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THE CONVERGENCE OF HAMILTONIAN STRUCTURES IN THE SHALLOW WATER APPROXIMATION

ZHONG GE, HANS PETER KRUSE, JERROLD E. MARSDEN AND CLINT SCOVEL

ABSTRACT. It is shown that the Hamiltonian structure of the shallow water equations is, in a precise sense, the limit of the Hamiltonian structure for that of a three-dimensional ideal fluid with a free boundary problem as the fluid thickness tends to zero. The procedure fits into an emerging general scheme of convergence of Hamiltonian structures as parameters tend to special values. The main technical difficulty in the proof is how to deal with the condition of incompressibility. This is treated using special estimates for the solution of a mixed Dirichlet-Neumann problem for the Laplacian in a thin domain.

1. Introduction. Whereas Hamiltonian methods have been very successful for a long time in stability and bifurcation analysis of a variety of systems, including fluid mechanical ones, (see, for example, [**12**] and references therein), the study of "Hamiltonian asymptotics" is still quite young. One area where this has already received some attention is that of semi-classical quantum mechanics with the small parameter being Planck's constant (see, for example, [**1, 11, 20**] and references therein). The beginnings of a suitable framework for Hamiltonian asymptotics for classical systems, such as the ones considered here, was laid in [**5, 13, 17, 18, 19**]. In the first three references, the problem of convergence of dynamical trajectories is discussed in the context of singular limits of the potential energy and the case of the convergence of compressible flow to incompressible flow as the speed of sound converges to infinity is discussed specifically (see also [**1**] for a discussion). In Marsden and Weinstein [**13**] the problem of convergence of Poisson structures for the case of the Maxwell-Vlasov equations to

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the Poisson-Vlasov equations as the velocity of light tends to infinity is discussed, and in Weinstein [20] the convergence of the three body problem to the restricted three body problem is examined.

The above problems may all be thought of as being in the area of Hamiltonian asymptotics. Useful alternative points of view on the problem have been given by Olver ([15] and references therein). In this context, the work of Camassa and Holm [2] is especially interesting in that it was able to produce a new integrable system with peaked soliton solutions by paying careful attention to asymptotics (starting with the Green-Naghdi equation) that preserves the Hamiltonian structure. The present paper is in the same spirit as these works, but concentrating on the limiting Poisson bracket structure and the limiting Hamiltonian. We also note that recent work of Levermore and his collaborators on the zero forcing-dissipation limit of the complex Ginzburg-Landau equation to the nonlinear Schrödinger equation uses a variant of the limit of Hamiltonian structure idea of the present work, through the observation that the forcing-dissipation can be captured in a Poisson bracket like form. This Hamiltonian structure turns out to be very useful for the question of which periodic solutions of the NLS equation can be limits of periodic solutions of the CGL equation; the answers turn out to be very interesting and suggest that one attempt similar things in other problems involving limits of Hamiltonian like structures.

We note that our line of research is different from that of Kano and Nishida [2] who show that solutions of the three-dimensional Euler equations converge to the solutions of the shallow water equations in the special case of two-dimensional irrotational flow. (We do not make the irrotational restriction in the present paper.) In this paper we do not address the question of convergence of individual solutions but believe that the geometric and analytic results here may be useful for that question (for example, combined with the methods of Ebin and Marsden [6] and the paper of Ebin [5] mentioned above).

In [7] we examine the convergence of Hamiltonian structures in elasticity, specifically the limit of three-dimensional elasticity to rod and shell theories. This is a subject of current interest for which the geometry of the asymptotic expansions is somewhat more complicated than here. Nonetheless, some of the same philosophy that is developed in this paper proves to be very useful in the elasticity problem.

We consider an inviscid, homogeneous, incompressible fluid moving in $\mathbf{R}^3$ with a free surface, but with no surface tension. We use coordinates denoted by (x_1, x_2, y), where the free surface has the form $y = h(x_1, x_2, t)$ and $y = 0$ is a fixed surface. Euler's equations for the spatial velocity field $V(x_1, x_2, y, t)$ are

$$\begin{aligned} &\frac{\partial V}{\partial t} + (V \cdot \nabla)V = -\nabla(p + gy), \\ &\frac{\partial h}{\partial t} = V \cdot \nabla(y - h)|_{y=h(x_1,x_2,t)}, \\ &\nabla \cdot V = 0, \end{aligned} \tag{1.1}$$

together with the condition that V be parallel to the fixed surface $y = 0$, where g is the acceleration due to gravity, p is the pressure, which vanishes at $y = h(x_1, x_2)$, and n is the unit normal vector to that free surface $y = h(x_1, x_2)$. For a discussion of the Hamiltonian structure for these equations, see [**10**]. Our purpose is to study the shallow water approximation (or the SW approximation for short), namely,

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla_x)u + g\nabla h = 0, \\ &\frac{\partial h}{\partial t} + \nabla_x \cdot (hu) = 0, \end{aligned}$$

where $u = (u_1, u_2)$ is the spatial velocity, which corresponds to the horizontal components of the three-dimensional velocity, V. See, for example, [**16**] for a classical derivation of the SW equation. Recall that one assumes in this derivation that the hydrostatic approximation $p = h - gy$ is valid and that the horizontal components are independent of y; one eliminates the third component using a vertical integration of the divergence free condition. The SW equation has many applications, especially in geophysics as an important ingredient in ocean and atmospheric dynamics.

It is well known that both the original 3-D fluid and the SW equation are Hamiltonian systems. However, the Hamiltonian nature of the approximation procedure is not so clear from the classical derivation. One can ask questions such as: does the Poisson bracket for the 3-D fluid with a free boundary converge (in some sense to be made precise) to the SW bracket? Does the energy (Hamiltonian) converge as well? In this paper we will answer these questions.

We would like to emphasize that this problem is not to be confused with that of imposing constraints on a given system. Rather, it is that of the convergence of the Poisson bracket of a family of Hamiltonian systems to a limiting one. In this sense it is rather different from the line of investigation started by Rubin and Ungar mentioned above.

There is another result of independent interest, which is in some sense a dual to the main theorem and that is proved by similar techniques, namely, that there is an almost Poisson map from the shallow water bracket to the 3-D bracket (see Theorem 4.2). This theorem is, in spirit, very close to the philosophy of Marsden and Weinstein [12] and in [20].

The main technical difficulty in the proof of our result arises from the fact that in incompressible fluid mechanics the spatial velocity field is divergence free and, correspondingly, the functional derivatives in the Poisson bracket of the Hamiltonian formulation are divergence free. One must deal with the question of how the Helmholtz-Hodge decomposition of an arbitrary vector field on a domain into its divergence-free and its gradient part behaves when one dimension of the domain tends to zero. This is complicated by the fact that the boundary conditions are mixed, with Dirichlet conditions on some portions of the boundary and Neumann on the remainder. To deal with this, we show that when the thickness of the fluid layer goes to zero, the projection onto the space of divergence-free vector fields is an "almost identity map" with a sharp error estimate given in terms of the thickness. This amounts to showing that the solution of a Poisson equation $\nabla^2 g = f$ with homogeneous boundary condition goes to zero (in the Sobolev H^3 norm) as the first power of the thickness. The proof of this appears to be new (we thank Genevieve Raugel for her advice in this matter).

In this paper we only consider the case in which x_1 and x_2 are periodic variables, so may be regarded as residing on the two torus, T^2. We believe that similar results also hold for flow on a (possibly rotating) sphere in $\mathbf{R}^3$, in which case one should also add Coriolis terms; cf. [3, 4].

This paper is organized as follows. In Section 2 we state our version of the Poisson bracket for a 3-D fluid with a free surface, and in Section 3 the SW Poisson bracket is recalled. In Section 4 we state and prove the main results.

2. The dynamics of a 3-D fluid with a free surface. A Hamiltonian structure for a physical system is usually determined by choosing a phase space, a Poisson bracket, and a Hamiltonian function. In this section, we will describe in detail the Lagrangian phase space of 3-D incompressible fluid flow with a free boundary. In the first subsection we describe this structure for the Eulerian representation. Following this, we shall describe it for the Lagrangian representation, and show how these two representations are related by reduction from the Lagrangian to the Eulerian viewpoint.

2.1. *The Poisson bracket in Eulerian representation.* Throughout this paper we consider a fluid contained in a 3-dimensional domain over a two torus T^2, consisting of points (x_1, x_2, y), $x = (x_1, x_2) \in T^2$, $0 \le y \le h(x)$. That is, x_1 and x_2 are periodic variables. The hyperplane $y = 0$ is a fixed boundary, $y = h(x)$ is the free boundary, and $h(x)$ is the height function. We will write the velocity field of the fluid as $V = (u_1, u_2, v)$ or simply $V = (u, v)$, where $u = (u_1, u_2)$.

Let $dA = d^2x = dx_1 dx_2$ be the standard area form on T^2, and $dv = dx_1 dx_2\, dy$ the volume-form. If $\bar{x} = F(x) : T^2 \to T^2$ is a transformation, denote the determinant of the Jacobian relative to the standard flat Riemannian metric on T^2 by $\det DF(x)$.

We next recall the Poisson bracket for a 3-D fluid with a free surface, which was derived in [10]. We shall take a slight variant of their bracket as our starting point. Because we do not deal with surface tension effects and because the geometry of the domain is simpler, our derivation of the bracket is a little easier and avoids some delicate issues with delta functions that occur in the general case. In addition, our boundary conditions and our choice of variations are a little different. Notwithstanding, our results also apply to systems with surface tension, a remark we will elaborate on below.

Let N_3 be the space of Eulerian variables, that is, the set of pairs (V, h) where V is a spatial divergence-free velocity field on the domain $0 \le y < h(x)$, that is, tangent to the boundary $y = 0$. We put on the space N_3 the following Sobolev topology. The vector field V should be of class H^s where $s \ge 3$ and the boundary surface h is of class $H^{s-1/2}$, consistent with the Sobolev trace theorems. We shall not go into the details of the Sobolev structure here, but refer to, for example,

[6] for details, and how to put the corresponding Sobolev structures on the space of embeddings that will be used below in the Lagrangian description.

We assume that the functionals we are dealing with have functional derivatives in the following sense:

If F is a functional on N_3, the functional derivative $\delta F/\delta V$ is a divergence-free vector field, tangent to the boundary $y = 0$, defined in the following way: if δV is a variation, then

$$\begin{aligned} DF \cdot \delta V &= \int_{T^2} \int_0^h \frac{\delta F}{\delta V} \cdot \delta V \, dv \\ &= \lim_{t \to 0} \frac{F(V + t\delta V, h) - F(V, h)}{t}, \end{aligned}$$

assuming that the righthand side exists, and where $\int_{T_2} \int_0^h$ denotes the three-dimensional integral over the region $0 \leq y \leq h(x)$. The derivative $\delta F/\delta h$, a function on T^2, is defined similarly.

Caution. The definition of the functional derivative depends on the function spaces chosen. In the present case, this means that the constraint of being divergence free is built into the definition. If one prefers, the functional derivative here is the Helmholtz-Hodge projection of the unconstrained (or usual) functional derivative. This constraint must be taken into account when doing the calculations; in particular, if one takes the usual functional derivative in this bracket, one does not get the correct Euler equations for the problem.

As an example, consider a function of the form

$$F = \int_{T^2} \int_0^h f(V) \, dv,$$

where f is smooth. By the Hodge decomposition, any vector field A on the domain $0 \leq y \leq h$ can be written as

$$A = \nabla \phi + A_0, \tag{2.1}$$

where A_0 is divergence-free and ϕ satisfies $\nabla^2 \phi = \operatorname{div} A$, along with the boundary conditions

$$\phi|_{y=h(x)} = 0, \qquad \left.\frac{\partial \phi}{\partial y}\right|_{y=0} = A \cdot (0,0,1)|_{y=0}.$$

Note that the second boundary condition is equivalent to requiring that A_0 be tangent to the fixed boundary $y = 0$, and the first boundary condition implies that the decomposition (2.1) be orthogonal with respect to the L^2-inner product.

Letting P be the L^2-orthogonal projection that maps A to A_0, then one checks that

$$\frac{\partial F}{\partial V} = P\left(\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial v}\right).$$

We let $\delta F/\delta n$ be the function over T^2 defined by

$$\frac{\delta F}{\delta n}(x) = \left(\frac{\delta F}{\delta V}(x, h(x))\right) \cdot n,$$

where n is the outer unit normal vector to the surface $y = h(x)$.

Lemma 2.1. *Given $F_1, F_2 : N_3 \to \mathbf{R}$, set*

$$\text{(2.2)} \quad \{F_1, F_2\}_{3D} = \int_{T^2}\int_0^h \omega \cdot \left(\frac{\delta F_1}{\delta V} \times \frac{\delta F_2}{\delta V}\right) dv + \int_{T^2}\left(\frac{\delta F_1}{\delta h}(x)\frac{\delta F_2}{\delta n}(x) - \frac{\delta F_1}{\delta n}(x)\frac{\delta F_2}{\delta h}(x)\right)\sqrt{1+(\nabla_x h)^2}\, d^2x,$$

where $\omega = \nabla \times V$ is the vorticity. Then $\{,\}_{3D}$ is a Poisson bracket. Moreover, equation (1.1) *is a Hamiltonian system with the Hamiltonian*

$$E_3(v, h) = \frac{1}{2}\left(\int_{T^2}\int_0^h ||V||^2\, dv + \int_{T^2} gh^2\, d^2x\right).$$

This lemma is proved as in [**10**].

2.2. *The Poisson structure in Lagrangian representation.* Now we turn our attention to the Hamiltonian structure for the Lagrangian representation. This structure will then be related to that in the Eulerian representation described above by the process of Poisson reduction which, in this case, is simply the relation between the Lagrangian and Eulerian representations of the fluid. In the next section we will describe the phase space, the Poisson structure and the Hamiltonian of the shallow water equation. In Section 4 we show how the 3-D and shallow water Hamiltonian structures are precisely related.

We consider a fluid moving in the region R_h between a dynamic free-surface $y = h(x)$, and the fixed hyperplane $y = 0$. The reference region corresponds to the region R_{h_0} obtained by putting $h = h_0$. The Lagrangian representation configuration space for the 3-D fluid, denoted by D_3, consists of a collection of (H^s) volume-preserving embeddings

$$T : R_{h_0} \to T^2 \times [0, \infty),$$

which fix the lower boundary $y = 0$ setwise. We only allow, as elements of D_3, those embeddings T whose image is one of the sets R_h described above.

Write $T : (x, y) \mapsto (\bar{x}, \bar{y}) = (X(x, y), Y(x, y))$, where $(\bar{x}, \bar{y})$ is a spatial point and (x, y) a reference point. The fixed boundary condition is $Y(x, 0) = 0$. Note that T can map $y = h_0$ onto any surface $y = h(x)$ as long as this surface together with $y = 0$ encloses the same volume as $0 \leq y \leq h_0$ (cf. [14]). The space of all height functions will be denoted by B. (By the Sobolev trace theorems, the appropriate Sobolev class for the height functions is $H^{s-1/2}$).

The energy functional is the kinetic energy plus the gravitational potential energy. At an element $T = (X, Y)$ the potential energy can be written as

$$V(T) = \int_{T^2} \int_0^{h_0} gY \, d^2x \, dy. \tag{2.3}$$

In the infinite-dimensional case, symplectic structures on cotangent bundles are often defined with the help of metrics, which give a pairing between tangent and cotangent bundles. When one defines the corresponding Poisson bracket, the metric occurs again in the definition of functional derivatives.

The phase space in the present problem is $P := TD_3$ with the standard Poisson bracket defined with the help of the L^2 metric. As a set, the space P consists of pairs (T, μ), where T is a particle placement field as before and μ is a divergence-free vector field over T; i.e., to each reference point (x, y), μ assigns a vector on $\mathbf{R}^3$ based at the spatial point $(\bar{x}, \bar{y}) = T(x, y)$ such that $\mu \circ T^{-1}$ is a divergence free vector field. The Poisson bracket on P is

$$\{F_1, F_2\} = \int_{T^2} \int_0^{h_0} \left(\frac{\delta F_1}{\delta T} \cdot \frac{\delta F_2}{\delta \mu} - \frac{\delta F_2}{\delta T} \cdot \frac{\delta F_1}{\delta \mu} \right) dv.$$

The energy functional can be written as

$$E_3(T, \mu) = \frac{1}{2} \int_{T^2} \int_0^h \langle \mu \circ T^{-1}, \mu \circ T^{-1} \rangle \, dv + \frac{1}{2} \int_{T^2} g h^2 \, d^2 x,$$

where h is the height function associated with the embedding T, as described above.

The symmetry group G_3 of the 3-D fluid consists of (the identity component of) the set of all (H^s) volume-preserving diffeomorphisms of the reference configuration that setwise fix the surfaces $y = 0$ and $y = h_0$. This group acts on D_3 on the right by composition and leaves the energy invariant.

The passage from the Lagrangian description to the Eulerian description is done by using the map $\Pi : T^* D_3 \to N_3$ defined by

$$(T, \mu) \to (\mu \circ T^{-1}, h), \tag{2.4}$$

where h is the height function associated to the embedding T. The map Π is invariant under the right action of G_3, and so induces a diffeomorphism

$$\bar{\Pi} : T^* D_3 / G_3 \to N_3.$$

Thus, N_3 inherits a Poisson bracket, which is exactly the one in Lemma 2.2. This is proved as in [10]. Thus, by construction, Π is a Poisson map, which relates the Poisson bracket in the Lagrangian representation to that in the Eulerian representation.

3. The shallow water fluid model. After an initial motivation based on an asymptotic expansion, we introduce the shallow water Hamiltonian structure in both the Lagrangian and the Eulerian representations and, following the pattern of the last section, relate these two structures. In the next section we will give the precise relations between the 3-D fluid bracket and the shallow water bracket.

3.1. *The shallow water system in Lagrangian representation.* To motivate our choice of the configuration space for the SW fluid, we introduce a parameter ε measuring the fluid thickness and consider a modification of the configuration space of the three-dimensional model. We do this by choosing $h_0 = \varepsilon$ in the reference configuration. Thus, we consider a configuration of the 3-D fluid with a free surface to be a volume preserving map $T : (x, y) \mapsto (\bar{x}, \bar{y})$, from $T^2 \times [0, \varepsilon]$ to $T^2 \times \mathbf{R}^+$. We also introduce a rescaling map

$$S_\varepsilon : (x, y) \mapsto \left(x, \frac{y}{\varepsilon}\right)$$

from $T^2 \times \mathbf{R}^+$ to itself. We conjugate an element T of D_3 by S_ε to obtain $S_\varepsilon \circ T \circ S_\varepsilon^{-1}$ which is defined on the fixed domain $T^2 \times [0, 1]$. Note that even though S_ε is not volume-preserving, the conjugation of T by S_ε is.

The configuration space of the 3-D fluid after this conjugation by S_ε converges to the following space of mappings $\eta : T^2 \times [0, 1] \to T^2 \times \mathbf{R}^+$:

$$D_{sw} := \{\eta = (F, \det DF^{-1}y) \mid F : T^2 \to T^2\}$$

in the following sense:

$$S_\varepsilon \circ T \circ S_\varepsilon^{-1}(x, y) = (F(x), \det DF^{-1}(x)y) + O(\varepsilon)$$

where $(F(x), 0) = T(x, 0)$, which can be easily verified by using a first order approximation.

This suggests that as the configuration space for the SW fluid, we take the subset $D_{SW} \subset D_3$. To represent a tangent vector $\mu \in T_\eta D_{SW}$, choose a one-parameter family $r \mapsto \eta_r \in D_{SW}$ with $\eta_0 = \eta$ and take the derivative with respect to r at $r = 0$. One gets

$$\mu = \frac{d}{dr}\bigg|_{r=0} (F_r, \det DF_r^{-1}y) = (\delta F, -\det(DF^{-1})\operatorname{div}(\delta F \circ F^{-1})y),$$

where $\delta F = dF_r/dr|_{r=0}$. If $\mu' = (\delta F', -\det(DF)^{-1}\operatorname{div}(\delta F' \circ F^{-1})y)$ is another tangent vector to D_{SW} at η, we want to define the scalar product of μ and μ'. To do so, we consider the corresponding tangent vectors $S_\varepsilon^{-1} \circ \mu \circ S_\varepsilon$ and $S_\varepsilon^{-1} \circ \mu' \circ S_\varepsilon$ to the three-dimensional configuration space and take their inner product (the kinetic energy inner product). One gets

$$\int_{T^2} \int_0^\varepsilon (S_\varepsilon^{-1} \circ \mu \circ S_\varepsilon) \cdot (S_\varepsilon^{-1} \circ \mu' \circ S_\varepsilon)\, dv$$
$$= \varepsilon \int_{T^2} \delta F \cdot \delta F'\, dA + \text{terms cubic in } \varepsilon.$$

This discussion suggests that, for the definition of the Poisson bracket in the shallow water approximation, we should use the following metric

$$\langle\langle \mu, \mu' \rangle\rangle := \int_{T^2} \delta F \cdot \delta F'\, dv = \int_{T^2} P_2(\mu) \cdot P_2(\mu')\, dv, \tag{3.1}$$

where P_2 denotes the projection onto the first two components.

Observe that, by definition of D_{SW}, $P_2(\mu)$ is independent of the y-variable and that the metric (3.1) is invariant under the action of the group D_2^{vol} of volume-preserving diffeomorphisms of the two torus, which acts naturally on D_{SW} by composition on the right.

We proceed as usual to define a Poisson bracket on TD_{SW}. We denote elements of TD_{SW} by (η, μ). Consider functions $F : TD_{SW} \to \mathbf{R}$ which possess functional derivatives $\delta F/\delta\eta$, $\delta F/\delta\mu$ such that

$$DF \cdot \delta\eta = \left\langle\left\langle \frac{\delta F}{\delta \eta}, \delta\eta \right\rangle\right\rangle,$$
$$DF \cdot \delta\mu = \left\langle\left\langle \frac{\delta F}{\delta \mu}, \delta\mu \right\rangle\right\rangle.$$

The Poisson bracket of two such functions is given by

$$\{E, H\} := \left\langle\left\langle \frac{\delta E}{\delta \eta}, \frac{\delta H}{\delta \mu} \right\rangle\right\rangle - \left\langle\left\langle \frac{\delta H}{\delta \eta}, \frac{\delta E}{\delta \mu} \right\rangle\right\rangle$$
$$= \int_{T^2} \left[P_2\left(\frac{\delta E}{\delta \eta}\right) P_2\left(\frac{\delta H}{\delta \mu}\right) - P_2\left(\frac{\delta H}{\delta \eta}\right) P_2\left(\frac{\delta E}{\delta \mu}\right) \right] dA.$$

We will write d^2x when writing integrals over the current configuration, and dA when writing integrals over the reference configuration.

Note that the structure of D_{SW} is very similar to D_3. In fact, D_{SW} has the structure of a principal bundle over B, the structure group being its group D_2^{vol} of area-preserving transformations of T^2. In other words, the approximation of D_3 by D_{SW} is analogous to the reduction of a structure group.

The group D_2^{vol} acts on the space D_{SW} by composition on the right, and hence it induces an action on TD_{SW} by Poisson maps. Denote the tangent space to D_{SW} at the identity embedding by d_{SW}. The elements of d_{SW} are of the form

$$v = (u, -(\operatorname{div} u)y),$$

where $u : T^2 \to \mathbf{R}^2$.

Let $\mathcal{F}^*$ denote the space of densities on T^2. We will identify these with functions h by identifying H and hdx. The map

$$\pi : T^* D_{SW} \to \mathcal{F}^* \times d_{SW}$$

defined by

$$(\eta, \mu) \mapsto (\det DF^{-1}, \mu \circ \eta^{-1}),$$

induces a map

$$\bar{\pi} : T^* D_{SW}/D_2^{\mathrm{vol}} \to \mathcal{F}^* \times d_{SW}.$$

It is easy to see that $\bar{\pi}$ is well defined and injective. By a theorem of Moser [**14**], it is also surjective.

The space $TD_{SW}/D_2^{\mathrm{vol}}$ inherits a natural Poisson structure such that $\bar{\pi}$ is a Poisson map; that is, $\bar{\pi}$ carries this structure to $\mathcal{F}^* \times d_{SW} =: M$.

Next we compute the induced Poisson bracket. Denote elements of $\mathcal{F}^* \times d_{SW}$ by (h, v). Consider functionals $E : M \to \mathbf{R}$ that possess functional derivatives $\delta E(h,v)/\delta h : T^2 \to \mathbf{R}$, so that $\delta E(h,v)/\delta v \in d_{SW}$ is given by

$$\frac{\delta E}{\delta v} =: \left(\frac{\delta E}{\delta u}, -\operatorname{div}\left(\frac{\delta E}{\delta u}\right)y\right),$$

where

$$DE \cdot \delta h = \int_{T^2} \frac{\delta E}{\delta h} \cdot \delta h d^2x$$

and

$$DE \cdot \delta v = \int_{T^2} P_2\left(\frac{\delta E}{\delta v}\right) \cdot P_2(\delta v) d^2x.$$

Letting $\overline{E} = E \circ \pi$, we have

$$\begin{aligned}\delta\overline{E}\cdot\delta\mu &= DE(Dv\,d\mu) = DE(\delta\mu\circ\eta^{-1})\\ &= \int_{T^2} P_2\left(\frac{\delta E}{\delta v}\right)\cdot P_2(\delta\mu\circ\eta^{-1})\,d^2x\\ &= \int_{T^2}\left[P_2\left(\frac{1}{h}\frac{\delta E}{\delta v}\right)\circ F\right]\cdot P_2(\delta\mu)\,dA.\end{aligned}$$

Thus,

$$P_2\left(\frac{\delta\overline{E}}{\delta\mu}\right) = P_2\left(\frac{1}{h}\frac{\delta E}{\delta v}\circ F\right).$$

Writing $v = (u, -(\operatorname{div} u)y)$, we have

$$D\overline{E}\cdot\delta\eta = D_hE\cdot(Dh\cdot\delta\eta) + D_vE\cdot(Dv\cdot\delta\eta).$$

Since

$$Dv\cdot\delta\eta = -(\delta\eta\circ\eta^{-1})\cdot\nabla v$$

and

$$Dh\cdot\delta\eta = -h\operatorname{div}(\delta F\circ F^{-1}),$$

where $\eta = (F, \det DF^{-1}y)$ and $\delta\eta = (\delta F, -\det(DF^{-1})\operatorname{div}(\delta F\circ F^{-1})y)$, it follows that

$$\begin{aligned}D\overline{E}\cdot\delta\eta &= -\int_{T^2}\left(h\frac{\delta E}{\delta h}\operatorname{div}(\delta F\circ F^{-1}) + \frac{\delta E}{\delta u}\cdot(\delta F\cdot F^{-1}\cdot\nabla u)\right)d^2x\\ &= \int_{T^2}\left(\frac{1}{h}\nabla\left(\frac{\delta E}{\delta h}h\right)\circ F\cdot\delta F - \frac{1}{h}(\nabla u)^T\frac{\delta E}{\delta u}\circ F\cdot\delta F\right)dA.\end{aligned}$$

Thus,

$$P_2\left(\frac{\delta\overline{E}}{\delta\eta}\right) = \frac{1}{h}\nabla\left(\frac{\delta E}{\delta h}h\right)\circ F - (\nabla u)^T\frac{\delta E}{\delta u}\circ F,$$

and for $(h,v)=\pi(\eta,\mu)$, one has

$$
\begin{aligned}
\{E,H\}_{(h,v)} &= \{\overline{E},\overline{H}\}_{(\eta,\mu)} \\
&= \int_{T^2}\left[\frac{1}{h}\nabla\left(\frac{\delta E}{\delta h}h\right)\circ F\cdot F\cdot\frac{1}{h}\frac{\delta H}{\delta u}\cdot F \right.\\
&\qquad\left. -\frac{1}{h}\nabla\left(\frac{\delta H}{\delta h}h\right)\cdot\frac{1}{h}\frac{\delta E}{\delta u}\circ F\right] dA \\
&\quad+\int_{T^2}\frac{1}{h^2}\left(-(\nabla u)^T\frac{\delta E}{\delta u}\circ F\cdot\frac{\delta H}{\delta u}\circ F\right.\\
&\qquad\left.+(\nabla u)^T\frac{\delta H}{\delta u}\circ F\cdot\frac{\delta E}{\delta u}\circ F\right) dA \\
&= \int_{T^2}\left[\frac{1}{h}\nabla\left(\frac{\delta E}{\delta h}h\right)\cdot\frac{\delta H}{\delta u}-\frac{1}{h}\nabla\left(\frac{\delta H}{\delta h}h\right)\cdot\frac{\delta E}{\delta u}\right] d^2x \\
&\quad+\int_{T^2}\frac{1}{h}\left(\frac{\delta H}{\delta u}\cdot\nabla u\frac{\delta E}{\delta u}-\frac{\delta E}{\delta u}\cdot\nabla u\frac{\delta H}{\delta u}\right) d^2x \\
&= \int_{T^2}\left(\frac{\delta H}{\delta u}\cdot\nabla\frac{\delta E}{\delta h}-\frac{\delta E}{\delta u}\cdot\nabla\frac{\delta H}{\delta h}\right) d^2x \\
&\quad+\int_{T^2}\frac{1}{h}\left(\frac{\delta H}{\delta u_i}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)\frac{\delta E}{\delta u_j}\right) d^2x,
\end{aligned}
$$

which is the shallow water bracket.

As the energy for the SW fluid, we take

$$
E_{SW}(F,\dot{F}) := \frac{1}{2}\int_{T^2}\dot{F}\cdot\dot{F}\,dA+\frac{g}{2}\int_{T^2}(\det(DF))^{-1}\,dA, \tag{3.2}
$$

where the second term (the potential energy) is the pullback of potential energy (2.3), using the inclusion map $i: D_{SW}\to D_3$. When pulled back to the Eulerian representation, it can be written as

$$
E_{SW}(v,h)=\frac{1}{2}\int_{T^2}(hu\cdot u+gh^2)\,d^2x. \tag{3.3}
$$

4. The SW Poisson bracket as an approximation of the 3-D bracket.

4.1. *The main results.* We assume from now on that the fluid is contained in a region under a graph. Since the reference height is of order ε, we can describe the domain as $0 \le z \le h$ where h is of order ε. Since h is of order ε, we write it as $h = \varepsilon h$.

Consider a functional that has the form of a generalized energy function:

$$F = \int_{T^2} \int_0^{\varepsilon \bar{h}} f(V(x,y))\, dv + \int_{T^2} g(\varepsilon \bar{h}(x))\, d^2x. \tag{4.1}$$

As we will see later, as $\varepsilon \to 0$, F converges, in a sense we have to make precise, to

$$F_{SW} = \int_{T^2} f(V(x,0))\varepsilon \bar{h}(x)\, d^2x + \int_{T^2} g(\varepsilon \bar{h}(x))\, d^2x. \tag{4.2}$$

The Poisson bracket of two functionals of the form (4.1) leads to integrals of more general type, so we need to consider functionals that have the following form involving an integral over the domain $0 \le y \le \varepsilon \bar{h}$:

$$\begin{aligned} G = &\int_0^{\varepsilon \bar{h}} f_1(V(x,y), \ldots, D^\alpha V(x,y), \ldots)\, dv \\ &+ \int_{T^2} g_1(V(x, \varepsilon \bar{h}(x)), \ldots, D^\alpha V(x, \varepsilon \bar{h}(x)), \ldots, \varepsilon \bar{h}(x))\, d^2x, \end{aligned}$$

and their truncations

$$\begin{aligned} G_{SW} = &\int_{T^2} \bar{h}\varepsilon f_1(V(x,0), \ldots, D^\alpha V(x,0), \ldots)\, d^2x \\ &+ \int_{T^2} g_1(V(x,0), \ldots D^\alpha V(x,0), \ldots, \varepsilon \bar{h}(x))\, d^2x. \end{aligned}$$

We need to compare G with G_{SW}.

We say that *$G - G_{SW}$ belongs to the class $O(m,n)$* or $G = G_{SW}$ mod $O(m,n)$ or $G - G_{SW} \in O(m,n)$, if the first and the second integrals of $G - G_{SW}$ are, respectively, of order ε^m and ε^n, i.e.,

$$f_1(V(x,\varepsilon y), \ldots, D^\alpha V(x, \varepsilon y), \ldots) - f_1(V(x,0), \ldots, D^\alpha V(x,0), \ldots) = O(\varepsilon^{m-1}),$$

and

$$g_1(V(x,\varepsilon\bar{h}),\ldots,D^\alpha V(x,\varepsilon\bar{h}),\ldots,\varepsilon\bar{h}) - g_1(V(x,0), \ldots, D^\alpha V(x,0),\ldots,\varepsilon\bar{h}) = O(\varepsilon^n).$$

For example, for the energy of the 3-D fluid, we have

$$E - E_{sw} \in O(2,\infty).$$

The approximation of F by F_{sw} is not unique. For example, in Green and Naghdi [8], the following truncation of the Hamiltonian is used

$$F_{GN} = \int_{T^2}\int_0^{\varepsilon\bar{h}} f_1(u(x,0) - y\nabla u(x,0))\,dy\,d^2x + \int_{T^2} g_1(V(x,0))\,d^2x,$$

which is obtained by setting $u(x,y) = u(x,0), v(x,y) = -y\nabla_x u(x,0)$ in (4.1). This truncation has essentially the same order of approximation as the shallow water truncation.

Our main result is

Theorem 4.1. *Let F_i be two functionals of the form* (4.1). *Then*

$$\{F_1,F_2\}_{3D} = \{F_{1,SW},F_{2,SW}\}_{SW} \bmod O(2,2).$$

This theorem implies, in particular, that

$$\{F_1,F_2\}_{3D} - \{F_{1,SW},F_{2,SW}\}_{SW} = O(\varepsilon^2).$$

Remark. To include surface tension effects, one has to add terms of the form

$$\int_{T^2} g(\varepsilon\bar{h}(x),\varepsilon\bar{h}_x(x))\,d^2x$$

to the functionals F_i. Our Theorem 4.1 remains true for these more general functionals. To keep the exposition as clear as possible, we do not include surface tension terms in the derivation of our results.

There is a dual version of Theorem 4.1 which roughly says that the embedding $A : (u(x), h(x)) \to ((u(x), -\nabla_x u(x)y), h(x))$ is an almost Poisson map.

Theorem 4.2. *The embedding A is an almost Poisson transformation, i.e., for F_i of the form* (4.1),

$$\{F_1 \circ A, F_2 \circ A\}_{SW} = \{F_1, F_2\}_{3D} \circ A + O(\varepsilon^2).$$

The proof of Theorem 4.2 is almost the same as that of Theorem 4.1, so we will only prove Theorem 4.1; we do this in the next subsection.

The following result concerns a type of weak convergence of the 3-D fluid flow to the shallow water flow. We shall tacitly assume in this result that one already has proven an existence theorem and that one can show, at least locally in function space, that the solutions of the 3-D equations remain in a neighborhood on which a given functional F is bounded and smooth as $\varepsilon \to 0$. Our approach is consistent with a common situation in weak convergence in which one first proves boundedness in a strong sense and then derives convergence in a weak sense.

Corollary 4.3. *Let $\Theta^t_{3D,\varepsilon}$ and Θ^t_{SW} be the phase flows (the flows on the Eulerian function spaces) of the* 3D *fluid and shallow water fluid, respectively. Then*

$$\Theta^t_{3D,\varepsilon} \circ A = A \circ \Theta^t_{SW} + O(\varepsilon^2),$$

in the sense that for any function F of the form (4.1),

$$(\Theta^t_{3D,\varepsilon} \circ A)^* F = (A \circ \Theta^t_{SW})^* F + O(\varepsilon^2).$$

We caution that the constant implicit in the notation $o(\varepsilon^2)$ may depend on the functional F.

Proof. Let X_{3D}, X_{SW} be the phase space vector fields generating the 3D fluid and shallow water fluid, respectively. Then, by Theorem 4.2,

$$\begin{aligned}\$_{X_{SW}}(F \circ A) &= \{F \circ A, E_{SW}\} \\ &= \{F \circ A, E_{3D} \circ A\} + O(\varepsilon^2) \\ &= \{F, E_{3D}\} \circ A + O(\varepsilon^2) \\ &= \{\$_{X_{3D}} F) \circ A + O(\varepsilon^2).\end{aligned}$$

Hence,

$$\frac{d}{dt}(F \circ \Theta^t_{3D,\varepsilon} \circ A - F \circ A \circ \Theta^t_{SW})\Big|_{t=0} = O(\varepsilon^2). \tag{4.3}$$

Introduce

$$F(s,t) = F \circ \Theta^s_{3D,\varepsilon} \circ A \circ \Theta^t_{SW},$$

then, from equation (4.3), we obtain

$$\frac{\partial F(s,t)}{\partial s} - \frac{\partial F(s,t)}{\partial t} = O(\varepsilon^2).$$

Introduce new coordinates $u = t + s$, $v = t - s$. Then the above inequality can be written as

$$\frac{\partial F}{\partial v} = O(\varepsilon^2).$$

Hence, we have $F(0,t) = F(t,0) + O(\varepsilon^2)$. The lemma now follows. □

A word of caution is in order here. We note that the phase flows in this corollary will lose derivatives when the t-derivative is taken. For example, the time derivative of initial data of class H^s will in general be only of class H^{s-1} since the governing PDEs are first order in the spatial variables. However, the chain rule and the equations of motion in Poisson bracket form can still be applied since the functionals in question are differentiable in both the H^s and the H^{s-1} topologies. We do note, however, that the terms that are $O(\varepsilon^2)$ in the statement depend on the F that is chosen. We presume that results like this can be strengthened by a more careful analysis of the PDEs in question, but this was not our goal in the present paper.

4.2. *Proof of Theorem* 4.1. Let $V = (U_1, U_2, U_3)$ be a vector field over the domain $0 \le y \le \varepsilon \bar{h}(x)$. By the Hodge decomposition, we have $V = \nabla\phi + V_0$, where V_0 is a divergence-free vector field and ϕ satisfies

$$\phi|_{y=\varepsilon\bar{h}} = 0, \qquad \frac{\partial \phi}{\partial y}\Big|_{y=0} = U_3(x,0). \tag{4.4}$$

Let P_ε be the map defined by $P_\varepsilon(V) = V_0$, which is the orthogonal projection onto the space of divergence-free vector fields tangent to $y = 0$.

The Hodge decomposition is global in nature in that it depends on the solution of a Laplacian. However, when $\varepsilon \to 0$, it becomes local as it degenerates to the decomposition into horizontal and vertical components. This is the content of the following result, which will be proved in the next subsection.

Lemma 4.4. *Let V, h be fixed, then*

$$||V - P_\varepsilon(V) - (0, 0, U_3(x, 0))||_{C^0} = O(\varepsilon).$$

Proof of Theorem 4.1. Note that the vorticity is

$$\omega = \left(\frac{\partial u_2}{\partial y} - \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_1} - \frac{\partial u_1}{\partial y}, \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right).$$

By Lemma 4.4, we have

$$\frac{\delta F_i}{\delta V}(x, \varepsilon y) = \left(\frac{\partial f_i(x,0)}{\partial x_1}, \frac{\partial f_i(x,0)}{\partial x_2}, 0\right) + O(\varepsilon).$$

Equation (4.2) shows that $\delta F_{SW}/\delta u = \varepsilon\bar{h}\partial f(u,0)/\partial x$, so

$$\begin{aligned}
&\omega \cdot \left(\frac{\delta F_1}{\delta V} \times \frac{\delta F_2}{\delta V}\right)(x, \varepsilon y) \\
&= (\varepsilon\bar{h})^{-2}\left(\frac{\partial F_{2,SW}}{\partial u_i}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\frac{\partial F_{1,SW}}{\partial u_j}\right)(x, 0) \\
&\quad + O(\varepsilon).
\end{aligned}$$

This shows that the first integral in $\{F_1, F_2\}_{3D}$ converges to the corresponding term in $\{F_{1,SW}, F_{2,SW}\}_{SW}$.

Now we consider the second integral in $\{F_1, F_2\}_{3D}$. By definition,

$$\begin{aligned}
\frac{\delta F}{\delta n}(x, \varepsilon\bar{h})\sqrt{1 + \varepsilon^2 \nabla_x \bar{h} \cdot \nabla_x \bar{h}} &= -\frac{\delta F}{\delta u_1}(x, \varepsilon\bar{h})\varepsilon(\bar{h})_{x_1}(x) \\
&\quad - \frac{\delta F}{\delta u_2}(x, \varepsilon\bar{h})\varepsilon(\bar{h})_{x_2}(x) \qquad (4.5) \\
&\quad + \frac{\delta F}{\delta v}(x, \varepsilon\bar{h}).
\end{aligned}$$

The last term above can be written, using the fact that $\delta F/\delta V = (\delta F/\delta u_1, \delta F/\delta u_2, \delta F/\delta v)$ is divergence-free and that $\delta F/\delta v$ vanishes at $y = 0$, as

$$\begin{aligned}\frac{\delta F}{\delta v}(x, \varepsilon\bar{h}) &= \varepsilon\bar{h}\frac{\partial}{\partial y}\frac{\delta F}{\delta v}(x,0) + O(\varepsilon^2) \\ &= -\varepsilon\bar{h}\left(\frac{\partial}{\partial x_1}\frac{\delta F}{\delta u_1}(x,0) + \frac{\partial}{\partial x_2}\frac{\delta F}{\delta u_2}(x,0)\right) \\ &\quad + O(\varepsilon^2).\end{aligned}$$

Hence, equation (4.5) can be written as

$$\frac{\delta F}{\delta n}(x, \varepsilon h_0)\sqrt{1+\varepsilon^2\nabla\bar{h}\cdot\nabla\bar{h}} = -\nabla_x \cdot \left(\varepsilon h_0 \frac{\delta F}{\delta u}\right)(x,0) + O(\varepsilon^2).$$

Now by equation (4.2) we compute

$$\frac{\delta F_{i,SW}}{\delta u} = \varepsilon\bar{h}\frac{\partial f_i}{\partial u}(x,0),$$

and

$$\frac{\delta F_{i,SW}}{\delta(\varepsilon\bar{h})}(x) = f_i(x,0) + \frac{\partial g_i}{\partial(\varepsilon\bar{h})}(x),$$

so

$$\begin{aligned}&\frac{\delta F_1}{\delta(\varepsilon\bar{h})}\frac{\delta F_2}{\delta n}(x,\varepsilon\bar{h})\sqrt{1+\varepsilon^2\nabla_x\bar{h}\cdot\nabla_x\bar{h}} \\ &\qquad = -\frac{\delta F_{1,SW}}{\delta(\varepsilon h_0)}\nabla_x\left(\frac{\delta F_{2,SW}}{\delta u}\right)(x,0) + O(\varepsilon^2).\end{aligned}$$

This proves the theorem. □

4.3. *Projection to the divergence-free vector fields.* We now will prove Lemma 4.4. We first consider the case $U_3(x, 0) = 0$. From the Hodge decomposition, $V - P_\varepsilon(V) = \nabla\phi$, where ϕ satisfies

$$\nabla^2\phi = \nabla \cdot V, \tag{4.6}$$

and the boundary conditions (4.4). The main idea of the proof is to rescale the domain $0 \le y \le \varepsilon\bar{h}(x)$ to the standard one $0 \le y \le 1$ by a change of variables

$$\bar{x}_1 = x_1, \qquad \bar{x}_2 = x_2, \qquad \bar{y} = \frac{y}{\varepsilon\bar{h}(x)}.$$

Introduce

$$X_1 = \frac{\partial}{\partial x_1} - \frac{(\bar{h})_{x_1}}{\varepsilon \bar{h}^2}\frac{\partial}{\partial \bar{y}},$$
$$X_2 = \frac{\partial}{\partial x_2} - \frac{(\bar{h})_{x_2}}{\varepsilon \bar{h}^2}\frac{\partial}{\partial \bar{y}},$$
$$Y = \frac{1}{\varepsilon \bar{h}}\frac{\partial}{\partial \bar{y}}.$$

In the new coordinates, ∇^2 and ∇ can be written as $\nabla^2 = X_1^2 + X_2^2 + Y^2$ and $\nabla = (X_1, X_2, Y)$. Then equation (4.6) can be rewritten as

$$\nabla^2 \phi = \nabla \cdot V, \tag{4.7}$$

with boundary condition

$$\phi|_{\bar{y}=1} = 0, \qquad \left.\frac{\partial \phi}{\partial \bar{y}}\right|_{\bar{y}=0} = 0. \tag{4.8}$$

In what follows we will write the integral $\int_0^1$ simply as $\int$, and the Sobolev norm as

$$||\phi||_{H^k} = \int \sum_{|\alpha|=k} |D^\alpha \phi|^2 + \int |\phi|^2.$$

Lemma 4.5. *Let ϕ be the solution of equation* (4.7) *with boundary conditions* (4.8), *then*

$$||\phi||_{H^k} \leq C(\bar{h})\varepsilon^2 ||V||_{H^k}, \qquad k = 0, 1, 2, 3,$$

where $C(\bar{h})$ is a constant only depending on $\bar{h}$.

Proof. We first prove the case $k = 0$. From (4.7) we obtain by integration by parts,

$$\int \nabla \phi \cdot \nabla \phi = -\int \phi \nabla^2 \phi = -\int \phi \nabla \cdot V,$$

so by the Schwarz inequality

$$\int \nabla\phi\cdot\nabla\phi = \int \nabla\phi\cdot V \le \left(\int \nabla\phi\cdot\nabla\phi\right)^{1/2}\left(\int V\cdot V\right)^{1/2},$$

or

$$\int \nabla\phi\cdot\nabla\phi \le \int V\cdot V, \tag{4.9}$$

and

$$\int (Y(\phi))^2 \le \int \nabla\phi\cdot\nabla\phi \le \int V\cdot V. \tag{4.10}$$

Since $\phi(x,1)=0$,

$$\phi(x,\bar{y}) = -\int_{\bar{y}<t<1} \frac{\partial\phi}{\partial\bar{y}}(x,t)\,dt.$$

By the Schwarz inequality, we obtain

$$\int \phi^2 \le C\int\left(\frac{\partial\phi}{\partial\bar{y}}\right)^2 = C\int h_0^2\varepsilon^2(Y(\phi))^2 \le C(\bar{h})\varepsilon^2\int V\cdot V.$$

This proves the case $k=0$.

Next we estimate the derivatives of ϕ. Note that $\phi_1 := \partial\phi/\partial x_1$ satisfies

$$\nabla^2\phi_1 + L_1 = \frac{\partial}{\partial x_1}(\nabla V), \tag{4.11}$$

where

$$L_1 = (a_1X_1 + a_2X_2 + a_3Y)\phi_1 + (a_4X_1 + a_5X_2 + a_6Y)\phi,$$

a_i are functions of $x=(x_1,x_2)$, and the same boundary conditions apply as in (4.8).

Equation (4.11) yields

$$\int (\nabla\phi_1,\nabla\phi_1) - (L_1,\phi_1) = -\int \phi_1\frac{\partial}{\partial x_1}(\nabla V). \tag{4.12}$$

Note that

$$-\int (L_1, \phi_1) \le \frac{1}{2}\int (\nabla\phi_1, \nabla\phi_1) + C\int \phi_1^2 + C\int |\nabla\phi|^2,$$

so we obtain from equations (4.12) and (4.9) that

$$\int (\nabla\phi_1, \nabla\phi_1) \le C\Big(||V||_{H^1}^2 + \int \phi_1^2 + \int |\nabla\phi|^2\Big) < C||V||_{H^1}^2.$$

Therefore,

$$\int (Y(\phi_1))^2 \le \int (\nabla\phi_1, \nabla\phi_1) \le ||V||_{H^1}^2. \tag{4.13}$$

Since $\phi_1(x, 1) = 0$, as in the case of $k = 0$ we obtain

$$\int \left(\frac{\partial\phi}{\partial x_1}\right)^2 \le C(\bar{h})\varepsilon^2 ||V||_{H^1}^2.$$

Using a similar method, we have the same estimate for $\partial\phi/\partial x_2$. This and (4.10) prove the case $k = 1$.

Now we turn to the case $k = 2$. We estimate the second derivatives in a like manner: $\partial^2\phi/\partial x_1^2, \partial^2\phi/\partial x_2^2$: First, by differentiating (4.7) twice with respect to x_1, we obtain an equation similar to equation (4.11) for $\phi_2 = \partial^2\phi/\partial x_1^2$, and then by the same method as above,

$$\int (Y(\phi_2))^2 \le C||V||_{H^2}^2. \tag{4.14}$$

Then, using the fact $\phi_2(x, 1) = 0$, as above we obtain

$$\int \phi_2^2 \le C(\bar{h})\varepsilon^2 ||V||_{H^2}^2. \tag{4.15}$$

To complete the proof for the case $k = 2$, we need to estimate $\partial^2\phi/\partial\bar{y}^2$. To do this, we obtain from equation (4.7),

$$Y^2(\phi) = \sum b_{ij}(x)\frac{\partial^2\phi}{\partial x_i \partial x_j} + c_i(x)Y\left(\frac{\partial\phi}{\partial x_i}\right) + d(x)Y(\phi),$$

then by equations (4.10), (4.15) and (4.13), it follows that

$$\int (Y^2(\phi))^2 \leq C \int \sum \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)^2 + \left(Y\left(\frac{\partial \phi}{\partial x_i}\right)\right)^2 + (Y(\phi))^2$$
$$\leq C(\bar{h})||V||^2_{H^2},$$

and hence,

$$\int \left(\frac{\partial \phi}{\partial \bar{y}}\right)^2 = \int \varepsilon^2 h^2 (Y^2(\phi))^2 \leq C(h)\varepsilon^2 ||V||^2_{H^2}.$$

This completes the proof of the case $k = 2$.

Finally we prove the case $k = 3$. First note that from equation (4.7) we have

$$Y^3(\phi) + YX_1^2(\phi) + YX_2^2(\phi) = Y(\nabla V),$$

then we estimate $\partial^3\phi/\partial\bar{y}^3$ using the same method as in the estimate of $\partial^2\phi/\partial\bar{y}^2$. Next we estimate $\partial^3\phi/\partial x_1^3$. Differentiating (4.7) three times with respect to x_1, and repeating the same argument as in the case $k = 2$, we obtain an equation similar to equation (4.11) for $\phi_3 = \partial^3\phi/\partial x_1^3$, and hence we can estimate $Y(\phi_3)$ and hence ϕ_3. □

Proof of Lemma 4.4. We first consider the case $U_3(x, 0) = 0$. Sobolev's inequaltiy

$$||\phi||_{C^1} \leq C(\bar{h})||\phi||_{H^3} \leq C(\bar{h})\varepsilon^2||V||_{H^3}$$

gives the result in case $U_3(x, 0) = 0$.

We now consider the more general case. Let ϕ_ε be the solution of $\nabla^2\phi = \nabla \cdot V$ with boundary condition

$$\phi|_{y=\varepsilon h} = 0, \qquad \left.\frac{\partial \phi}{\partial y}\right|_{y=0} = U_3(x, 0).$$

Then ϕ_ε can be written as $\phi_\varepsilon = U_3(x,0)(y - \varepsilon\bar{h}(x)) + \phi_2$, where ϕ_2 is the solution of

$$\nabla^2\phi_2 = \nabla \cdot V - \nabla^2(U_3(x, 0)(y - \varepsilon\bar{h}(x))),$$

with the boundary condition

$$\phi_2|_{y=\epsilon\bar{h}} = 0, \qquad \left.\frac{\partial \phi_2}{\partial y}\right|_{y=0} = 0.$$

As before, we see that the C^1-norm of ϕ_2 is of order $O(\varepsilon)$, so

$$||\phi_\epsilon - U_3(x,0)(y - \varepsilon\bar{h}(x))||_{C^1} = O(\varepsilon),$$

and

$$P_\epsilon(V) = V - (0,0,U_3(x,0)) + O(\varepsilon). \qquad \square$$

REFERENCES

1. V.I. Arnold, *Mathematical methods in classical mechanics*, 2nd edition, Springer-Verlag, New York, 1989.

2. R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), 1661–1664.

3. S.J. Chern, *Fluid dynamics on rotating spheres*, thesis, Cornell University, 1991.

4. S.J. Chern and J.E. Marsden, *A note on symmetry and stability for fluid flows*, Geophys. Astrophys. Fluid. Dynamics **51** (1990), 1–4.

5. D.G. Ebin, *The motion of slightly compressible fluids viewed as a motion with strong constraining forces*, Ann. Math. **105** (1977), 141–200.

6. D.G. Ebin and J.E. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. Math. **92** (1990), 102–163.

7. Z. Ge, H.P. Kruse and J.E. Marsden, *The limits of Hamiltonian structures in three dimensional elasticity, shells and rods*, preprint, 1995.

8. A.E. Green and P.M. Naghdi, *A derivation of equations for wave propagation in water of variable depth*, J. Fluid Mech. **78** (1976), 237–246.

9. T. Kano and T. Nishida, *Sur les ondes de surface de l'eau avec une justification mathematique des equations des ondes en eau peu profonde*, J. Math. Kyoto Univ. **19** (1979), 335–370.

10. D. Lewis, J.E. Marsden, R. Montgomery and T.S. Ratiu, *The Hamiltonian structure for dynamic free boundary problems*, Physica D **18** (1986), 391–404.

11. A. Lichnerowicz, *Deformations d'algebres associees a une variete symplectique (les ⋆-produits)*, Ann. Inst. Fourier (Grenoble) **32** (1982), 157–209.

12. J.E. Marsden, *Lectures on mechanics*, London Math. Soc. Lecture Note Ser. **174**, Cambridge Univ. Press, Cambridge-New York, 1992.

13. J.E. Marsden and A. Weinstein, *The Hamiltonian structure of the Maxwell-Vlasov equations*, Physica D **4** (1982), 394–406.

14. J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294.

15. P. Olver, *Hamiltonian perturbation theory and water waves*, in *Fluid and plasmas: Geometry and dynamics*, Contemp. Math. **28** (1983), 231–249.

16. J. Pedlosky, *Geophysical fluid dynamics*, Springer, New York, 1979.

17. H. Rubin and P. Ungar, *Motion under a strong constraining force*, Comm. Pure Appl. Math. 10 (1967), 65–87.

18. N.G. van Kampen and J.J. Lodder, *Constraints*, Amer. J. Phys. **52** (1984), 419–424.

19. A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. 18 (1983), 523–557.

20. N.M.J. Woodhouse, *Geometric quantization*, 2nd edition, Oxford University Press, Oxford, 1992.

THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES, 185 COLUMBIA ST. WEST, WATERLOO, ONTARIO N2L 5Z5

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720

CONTROL AND DYNAMICAL SYSTEMS, CALTECH 104-44, PASADENA, CA 91125

COMPUTER RESEARCH GROUP, C-3, MS-B265, LOS ALAMOS NATIONAL LABORATORY, LOS ALAMOS, NEW MEXICO 87545