beta^2(\gamma^2 - 1)(S-2)^2} + \mathcal{O}(\varepsilon^2) \tag{50}$$

where

$$\begin{aligned}
e_0 &= 48\alpha\beta^2\gamma(\gamma^2 - 1)\\
e_1 &= 28\alpha\beta^2\gamma - 68\alpha\beta^2\gamma^3 - 16\alpha^2\delta - 40\beta^2\gamma^2\delta + 32\alpha^2\gamma^2\delta - 16\alpha^2\gamma^4\delta\\
e_2 &= 4\beta^2\gamma^4\delta - 2\alpha\beta^2\gamma - 16\alpha\gamma\delta^2 - 24\alpha^2\gamma^2\delta + 4\alpha^2\delta + 20\alpha^2\gamma^4\delta + 26\beta^2\gamma^2\delta\\
&\quad + 32\alpha\beta^2\gamma^3 + 16\alpha\gamma^3\delta^2\\
e_3 &= \gamma(4\alpha^2\gamma\delta - 4\gamma\delta^3 - 5\alpha\beta^2\gamma^2 - \beta^2\gamma\delta - 8\alpha^2\gamma^3\delta - 12\alpha\gamma^2\delta^2 + 4\alpha\delta^2 - 4\beta^2\gamma^3\delta)\\
e_4 &= \delta\gamma^2(\delta^2 - \beta^2 + \alpha^2\gamma^2 + 2\alpha\gamma\delta + \beta^2\gamma^2)
\end{aligned} \tag{51}$$

The determinant can similarly be expressed in terms of S. However, in this case the expression is quite unwieldy. For example, in the special case that $\alpha = 0$, we find that the determinant is of the form

$$\begin{aligned}
\mathrm{Det}(\mathbf{M}) &= \varepsilon^2 \frac{f_i^2 S(m_7 S^7 + m_6 S^6 + m_5 S^5 + m_4 S^4 + m_3 S^3 + m_2 S^2 + m_1 S + m_0)}{\beta^2 (S-2)^3 (\gamma^2 - 1)^3}\\
&\quad + \mathcal{O}(\varepsilon^3)
\end{aligned} \tag{52}$$

where the coefficients $m_0, \ldots, m_7$ are polynomials in β, γ and δ. We omit the explicit expressions, but pursue this example further in the next subsection. In any case, so long as the order ε terms in $\mathrm{Tr}(\mathbf{M})$ and the order ε^2 terms in $\mathrm{Det}(\mathbf{M})$ are not identically zero, we see that we do in fact have the desired dependence of the eigenvalues on ε, i.e. $p = q = 1$. We note that this is the case for the example given in the next subsection, and hence for an open set of the $(\alpha, \beta, \gamma, \delta, f_i)$-coefficient space.

These stability results have been calculated for three-tori that are solutions to equations (3), the cubic truncation of the normal form. Because the three-tori in the normal form are hyperbolic, the stability results carry over to the corresponding three-tori that exist in the full $D_2 \times_s T^2$ bifurcation problem.

4.4 *An example*

In this subsection, we give an example that demonstrates that three-tori can exist as stable solution branches. We set $\alpha = 0$, $\beta = 1$, $\delta = 2$, and consider a range of values of γ. Our conclusions are independent of the parameter f_i, provided $f_i \neq 0$.

In Section 3.3, we showed that for the chosen values of α, β, δ and $\gamma \in (1, 50]$, the solutions of the unperturbed problem, corresponding to $S = 2 \pm 4/\sqrt{\gamma^2 + 3}$, persist under small non-Hamiltonian perturbations. Substituting these values of S into equation (39) yields

$$w_0^2 = \frac{8f_i^2}{\gamma^2+3}(7\gamma^2 - 3 \pm 2\sqrt{\gamma^2+3})$$

Both choices of sign give positive expression for w_0^2 when $\gamma^2 > 1$, and this ensures that both equilibria correspond to elliptic fixed points in the Hamiltonian limit.

The movement of the purely imaginary eigenvalues under perturbation can be calculated from equation (44). Figure 5(a) plots the quantity $B_{11}+B_{22}$ for the equilibria corresponding to $S=2-4/\sqrt{\gamma^2+3}$ and for $\gamma \in [2, 50]$. We see that $B_{11}+B_{22}<0$ for these values of the coefficients; for $\varepsilon>0$ sufficiently small, the purely imaginary eigenvalues move into the left half plane uder perturbation. Figure 5(b) shows the order one terms of $\mathrm{Tr}(\mathbf{M})/\varepsilon - (B_{11}+B_{22})$ and Fig. 5(c) plots the order one terms of $\mathrm{Det}(\mathbf{M})/(f_i^2\varepsilon^2)$. We see that for $\varepsilon>0$ sufficiently small and $\gamma \in [2, 50]$, the zero eigenvalues perturb to μ_1, μ_2, where $\mu_1+\mu_2<0$ while $\mu_1\mu_2>0$; the zero eigenvalues also move into the left half-plane under perturbation. Thus, in the case $\alpha=0$, $\beta=1$, $\delta=2$, $\gamma \in [2, 50]$, $f_i \neq 0$, and for ε sufficiently small and positive, the equilibrium corresponding to $S=2-4/\sqrt{\gamma^2+3}$ becomes asymptotically stable under the addition of the non-Hamiltonian terms $\varepsilon \mathbf{f}_d$ in equations (25). This equilibrium corresponds to a stable branch of three-tori existing in the full system for $\lambda>0$ so long as $\sigma<0$. In terms of our original system of equations (3), we have shown that there is an open region of the parameter space with $f_r \approx 0$, $a_r \approx b_r \approx c_r \approx d_r$, $a_r + c_r < 0$, for which a branch of stable three-tori bifurcates supercritically from the equilibrium at the origin when $\lambda=0$.

Similar calculations for the choice $S=2+4/\sqrt{\gamma^2+3}$ show that, to lowest order in ε, $B_{11}+B_{22}>0$, $\mathrm{Tr}(\mathbf{M})/\varepsilon-(B_{11}+B_{22})>0$ and $\mathrm{Det}(\mathbf{M})>0$, i.e. when $\varepsilon>0$ the purely imaginary eigenvalues and the zero eigenvalues all move into the right half-plane under perturbation. Thus the corresponding branch of three-tori in the full system is unstable.

As a demonstration of the above result, we return to the original system of equations (3) for $(w_1, w_2, w_3, w_4) \in \mathbb{C}^4$. We numerically integrate equations (3) starting from a single initial condition and with the choice of parameters

$$\mu=0.1+i, \quad a=-0.95+i, \quad b=-0.9-i, \quad c=-0.95+2i, \quad d=-1-2i, \quad f=0.2+i \tag{53}$$

This choice of parameters gives

$$\alpha=0, \quad \beta=1, \quad \delta=2, \quad \gamma=6, \quad \sigma=-1.9, \quad \varepsilon=0.1 \tag{54}$$

If we neglect the $\mathcal{O}(\varepsilon)$ corrections, we expect the branch of stable three-tori to satisfy $L=-\lambda/2\sigma$, $S=2-2|\delta|/\sqrt{(\gamma^2-1)\beta^2+\delta^2}$, and equations (28)–(29), i.e. to satisfy

$$L\approx 0.026, \quad x\approx 0.15, \quad y\approx 0.0, \quad z^2\approx 0.077, \quad jk\approx 0.65, \quad S\approx 1.36 \tag{55}$$

The results of our numerical integration are presented in Figs 6 and 7. In Fig. 6, we plot $\mathrm{Re}(w_i)$ as functions of t, $i=1,\ldots,4$. The plots of $\mathrm{Im}(w_i)$ as functions of t are similar. From this data, we can generate plots of L, x, y, z, j, k as functions of t, and these are presented in Fig. 7. Note that in drawing Fig. 6, we discard the initial transient solution, but we retain the transient for Fig. 7. We find that the final values of L, x, y, z, j, k are (cf. equation (55))

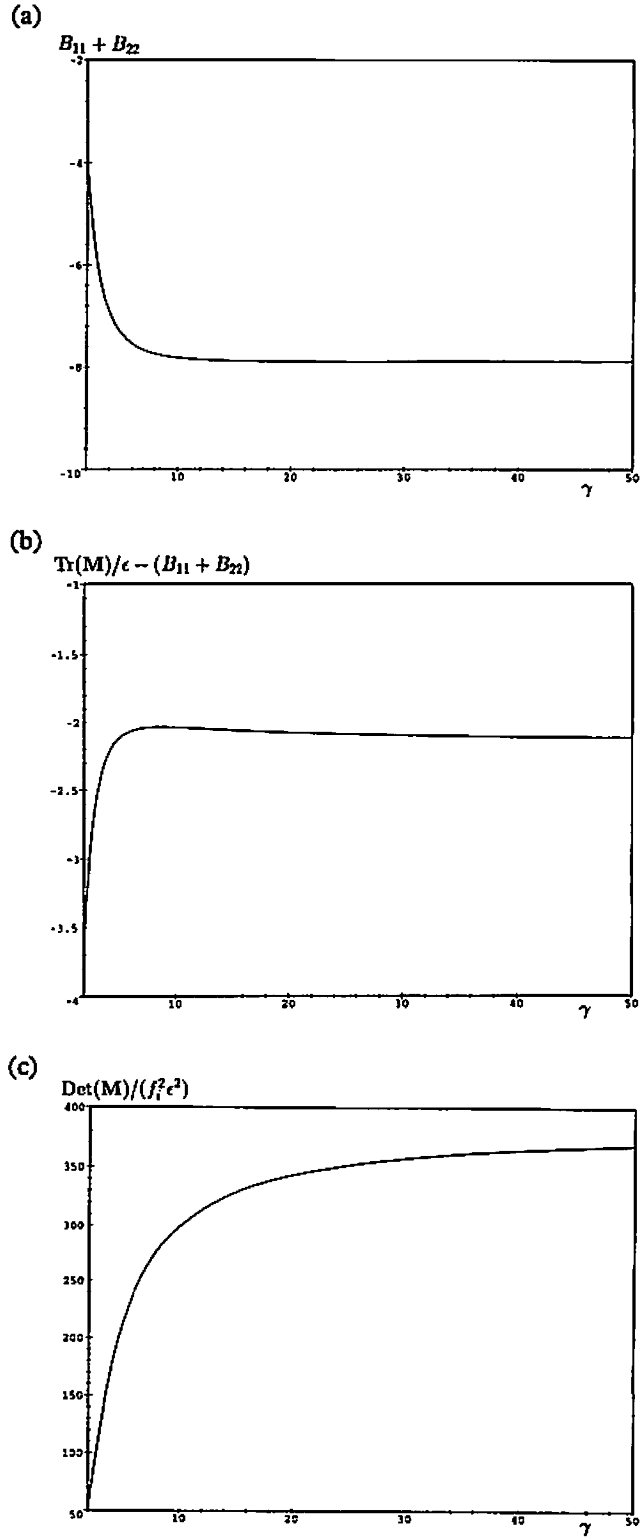

Fig. 5. For the case $\alpha=0$, $\beta=1$, $\delta=2$, $\gamma \in (2, 50]$, and with $S=2-4/\sqrt{\gamma^2+3}$: (a) $B_{11}+B_{22}$; (b) order one terms of $\mathrm{Tr}(M)/\varepsilon-(B_{11}+B_{22})$; (c) order one terms of $\mathrm{Det}(M)/(f_i^2\varepsilon^2)$.

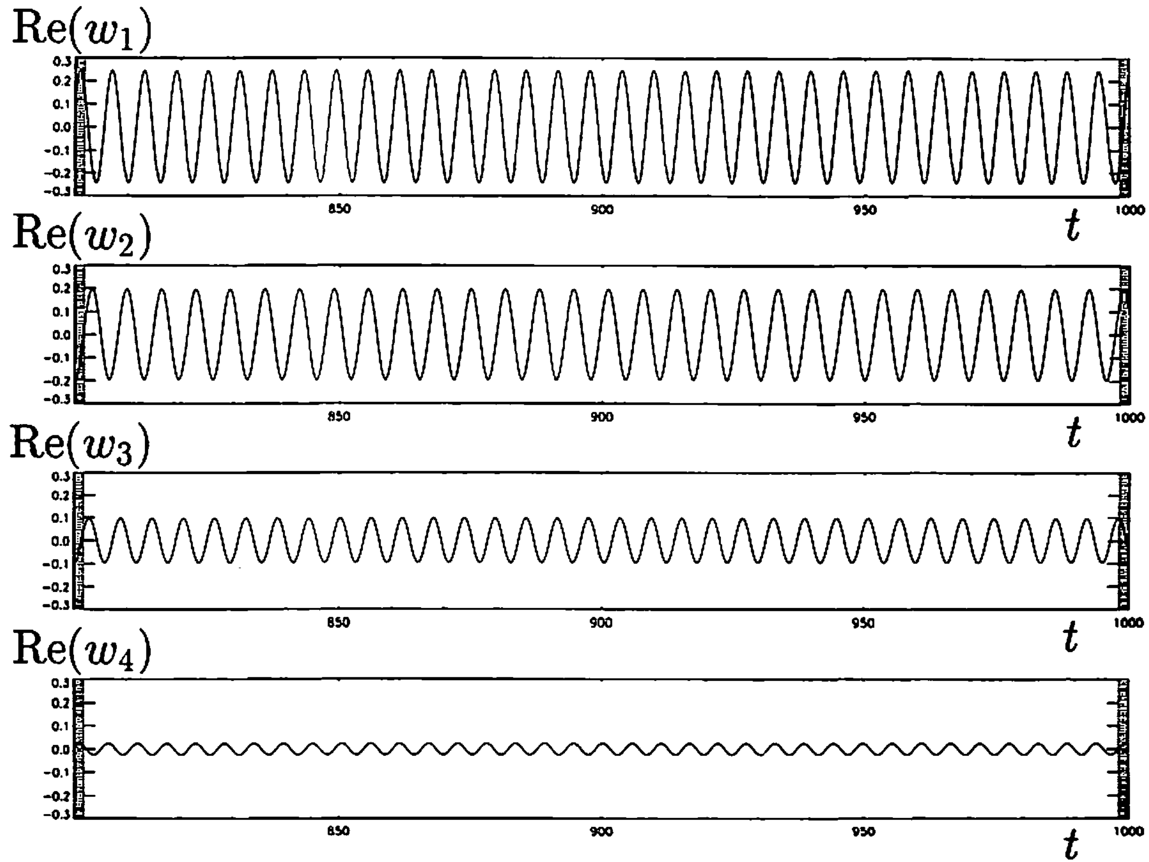

Fig. 6. Re(w_i) plotted as functions of t, $i=1, \ldots, 4$ for equations (3) with parameters $\mu=0.1+i$, $a=-0.95+i$, $b=-0.9-i$, $c=-0.95+2i$, $d=-1-2i$, $f=0.2+i$ and initial condition $w_1=0.23$, $w_2=0.2+0.01i$, $w_3=0.1-0.01i$, $w_4=0.05+0.005i$. The solution is displayed after the transient has died away.

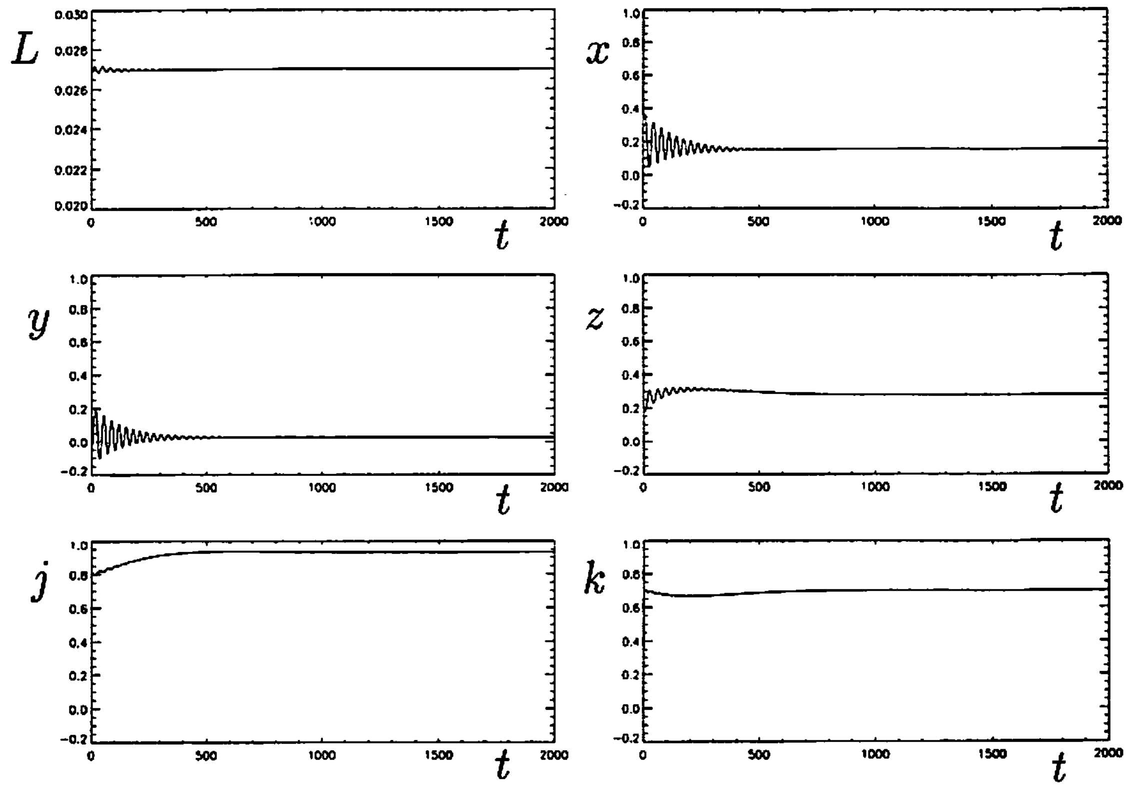

Fig. 7. Evolution of the invariants L, x, y, z, j, k with time for the solution of (3) shown in Fig. 6. The transient behavior is shown in this figure.

$$L \approx 0.027, \quad x \approx 0.16, \quad y \approx 0.03, \quad z^2 \approx 0.078, \quad jk \approx 0.65, \quad S \approx 1.35 \quad (56)$$

The solution plotted in Figs 6 and 7 appears to approach an equilibrium with $xz(|j|+|k|) \neq 0$, as is required for a non-symmetric three-torus. Furthermore, the amplitudes of oscillation for the components w_i are all different from one another, as is also required for a non-symmetric three-torus.

5 Summary

We have shown the existence of a branch of stable three-tori, bifurcating from the origin, in a Hopf bifurcation problem on $\mathbb{C}^4$ with $D_2 \times_s T^2$ symmetry. The three-tori completely break the symmetry of the problem, and correspond to relative equilibria for the normal form of the bifurcation problem. For the applications where this symmetry group and bifurcation problem arose, the three-tori represent three-frequency spatially periodic waves (Clune & Knobloch, 1994; Silber *et al.*, 1992). The existence of the branch of three-tori is shown in a regime of the coefficient space that is close to a Hamiltonian limit of the bifurcation problem. We expect the approach taken here may apply to other equivariant bifurcation problems that have integrable Hamiltonian limits by virtue of their continuous symmetries. The main steps in our analysis can be summarized as follows:

- We first posed the bifurcation problem in the orbit space of the continuous symmetries by introducing, as coordinates, invariants for the $T^2 \times S^1$ symmetry. This decoupled three phases from the rest of the dynamics, reducing the dimension of the problem from eight to five. The three-tori of interest are then equilibria in the orbit space.
- Next, we reduce the problem by one more dimension by rescaling time and our coordinates by an appropriate power of $L \equiv \frac{1}{4}|\mathbf{w}|^2$, where $\mathbf{w} \in \mathbb{C}^4$. This caused the $\dot{L}$ equation to decouple from the others. Moreover, the bifurcation parameter appeared only in the $\dot{L}$ equation with the consequence that the dynamics of the reduced problem were independent of the bifurcation parameter. This reduction worked because the nonlinear terms in the truncated normal form (3) are homogeneous (see Field & Swift, 1994).
- The remainder of our analysis was applied to the four-dimensional problem. The equilibria for this problem are readily identified in a Hamiltonian limit, although they are not isolated due to the integrable structure. We perturbed from this limit, and looked for equilibria that persisted under the non-Hamiltonian perturbations. The difficulty with equilibria not being isolated in the Hamiltonian limit was removed by rewriting the fixed point equations so that the implicit function theorem could be used to deduce persistence.
- We addressed the stability of the three-tori by computing the eigenvalues of the corresponding equilibria of the four-dimensional problem in the Hamiltonian limit. For the elliptic points, we used an eigenvalue movement formula derived in Appendix B to determine how the purely imaginary eigenvalues moved under the non-Hamiltonian perturbations.
- The movement of the two remaining eigenvalues was then calculated directly by a perturbation argument to finish the stability calculation.

It may be possible to use the framework presented here to also prove existence of a primary branch of four-tori. Based on what we have done, it is reasonable to expect that there are parameter values where the complex eigenvalues for an equilibrium in the four-dimensional problem remain purely imaginary under perturbation. To calculate the values of α, β, γ, δ for which this occurs, we would determine where $B_{11}+B_{22}=0$. If this condition can be met with α, β, δ not all zero, i.e. without reducing to the Hamiltonian limit, and we can show that $B_{11}+B_{22}$ changes sign as we vary these parameters, then we are in a setting where we may be able to apply the Hopf bifurcation theorem. The bifurcation parameter in this case would be a function of the nonlinear coefficients α, β, γ, δ; it would not depend on the value of the original bifurcation parameter λ, thereby ensuring that the four-tori appear as a primary branch. Four-tori obtained in this manner would correspond to relative periodic orbits in the original normal form. We would interpret this as meaning that one of the periodic orbits of the reduced Hamiltonian problem survives the non-Hamiltonian perturbation (see Fig. 1).

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Appendix A: Invariants and resonances

For the problem considered in this paper, symmetry forces a 1:1:1:1 resonance at the Hopf bifurcation point. In this appendix, we recall some results of Kummer (1990) that explain the Hamiltonian structure of our problem and similar problems. We shall first describe the situation relevant for simple resonances of two oscillators and then indicate how it may be generalized.

A.1 Resonances of two oscillators

We start by considering the action of S^1 on $\mathbb{C}^2$, pertinent to the $k{:}l$ resonance, given by

$$(w_1, w_2) \mapsto (e^{ik\theta} w_1, e^{il\theta} w_2) \tag{A1}$$

where k and l are integers.

Hamiltonian reduction by such a group action can be a useful tool in studying the associated Hamiltonian system, and its dissipative perturbations. For instance, Hamiltonian reduction led to the egg dynamics in Section 2.4. Here we describe this reduction from both the Poisson and the symplectic points of view. For the 1:1 resonance, this situation was developed by, among others, Cushman and Rod (1982) and Marsden (1987). Of course, resonances occupy a large body of literature, but two other references relevant to the reduction point of view are Holm (1989) (see also David *et al.*, 1990; David & Holm, 1992) and Kummer (1990).

The action (A1) is symplectic with respect to the symplectic form on $\mathbb{C}^2$ given by

$$\Omega((w_1, w_2), (\tilde{w}_1, \tilde{w}_2)) = -\frac{1}{k}\mathrm{Im}(w_1\overline{\tilde{w}_1}) - \frac{1}{l}\mathrm{Im}(w_2\overline{\tilde{w}_1}) \tag{A2}$$

which is a convenient rescaling of the canonical symplectic structure. Apart from this scaling, the Hamiltonian structure we use is the standard one obtained by taking the real and imaginary parts of w_1 and w_2 as conjugate variables. For example, with k and l equal to one and writing $w_1 = q_1 + ip_1$, Hamilton's equations have the standard form

$$\dot{q}_1 = \partial H/\partial p_1, \qquad \dot{p}_1 = -\partial H/\partial q_1$$

In complex notation, Hamilton's equations are $\dot{z}_k = -2i\partial H/\partial \bar{z}_k$.

Associated with the symmetry there is a corresponding conserved quantity, or momentum mapping, given in this case by

$$M(w_1, w_2) = \frac{1}{2}(|w_1|^2 + |w_2|^2) \tag{A3}$$

Note that if we had not put the scaling factors in the symplectic structure, they would appear in the momentum map.

The momentum map M is invariant under the S^1 action. Other invariant functions are given by

$$Z = \frac{1}{2}(|w_1|^2 - |w_2|^2) \tag{A4}$$

$$X + iY = \overline{w_1}^{\,l} w_2^k \tag{A5}$$

Notice that $-M \leq Z \leq M$. Also, note that these invariants are related by

$$X^2 + Y^2 = (M+Z)^l(M-Z)^k \tag{A6}$$

In performing Poisson reduction, one normally constructs the quotient space $\mathbb{C}^2/S^1$ and calculates its naturally induced Poisson bracket. However, except for the case of $k=1$ and $l=1$, the action, while locally free (apart from the origin), is not free (i.e. there are non-identity elements of the group that leave some points fixed), and so one has to be careful about singularities in the quotient space. For example, for $k=1$ and $l=2$, the action of the group element $e^{i\pi}$ leaves points of the form $(0, w_2) \in \mathbb{C}^2$ invariant. Nevertheless, there can often be situations where the quotient in the Poisson context can be singularity free, while the symplectic context has singularities.

For each real number m, define the map $\phi_m : \mathbb{R}^3 \to \mathbb{R}$ by

$$\phi_m = X^2 + Y^2 - (m+Z)^l(m-Z)^k \tag{A7}$$

Notice that the relation (A6) between the variables X, Y, Z, M can be written as $\phi_M(X, Y, Z) = 0$.

Proposition A1. The quotient $\mathbb{C}^2/S^1$ is identifiable with $\mathbb{R}^3$ coordinatized by (X, Y, Z) and it carries the quotient Poisson structure given as follows. Let F and G be given functions of X, Y, Z and let (X, Y, Z) lie on the set $\phi_m(X, Y, Z) = 0$. Then

$$\{F, G\}(X, Y, Z) = kl\nabla\phi_m \cdot (\nabla F \times \nabla G) \tag{A8}$$

Proof. This is proved as follows. Define $f = F \circ \pi$ where π is the map sending $(w_1, w_2) \mapsto (X, Y, Z)$. The Poisson bracket on $\mathbb{C}^2$ associated to the (scaled) canonical symplectic structure is given by

$$\{f, g\} = -k\mathrm{Im}\langle \nabla_{w1} f, \nabla_{w1} g\rangle - l\mathrm{Im}\langle \nabla_{w2} f, \nabla_{w2} g\rangle \tag{A9}$$

where the gradient is the real gradient, taken with respect to the real inner product. One now computes $\{f, g\}$ using the chain rule and one gets the bracket stated. It is a straightforward, although slightly lengthy, computation. ∎

Next, one shows that the symplectic leaves in the above Poisson structure are given by the symplectic reduced spaces, namely by the sets $\phi_m = 0$ corresponding to the symplectic-reduced spaces $M^{-1}(m)/S^1$. The bracket (A8) is the Poisson bracket associated with these leaves. The leaves $\phi_m = 0 \subset \mathbb{R}^3$ are, in general, 'pinched spheres', with the 'pinch' referring to the singularities in these sets, as mentioned above. For example, for the 2:1 resonance, this set is pear shaped with one conical singularity.

If H is a Hamiltonian on $\mathbb{C}^2$ that is invariant under the action of S^1, then it induces a function H on $\mathbb{R}^3$ and the reduced equations on the pinched sphere $\phi_m = 0$ are given by the (Euler-like) equations

$$\dot{V} = kl\nabla H \times \nabla\phi_m \tag{A10}$$

where $V = (X, Y, Z)$.

In deriving normal forms for bifurcations of equilibria in Hamiltonian systems, one naturally obtains systems that have a symmetry inherited from the flow of the linear part. Such normal forms may be integrable systems and be subject to an analysis as above, as they are in the present paper and in the following section. We also remark that one can derive a geometric phase formula associated with this situation that is analogous to Montgomery's geometric phase formula for the rigid body (see Marsden and Ratiu (1994) for a discussion and references).

A.2 Multiple resonances

The case of multiple resonances proceeds in a way similar to the case of simple resonances in the preceding subsection. The details depend on the nature of the symmetry and the resonance. We make a few remarks relevant to our case.

Start with the action of $\mathrm{T}^2 \times \mathrm{S}^1$ on $\mathbb{C}^4$ given by

$$\begin{aligned} (\theta_1, \theta_2) &: \mathbf{w} \mapsto (e^{i\theta_1} w_1, e^{i\theta_2} w_2, e^{-i\theta_1} w_3, e^{-i\theta_2} w_4) \\ \phi &: \mathbf{w} \mapsto e^{i\phi}\mathbf{w} \end{aligned} \tag{A11}$$

The canonical Poisson bracket on $\mathbb{C}^4$ is given by

$$\begin{aligned} \{f, g\}(\mathbf{w}, \bar{\mathbf{w}}) = {} & 2i\left(\frac{\partial f}{\partial \bar{w}_1}\frac{\partial g}{\partial w_1} - \frac{\partial f}{\partial w_1}\frac{\partial g}{\partial \bar{w}_1}\right) + 2i\left(\frac{\partial f}{\partial \bar{w}_2}\frac{\partial g}{\partial w_2} - \frac{\partial f}{\partial w_2}\frac{\partial g}{\partial \bar{w}_2}\right) \\ & + 2i\left(\frac{\partial f}{\partial \bar{w}_3}\frac{\partial g}{\partial w_3} - \frac{\partial f}{\partial w_3}\frac{\partial g}{\partial \bar{w}_3}\right) + 2i\left(\frac{\partial f}{\partial \bar{w}_4}\frac{\partial g}{\partial w_4} - \frac{\partial f}{\partial w_4}\frac{\partial g}{\partial \bar{w}_4}\right) \end{aligned} \tag{A12}$$

where w_j and $\bar{w}_j$ ($j=1, \ldots, 4$) are conjugate variables. Hamilton's equations are

$$\dot{w}_i = -2i \frac{\partial h}{\partial \bar{w}_i}, \qquad i=1, \ldots, 4 \tag{A13}$$

The momentum map is given by $(\mathcal{J}, K, L)$ as defined in (4) in the main body of the paper, and the other invariants are given by X, Y, Z, also as defined in (4). Define the map $\phi_{j,k,l}: \mathbb{R}^3 \to \mathbb{R}$ by

$$\phi_{j,k,l}(X, Y, Z) = (Z+l+j)(Z+l-j)(Z-l-k)(Z-l+k) - X^2 - Y^2 \tag{A14}$$

where $\phi_{J,K,L}(X, Y, Z) = 0$ on the reduced manifold defined by (7). A tedious calculation analogous to the simple resonance case then determines that the reduced bracket is a bracket of 'Nambu' type:

$$\{F, G\}(X, Y, Z) = \nabla\phi_{j,k,l} \cdot (\nabla F \times \nabla G) \tag{A15}$$

and the reduced vector field is

$$\dot{V} = \nabla H \times \nabla \phi_{j,k,l} \tag{A16}$$

where $V = (X, Y, Z)$. This Hamiltonian structure has of the form of a generalized rigid body, as with the case of simple resonances (see Churchill *et al.*, 1983; Haller & Wiggins, 1996).

Appendix B: An eigenvalue movement formula

This appendix derives a formula for the movement of eigenvalues that is relevant for dissipative perturbations of Hamiltonian systems. This formula is a modification of formulae for the movement of eigenvalues that go back to Krein, see MacKay (1991) and Bloch *et al.* (1994) and references therein. Our version of the formula allows the symplectic form to be degenerate, which is appropriate for the situation in this paper.

B.1 Notation

Start with a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \varepsilon)$ where $\mathbf{x} \in \mathbb{R}^n$ and ε is a small parameter. Assume that $\mathbf{x}(\varepsilon)$ is a curve of equilibria:

$$\mathbf{f}(\mathbf{x}(\varepsilon), \varepsilon) = \mathbf{0}$$

and that for $\varepsilon = 0$, the linearization in $\mathbf{x}$,

$$\mathbf{D}_x \mathbf{f}(\mathbf{x}(0), 0)$$

has purely imaginary eigenvalues $\lambda_0 = i w_0$ and $\overline{\lambda_0} = -i w_0$. We will assume these eigenvalues are simple so there are corresponding (smooth) eigenvalues $\lambda(\varepsilon)$ and eigenvectors $\mathbf{v}(\varepsilon)$ for $\mathbf{D}_x \mathbf{f}(\mathbf{x}(\varepsilon), \varepsilon)$ for small $|\varepsilon|$. Write

$$\mathbf{x}(\varepsilon) = \mathbf{x}_0 + \varepsilon \mathbf{x}_1 + \ldots$$

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon \lambda_1 + \ldots$$

$$\mathbf{v}(\varepsilon) = \mathbf{v}_0 + \varepsilon \mathbf{v}_1 + \ldots$$

Our goal is to find a formula for $\mathrm{Re}(\lambda_1)$. Note that

$$\lambda_1 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda(\varepsilon)|_{\varepsilon=0}$$

so that $\mathrm{Re}(\lambda_1)$ tells us to first order how the eigenvalues are moving transverse to the imaginary axis as ε is varied.

Write

$$\mathbf{f} = \mathbf{f}_h + \varepsilon \mathbf{f}_d$$

where $\mathbf{f}_h$ is Hamiltonian and $\mathbf{f}_d$ is dissipative (non-Hamiltonian). Explicitly, assume we have a skew bilinear form Ω on $\mathbb{R}^n$ relative to which $\mathbf{D}_x\mathbf{f}_h$ is skew. We do not assume Ω is non-degenerate. We write, relative to a set of coordinates,

$$\Omega(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T\mathbf{J}\mathbf{w}$$

where $\mathbf{v}^T$ is the transpose of $\mathbf{v}$, so $\mathbf{v}^T$ is a row vector and $\mathbf{J}$ is a real matrix.

Let $\mathbf{v}_0 = \mathbf{v}_r + i\mathbf{v}_i$ be the real and imaginary parts of $\mathbf{v}_0$. We will assume that

$$\mathbf{v}_r^T\mathbf{J}\mathbf{v}_i \neq 0$$

Define a linear operator $\mathbf{B}_d$ by

$$\mathbf{B}_d \cdot \mathbf{w} \equiv \mathbf{D}_x \mathbf{f}_d(\mathbf{x}_0, 0) \cdot \mathbf{w}$$

and note that this operator depends on $\mathbf{f}_d$ but not on $\mathbf{x}_1$.

B.2 *The formula*

Proposition B1. The following formula holds

$$\mathrm{Re}(\lambda_1) = \frac{\mathbf{v}_r^T\mathbf{J}\mathbf{B}_d\mathbf{v}_i - \mathbf{v}_i^T\mathbf{J}\mathbf{B}_d\mathbf{v}_r}{2\mathbf{v}_r^T\mathbf{J}\mathbf{v}_i}$$

Note especially that the formula does not involve $\mathbf{v}_1$ or $\mathbf{x}_1$ which is what makes it so useful.

B.3 *Proof of the formula*

We start by differentiating the defining condition

$$\mathbf{D}_x \mathbf{f}(\mathbf{x}(\varepsilon), \varepsilon)\mathbf{v}(\varepsilon) - \lambda(\varepsilon)\mathbf{v}(\varepsilon) = 0$$

with respect to ε at $\varepsilon = 0$ to give

$$\mathbf{D}_x \mathbf{f}(\mathbf{x}_0, 0)\cdot\mathbf{v}_1 + \mathbf{D}_x^2 \mathbf{f}(\mathbf{x}_0, 0)(\mathbf{x}_1, \mathbf{v}_0) + \mathbf{D}_{x\varepsilon}^2 \mathbf{f}(\mathbf{x}_0, 0)\cdot\mathbf{v}_0 = \lambda_0\mathbf{v}_1 + \lambda_1\mathbf{v}_0$$

Now we apply the operation $\Omega(\bar{\mathbf{v}}_0, \cdot)$ to each side of this equation, where the overbar denotes complex conjugation. Here we need to keep in mind that Ω is regarded as a real bilinear form extended to complex vectors and that $\mathbf{D}_x\mathbf{f}(\mathbf{x}_0, 0)$ is a real linear transformation. Since $\mathbf{D}_x\mathbf{f}(\mathbf{x}_0, 0) = \mathbf{D}_x\mathbf{f}_h(\mathbf{x}_0)$ is Ω-skew and since $\bar{\lambda}_0 = -\lambda_0$ because λ_0 is pure imaginary, the terms involving $\mathbf{v}_1$ drop out and we get

$$\Omega(\bar{\mathbf{v}}_0, \mathbf{D}_x^2\mathbf{f}(\mathbf{x}_0, 0)\cdot(\mathbf{x}_1, \mathbf{v}_0)) + \Omega(\bar{\mathbf{v}}_0, \mathbf{D}_{x\varepsilon}^2\mathbf{f}(\mathbf{x}_0, 0)\cdot\mathbf{v}_0) = \lambda_1\Omega(\bar{\mathbf{v}}_0, \mathbf{v}_0) \qquad \text{(B1)}$$

Next, we use the fact that $\mathbf{D}_x\mathbf{f}_h(\mathbf{x}(\varepsilon))$ is Ω-skew:

$$\Omega(\mathbf{D}_x\mathbf{f}_h(\mathbf{x}(\varepsilon))\cdot\mathbf{u},\mathbf{v}) = -\Omega(\mathbf{u},\mathbf{D}_x\mathbf{f}_h(\mathbf{x}(\varepsilon))\cdot\mathbf{v}) \tag{B2}$$

Differentiate (B2) with repect to ε at $\varepsilon=0$ to get

$$\Omega(\mathbf{D}_x^2\mathbf{f}_h(\mathbf{x}_0)\cdot(\mathbf{x}_1,\mathbf{u}),\mathbf{v}) = -\Omega(\mathbf{u},\mathbf{D}_x^2\mathbf{f}_h(\mathbf{x}_0)\cdot(\mathbf{x}_1,\mathbf{v})) \tag{B3}$$

This implies that the first term in (B1) is real. Thus, taking the imaginary part of (B1) and using $\mathbf{f}=\mathbf{f}_h+\varepsilon\mathbf{f}_d$ gives

$$\mathrm{Im}(\Omega(\bar{\mathbf{v}}_0,\mathbf{D}_x\mathbf{f}_d(\mathbf{x}_0,0)\cdot\mathbf{v}_0)) = \mathrm{Im}(\lambda_1\Omega(\bar{\mathbf{v}}_0,\mathbf{v}_0)) \tag{B4}$$

Since $\Omega(\bar{\mathbf{v}}_0,\mathbf{v}_0)$ is pure imaginary, this becomes

$$\mathrm{Im}(\Omega(\bar{\mathbf{v}}_0,\mathbf{B}_d\cdot\mathbf{v}_0)) = \mathrm{Re}(\lambda_1)\mathrm{Im}(\Omega(\bar{\mathbf{v}}_0,\mathbf{v}_0)) \tag{B5}$$

Solving for $\mathrm{Re}(\lambda_1)$, using the expression for Ω in terms of $\mathbf{J}$ and writing $\mathbf{v}_0=\mathbf{v}_r+i\mathbf{v}_i$, we get the stated formula.