Remarks on Geometric Mechanics

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1 Introduction

This paper gives a few new developments in mechanics, as well as some remarks of a historical nature. To keep the discussion focussed, most of the paper is confined to equations of "rigid body", or "hydrodynamic" type on Lie algebras or their duals. In particular, we will develop the variational structure of these equations and will relate it to the standard variational principle of Hamilton.

Even this small area of mechanics is fascinating from the historical point of view. In fact, it is quite surprising how long it can sometimes take for fundamental results of the masters to be tied together and to filter into the main literature and to become "well-known". In particular, part of our story follows a few fragments of a thread through the works of Euler, Lagrange, Lie, Poincaré, Clebsch, Ehrenfest, Hamel, Arnold, and many others.

Although Newton's discoveries were directly motivated by planetary motion, the realm of mechanics expanded well beyond particle mechanics with the work of Euler, Lagrange, and others to include fluid and solid mechanics. Today we see its methods permeating large areas of physical phenomena besides these, including electromagnetism, plasma physics, classical field theories, general relativity, and quantum mechanics. Part of what makes this unified point of view possible is the abstraction, often in a geometric way, of the underlying structures in mechanics.

Two general points of view emerged early on concerning the basic structures in mechanics. One, which is commonly referred to as "Lagrangian mechanics" can be based in variational principles, and the other, "Hamiltonian mechanics", rests on symplectic and Poisson geometry. As we shall see shortly, the history of this development is actually quite complex.

How rigid body mechanics, fluid mechanics and their generalizations fit into this story is quite interesting because of the way their equations fit into the schemes of Lagrange and Hamilton. For example, the way the equations are normally presented (in body representation for the rigid body, and in spatial representation for ideal fluids), they do not *literally* fit in as written. However, through a process of *reduction*, whereby the quotient by a Lie group of symmetries is taken, one gets in either picture, a clear understanding of how the variational and symplectic (or Poisson) structures descend to the quotient space. Since the reduction of variational principles has received less attention in the literature than that of symplectic and Poisson reduction, the paper spends more time on that aspect. Indeed, although the results here are very simple, they do appear to be new. A more general reduction procedure for Lagrangian systems that will also be sketched, is due to Marsden and Scheurle [1993].

A specific instance of this reduction procedure, which the paper will focus on, for both simplicity of exposition and its historical relevance, is that of equations on Lie algebras $\mathfrak{g}$ or their duals $\mathfrak{g}^*$. The equations on $\mathfrak{g}$ fit into the "Lagrangian mechanics" scheme, while those on $\mathfrak{g}^*$ fit into that of "Hamiltonian mechanics". The equations on $\mathfrak{g}$ will be called the ***Euler-Poincaré equations***, while those on $\mathfrak{g}^*$ will be called the ***Lie-Poisson equations***.

Mechanics has not only undergone considerable internal maturation, but its links with other areas of science and mathematics have strengthened considerably. For example, in engineering and physics, we have come to a much deeper understanding of stability, bifurcation and pattern formation through the maturation of mechanics and concurrent developments in dynamical systems. Perhaps the best known example of how mechanics links with mathematics is the use of symplectic techniques in representation theory through the work of Kostant, Kirillov, Guillemin, Sternberg, and many others. There are of course many other examples of deep links with mathematics and these mathematical bonds appear to be strengthening.

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2 Some basic Principles of Mechanics

Let Q be an n-manifold and TQ its tangent bundle. Coordinates $q^i, i = 1, \ldots, n$ on Q induce coordinates $(q^i, \dot{q}^i)$ on TQ, called ***tangent coordinates***. A mapping $L : TQ \to \mathbb{R}$ is called a ***Lagrangian***. Often we choose L to be $L = K - V$ where $K(v) = \frac{1}{2}\langle v, v\rangle$ is the ***kinetic energy*** of the given mechanical system, and that thus defines a Riemannian metric and where $V : Q \to \mathbb{R}$ is the ***potential energy***.

The ***variational principle of Hamilton*** singles out particular curves $q(t) \in Q$ by the condition

$$\delta \int_b^a L(q(t), \dot{q}(t))dt = 0, \tag{2.1}$$

where the variation is over smooth curves in Q with fixed endpoints. Note that (2.1) is unchanged if we replace the integrand by $L(q, \dot{q}) - \frac{d}{dt}S(q,t)$ for any function $S(q,t)$.

This reflects the ***gauge invariance*** of classical mechanics and is closely related to Hamilton-Jacobi theory.

If one prefers, Hamilton's variational principle states that the map I defined by

$$I(q(\cdot)) = \int_a^b L(q(t), \dot{q}(t))dt$$

from the space of curves with prescribed endpoints in Q to $\mathbb{R}$ has a critical point at the curve in question. In any case, a basic, but elementary result of the calculus of variations is that Hamilton's variational principle for a curve $q(t)$ is equivalent to the condition that this curve satisfy the ***Euler-Lagrange equations***:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0. \tag{2.2}$$

The ***Maupertuis principle of critical action***, which is closely related to Hamilton's principle, states that the integral of the canonical one form be stationary relative to curves with the energy constrained to a fixed value and with temporal variations of the endpoints possible.

Let us recall a few other basic results about this formalism. Given $L : TQ \to \mathbb{R}$, let $\mathbb{F}L : TQ \to T^*Q$, called the ***fiber derivative***, be the derivative of L in the fiber direction. In coordinates,

$$(q^i, \dot{q}^j) \mapsto (q^i, p_j)$$

where $p_j = \partial L/\partial \dot{q}^j$. A Lagrangian L is called ***hyperregular*** if $\mathbb{F}L$ is a diffeomorphism. If L is a hyperregular Lagrangian, we define the corresponding ***Hamiltonian*** by

$$H(q^i, p_j) = p_i \dot{q}^i - L.$$

The change of data from L on TQ to H on T^*Q is called the ***Legendre transform***.

For the hyperregular case, the Euler-Lagrange equations for L are equivalent to Hamilton's equations for H, namely,

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \tag{2.3}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2.4}$$

These equations define a vector field X_H on T^*Q that is related to the canonical symplectic form

$$\Omega = \sum_{i=1}^n dq^i \wedge dp_i$$

by

$$\mathbf{i}_{X_H}\Omega = dH,$$

where $\mathbf{i}$ denotes the interior product. They can also be written in Poisson bracket form $\dot{F} = \{F, H\}$ where

$$\{F, K\} = \sum_{i=1}^n \frac{\partial F}{\partial q_i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q^i}\frac{\partial F}{\partial p_i}$$

is the canonical Poisson bracket. One can, as is well known, also cast Hamilton's equations directly into a variational form on phase space (unlike Hamilton's principle, which is presented on configuration space).

In a relativistic context one finds that the two conditions $p_j = \partial L/\partial \dot{q}^j$ and $H = p_i \dot{q}^i - L$, defining the Legendre transform, fit together as the spatial and temporal components of a single object. Suffice it to say that the formalism developed here is useful in the context of relativistic fields.

3 Some Early History of the Euler-Lagrange Equations and Symplectic Geometry

In this section we make a few remarks concerning the history of the Euler-Lagrange equations.[1] Naturally, much of the story focuses on Lagrange. Section V of Lagrange's *Mecanique Analytique* contains the equations of motion in Euler-Lagrange form (2.2). Lagrange writes $Z = T - V$ for what we would call the Lagrangian today. In the preceding section of *Mecanique Analytique*, Lagrange came to these equations by asking for a coordinate invariant (*i.e.,* a covariant) expression for mass times acceleration. His conclusion is that it is given (in abbreviated notation) by $(d/dt)(\partial T/\partial v) - \partial T/\partial q$, which transforms under changes of configuration variables as a 1-form. This approach is closely related to Lagrange's introduction of generalized coordinates, which we would today refer to by saying that the configuration space is a differentiable manifold.

Interestingly, Lagrange does *not* recognize the equations of motion as being equivalent to the variational principle

$$\delta \int L \, dt = 0$$

. In fact, this principle was observed only a few decades later by Hamilton. The peculiar fact about this is that Lagrange *did* know the general form of the differential equations for variational problems and he actually had commented on Euler's proof of this—his early work on this in 1759 was admired very much by Euler. He immediately applied it to give a proof of the Maupertuis principle of least action, as a consequence of Newton's equations of motion. This principle, apparently having its roots in early work of Leibnitz, is a less natural principle in the sense that the curves are only varied over those which have a constant energy. It is also Hamilton's principle that applies in the *time dependent* case, when H is *not* conserved and which also generalizes to allow for certain external forces as well.

This discussion in the *Mecanique Analytique precedes* the equations of motion in general coordinates, and correspondingly is written in the case that the kinetic energy is of the form $\sum_i m_i v_i^2$, with constant $m_i's$. Wintner [1941] is also amazed by the fact that the more complicated Maupertuis principle historically precedes

[1] Many of these interesting historical points were conveyed by Hans Duistermaat. The reader can profitably consult with the standard texts such as those of Whittaker, Wintner, and Lanczos listed in the bibliography for additional information.

Hamilton's principle. One possible explanation is that Lagrange did not consider L as an interesting physical quantity; for him it was only a convenient function for writing down the equations of motion in a coordinate-invariant fashion. The time span between his work on variational calculus and the *Mecanique Analytique* (1788, 1808) could also be part of the explanation; he may have not been thinking of the variational calculus at the time he addressed the question of a coordinate invariant formulation of the equations of motion.

Section V starts by discussing the fact that the position and velocity at time t depend on the initial position and velocity, which can be chosen freely. We write this as (suppressing the coordinate indices for simplicity): $q = q(t, q_0, v_0), v = v(t, q_0, v_0)$, and in modern terminology we would talk about the flow in $x = (q, v)$-space. One problem in reading Lagrange is that he does not explicitly write the variables on which his quantities depend. In any case, he then makes an infinitesimal variation in the initial conditions and looks at the corresponding variations of position and velocity at time t. In our notation we would write $\delta x = (\partial x/\partial x_0)(t, x_0)\delta x_0$ and we would say that he considers the tangent mapping of the flow on the tangent bundle of $X = TQ$. Now comes the first interesting result. He takes two such variations, one denoted by δx and the other by Δx, and he writes down a bilinear form $\omega(\delta x, \Delta x)$, in which we recognize ω as the pull-back of the canonical symplectic form on the cotangent bundle of Q, by means of the fiber derivative $\mathbb{F}L$. What he then shows is that this symplectic product is constant as a function of t. This is nothing else than the *invariance of the symplectic form* ω *under the flow in* TQ.

It is striking that Lagrange gets the invariance of the symplectic form in TQ and *not* in T^*Q. In fact, Lagrange does *not* look at the equations of motion in the cotangent bundle via the transformation $\mathbb{F}L$; again it is Hamilton who observes that these take the Hamiltonian form (2.3). This is retrospectively puzzling since, later on in section V, Lagrange states very explicitly that it useful to pass to the (q, p)-coordinates by means of the coordinate transformation $\mathbb{F}L$ and one even sees written down a system of ordinary differential equations *in Hamiltonian form*, but with the total energy function H replaced by some other mysterious function $-\Omega$. Lagrange does use the letter H for the constant value of the energy, apparently in honor of Huygens. He also knew about the conservation of momentum as a result of translational symmetry.

The part where he discusses the Hamiltonian form deals with the case in which he modifies the system by perturbing the potential from $V(q)$ to $V(q) - \Omega(q)$, leaving the kinetic energy unchanged. To this perturbation problem, he applies his famous method of variation of constants, which is presented here in a truly nonlinear framework! In our notation, he keeps $t \mapsto x(t, x_0)$ as the solution of the unperturbed system, and then looks at the differential equations for $x_0(t)$ that make $t \mapsto x(t, x_0(t))$ a solution of the perturbed system. The result is that, if V is the vector field of the unperturbed system and $V + W$ is the vector field of the perturbed system, then $dx_0/dt = ((e^{tV})^*W)(x_0)$. Thus, $x_0(t)$ is the solution of the time dependent system, the vector field of which is obtained by pulling back W by means of the flow of V after time t. In the case Lagrange considers, the dq/dt-component of the perturbation is equal to zero, and the dp/dt-component is equal $\partial\Omega/\partial q$. Thus, it is obviously in a Hamiltonian form; this discussion does not use anything about Legendre-transformations (which Lagrange does not seem to know). But Lagrange knows already that the flow of the unperturbed system preserves the symplectic form, and he shows that the pull-back of his W under such a transformation is a vector field in Hamiltonian form. This is a time-dependent vector field defined by the function $G(t, q_0, p_0) = -\Omega(q(t, q_0, p_0))$. A potential point of confusion is that Lagrange denotes this by just $-\Omega$, and writes down expressions like $d\Omega/dp$, and one might first think these are zero because Ω was assumed to depend only on q. Lagrange presumably means that $dq_0/dt = \partial G/\partial p_0, dp_0/dt = -\partial G/\partial q_0$.

Most classical textbooks on mechanics, for example Routh, correctly point out that Lagrange has the invariance of the symplectic form in (q, v) coordinates (rather than in the canonical (q, p) coordinates). Less attention is paid to the equations obtained by the method of variation of constants that he wrote in Hamiltonian form. We do note, however, that this point is discussed in Weinstein [1981]. In fact, we should point out that the whole question of linearizing the Euler-Lagrange and Hamilton equations and retaining the mechanical structure is remarkably subtle (see Marsden, Ratiu, and Raugel [1991], for example).

Lagrange continues by introducing the *Poisson brackets* for arbitrary functions, arguing that these are useful in writing the time derivative of arbitrary functions of arbitrary variables, along solutions of systems in Hamiltonian form. He also continues by saying that if Ω is small, then $x_0(t)$ in zero order approximation is a constant and he obtains the next order approximation by an integration over t; here Lagrange introduces the first steps of the so-called *method of averaging*. When Lagrange discovered (in 1808) the invariance of the symplectic form, the variations-of-constants equations in Hamiltonian form and the Poisson brackets, he was already 73 years old. It is quite probable that Lagrange shared some of his ideas on brackets with Poisson at this time. In any case, it is clear that Lagrange had a surprisingly large part of the symplectic picture of classical mechanics.

4 Some History of Poisson Structures

Following from the work of Lagrange and Poisson mentioned above, the general concept of Poisson manifold probably should be credited to Sophus Lie in his treatise on transformation groups about 1880 in the chapter on "function groups". As was pointed out in Weinstein [1983], he also defined quite explicitly, a Poisson structure on the dual of a general Lie algebra; because of this, Marsden and Weinstein [1983] coined the phrase "Lie-Poisson bracket" for this object, and this terminology is now in common use. We recall the definition at the start of the next section. However, it is not clear that Lie realized that the Lie-Poisson bracket is obtained by a simple reduction process, namely that it is induced from the canonical cotangent Poisson bracket on T^*G by passing to $\mathfrak{g}$ regarded as the quotient T^*G/G, as will be explained in the next section. (This fact seems to have been first noted for the corresponding symplectic context by Marsden and Weinstein [1974]).

As noted by Weinstein [1983], Lie seems to have come very close, and may have even understood implicitly, the general concepts of momentum map and coadjoint orbit. The link between the closedness of the symplectic form and the Jacobi identity is a little harder to trace explicitly; some comments in this direction are given in Souriau [1970].

Lie starts by taking functions $F_1, \ldots, F_r$ on a symplectic manifold M, with the property that there exist functions G_{ij} of r variables, such that

$$\{F_i, F_j\} = G_{ij}(F_1, \ldots, F_r).$$

In Lie's time, functions were implicitly assumed to be analytic. The collection of all functions ϕ of $F_1, \ldots, F_r$ is the "function group" and is provided with the bracket

$$[\phi, \psi] = \sum_{ij} G_{ij} \phi_i \psi_j, \tag{4.5}$$

where

$$\phi_i = \frac{\partial \phi}{\partial F_i} \qquad \text{and} \qquad \psi_j = \frac{\partial \psi}{\partial F_j}.$$

Considering $F = (F_1, \ldots, F_r)$ as a map from M to an r-dimensional space P and ϕ and ψ as functions on P, one may formulate this as: $[\phi, \psi]$ is a Poisson structure on P, with the property that

$$F^*[\phi, \psi] = \{F^*\phi, F^*\psi\}.$$

Lie writes down the equations for the G_{ij} that follow from the antisymmetry and the Jacobi identity for the bracket $\{\,,\}$ on M. He continues with the question: suppose we have given a system of functions G_{ij} in r variables that satisfy these equations, is it induced as above from a function group of functions of $2n$ variables? He shows that under suitable rank conditions the answer is yes. As we shall see below, this result is the precursor to many of the fundamental results about the geometry of Poisson manifolds.

It is obvious that if G_{ij} is a system that satisfies the equations that Lie writes down, then (4.5) is a Poisson structure in the r-dimensional space. Vice versa, for any Poisson structure $[\phi, \psi]$, the functions

$$G_{ij} = [F_i, F_j]$$

satisfy Lie's equations.

Lie continues with more remarks on local normal forms of function groups (*i.e.,* of Poisson structures), under suitable rank conditions, which are not always stated as explicitly as one would like. These amount to the statement that a Poisson structure of constant rank is determined from a foliation by symplectic leaves. It is this characterization that Lie uses to get the symplectic form on coadjoint orbits. On the other hand, Lie does not apply the symplectic form on the coadjoint orbits to representation theory—representation theory of Lie groups started only later with Schur on GL_n, Elie Cartan on representations of semisimple Lie algebras and much later, in the 1930's by Weyl for compact Lie groups. The coadjoint orbit symplectic structure was connected with representation theory in the work of Kirillov and

Kostant. On the other hand, Lie *did* apply the Poisson structure on the dual of the Lie algebra to prove that every abstract Lie algebra can be realized as a Lie algebra of Hamiltonian vector fields, or as a Lie subalgebra of the Poisson algebra of functions on some symplectic manifold. This is "Lie's third fundamental theorem" in the form as given by Lie.

Of course, in geometry, people like Engel, Study and in particular Elie Cartan studied Lie's work intensely and propagated it very actively. However, through the tainted glasses of retrospection, Lie's work on Poisson structures did not appear to receive as much attention in mechanics; for example, even though Cartan himself did very important work in mechanics, he did not seem to realize that the Lie-Poisson bracket was central to the Hamiltonian description of some of the rotating fluid systems he was himself studying. However, others, such as Hamel [1904, 1949] did study Lie intensively and used it to make substantial contributions and extensions (such as to the study of nonholonomic systems, including rolling constraints), but many other active schools seem to have missed it. Even more surprising in this context is the contribution of Poincaré [1901, 1910] to the Lagrangian side of the story, a tale that we shall come to shortly. But we are getting ahead of ourselves—before telling this part of the story let us study some of the theory of mechanics and Lie algebras from the Hamiltonian point of view.

5 Lie-Poisson Structures and the Rigid Body

We now summarize a few topics in the dynamics of systems associated with Lie groups from a modern point of view to put the preceding historical comments in perspective.

Let G be a Lie group and $\mathfrak{g} = T_eG$ its Lie algebra with $[\,,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ the associated Lie bracket. The dual space $\mathfrak{g}^*$ is a Poisson manifold with either of the two brackets

$$\{f, k\}_{\pm}(\mu) = \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu} \right] \right\rangle . \tag{5.6}$$

Here $\delta f/\delta\mu \in \mathfrak{g}$ is defined by

$$\left\langle \nu, \frac{\delta f}{\delta \mu} \right\rangle = \mathbf{D}f(\mu) \cdot \nu$$

for $\nu \in \mathfrak{g}^*$, where $\mathbf{D}$ denotes the Frechet derivative. (In the infinite dimensional case one needs to worry about the existence of $\delta f/\delta\mu$). See, for instance, Marsden and Weinstein [1982, 1983] for applications to plasma physics and fluid mechanics. The notation $\delta f/\delta\mu$ is used to conform to the functional derivative notation in classical field theory. In coordinates, $(\xi^1, \ldots, \xi^m)$ on $\mathfrak{g}$ and corresponding dual coordinates $(\mu_1, \ldots, \mu_m)$ on $\mathfrak{g}^*$, the ***Lie-Poisson bracket*** (5.6) is

$$\{f, k\}_{\pm}(\mu) = \pm \mu_a C^a_{bc} \frac{\partial f}{\partial \mu_b} \frac{\partial k}{\partial \mu_c}; \tag{5.7}$$

here, C^a_{bc} are the structure constants of $\mathfrak{g}$ defined by $[e_a, e_b] = C^c_{ab} e_c$, where $(e_1, \ldots, e_m)$ is the coordinate basis of $\mathfrak{g}$ and where, for $\xi \in \mathfrak{g}$, we write $\xi = \xi^a e_a$, and for

$\mu \in \mathfrak{g}^*, \mu = \mu_a e^a$, where $(e^1, \ldots, e^m)$ is the dual basis. As we mentioned earlier, formula (5.7) appears explicitly in Lie [1890] (see §75).

Which sign to take in (5.7) is determined by understanding ***Lie-Poisson reduction***, which can be summarized as follows. Let

$$\lambda : T^*G \to \mathfrak{g}^* \quad \text{be defined by} \quad p_g \mapsto (T_e L_g)^* p_g \in T_e^* G \cong \mathfrak{g}^*, \tag{5.8}$$

and

$$\rho : T^*G \to \mathfrak{g}^* \quad \text{be defined by} \quad p_g \mapsto (T_e R_g)^* p_g \in T_e^* G \cong \mathfrak{g}^*. \tag{5.9}$$

Then λ *is a Poisson map if one takes the* $-$ *Lie-Poisson structure on* $\mathfrak{g}^*$ *and* ρ *is a Poisson map if one takes the* $+$ *Lie-Poisson structure on* $\mathfrak{g}^*$. This procedure uniquely characterizes the Lie-Poisson bracket and is a basic example of Poisson reduction.

Every left invariant Hamiltonian and Hamiltonian vector field is mapped by λ to a Hamiltonian and Hamiltonian vector field on $\mathfrak{g}^*$. There is a similar statement for right invariant systems on T^*G. One says that the original system on T^*G has been ***reduced*** to $\mathfrak{g}^*$. The reason λ and ρ are both Poisson maps is perhaps best understood by observing that they are both equivariant momentum maps generated by the action of G on itself by right and left translations, respectively together with the fact that equivariant momentum maps are always Poisson maps (see, for example, Marsden et. al. [1983]).

The Euler equations of motion for rigid body dynamics are given by

$$\dot{\Pi} = \Pi \times \Omega, \tag{5.10}$$

where $\Pi = \mathbb{I}\Omega$ is the body angular momentum, $\mathbb{I}$ is the moment of inertia tensor, and Ω is the body angular velocity. Euler's equations are Hamiltonian relative to the minus Lie-Poisson structure. To see this, take $G = SO(3)$ to be the configuration space. Then $\mathfrak{g} \cong (\mathbb{R}^3, \times)$ and we identify $\mathfrak{g} \cong \mathfrak{g}^*$ using the standard inner product on Euclidean space. The corresponding (minus) Lie-Poisson structure on $\mathbb{R}^3$ is given by

$$\{f, k\}(\Pi) = -\Pi \cdot (\nabla f \times \nabla k). \tag{5.11}$$

For the rigid body one chooses the minus sign in the Lie-Poisson bracket because the rigid body Lagrangian (and hence Hamiltonian) is *left* invariant and so its dynamics pushes to $\mathfrak{g}^*$ by the map λ in (5.8) .

To understand the way the Hamiltonian function originates, it is helpful to recall some basic facts about rigid body dynamics. We regard an element $R \in SO(3)$ giving the configuration of the body as a map of a reference configuration $\mathcal{B} \subset \mathbb{R}^3$ to the current configuration $R(\mathcal{B})$; the map R takes a reference or label point $X \in \mathcal{B}$ to a current point $x = R(X) \in R(\mathcal{B})$. When the rigid body is in motion, the matrix R is time dependent and the velocity of a point of the body is $\dot{x} = \dot{R}X = \dot{R}R^{-1}x$. Since R is an orthogonal matrix, $R^{-1}\dot{R}$ and $\dot{R}R^{-1}$ are skew matrices, and so we can write

$$\dot{x} = \dot{R}R^{-1}x = \omega \times x, \tag{5.12}$$

which defines the ***spatial angular velocity vector*** ω. The corresponding *body angular velocity* is defined by

$$\Omega = R^{-1}\omega, \quad i.e., \quad R^{-1}\dot{R}v = \Omega \times v \tag{5.13}$$

so that Ω is the angular velocity relative to a body fixed frame. The kinetic energy is

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho(X)\|\dot{R}X\|^2 \, d^3X, \tag{5.14}$$

where ρ is a given mass density in the reference configuration. Since

$$\|\dot{R}X\| = \|\omega \times x\| = \|R^{-1}(\omega \times x)\| = \|\Omega \times X\|,$$

K is a quadratic function of Ω. Writing

$$K = \frac{1}{2}\Omega^T \mathbb{I}\Omega \tag{5.15}$$

defines the ***moment of inertia tensor*** $\mathbb{I}$, which, if the body does not degenerate to a line, is a positive definite 3×3 matrix, or equivalently, a quadratic form. This quadratic form can be diagonalized, and this defines the principal axes and moments of inertia. In this basis, we write $\mathbb{I} = \operatorname{diag}(I_1, I_2, I_3)$. The function K is taken to be the Lagrangian of the system on $TSO(3)$ and by means of the Legendre transformation we get the corresponding Hamiltonian description on $T^*SO(3)$. One observes that the Lagrangian and the Hamiltonian are left invariant functions and so can be expressed in body representation. In this way, we obtain the formula for the Hamiltonian in body representation $H(\Pi) = \frac{1}{2}\Pi \cdot (\mathbb{I}^{-1}\Pi)$. One can then verify directly from the chain rule and properties of the triple product that Euler's equations are equivalent to the following equation for all $f \in \mathcal{F}(\mathbb{R}^3)$:

$$\dot{f} = \{f, H\}. \tag{5.16}$$

If $(P, \{\,,\})$ is a Poisson manifold, a function $C \in \mathcal{F}(P)$ satisfying $\{C, f\} = 0$ for all $f \in \mathcal{F}(P)$ is called a *Casimir function*. In the case of the rigid body, every function $C : \mathbb{R}^3 \to \mathbb{R}$ of the form $C(\Pi) = \Phi(\|\Pi\|^2)$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a differentiable function, is a Casimir function, as is readily checked. Casimir functions are constants of the motion for *any* Hamiltonian since $\dot{C} = \{C, H\} = 0$ for any H. In particular, for the rigid body, $\|\Pi\|^2$ is a constant of the motion. Casimir functions and momentum maps play a key role in the stability theory of relative equilibria (see Marsden [1992] and references therein and for references and a discussion of the relation between Casimir functions and momentum maps.

As we have remarked, the maps λ and ρ induce Poisson isomorphisms between $(T^*G)/G$ and $\mathfrak{g}^*$ (with the $-$ and $+$ brackets respectively) and this is a special instance of Poisson reduction. The following result is one useful way of formulating the general relation between T^*G and $\mathfrak{g}^*$. We treat the left invariant case for simplicity.

Theorem 5.1 *Let G be a Lie group and $H : T^*G \to \mathbb{R}$ be a left invariant Hamiltonian. Let $h : \mathfrak{g}^* \to \mathbb{R}$ be the restriction of H to the identity. For a curve $p(t) \in T^*_{g(t)}G$, let $\mu(t) = (T^*_{g(t)}L) \cdot p(t) = \lambda(p(t))$ be the induced curve in $\mathfrak{g}^*$. Then the following are equivalent:*

- **i** *$p(t)$ is an integral curve of X_H; i.e., Hamilton's equations on T^*G hold,*
- **ii** *for any smooth function $F \in \mathcal{F}(T^*G), \dot{F} = \{F, H\}$, where $\{\,,\}$ is the canonical bracket on T^*G*
- **iii** *$\mu(t)$ satisfies the **Lie-Poisson equations***

$$\frac{d\mu}{dt} = \mathrm{ad}^*_{\delta h/\delta\mu}\mu \tag{5.17}$$

where $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}_\xi \eta = [\xi, \eta]$ and ad^*_ξ is its dual, i.e.,

$$\dot{\mu}_a = C^d_{ba}\frac{\delta h}{\delta \mu_b}\mu_d \tag{5.18}$$

- **iv** *for any $f \in \mathcal{F}(\mathfrak{g}^*)$, we have*

$$\dot{f} = \{f, h\}_- \tag{5.19}$$

where $\{\,,\}_-$ is the minus Lie-Poisson bracket.

We now make some remarks about the proof. First of all, the equivalence of **i** and **ii** is general for any cotangent bundle, as is well known. The equivalence of **ii** and **iv** follows from the fact that λ is a Poisson map and $H = h \circ \lambda$. Finally, we establish the equivalence of **iii** and **iv**. Indeed, $\dot{f} = \{f, h\}_-$ means

$$\begin{aligned}
\left\langle \dot{\mu}, \frac{\delta f}{\delta \mu} \right\rangle &= -\left\langle \mu, \left[\frac{\delta f}{\delta\mu}, \frac{\delta h}{\delta \mu}\right]\right\rangle \\
&= \left\langle \mu, \mathrm{ad}_{\delta h/\delta\mu}\frac{\delta f}{\delta\mu}\right\rangle \\
&= \left\langle \mathrm{ad}^*_{\delta h/\delta\mu}\mu, \frac{\delta f}{\delta\mu}\right\rangle .
\end{aligned}$$

Since f is arbitrary, this is equivalent to **iii**.

6 A Little History of the Equations of Mechanics on Lie algebras and their Duals

The above theory describes the adaptation of the concepts of Hamiltonian mechanics to the context of the duals of Lie algebras. This theory could easily have been given shortly after Lie's work, but evidently it was not observed for the rigid body or ideal fluids until the work of Pauli [1953], Martin [1959], Arnold [1966], Ebin and Marsden [1970], Nambu [1973], and Sudarshan and Mukunda [1974], all of whom were, it seems, unaware of Lie's work on the Lie-Poisson bracket. It would appear that even Elie Cartan was unaware of this aspect of Lie's work, which does seem surprising. Perhaps it is less surprising when one thinks for a moment about how many other things Cartan was involved in at the time. Nevertheless, one is struck by the amount of rediscovery and confusion in this subject. Evidently this situation is not unique to mechanics.

One can also write the equations directly on the Lie algebra, bypassing the Lie-Poisson equations on the dual. The resulting equations were first written down on a general Lie algebra by Poincaré [1901]; we refer to these as the ***Euler-Poincaré equations***. Arnold [1988] and Chetaev [1989] emphasized these equations as important in the recent literature. We shall develop them from a modern point of view in the next section. Poincaré [1910] goes on to study the effects of the deformation of the earth on its precession—he apparently recognizes the equations as Euler equations on a semi-direct product Lie algebra. In general, the command that Poincaré had of the subject is most impressive, and is hard to match in his near contemporaries, except perhaps Riemann and Routh. It is noteworthy that Poincaré [1901] has no bibliographic references and is only three pages in length, so it is rather hard to trace his train of thought or his sources–compare this style with that of Hamel [1904]! In particular, he gives no hints that he understood the work of Lie on the Lie-Poisson structure, but of course Poincaré understood the Lie group and Lie algebra concepts very well indeed.

Our derivation of the Euler-Poincaré equations in the next section is based on a reduction of variational principles, not on a reduction of the symplectic or Poisson structure, which is natural for the dual. We also show that the Lie-Poisson equations are related to the Euler-Poincaré equations by the "fiber derivative" in the same way as one gets from the ordinary Euler-Lagrange equations to the Hamilton equations. Even though this is relatively trivial in the present context, it does not appear to have been written down before.

In the dynamics of ideal fluids, the resulting variational principle is essentially what has been known as "Lin constraints". (See Cendra and Marsden [1987] for a discussion of this theory and for further references; that paper introduced a constrained variational principle closely related to that given here, but using Lagrange multipliers rather than the direct and simpler approach in this paper). Variational principles in fluid mechanics itself has an interesting history, going back to Ehrenfest, Boltzman, and Clebsch, but again, there was little if any contact with the heritage of Lie and Poincaré on the subject. Even as recently as Seliger and Witham [1968] it was remarked that "Lin's device still remains somewhat mysterious from a strictly mathematical view". It is our hope that the methods of the present paper remove some of this mystery.

One person who was well aware of the work of both Lie and Poincaré was Hamel. However, despite making excellent contributions, he seemed to miss the true simplicity of the situation, and instead got tangled up in the concept of "quasi-coordinates".

How does Lagrange fit into this story? In *Mecanique Analytique*, volume 2, equations A on page 212 are the Euler-Poincaré equations for the rotation group written out explicitly for a reasonably general Lagrangian. Of course, he must have been thinking of the rigid body equations as his main example. We should remember that Lagrange also developed the key concept of the Lagrangian representation of fluid motion, but it is not clear that he understood that both systems are special instances of one theory. Lagrange spends a large number of pages on his derivation of the Euler-Poincaré equations for $SO(3)$, in fact, a good chunk of volume 2 of *Mecanique Analytique*. His derivation is not as clean as we would give today, but it seems to have the right spirit of a reduction method. That is, he tries to get the equations from the Euler-Lagrange equations on $TSO(3)$ by passing to the Lie algebra.

Because of the above facts, one might argue that the term "Euler-Lagrange-Poincaré" equations is the right nomenclature for these equations. Since Poincaré noted the generalization to arbitrary Lie algebras, and applied it to interesting fluid problems, it is clear that his name belongs, but in light of other uses of the term "Euler-Lagrange", it seems that "Euler-Poincaré" is a reasonable choice.

Marsden and Scheurle [1992], [1993] and Weinstein [1993] have studied a more general version of Lagrangian reduction whereby one drops the Euler-Lagrange equations from TQ to TQ/G. This is a nonabelian generalization of the classical Routh method, and leads to a very interesting coupling of the Euler-Lagrange and Euler-Poincaré equations. This problem was also studied by Hamel [1904] in connection with his work on nonholonomic systems (see Koiller [1992] and Bloch, Krishnaprasad, Marsden and Murray [1993] for more information).

7 The Euler-Poincaré Equations

Above, we saw how to write the Lagrangian of rigid body motion as a function $L : TSO(3) \to \mathbb{R}$ and that the Lagrangian can be written entirely in terms of the body angular velocity. From the Lagrangian point of view, the relation between the motion in R space and that in body angular velocity (or Ω) space is as follows.

Theorem 7.1 *The curve $R(t) \in SO(3)$ satisfies the Euler-Lagrange equations for*

$$L(R, \dot{R}) = \frac{1}{2} \int_{\mathcal{B}} \rho(X) \|\dot{R}X\|^2 \, d^3X \tag{7.20}$$

if and only if $\Omega(t)$ defined by $R^{-1}\dot{R}v = \Omega \times v$ for all $v \in \mathbb{R}^3$ satisfies Euler's equations:

$$\mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega. \tag{7.21}$$

Moreover, this equation is equivalent to conservation of the spatial angular momentum:

$$\frac{d}{dt}\pi = 0 \tag{7.22}$$

where $\pi = R\mathbb{I}\Omega$.

One instructive way to prove this *indirectly* is to pass to the Hamiltonian formulation and use Lie-Poisson reduction, as outlined above. One way to do it *directly* is to use variational principles. By Hamilton's principle, $R(t)$ satisfies the Euler-Lagrange equations if and only if

$$\delta \int L\,dt = 0.$$

Let $l(\Omega) = \frac{1}{2}(\mathbb{I}\Omega) \cdot \Omega$ so that $l(\Omega) = L(R, \dot{R})$ if R and Ω are related as above. To see how we should transform Hamilton's principle, we differentiate the relation $R^{-1}\dot{R}v = \Omega \times v$ with respect to R to get

$$-R^{-1}(\delta R)R^{-1}\dot{R}v + R^{-1}(\delta\dot{R})v = \delta\Omega \times v. \tag{7.23}$$

Let the skew matrix $\hat{\Sigma}$ be defined by

$$\hat{\Sigma} = R^{-1}\delta R \tag{7.24}$$

and define the vector Σ by

$$\hat{\Sigma}v = \Sigma \times v. \tag{7.25}$$

Note that

$$\dot{\hat{\Sigma}} = -R^{-1}\dot{R}R^{-1}\delta R + R^{-1}\delta\dot{R},$$

so

$$R^{-1}\delta\dot{R} = \dot{\hat{\Sigma}} + R^{-1}\dot{R}\hat{\Sigma} \tag{7.26}$$

substituting (7.26) and (7.24) into (7.23) gives

$$-\hat{\Sigma}\hat{\Omega}v + \dot{\hat{\Sigma}}v + \hat{\Omega}\hat{\Sigma}v = \widehat{\delta\Omega}v$$

i.e.,

$$\widehat{\delta\Omega} = \dot{\hat{\Sigma}} + [\hat{\Omega}, \hat{\Sigma}]. \tag{7.27}$$

The identity $[\hat{\Omega}, \hat{\Sigma}] = (\Omega \times \Sigma)\hat{}$ holds by Jacobi's identity for the cross product, and so

$$\delta\Omega = \dot{\Sigma} + \Omega \times \Sigma. \tag{7.28}$$

These calculations prove the following

Theorem 7.2 *Hamilton's variational principle*

$$\delta \int_a^b L\,dt = 0 \tag{7.29}$$

on $SO(3)$ is equivalent to the ***reduced variational principle***

$$\delta \int_a^b l\,dt = 0 \tag{7.30}$$

on $\mathbb{R}^3$ where the variations $\delta\Omega$ are of the form (7.28) with $\Sigma(a) = \Sigma(b) = 0$.

To complete the proof of Theorem **7.1**, it suffices to work out the equations equivalent to the reduced variational principle (7.30). Since $l(\Omega) = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle$, and $\mathbb{I}$ is symmetric, we get

$$\begin{aligned}
\delta \int_a^b l\, dt &= \int_a^b \langle \mathbb{I}\Omega, \delta\Omega\rangle dt \\
&= \int_a^b \langle \mathbb{I}\Omega, \dot{\Sigma} + \Omega \times \Sigma\rangle dt \\
&= \int_a^b \left[\left\langle -\frac{d}{dt}\mathbb{I}\Omega, \Sigma \right\rangle + \langle \mathbb{I}\Omega, \Omega \times \Sigma\rangle \right] \\
&= \int_a^b \left\langle -\frac{d}{dt}\mathbb{I}\Omega + \mathbb{I}\Omega \times \Omega, \Sigma \right\rangle dt
\end{aligned}$$

where we have integrated by parts and used the boundary conditions $\Sigma(b) = \Sigma(a) = 0$. Since Σ is otherwise arbitrary, (7.30) is equivalent to

$$-\frac{d}{dt}(\mathbb{I}\Omega) + \mathbb{I}\Omega \times \Omega = 0,$$

which are Euler's equations. That these are equivalent to the conservation of spatial angular momentum is a straightforward calculation. Note that alternatively, one can use Noether's theorem to prove conservation of spatial angular momentum, and from this one can derive the Euler equations. ■

We now generalize this procedure to an arbitrary Lie group and later will make the direct link with the Lie-Poisson equations.

Theorem 7.3 *Let G be a Lie group and $L : TG \to \mathbb{R}$ a left invariant Lagrangian. Let $l : \mathfrak{g} \to \mathbb{R}$ be its restriction to the identity. For a curve $g(t) \in G$, let $\xi(t) = g(t)^{-1} \cdot \dot{g}(t)$; i.e., $\xi(t) = T_{g(t)}L_{g(t)^{-1}}\dot{g}(t)$. Then the following are equivalent*

i *$g(t)$ satisfies the Euler-Lagrange equations for L on G,*

ii *the variational principle*

$$\delta \int L(g(t), \dot{g}(t))dt = 0 \tag{7.31}$$

holds, for variations with fixed endpoints,

iii *the* ***Euler-Poincaré equations*** *hold:*

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_{\xi}\frac{\delta l}{\delta \xi}, \tag{7.32}$$

iv *the variational principle*

$$\delta \int l(\xi(t))dt = 0 \tag{7.33}$$

holds on $\mathfrak{g}$, using variations of the form

$$\delta\xi = \dot\eta + [\xi, \eta], \tag{7.34}$$

where η vanishes at the endpoints,

v *conservation of spatial angular momentum holds:*

$$\frac{d}{dt}\pi = 0 \tag{7.35}$$

where π is defined by

$$\pi = \mathrm{Ad}^*_{g^{-1}}\frac{\partial l}{\partial \xi}. \tag{7.36}$$

We comment on the proof. First of all, the equivalence of **i** and **ii** holds on the tangent bundle of any configuration manifold Q. Secondly, **ii** and **iv** are equivalent. To see this, one needs to compute the variations $\delta\xi$ induced on $\xi = g^{-1}\dot g = TL_{g^{-1}}\dot g$ by a variation of g. To calculate this, we need to differentiate $g^{-1}\dot g$ in the direction of a variation δg. If $\delta g = dg/d\epsilon$ at $\epsilon = 0$, where g is extended to a curve g_ϵ, then, roughly speaking,

$$\delta\xi = \frac{d}{d\epsilon}g^{-1}\frac{d}{dt}g$$

while if $\eta = g^{-1}\delta g$, then

$$\dot\eta = \frac{d}{dt}g^{-1}\frac{d}{d\epsilon}g.$$

It is thus plausible that the difference $\delta\xi - \dot\eta$ is the commutator, $[\xi, \eta]$. Above, we saw the explicit verification of this for the rigid body, and the same proof works for any matrix group. For a complete proof for the general case, see Bloch, Krishnaprasad, Marsden and Ratiu [1994b] (it also follows from formulas in Marsden, Ratiu and Raugel [1991]).

The proof that **iii** and **v** are equivalent is a straightforward verification. We also note that conservation of the spatial angular momentum follows from Noether's theorem (indeed the spatial angular momentum is the value of the momentum map for the left action of the group), so this can be used to give another derivation of the Euler-Poincaré equations.

To complete the proof, we show the equivalence of **iii** and **iv**. Indeed, using the definitions and integrating by parts,

$$\begin{aligned}
\delta\int l(\xi)dt &= \int \frac{\delta l}{\delta\xi}\delta\xi\, dt \\
&= \int \frac{\delta l}{\delta\xi}(\dot\eta + \mathrm{ad}_\xi\eta)dt \\
&= \int \left[-\frac{d}{dt}\left(\frac{\delta l}{\delta\xi}\right) + \mathrm{ad}^*_\xi\frac{\delta l}{\delta\xi}\right]\eta\, dt
\end{aligned}$$

so the result follows. ■

Since the Euler-Lagrange and Hamilton equations on TQ and T^*Q are equivalent, it follows that the Lie-Poisson and Euler-Poincaré equations are also equivalent. To see this *directly*, we make the following Legendre transformation from $\mathfrak{g}$ to $\mathfrak{g}^*$:

$$\mu = \frac{\delta l}{\delta \xi}, \quad h(\mu) = \langle \mu, \xi \rangle - l(\xi).$$

Note that

$$\frac{\delta h}{\delta \mu} = \xi + \left\langle \mu, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \frac{\delta l}{\delta \xi}, \frac{\delta \xi}{\delta \mu} \right\rangle = \xi$$

and so it is now clear that the Lie-Poisson and Euler-Poincaré equations are equivalent.

8 The Reduced Euler-Lagrange Equations

As we have mentioned, the Lie-Poisson and Euler-Poincaré equations occur for many systems besides the rigid body equations. They include the equations of fluid and plasma dynamics, for example. For many other systems, such as a rotating molecule or a spacecraft with movable internal parts, one can use a combination of equations of Euler-Poincaré type and Euler-Lagrange type. Indeed, on the Hamiltonian side, this process has undergone development for quite some time, and is discussed briefly below. On the Lagrangian side, this process is also very interesting, and has been recently developed by, amongst others, Marsden and Scheurle [1993]. The general problem is to drop Euler-Lagrange equations and variational principles from a general velocity phase space TQ to the quotient TQ/G by a Lie group action of G on Q. If L is a G-invariant Lagrangian on TQ, it induces a reduced Lagrangian l on TQ/G.

An important ingredient in this work is to introduce a connection A on the principal bundle $Q \to S = Q/G$, assuming that this quotient is nonsingular. For example, the mechanical connection (see Kummer [1981], Marsden [1992] and references therein), may be chosen for A. This connection allows one to split the variables into a horizontal and vertical part.

We let x^α, also called "internal variables", be coordinates for shape space Q/G, η^a be coordinates for the Lie algebra $\mathfrak{g}$ relative to a chosen basis, l be the Lagrangian regarded as a function of the variables $x^\alpha, \dot{x}^\alpha, \eta^a$, and let C^a_{db} be the structure constants of the Lie algebra $\mathfrak{g}$ of G.

If one writes the Euler-Lagrange equations on TQ in a local principal bundle trivialization, using the coordinates x^α introduced on the base and η^a in the fiber, then one gets the following system of ***Hamel equations***

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} = 0 \tag{8.37}$$

$$\frac{d}{dt}\frac{\partial l}{\partial \eta^b} - \frac{\partial l}{\partial \eta^a} C^a_{db}\eta^d = 0. \tag{8.38}$$

However, this representation of the equations does not make global intrinsic sense (unless $Q \to S$ admits a global flat connection). The introduction of a connection allows one to intrinsically and globally split the original variational principle relative to horizontal and vertical variations. One gets from one form to the other by means of the velocity shift given by replacing η by the vertical part relative to the connection:

$$\xi^a = A^a_\alpha \dot{x}^\alpha + \eta^a$$

Here, A^d_α are the local coordinates of the connection A. This change of coordinates is well motivated from the mechanical point of view since the variables ξ have the interpretation of the locked angular velocity and they often complete the square (help to diagonalize) the kinetic energy expression. The resulting ***reduced Euler-Lagrange equations*** have the following form:

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} = \frac{\partial l}{\partial \xi^a}\left(B^a_{\alpha\beta}\dot{x}^\beta + B^a_{\alpha d}\xi^d\right) \tag{8.39}$$

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^b} = \frac{\partial l}{\partial \xi^a}(B^a_{\alpha b}\dot{x}^\alpha + C^a_{db}\xi^d) \tag{8.40}$$

In these equations, $B^a_{\alpha\beta}$ are the coordinates of the curvature B of A, $B^a_{\alpha d} = c^a_{bd}A^b_\alpha$ and $B^a_{d\alpha} = -B^a_{\alpha d}$.

It is interesting to note that the matrix

$$\begin{bmatrix} B^a_{\alpha\beta} & B^a_{\alpha d} \\ B^a_{\alpha d} & c^a_{bd} \end{bmatrix}$$

is itself the curvature of the connection regarded as residing on the bundle $TQ \to TQ/G$.

The variables ξ^a may be regarded as the rigid part of the variables on the original configuration space, while x^α are the internal variables. As in Simo, Lewis, and Marsden [1991], the division of variables into internal and rigid parts has deep implications for both stability theory and for bifurcation theory, again, continuing along lines developed originally by Riemann, Poincaré and others. The main way this new insight is achieved is through a careful split of the variables, using the (mechanical) connection as one of the main ingredients. This split puts the second variation of the augmented Hamiltonian at a relative equilibrium as well as the symplectic form into "normal form". It is somewhat remarkable that they are *simultaneously* put into a simple form. This link helps considerably with an eigenvalue analysis of the linearized equations, and in Hamiltonian bifurcation theory–see for example, Bloch, Krishnaprasad, Marsden and Ratiu [1994a].

One of the key results in Hamiltonian reduction theory says that the reduction of a cotangent bundle T^*Q by a symmetry group G is a bundle over T^*S, where $S = Q/G$ is shape space, and where the fiber is either $\mathfrak{g}^*$, the dual of the Lie algebra of G, or is a coadjoint orbit, depending on whether one is doing Poisson or symplectic reduction. We refer to Montgomery, Marsden, and Ratiu [1984] and Marsden [1992] for details and references. The reduced Euler-Lagrange equations give the analogue of this structure on the tangent bundle.

Remarkably, equations (8.39) are formally identical to the equations for a mechanical system with classical nonholonomic velocity constraints (see Neimark and

Fufaev [1972] and Koiller [1992].) The connection chosen in that case is the one-form that determines the constraints. This link is made precise in Bloch, Krishnaprasad, Marsden and Murray [1994]. In addition, this structure appears in several control problems, especially the problem of stabilizing controls considered by Bloch, Krishnaprasad, Marsden, and Sanchez [1992].

For systems with a momentum map $\mathbf{J}$ constrained to a specific value μ, the key to the construction of a reduced Lagrangian system is the modification of the Lagrangian L to the Routhian R^μ, which is obtained from the Lagrangian by subtracting off the mechanical connection paired with the constraining value μ of the momentum map. On the other hand, a basic ingredient needed for the reduced Euler-Lagrange equations is a velocity shift in the Lagrangian, the shift being determined by the connection, so this velocity shifted Lagrangian plays the role that the Routhian does in the constrained theory.

Conclusions

The current vitality of mechanics, including the investigation of fundamental questions, is quite remarkable, given its long history and development. This vitality comes about through rich interactions with both pure mathematics (from topology and geometry to group representation theory) and through new and exciting applications to areas such as control theory. It is perhaps even more remarkable that absolutely fundamental points, such as a clear and unambiguous linking of Lie's work on the Lie-Poisson bracket on the dual of a Lie algebra and Poincaré's work on the Euler-Poincaré equations on the Lie algebra itself, with the most basic of examples in mechanics, such as the rigid body and the motion of ideal fluids, took nearly a century to complete. The attendant lessons to be learned about communication between mathematics and the other mathematical sciences are, hopefully, obvious.

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