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Semiclassical Monodromy and the Spherical Pendulum as a Complex Hamiltonian System

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1 Introduction

In Alber and Marsden [1992, 1994a, 1994b], we introduced a new method for obtaining geometric phase phenomena for soliton equations, including the familiar phase shift of interacting solitons—see for example, Ablowitz and Segur [1981]. Prior to this work, the phase spaces of integrable systems were viewed as being foliated by invariant tori; however, for soliton phase spaces and to get formulas for the geometric phases, we have shown that a foliation by noncompact varieties is essential. The method is based on what we called *asymptotic reduction*. This procedure is applied to a new complex angle representation on a noncompact invariant variety and it leads to a description of geometric phases in terms of the monodromy of the phase function at singularities on hyperelliptic Jacobian fibrations. Monodromy and action-angle variables for Hamiltonian systems were investigated in Duistermaat [1980], Cushman and Duistermaat [1988], Ercolani [1989], Baider, Churchill and Rod [1990], Alber and Marsden [1994a], Bates and Zou [1993], amongst others.

In the present paper, we apply our approach to effects that are related to the problem of semiclassical monodromy. A crucial point in doing this is to consider the complexification of the system. In particular, we describe new complex angle representations and Hamiltonians for the classical simple spherical pendulum (our angle representations hold for the n-dimensional case and are different from those of the above mentioned authors even in the real case, and are based on the Abel–Jacobi map). In particular, this yields new exponential Hamiltonians and angle representations on homoclinic varieties and leads to the introduction of Maslov indices of closed curves in Lagrangian submanifolds of the cotangent bundle of the configuration space. These Lagrangian submanifolds are defined by the first

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integrals of the problem and will be described below. Then we develop complex geometric asymptotics with corresponding quantum conditions. These quantum conditions include classical and complex monodromy together with phase shifts that are related to Maslov indices after transporting the system along certain closed curves in the space of parameters. We refer to these types of phase shifts which are associated to the quantum conditions and also phase shifts that are associated to singularities in the space of parameters as *semiclassical monodromy.*

Cushman and Duistermaat [1988] used Bohr-Sommerfeld quantum conditions in the 2-dimensional case (i.e., a pendulum moving in 3-space) and, using a numerical investigation, detected monodromy in the semiclassical spectrum of the Schrödinger operator. They also pointed out that the main difficulty is related to the construction of action-angle variables at the singular points. Guillemin and Uribe [1989] suggested that one should relate monodromy to Maslov effects and asked if one can "hear" monodromy for the spherical pendulum. In this paper, we establish this relation and show that indeed one can "hear" the semiclassical monodromy (in the sense that one can hear similar semiclassical modes in the acoustic problem).

To carry out this task, we obtain exact formulae for complex modes, the main ingredients in geometric asymptotics, which are defined on the covering space of the Riemann surface under consideration. (These modes can be treated in a way similar to the case of acoustic modes in the whispering gallery phenomenon. For a detailed description of this phenomenon see Keller and Rubinow [1960] and Alber [1989, 1991]). Then we obtain quantum conditions of Bohr-Sommerfeld-Keller (BSK) type, which include Maslov indices. We investigate these quantum conditions on the bundle of Riemann surfaces over the base space consisting of the spectral data, considered as parameters of the problem. Finally, we describe semiclassical monodromy which has contributions from classical (real) monodromy and complex monodromy, as shifts in the quantum conditions resulting from the transport of a semiclassical mode around certain closed curves in the space of parameters.

2 Monodromy and Complex Angle Representations

In this section we obtain complex Hamiltonians for the spherical pendulum. The method works in the general n-dimensional case and is presented in the Appendix; in the main text we focus on the two dimensional case (that is, the ordinary spherical pendulum in three-space) and just specialize the relevant results from the appendix to this case.

Our approach resolves the issue of multivaluedness for the angle representations through the introduction of Riemann surfaces. The associated Riemann surfaces depend parametrically on values of the first integrals of the problem and have singularities, as described below; these manifolds are Lagrangian submanifolds away from singularities. If one fixes values of the first integrals, then for each closed curve in phase space (normally chosen to be in the real phase space and passing through these Lagrangian submanifolds), one can define an associated Maslov index. This index plays a crucial role in the semiclassical geometric asymptotics as we shall see below.

As we shall see, the main difference between the spherical pendulum and the problem of geodesics on a sphere is the presence of additional singularities of these Lagrangian submanifolds. We will show that the angle representation in this case is similar to that for umbilic geodesics (see Alber, Camassa, Holm and Marsden [1995]).

2.1 Some Complex Geometry for the 2-dimensional Spherical Pendulum The Hamiltonian for the spherical pendulum in Cartesian coordinates Q_j, $j = 1, 2, 3$, and their conjugate momenta P_j has the form

$$H = \frac{1}{2}\left(\sum_{j=1}^{3} P_j^2 - \left(\sum_{j=1}^{3} P_j Q_j\right)^2\right) + Q_3. \tag{2.2.1}$$

Here the acceleration due to gravity and constraint of the length of Q have been set equal to one. In this Hamiltonian, the constraint

$$\sum_{j=1}^{3} Q_j^2 = 1 \tag{2.2.2}$$

has been enforced by the extra term

$$\left(\sum_{j=1}^{3} P_j Q_j\right)^2 .$$

Hamilton's equations on $\mathbb{R}^6$ for (2.2.1) *automatically preserve the set* (2.2.2). Notice also that this extra term vanishes on the constrained phase space, so the Hamiltonian becomes the usual expression of kinetic energy plus potential energy on this constrained phase space. This construction, which also holds in the n-dimensional case, provides an example of a Hamiltonian that automatically preserves a desired set of constraints.

Now we change to new variables z_1 and z_2 and their conjugate momenta P_{z_1}, P_{z_2} as follows. Let θ_1 and θ_2 be spherical coordinates, so that $Q_1 = \cos\theta_1 \sin\theta_2$, $Q_2 = \sin\theta_1 \sin\theta_2$, and $Q_3 = \cos\theta_2$, and let P_{θ_1} and P_{θ_2} be their conjugate momenta. Let

$$\begin{aligned} z_1 &= \cos^2\theta_1, \\ z_2 &= \cos\theta_2, \\ P_{z_1}^2(1 - z_1)z_2 &= P_{\theta_1}^2, \\ P_{z_2}^2(1 - z_2^2) &= P_{\theta_2}^2. \end{aligned} \tag{2.2.3}$$

This change of variables is canonical and in terms of them, the Hamiltonian becomes

$$H = \frac{1}{2(1 - z_2^2)} P_{z_1}^2 (1 - z_1) z_1 + \frac{1}{2} P_{z_2}^2 (1 - z_2^2) + z_2. \tag{2.2.4}$$

The n-dimensional version is given in the appendix.

Spherical coordinates provide a convenient "nested" structure that leads to the first integrals, whereas the variables z_j provide variables on the Riemann surfaces determined by the first integrals. We will demonstrate this below explicitly for the spherical pendulum.

2.2 The Angle Representation in the 2-dimensional Case First integrals for the spherical pendulum are the angular momentum about the z-axis and the energy, which we denote β_1 and β_2. A computation establishes the identities

$$\begin{aligned} P_{z_1}^2 &= \frac{\beta_1^2}{(1-z_1)z_1}, \\ P_{z_2}^2 &= 2\frac{\beta_2^2}{(1-z_2^2)} - \frac{\beta_1^2}{(1-z_2^2)^2} - 2\frac{z_2}{(1-z_2^2)} \\ &= \frac{2(\beta_2^2 - z_2)(1-z_2^2) - \beta_1^2}{(1-z_2^2)^2}. \end{aligned} \tag{2.2.5}$$

This yields the following expression for the action function

$$\begin{aligned} S &= S_1(z_1) + S_2(z_2) \\ &= \int_{z_1^0}^{z_1} \sqrt{\frac{\beta_1^2}{(1-z_1)z_1}}\, dz_1 + \int_{z_2^0}^{z_2} \sqrt{\frac{M(z_2)}{(1-z_2^2)^2}}\, dz_2, \end{aligned} \tag{2.2.6}$$

which will be regarded as the generating function of a canonical transformation from coordinates (z_j, P_{z_j}) to (β_k, α_k) to be described below. Here

$$M(z) = 2\left(\beta_2^2 - z\right)\left(1 - z^2\right) - \beta_1^2 \tag{2.2.7}$$

is a basic polynomial of the hyperelliptic curve (Riemann surface) given by

$$\begin{aligned} W^2 = M(z) &= 2\left(\beta_2^2 - z\right)\left(1 - z^2\right) - \beta_1^2 \\ &= (z - m_1)(z - m_2)(z - m_3), \end{aligned} \tag{2.2.8}$$

where, $-1 < m_3 < 0 < m_2 < 1 < m_1$. We choose β_1 and β_2 as action variables of the problem and introduce the following conjugate variables

$$\begin{aligned} \alpha_1 &= -\frac{\partial S}{\partial \beta_1} = -\int_{z_1^0}^{z_1} \frac{1}{\sqrt{(1-z_1)z_1}}\, dz_1 + \int_{z_2^0}^{z_2} \frac{\beta_1 dz_2}{(1-z_2^2)\sqrt{M(z_2)}} = \alpha_1^0, \\ \alpha_2 &= -\frac{\partial S}{\partial \beta_2} = -2\beta_2 \int_{z_2^0}^{z_2} \frac{dz_2}{\sqrt{M(z_2)}} = t + \alpha_2^0. \end{aligned} \tag{2.2.9}$$

Here $\alpha^0 = (\alpha_1^0, \alpha_2^0)$ is the base point of the angle representation $\alpha = (\alpha_1, \alpha_2)$ and t is time in the original Hamiltonian system. In the real case, z_1 and z_2 move along cycles l_1 and l_2 over the basic cuts $[0, 1]$ and $[m_3, m_2]$ on the corresponding Riemann surfaces $\mathfrak{R}_1$ and $\mathfrak{R}_2$ defined by

$$\mathfrak{R}_1 : W^2 = (1 - z_1)z_1, \quad \mathfrak{R}_2 : W^2 = M(z_2). \tag{2.2.10}$$

The pair (α_1, α_2) can be considered as an Abel–Jacobi map with singularities. These singularities result in an additional monodromy in the space of parameters.

Equations (2.2.5) define Lagrangian submanifolds $\mathcal{L} = \mathfrak{R}_1 \times \mathfrak{R}_2$ as level sets of the first integrals in the phase space $\mathbb{C}^4$ except at singular points, namely at points where z_1 is 0 or 1 and z_2 is ± 1.

Definition 2.1 *We call the Hamiltonian system with Hamiltonian* (2.2.4) *and first integrals* (2.2.5) *on the Lagrangian submanifolds* $\mathcal{L}$ *the complex Hamiltonian system associated with the* 2*-dimensional spherical pendulum.*

This family of parameterized Lagrangian submanifolds will be used to define semiclassical monodromy and Maslov phases below.

The second expression in (2.2.9) can be considered as a Jacobi inversion problem on the Riemann surface (2.2.8). This shows that z_2 is a periodic function defined on this Riemann surface. The first integral from the expression for α_1 is equal to θ_1. This shows that

$$\begin{aligned}\theta_1 &= \frac{1}{2}\int_{z_2^0}^{z_2} \frac{\beta_1 dz_2}{(1-z_2^2)\sqrt{M(z_2)}} - \alpha_1^0 \\ \theta_2 &= \arccos z_2.\end{aligned} \tag{2.2.11}$$

Here the function z_2, the upper limit of the integral in the expression for θ_1, is considered on the Riemann surface $\Re$ given by (2.2.8). Therefore, θ_1 is defined on the covering space of $\Re$. In the real case, these two angles (θ_1, θ_2) are the usual spherical coordinates of the pendulum.

In conclusion, note especially that the construction presented here resolves the problem of multivaluedness for the angle representations for the two dimensional spherical pendulum only for the fixed values of parameters β_1 and β_2. First, averaged values of α_1 and α_2 are multivalued functions of β_1 and β_2 due to presence of classical monodromy. (For details concerning classical monodromy see Duistermaat [1980], Cushman and Duistermaat [1988] and Bates and Zou [1993]). Second, the additional monodromy of the singularities in the space of parameters is related to the symplectic representation of the braid group in a way described in Alber [1991b] and Alber and Marsden [1992]. These two types of monodromy are obtained when the system goes through the singularities ($\beta_1 = 0, \beta_2 = \pm 1$) in the first case and $m_3 = m_2$ in the second case. The first type of singularity can be resolved in the real context. In the second case, complexification is essential. We notice that in the multidimensional case $n \geq 3$, one gets additional monodromy due to the presence of additional singularities.

3 The Homoclinic Variety for the Spherical Pendulum

In this section, we will investigate the dependence of the Hamiltonian system defined by the spherical pendulum, on parameters. We will choose the parameters, as in the previous section, to be the β_j introduced in the Appendix for the n dimensional case. In the two dimensional case, the parameters, β_1 and β_2 are the angular momentum and the energy, as in Cushman and Duistermaat [1988], as we mentioned earlier.

This will lead, in particular, to the description of complex exponential Hamiltonians. We will also construct explicit homoclinic action-angle variables and describe the corresponding monodromy. In the next section, this monodromy will be shown to be one of the components of the semiclassical monodromy. This is done by investigating the dependence of the complex modes and the quantum conditions on the parameters of the system.

3.1 Homoclinic Angle Representations We now will choose special values of β_1 and β_2 that correspond to choosing special values of the constants of motion that put one on the homoclinic variety. This yields the following form of the basic polynomial associated with the homoclinic variety

$$M(z) = (1-z)^2(1+z). \tag{3.3.1}$$

Namely, we consider the following limiting process: in the polynomial (2.2.8), we let β_1^2 tend to zero and let β_2^2 tend to one so that the polynomial develops a double root. Recall that this corresponds to a special value of the conserved quantities. In this case, the system of angle variables can be represented as follows,

$$\begin{aligned} \alpha_1 &= -\int_{z_1^0}^{z_1} \frac{1}{\sqrt{(1-z_1)z_1}}\,dz_1 = \alpha_1^0, \\ \alpha_2 &= -\sqrt{2}\int_{z_2^0}^{z_2} \frac{dz_2}{(1-z_2)\sqrt{1+z_2}} = t + \alpha_2^0. \end{aligned} \tag{3.3.2}$$

The problem of inversion for (3.3.2) leads to the expression for the homoclinic orbit of the usual pendulum. This means that homoclinic manifold for the spherical pendulum splits into 1-dimensional homoclinic orbits on the great circles obtained by intersecting S^2 and a plane through the poles parameterized by the angle $\theta_1 = \alpha_1^0 = \text{const}$.

Lastly, we construct exponential Hamiltonians on the homoclinic varieties.

Theorem 3.1 *The spherical pendulum on the homoclinic variety defined above is a Hamiltonian system with a Hamiltonian of exponential type.*

Proof We introduce the following action function

$$S = \int_{z_1^0}^{z_1} \sqrt{(1-z_1)z_1 + b_1}\,dz_1 + \int_{z_2^0}^{z_2} \frac{\log(b_2 - z_2)}{\sqrt{1+z_2}}\,dz_2. \tag{3.3.3}$$

This corresponds to a new choice of the first integrals

$$\begin{aligned} P_1 &= \sqrt{(1-z_1)z_1 + b_1}, \\ P_2 &= \frac{\log(b_2 - z_2)}{\sqrt{1+z_2}}, \end{aligned} \tag{3.3.4}$$

for the system of differential equations

$$\begin{aligned} z_1' &= \sqrt{(1-z_1)z_1 + b_1} \\ z_2' &= (b_2 - z_2)\sqrt{1+z_2}. \end{aligned} \tag{3.3.5}$$

This is obtained from the Hamiltonian system with Hamiltonian (5.5.1) by setting $\beta_1 = 0$ and $\beta_2 = 1$ and by introducing parameters b_1 and b_2. System (3.3.5) is a Hamiltonian system with the Hamiltonian

$$H = \left(P_1^2 - ((1-z_1)\,z_1 + b_1)\right) + \left(e^{\sqrt{1+z_2}P_2} - (b_2 - z_2)\right). \tag{3.3.6}$$

Now we consider the following system of action-angle variables

$$\begin{aligned} I_1 &= b_1, \quad \alpha_1 = -\frac{\partial S}{\partial b_1}, \\ I_2 &= b_2, \quad \alpha_2 = -\frac{\partial S}{\partial b_2}. \end{aligned} \tag{3.3.7}$$

Lastly, one obtains angle representation (3.3.2) by setting $b_1 = 0$ and $b_2 = 1$ in (3.3.7). □

4 Semiclassical Monodromy

In what follows, we recall from Alber [1989, 1991] and Alber and Marsden [1992], a method of complex geometric asymptotics for integrable Hamiltonian flows on Riemann surfaces. We will use geometric asymptotics to describe the quantization conditions of Bohr-Sommerfeld-Keller (BSK) type. Then we will investigate the dependence of these conditions on the parameters (i.e., the first integrals) of the system. This dependence near singularities produces effects caused by the classical and semiclassical monodromy.

Let us consider quadratic complex Hamiltonians of the following form:

$$H = \frac{1}{2}\sum_{j=1}^{n} g^{jj} P_{z_j}^2 + V(z_1, \ldots, z_n) \tag{4.4.1}$$

defined on $\mathbb{C}^{2n}$. We think of $\mathbb{C}^{2n}$ as being the cotangent bundle of $\mathbb{C}^n$, with configuration variables $\mu_1, \ldots, \mu_n$ and with canonically conjugate momenta $P_1, \ldots, P_n$.

Notice that the systems considered earlier are indeed of this form. We consider the functions g^{jj} as components of a (diagonal) Riemannian metric, construct the associated Laplace-Beltrami operator, and then the stationary Schrödinger equation

$$\nabla^j \nabla_j U + w^2(E - V)U = 0, \tag{4.4.2}$$

defined on the n-dimensional complex Riemannian manifold $\mathbb{C}^n$. Here ∇^j and ∇_j are covariant and contravariant derivatives defined by the tensor g^{jj} and w (which is the inverse of Planck's constant $\hbar$) and E (the energy eigenvalue) are parameters. Note also that in general, the metric tensor is not constant, and even may have singularities, so that the kinetic term in the expression for H is *not* purely quadratic.

Now we establish a link between equation (4.4.2) and the Hamiltonian system (4.4.1) by means of geometric asymptotics; namely, we consider the following function that is similar to the well known Ansatz from WKB theory:

$$\begin{aligned} U(z_1, \ldots, z_n) &= \sum_k A_k(z_1, \ldots, z_n) \exp[iwS_k(z_1, \ldots, z_n)] \\ &= \sum_k \prod_{j=1}^{n} U_{kj}(z_j) \\ &= \sum_k \prod_{j=1}^{n} (A_{kj}(z_j) \exp[iwS_{kj}(z_j)]), \end{aligned} \tag{4.4.3}$$

which is a multivalued function of several complex variables defined on $\mathbb{C}^n$. If, instead, one considers U to be defined on the covering space of the Jacobi variety of the problem, then U becomes single valued. (The Jacobi variety was described in Section 3.) The functions present in this expression together with r, which denotes a vector of Maslov indices, will be determined below.

Substituting (4.4.3) in (4.4.2), equating coefficients for w and w^2 and integrating, we obtain the amplitude function A, which is a solution of the transport equation, of the form

$$A = \frac{A_0}{\sqrt{(D \det J)}}. \tag{4.4.4}$$

Here D is the volume element of the metric: $D = \sqrt{\prod_{i=1}^{n} g_{ii}}$ and J is the Jacobian of the change of coordinates from the z-representation to the angle (α) representation. We also find that the phase function S is a solution of the Hamilton-Jacobi equation

$$\Delta^j S \Delta_j S - V = E, \tag{4.4.5}$$

meaning that it coincides with the action function.

Now we can apply the above construction to the 2-dimensional spherical pendulum. Note that in this case the action function (2.2.6) can be represented in terms of angle variables (2.2.9) as follows

$$S = -\beta_1 \alpha_1^0 - \beta_2^2 \alpha_2 - 2 \int_{z_2^0}^{z_2} \frac{z_2 dz_2}{\sqrt{M(z_2)}}. \tag{4.4.6}$$

The last two terms correspond to the holomorphic and meromorphic parts of the action function. The holomorphic part is proportional to the angle variable of the classical problem. The amplitude A can be found after calculating D and J

$$D = \sqrt{g_{11} g_{22}} = \frac{1}{\sqrt{(1 - z_1) z_1}} \tag{4.4.7}$$

and

$$\det J^{-1} = \left| \frac{\partial \alpha_i}{\partial z_j} \right| = -\frac{2\beta_2}{\sqrt{M(z_2) z_1 (1 - z_1)}}. \tag{4.4.8}$$

This results in the following form of the function U:

$$U = \sum_{k=(k_1,k_2)} A_0 \sqrt{2\beta_2} (M(z_2))^{-\frac{1}{4}} \exp \left[iw \sum_{j=1}^{2} S_{kj}(z_j) \right], \tag{4.4.9}$$

where

$$S_{k1}(z_1) = \int_{z_1^0}^{z_1} \sqrt{\frac{\beta_1^2}{(1 - z_1) z_1}}\, dz_1 + k_1 T_1, \tag{4.4.10}$$

and

$$S_{k2}(z_2) = \int_{z_2^0}^{z_2} \sqrt{\frac{M(z_2)}{(1 - z_2^2)^2}}\, dz_2 + k_2 T_2 + \frac{\pi}{2} r_2, \tag{4.4.11}$$

where r_2 is Maslov index, and

$$\begin{aligned} T_1 &= \oint_{l_1} \sqrt{\frac{\beta_1^2}{(1 - z_1) z_1}}\, dz_1, \\ T_2 &= 2\beta_2^2 \oint_{l_2} \frac{dz_2}{\sqrt{M(z_2)}} - 2 \oint_{l_2} \frac{z_2 dz_2}{\sqrt{M(z_2)}} - \beta_1^2 \oint_{l_2} \frac{dz_2}{(1 - z_2^2)\sqrt{M(z_2)}}. \end{aligned} \tag{4.4.12}$$

The amplitude A has singularities at the branch points $z_2 = m_1, m_2, m_3$ of the Riemann surface (2.2.8). Each time a trajectory approaches one of these singularities, we continue in complex time and go around a small circle in complex plain enclosing the singularity. This results in a phase shift $(\pm i\pi/2)$ of the phase function S, which is common in geometric asymptotics. The indices k_1 and k_2 keep track of the number of oriented circuits for z_1 and z_2 around l_1 and l_2. The complex mode (4.4.9) is defined on the covering space of the complex Jacobi variety. Note that in the real case, it is defined on the covering space of a real subtorus. Keeping this in mind, quantum conditions of BSK type can be imposed as conditions on the number of sheets of the covering space of the corresponding Riemann surface for each coordinate z_j:

$$\begin{aligned} wk_1T_1 &= 2\pi N_1, \\ \frac{\pi}{2}r_2 + wk_2T_2 &= 2\pi N_2. \end{aligned} \tag{4.4.13}$$

Here N_1, N_2 are integer quantum numbers. The quantum conditions (4.4.13) include a monodromy part after transport along a closed loop in the space of parameters (β_1 and β_2). This semiclassical monodromy consists of a classical part as well as a contribution from complex monodromy and the Maslov phase.

Classical monodromy may be explained briefly as follows. We consider two different cases, namely the case $-1 < \beta_2^2 < 1$ and $\beta_2^2 > 1$. In the first case, one considers a cycle l_2 over the cut $[-1, \beta_2^2]$ and in the second case, one considers a cycle l_2 over the cut $[-1, 1]$. There is a closed curve in the space of parameters that leads one from one case to the other. Evidently, there is a difference in the values of the third integral in the expression for T_2 between the two cases that is given by the residue of the integrand at $z_2 = 1$.

The complex monodromy can be demonstrated if one of the roots m_1 and m_2 of the basic polynomial $M(z)$ approach each other. This singularity can be resolved by interchanging these two roots in the complex plane so as to avoid a real singularity. This leads to the change of orientation of the cycle l_2 and in the general case, can be described by the generator of the symplectic representation of the braid group. It results in additional shift in the quantum conditions.

The third type of shift in the quantum conditions comes from the integral representation for the Maslov class.

The complex mode (4.4.9), which corresponds to a particular choice of parameters in (4.4.13) is similar to an acoustic mode that occurs in the whispering gallery phenomenon described in Keller and Rubinow [1960] and Alber [1989, 1991].

The n-dimensional system can be treated in a similar way. A complex mode U has the form

$$U = \sum_{k=(k_1,\ldots,k_n)} \frac{A_0\sqrt{2\beta_2\ldots\beta_n}}{((-1)^n K_2(z_2)\ldots K_n(z_n)(1-z_3)(1-z_4)^2\ldots(1-z_n)^{n-2})^{\frac{1}{4}}} \times \exp\left[iw\sum_{j=1}^{n} S_{kj}(z_j)\right]. \tag{4.4.14}$$

which yields following quantum conditions

$$\begin{aligned} wk_1T_1 &= 2\pi N_1, \\ \frac{\pi}{2}r_2 + wk_2T_2 &= 2\pi N_2, \\ &\cdots \\ \frac{\pi}{2}r_n + wk_nT_n &= 2\pi N_n. \end{aligned} \tag{4.4.15}$$

Since the form of $T_j, j = 2, \ldots, n-1$ is different from both T_1 and T_n, one gets additional new types of monodromy in the n-dimensional case.

Lastly, the results of Section 4 yield a construction of the complex mode defined on the homoclinic varieties despite the fact that the Hamiltonians are not quadratic, but are exponential in this case.

5 Appendix: The n-dimensional Spherical Pendulum

Recall that a Riemann surface $\mathfrak{R}$ is a connected two-dimensional topological manifold with a complex-analytic structure. Compact Riemann surfaces are determined by nonsingular algebraic curves of the form $R(W, E) = 0$, where (W, E) are points in $\mathbb{C}^2$, and $R(W, E)$ is a polynomial with no multiple roots. We shall also need notions about complex Jacobi varieties and Abel–Jacobi maps; for details see Ercolani and McKean [1990].

A.1 The Hamiltonian and First Integrals In what follows we will handle the spherical pendulum in a way similar to the problem of geodesics on an n-dimensional sphere in $\mathbb{R}^{n+1}$.

We recall from Alber and Alber [1985] the following form of the Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{n} P_{z_j}^2 (1 - z_j) z_j \left(\prod_{k=j+1}^{n} \frac{1}{1 - z_k} \right). \tag{5.5.1}$$

for the problem of geodesics on the n-sphere of radius 1:

$$\sum_{j=1}^{n+1} Q_j^2 = 1.$$

To explain the notation in this expression for the Hamiltonian, we consider n-dimensional spherical coordinates on the unit n-sphere defined as follows

$$\begin{aligned} Q_1 &= (\cos\theta_1)(\sin\theta_2)\ldots(\sin\theta_n), \\ Q_2 &= (\sin\theta_1)(\sin\theta_2)\ldots(\sin\theta_n), \\ &\ldots \\ Q_n &= (\cos\theta_{n-1})(\sin\theta_n), \\ Q_{n+1} &= \cos\theta_n. \end{aligned} \tag{5.5.2}$$

In (5.5.1) the variables z_j and P_{z_j}, the momentum conjugate to z_j, are related to spherical coordinates and momentum (θ_j, P_{θ_j}) on the sphere by

$$\begin{aligned} z_j &= \cos^2\theta_j, \\ P_{z_j}^2 (1 - z_j) z_j &= P_{\theta_j}^2, \end{aligned} \tag{5.5.3}$$

where $j = 1, \ldots, n$. The Hamiltonian is taken to be the standard one, namely the kinetic energy. When the Hamiltonian is transformed to the variables z_j and P_{z_j} one gets the expression (5.5.1). Spherical coordinates provide a convenient "nested" structure that leads to the first integrals, whereas the variables z_j provide variables on the Riemann surfaces determined by the first integrals. We will demonstrate this below explicitly for the spherical pendulum.

The Hamiltonian of the n-dimensional spherical pendulum in Cartesian coordinates Q_j and their conjugate momenta P_j has the form

$$H = \frac{1}{2}\left(\sum_{j=1}^{n+1} P_j^2 - \left(\sum_{j=1}^{n+1} P_j Q_j\right)^2\right) + Q_{n+1}. \tag{5.5.4}$$

Here the acceleration due to gravity and constraint of the length of Q have been set equal to one.

Remark 5.1 In this Hamiltonian the constraint

$$\sum_{j=1}^{n+1} Q_j^2 = 1 \tag{5.5.5}$$

has been enforced by the extra term

$$\left(\sum_{j=1}^{n+1} P_j Q_j\right)^2 .$$

Hamilton's equations on $\mathbb{R}^{2(n+1)}$ for (5.5.4) automatically preserve the set (5.5.5). Notice also that this extra term vanishes on the constrained phase space, so the Hamiltonian becomes the usual expression of kinetic energy plus potential energy on this constrained phase space. This construction provides an example of a Hamiltonian that automatically preserves a desired set of constraints.

The same Hamiltonian in the n-dimensional spherical coordinates (5.5.2) can be expressed in the following "nested" form

$$H = \frac{1}{2R^2}\sum_{j=1}^{n} P_{\theta_j}^2 \left(\prod_{k=j+1}^{n} \frac{1}{(\sin\theta_k)^2}\right) + R\cos\theta_n, \tag{5.5.6}$$

or

$$\begin{aligned} H = \frac{1}{2R^2}\left(\frac{1}{\sin^2\theta_n}\left(P_{\theta_{n-1}}^2 + \frac{1}{\sin^2\theta_{n-1}}\left(P_{\theta_{n-2}}^2 + \frac{1}{\sin^2\theta_{n-2}}\right.\right.\right. \\ \left.\left.\left.\left(\cdots + \frac{1}{\sin^2\theta_3}\left(P_{\theta_2}^2 + \frac{1}{\sin^2\theta_2} P_{\theta_1}^2\right)\right)\cdots\right)\right)\right) \\ + \frac{1}{2R^2} P_{\theta_n}^2 + R\cos\theta_n. \end{aligned} \tag{5.5.7}$$

The change of coordinates

$$\begin{aligned} z_j &= \cos^2\theta_j, \quad P_{z_j}^2(1-z_j)z_j = P_{\theta_j}^2, \qquad j = 1,\ldots,(n-1); \\ z_n &= \cos\theta_n, \quad P_{z_n}^2(1-z_n^2) = P_{\theta_n}^2. \end{aligned} \tag{5.5.8}$$

results in the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n-1} P_{z_j}^2 (1 - z_j) z_j \left(\prod_{k=j+1}^{n-1} \frac{1}{(1 - z_k)} \right) \frac{1}{(1 - z_n^2)} + \frac{1}{2} P_{z_n}^2 (1 - z_n^2) + z_n. \quad (5.5.9)$$

Here $R = 1$ and (z_j, P_{z_j}) and (θ_j, P_{θ_j}) are pairs of conjugate variables.

The nested structure of the Hamiltonian (5.5.7) shows that one has the following first integrals for the n-dimensional pendulum

$$\begin{aligned} P_{\theta_1}^2 &= \beta_1^2, \\ P_{\theta_2}^2 + \frac{\beta_1^2}{\sin^2 \theta_2} &= \beta_2^2, \\ &\dots \\ P_{\theta_{n-1}}^2 + \frac{\beta_{n-2}^2}{\sin^2 \theta_{n-1}} &= \beta_{n-1}^2, \\ \frac{1}{2}\left(P_{\theta_n}^2 + \frac{\beta_{n-1}^2}{\sin^2 \theta_n} \right) + \cos\theta_n &= \beta_n^2. \end{aligned} \quad (5.5.10)$$

Here β_j are constants along solutions of the corresponding Hamiltonian system. Denote

$$K_j(z) = \beta_j^2 (1 - z) - \beta_{j-1}^2, \quad j = 2, \dots, n-1,$$

and

$$K_n(z) = 2\left(\beta_n^2 - z\right)\left(1 - z^2\right) - \beta_{n-1}^2.$$

Using the change of coordinates (5.5.8), the first integrals take the form

$$\begin{aligned} P_{z_1}^2 &= \frac{\beta_1^2}{z_1(1 - z_1)}, \\ P_{z_2}^2 &= \frac{\beta_2^2}{z_2(1 - z_2)} - \frac{\beta_1^2}{z_2(1 - z_2)^2} = \frac{K_2(z_2)}{z_2(1 - z_2)^2}, \\ &\dots \\ P_{z_{n-1}}^2 &= \frac{\beta_{n-1}^2}{z_{n-1}(1 - z_{n-1})} - \frac{\beta_{n-2}^2}{z_{n-1}(1 - z_{n-1})^2} = \frac{K_{n-1}(z_{n-1})}{z_{n-1}(1 - z_{n-1})^2}, \\ P_{z_n}^2 &= 2\frac{\beta_n^2}{(1 - z_n^2)} - \frac{\beta_{n-1}^2}{(1 - z_n^2)^2} - 2\frac{z_n}{(1 - z_n^2)} = \frac{K_n(z_n)}{(1 - z_n^2)^2}, \end{aligned} \quad (5.5.11)$$

where constants β_j^2 are positive real numbers. Therefore, we consider the real-valued solutions in which case one has a correspondence with the system in spherical coordinates.

A.2 The Complexified n-dimensional Spherical Pendulum Now we use the representation (5.5.11) to extend our system into the complex domain by considering β_j^2 as complex numbers and variables z_j defined on the associated Riemann surfaces:

$$\begin{aligned}
\Re_1 : W_1^2 &= \frac{\beta_1^2}{z_1(1-z_1)},\\
\Re_2 : W_2^2 &= \frac{K_2(z_2)}{z_2(1-z_2)^2},\\
&\dots \\
\Re_{n-1} : W_{n-1}^2 &= \frac{K_{n-1}(z_{n-1})}{z_{n-1}(1-z_{n-1})^2}\\
\Re_n : W_n^2 &= \frac{K_n(z_n)}{(1-z_n^2)^2}.
\end{aligned} \tag{5.5.12}$$

Equations (5.5.10) define Lagrangian submanifolds $\mathcal{L} = \Re_1 \times \cdots \times \Re_n$ as level sets of the first integrals in the phase space $\mathbb{C}^{2n}$ except at singular points, namely at points where z_j is 0 or ± 1.

Definition 5.2 *We call the Hamiltonian system with Hamiltonian* (5.5.9) *and first integrals* (5.5.11) *on the Lagrangian submanifolds $\mathcal{L}$ the complex Hamiltonian system associated with the n-dimensional spherical pendulum.*

A.3 Monodromy in the n-dimensional Case We construct an action function of canonical transformation from (z, P_z) to the new (α, β) coordinates to be described below, in the form

$$S = \sum_{j=1}^{n} \int_{z_j^0}^{z_j} P_{z_j} dz_j. \tag{5.5.13}$$

Here expressions for P_{z_j} are taken from (5.5.11). Notice that the function S is, at the same time, a generating equation of the Lagrangian submanifold in the phase space.

We choose β_j and $\alpha_j = -\partial S/\partial\beta_j$ as conjugate action-angle variables in the n-dimensional case. Thus,

$$\begin{aligned}
\alpha_1 &= -\int_{z_1^0}^{z_1} \frac{1}{\sqrt{(1-z_1)z_1}}\,dz_1 + \int_{z_2^0}^{z_2} \frac{\beta_1 dz_2}{(1-z_2)\sqrt{z_2K_2(z_2)}},\\
&\dots\\
\alpha_j &= -\int_{z_j^0}^{z_j} \frac{\beta_j dz_j}{\sqrt{z_jK_j(z_j)}} + \int_{z_{j+1}^0}^{z_{j+1}} \frac{\beta_j dz_{j+1}}{(1-z_{j+1})\sqrt{z_{j+1}K_{j+1}(z_{j+1})}},\\
&\dots\\
\alpha_{n-1} &= -\int_{z_{n-1}^0}^{z_{n-1}} \frac{\beta_{n-1} dz_{n-1}}{\sqrt{z_{n-1}K_{n-1}(z_{n-1})}} + \int_{z_n^0}^{z_n} \frac{\beta_{n-1} dz_n}{(1-z_n^2)\sqrt{K_n(z_n)}},\\
\alpha_n &= -\int_{z_n^0}^{z_n} \frac{2\beta_n dz_n}{\sqrt{K_n(z_n)}}.
\end{aligned} \tag{5.5.14}$$

where $j = 2, \ldots, n-2$. This vector function $\alpha_1, \ldots, \alpha_n$ defines the Abel–Jacobi map and its image (i.e., the space in which these angle variables are uniquely defined) is called the Jacobi variety of the problem.

Theorem 5.3 *The Hamiltonian flow of the n-dimensional spherical pendulum linearizes in terms of the action-angle variables* (5.5.14) *on each Jacobi variety determined by a choice of parameters $\beta_1, \ldots, \beta_n$.*

Proof We substitute in each equation of a Hamiltonian system $z_j' = \partial H/\partial P_{z_j}$ with the Hamiltonian (5.5.9) expression for momenta P_{z_j} from (5.5.11) to obtain the following system of differential equations on the corresponding Riemann surfaces

$$\begin{aligned}
z_1' &= \sqrt{(1-z_1)z_1}\left(\prod_{k=2}^{n-1}\frac{1}{(1-z_k)}\right)\frac{1}{(1-z_n^2)},\\
&\dots\\
z_j' &= \sqrt{K_j(z_j)z_j}\left(\prod_{k=j+1}^{n-1}\frac{1}{(1-z_k)}\right)\frac{1}{(1-z_n^2)},\\
&\dots\\
z_n' &= \sqrt{K_n(z_n)},
\end{aligned} \tag{5.5.15}$$

where $j = 1, \dots, n-1$. Transforming equations (5.5.15) and taking linear combination one obtains

$$\begin{aligned}
\frac{\beta_1 z'_2}{(1-z_2)\sqrt{z_2 K_2(z_2)}} - \frac{z'_1}{\sqrt{(1-z_1)z_1}} &= 0,\\
&\dots\\
\frac{\beta_j z'_{j+1}}{(1-z_{j+1})\sqrt{z_{j+1} K_{j+1}(z_{j+1})}} - \frac{\beta_j z'_j}{\sqrt{z_j K_j(z_j)}} &= 0,\\
&\dots\\
\frac{\beta_{n-1} z'_n}{(1-z_n^2)\sqrt{K_n(z_n)}} - \frac{\beta_{n-1} z'_{n-1}}{\sqrt{z_{n-1} K_{n-1}(z_{n-1})}} &= 0,\\
-\frac{2\beta_n z'_n}{\sqrt{K_n(z_n)}} &= \beta_n.
\end{aligned} \tag{5.5.16}$$

where $\beta_1, \dots, \beta_n$ are constants and $j = 2, \dots, n-2$.

After integrating (5.5.16) we obtain on the left-hand side expressions which coincide with the expressions (5.5.14) for the angle variables

$$\begin{aligned}
\alpha_j &= \alpha_j^0,\\
\alpha_n &= \beta_n t + \alpha_n^0.
\end{aligned} \tag{5.5.17}$$

where $j = 1, \dots, n-1$. □

A.4 Homoclinic Varieties in the n-dimensional case We fix an integer m and apply the following limiting processes to (5.5.14) in the given order:

$$\begin{aligned}
\beta_1 &\to 0,\\
&\dots\\
\beta_m &\to 0.
\end{aligned} \tag{5.5.18}$$

This yields the following expression for the angle representation

$$\begin{aligned}
\alpha_1 &= -\int_{z_1^0}^{z_1} \frac{dz_1}{\sqrt{(1-z_1)z_1}} = \alpha_1^0, \\
&\dots \\
\alpha_m &= -\int_{z_m^0}^{z_m} \frac{dz_m}{\sqrt{(1-z_m)z_m}} = \alpha_m^0, \\
\alpha_{m+1} &= -\int_{z_{m+1}^0}^{z_{m+1}} \frac{dz_{m+1}}{\sqrt{z_{m+1}(1-z_{m+1})}} \\
&\qquad + \int_{z_{m+2}^0}^{z_{m+2}} \frac{\beta_{m+1} dz_{m+2}}{(1-z_{m+2})\sqrt{z_{m+2}K_{m+2}(z_{m+2})}}, \\
&\dots \\
\alpha_j &= -\int_{z_j^0}^{z_j} \frac{\beta_j dz_j}{\sqrt{z_j K_j(z_j)}} \\
&\qquad + \int_{z_{j+1}^0}^{z_{j+1}} \frac{\beta_j dz_{j+1}}{(1-z_{j+1})\sqrt{z_{j+1}K_{j+1}(z_{j+1})}} = \alpha_j^0, \\
&\dots \\
\alpha_{n-1} &= -\int_{z_{n-1}^0}^{z_{n-1}} \frac{\beta_{n-1} dz_{n-1}}{\sqrt{z_{n-1}K_{n-1}(z_{n-1})}} \\
&\qquad + \int_{z_n^0}^{z_n} \frac{\beta_{n-1} dz_n}{(1-z_n^2)\sqrt{K_n(z_n)}} = \alpha_{n-1}^0, \\
\alpha_n &= -\int_{z_n^0}^{z_n} \frac{2\beta_n dz_n}{\sqrt{K_n(z_n)}} = \beta_n t + \alpha_n^0,
\end{aligned} \tag{5.5.19}$$

where $j = m+2, \dots, n-2$.

There are two different cases. For $m < n-1$, the system (5.5.19) describes an m-dimensional family of parameterized $(n-m)$-dimensional spherical pendula. For $m = n-1$ and $\beta_n = 1$, one obtains a family of 1-dimensional homoclinic orbits.

There have been many important developments in which the methods of complex and algebraic geometry have been used to investigate the eigenfunctions of Hill's operator in the context of integrable equations. In Alber and Marsden [1995] we link a new class of Hamiltonian systems on Riemann surfaces to systems of pde's using Bloch eigenfunctions for stationary Shrödinger equations with new types of potentials. In particular, this yields a system of pde's with monodromy.

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