Extended Abstract: Bifurcations in Hamiltonian Systems with Symmetry

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Introduction

This is an extended abstract for the lecture of J.E. Marsden to be presented at the ENOC-93 conference, Hamburg, August, 1993.

The goal of the lecture is to outline some recent work of the lecturer, M. Golubitsky, I. Stewart, E. Knobloch, A. Mahalov and T. Ratiu on bifurcations in Hamiltonian systems with symmetry. We will survey some earlier work with D. Lewis, T. Ratiu and others to set the stage for the problems to be discussed. Many of the basic results in bifurcation of mechanical systems is available in, for example, Marsden [1992] and Duistermaat [1983] and references therein.

Part of this theory involves an understanding of the generic movement of eigenvalues of Hamiltonian systems linearized at equilibrium points as parameters of the system are varied. For eigenvalues passing through zero, this theory was begun in Golubitsky and Stewart [1987] and for eigenvalues involved in a one to one resonance, it was studied by Dellnitz, Melbourne, and Marsden [1992]. The main point is that the movement of eigenvalues, such as the distinction between the cases when eigenvalues pass on the imaginary axis, or split into the right and left half planes depends very much on the nature of the symmetry group of the system and how this group acts on the phase space. Guckenheimer and Mahalov [1992] show how the situation of eigenvalues passing through the origin, but staying on the imaginary axis develops a window of instability when the symmetry of the system is broken from S^1 to a discrete group, and that this situation arises in some interesting fluid problems, such as rigid flow in a cylindrical container. Knobloch, Mahalov, and Marsden [1993] examine a similar situation when eigenvalues pass through a one to one resonance.

Interesting examples of bifurcation in mechanical systems include the following:

- rotating liquid drops as their angular momentum increases (Lewis et. al. [1987].
- bifurcations in planar coupled rigid bodies as system parameters vary (see Oh et. al. [1989],
- bifurcations in the double spherical pendulum (Marsden and Scheurle [1992])
- rotating fluid columns subject to symmetry breaking or Coriolis forces (Knobloch et. al. [1993])

One of the overall aims of this theory is to study to what extent the theory of bifurcation with symmetry, so successful in the theory of general dynamical systems, can be applied to mechanical systems, and when this is done, to study the relations between these two theories. There are (at least) two main approaches to the theory of bifurcation with symmetry that have been developed.

- The first is to use normal forms (averaging, amplitude equations) and center manifold methods
- The second is to look for special kinds of solutions, such as equilibria and periodic orbits using specialized tools like the Liapunov-Schmidt method.

For mechanical systems, the first method can have serious technical difficulties associated with the fact that equilibria typically have many eigenvalues on the imaginary axis, and so while normal form and invariant manifold methods are common and indeed powerful, in these situations they can be difficult to justify. On the other hand, the second methods, while powerful for locating specific types of solutions, do not often give the whole picture of the phase portrait. Typically they do not have the same technical difficulties as the first method.

In the lecture we will concentrate on the ideas and the examples; for the rest of the abstract, however, we give some of the more technical details for reference. We will be illustrating both of the techniques above, and will start with the first.

1 Hamiltonian Normal Forms

Following Knobloch, Mahalov, and Marsden [1993], we study a Hamiltonian normal form that comes up in resonance problems and in system symmetry breaking. Specifically, we consider Hamiltonian systems with $SO(2) \times SO(2)$ symmetry, such as the Euler equations for swirling fluid flow in a cylinder subject to periodic boundary conditions in the axial direction. We suppose that the system possesses a trivial (*i.e.*, $SO(2) \times SO(2)$ invariant) equilibrium. We are

interested in the stability of this flow with respect to perturbations that break both azimuthal and axial invariance. The trivial state is stable with respect to a single mode of this type, and this continues to be the case when the axial wavenumber k is chosen so that two such modes have identical frequencies. If the symmetry of the system is broken so that these two modes couple, then instability becomes possible. We consider first the interaction between two modes both with eigenvalues $i\omega$, one of which is axisymmetric with the other having azimuthal wavenumber $m = 1$. These modes appear to dominate the process of vortex breakdown.

If A_1 and A_2 denote the complex amplitudes of these modes, the corresponding linear eigenfunction takes the form

$$\psi(r, \phi, z) = \mathrm{Re}(A_1 e^{ikz} + A_2 e^{ikz+i\phi}). \tag{1.1}$$

The symmetry under translations $z \mapsto z + d$ in the axial direction acts by $(A_1, A_2) \mapsto e^{ikd}(A_1, A_2)$, while rotations $\phi \mapsto \phi+\theta$ act by $(A_1, A_2) \mapsto (A_1, A_2 e^{i\theta})$. These yield the symmetry $(A_1, A_2) \mapsto (A_1 e^{i\theta_1}, A_2 e^{i\theta_2})$. The resulting equivariant amplitude equations take the form

$$\begin{aligned} \frac{dA_1}{dt} &= i\omega A_1 + iA_1(s_{11}|A_1|^2 + s_{12}|A_2|^2) + h.o.t. \\ \frac{dA_2}{dt} &= i\omega A_2 + iA_2(s_{21}|A_1|^2 + s_{22}|A_2|^2) + h.o.t., \end{aligned} \tag{1.2}$$

where the s_{ij} are real constants and $h.o.t.$ denotes higher order terms.

When the symmetry with respect to rotations is reduced by system symmetry breaking with azimuthal wavenumber $m = 1$, e.g., by precession, the $SO(2)$ rotation symmetry is broken to the identity and the dominant symmetry breaking terms enter at linear order:

$$\begin{aligned} \frac{dA_1}{dt} &= i\omega A_1 + \epsilon p A_2 + iA_1(s_{11}|A_1|^2 + s_{12}|A_2|^2) + h.o.t. \\ \frac{dA_2}{dt} &= i\omega A_2 + \epsilon q A_1 + iA_2(s_{21}|A_1|^2 + s_{22}|A_2|^2) + h.o.t. \end{aligned} \tag{1.3}$$

Here, $\epsilon > 0$ is a measure of the departure from full symmetry (e.g., ϵ could be the strength of an external Coriolis force) These equations are invariant under the operation $(A_1, A_2) \mapsto (e^{i\theta}A_1, e^{i\theta}A_2)$. In the situation where the two modes are not exactly in resonance, we replace equations (1.3) by

$$\begin{aligned} \frac{dA_1}{dt} &= i\omega_1 A_1 + \epsilon p A_2 + iA_1(s_{11}|A_1|^2 + s_{12}|A_2|^2) + h.o.t. \\ \frac{dA_2}{dt} &= i\omega_2 A_2 + \epsilon q A_1 + iA_2(s_{21}|A_1|^2 + s_{22}|A_2|^2) + h.o.t., \end{aligned} \tag{1.4}$$

where $\omega_2 - \omega_1 \equiv \lambda$ is the detuning. Then at $\lambda = 0$, $\omega_1 = \omega_2 = \omega$. Finally, by going into the rotating frame $(A_1, A_2) \mapsto e^{i\omega_2 t}(A_1, A_2)$ the equations can be further simplified:

$$\begin{aligned} \frac{dA_1}{dt} &= i\lambda A_1 + \epsilon p A_2 + iA_1(s_{11}|A_1|^2 + s_{12}|A_2|^2) + h.o.t. \\ \frac{dA_2}{dt} &= \epsilon q A_1 + iA_2(s_{21}|A_1|^2 + s_{22}|A_2|^2) + h.o.t. \end{aligned} \tag{1.5}$$

By rescaling A_1 and A_2 and redefining ϵ we may set $p = q = 1$ (if $pq > 1$) or $p = -q = 1$ (if $pq < 1$).

The equations (1.4) can be written in the following two forms, depending on the sign of pq. First, if $pq > 0$, we set $z_1 = iA_1$ and $z_2 = A_2$ to get the $1 : -1$ resonance form

$$\begin{aligned} \dot{z}_1 &= i\omega_1 z_1 + i\epsilon p\bar{z}_2 + iz_1(s_{11}|z_1|^2 + s_{12}|z_2|^2) \\ \dot{z}_2 &= -i\omega_2 z_2 + i\epsilon q\bar{z}_1 - iz_2(s_{21}|z_1|^2 + s_{22}|z_2|^2). \end{aligned} \tag{1.6}$$

Second, if $pq < 0$, we let $\zeta_1 = z_1$ and $\zeta_2 = \bar{z}_2$ to get the $1 : 1$ resonance form

$$\begin{aligned} \dot{\zeta}_1 &= i\omega_1\zeta_1 + i\epsilon p\zeta_2 + i\zeta_1(s_{11}|\zeta_1|^2 + s_{12}|\zeta_2|^2) \\ \dot{\zeta}_2 &= i\omega_2\zeta_2 - i\epsilon q\zeta_1 + i\zeta_2(s_{21}|\zeta_1|^2 + s_{22}|\zeta_2|^2). \end{aligned} \tag{1.7}$$

As already mentioned, we can assume that $p = q = 1$ in (1.6) and that $p = -q = 1$ in (1.7). The former case corresponds to the case of splitting eigenvalues, while the latter corresponds to the passing case; cf Guckenheimer and Mahalov [1992].

The Hamiltonian structure of these systems is the standard one obtained by taking the real and imaginary parts of z_i and ζ_i as conjugate variables. For example, we write $z_1 = q_1 + ip_1$ and require $\dot{q}_1 = \partial H/\partial p_1$ and $\dot{p}_1 = -\partial H/\partial q_1$. It follows that:

(i) The system (1.6) is Hamiltonian if and only if $s_{12} = -s_{21}$ and $p = q$. In this case we can choose

$$\begin{aligned} H(z_1, z_2) &= \frac{1}{2}(\omega_2|z_2|^2 - \omega_1|z_1|^2) - \epsilon p\,\mathrm{Re}(z_1 z_2) \\ &\quad -\frac{s_{11}}{4}|z_1|^4 - \frac{s_{12}}{2}|z_1 z_2|^2 + \frac{s_{22}}{4}|z_2|^4. \end{aligned} \tag{1.8}$$

(ii) The system (1.7) is Hamiltonian if and only if $s_{12} = s_{21}$ and $p = -q$. In this case we can choose

$$\begin{aligned} H(\zeta_1, \zeta_2) = -\frac{1}{2}(\omega_1|\zeta_1|^2 + \omega_2|\zeta_2|^2) - \epsilon p\,\mathrm{Re}(\zeta_1\bar{\zeta}_2) \\ -\frac{s_{11}}{4}|\zeta_1|^4 - \frac{s_{12}}{2}|\zeta_1\zeta_2|^2 - \frac{s_{22}}{4}|\zeta_2|^4. \end{aligned} \tag{1.9}$$

Note that for (1.6) with $\epsilon = 0$ there are two separate S^1 actions acting on z_1 and z_2 independently; the corresponding conserved quantities are $|z_1|^2$ and $|z_2|^2$. However, for $\epsilon \neq 0$, the symmetry action is

$$(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2)$$

with the conserved quantity

$$J(z_1, z_2) = \frac{1}{2}(|z_1|^2 - |z_2|^2). \tag{1.10}$$

Likewise, for (1.7), the symmetry action is

$$(\zeta_1, \zeta_2) \mapsto (e^{i\theta} \zeta_1, e^{i\theta} \zeta_2)$$

leading to the conserved quantity

$$J(\zeta_1, \zeta_2) = \frac{1}{2}(|\zeta_1|^2 + |\zeta_2|^2). \tag{1.11}$$

In either case, it is clear that (1.6) and (1.7) are completely integrable systems, with the integrals being provided by the Hamiltonians and the corresponding conserved J. Note that in view of the conservation of the J the hamiltonians (1.8) and (1.9) can be further simplified. We choose not to do this in order to emphasize that the splitting and passing cases are distinguished by the structure of the corresponding hamiltonians even in the absence of additional symmetries. Close to the origin (1.8) is indefinite (splitting), while (1.9) is definite (passing). In fact, the eigenvalues μ of the linearization of both at the origin in $\mathbf{C}^2$ are given by

$$\mu = \frac{1}{2} i \left\{ \pm(\omega_1 + \omega_2) \pm \sqrt{\lambda^2 - 4\epsilon^2 pq} \right\}, \tag{1.12}$$

where $\lambda = \omega_1 - \omega_2$ is the detuning. In particular, if $pq > 0$, one gets splitting along the lines $\lambda = 2\epsilon\sqrt{pq}$, in agreement with the generic theory (see Dellnitz, Melbourne, and Marsden [1992]). In contrast, when $pq < 0$ all four eigenvalues remain imaginary. However, in this case the passing of eigenvalues for $\epsilon = 0$ changes to "bouncing" of eigenvalues for ϵ different from zero. Again this is consistent with the generic theory: when symmetry, other than normal form symmetry, is lost, passing is not generic.

It is convenient to introduce the variables r_1, r_2 and $\phi \equiv \phi_2 - \phi_1$, where $A_1 = r_1 \exp(i\phi_1)$ and $A_2 = r_2 \exp(i\phi_2)$. The truncated system (1.5) then takes the form

$$\begin{aligned} \frac{dr_1}{dt} &= \epsilon p r_2 \cos\phi \\ \frac{dr_2}{dt} &= \epsilon q r_1 \cos\phi \\ \frac{d\phi}{dt} &= -\lambda - \frac{\epsilon}{r_1 r_2}(q r_1^2 + p r_2^2) \sin\phi + a r_1^2 + b r_2^2, \end{aligned} \tag{1.13}$$

where $a = s_{21} - s_{11}$, $b = s_{22} - s_{12}$. This system has the following two integrals:

$$J = \frac{1}{2q}(qr_1^2 - pr_2^2), \quad E = \frac{\lambda}{2(p^2+q^2)}(pr_1^2 + qr_2^2) + \epsilon r_1 r_2 \sin\phi - \frac{a}{4p}r_1^4 - \frac{b}{4q}r_2^4 \quad (1.14)$$

Using these integrals one can obtain a single differential equation for $\rho \equiv r_1^2$. This equation takes the form

$$\left(\frac{d\rho}{dt}\right)^2 = P(\rho), \quad (1.15)$$

where

$$P(\rho) \equiv 4\epsilon^2 pq\rho(\rho - 2J) - 4p^2\left(E + \frac{\lambda q^2}{p(p^2+q^2)}J - \frac{\lambda}{2p}\rho + \frac{a}{4p}\rho^2 + \frac{bq}{4p^2}(\rho - 2J)^2\right)^2 \quad (1.16)$$

is a polynomial of degree four. Clearly, $P(\rho)$ may be thought of as the potential energy.

These results can be derived naturally by Hamiltonian reduction (cf. Marsden [1992]) as follows. Let $\phi = \theta_2 - \theta_1$, where $z_1 = r_1\exp(i\theta_1)$, $z_2 = r_2\exp(i\theta_2)$. We know that the Hamiltonian structure for (1.6) on $\mathbf{C}^2$ described above induces one on $\mathbf{C}^2/S^1$ and that the two integrals descend to the quotient space, as does the bracket. Now $\mathbf{C}^2/S^1$ is parametrized by (r_1, r_2, ϕ) and dropping the integrals from the previous subsection reproduces (1.14) apart from a scaling. But one can also drop the Poisson bracket. That is, the equations in (r_1, r_2, ϕ) can be cast in Hamiltonian form $\dot{F} = \{F, H\}$ for the induced Poisson bracket. It is obtained simply by using the chain rule to relate the complex variables and the polar coordinates. One finds that

$$\{F, K\}(r_1, r_2, \phi) = \frac{1}{r_1^2}\left(\frac{\partial F}{\partial r_1}\frac{\partial K}{\partial \phi} - \frac{\partial F}{\partial \phi}\frac{\partial K}{\partial r_1}\right) + \frac{1}{r_2^2}\left(\frac{\partial F}{\partial r_2}\frac{\partial K}{\partial \phi} - \frac{\partial F}{\partial \phi}\frac{\partial K}{\partial r_2}\right). \quad (1.17)$$

A similar procedure works for (1.7) with $\phi = \theta_2 + \theta_1$, where $\zeta_1 = r_1\exp(i\theta_1)$ and $\zeta_2 = r_2\exp(i\theta_2)$.

Equations (1.5) can be written in rigid body form and this form is important for resonances generally. We first recall some general theory for this situation. Consider the action of S^1 on $\mathbf{C}^2$ given by

$$(z_1, z_2) \mapsto (e^{ik\theta}z_1, e^{il\theta}z_2), \quad (1.18)$$

where k and l are integers. This action is Hamiltonian with respect to the symplectic form on $\mathbf{C}^2$ given by

$$\Omega((z_1, z_2), (\tilde{z}_1, \tilde{z}_2)) = -\mathrm{Im}(z_1\overline{\tilde{z}_1}) - \mathrm{Im}(z_2\overline{\tilde{z}_2}). \quad (1.19)$$

The momentum map for this action is given by

$$J(z_1, z_2) = \frac{1}{2}\left(k|z_1|^2 + l|z_2|^2\right). \tag{1.20}$$

The momentum map J is invariant under the S^1 action. Other invariant functions are given by

$$X + iY = \overline{z_1}^l z_2^k, \quad Z = \frac{1}{2}\left(k|z_1|^2 - l|z_2|^2\right). \tag{1.21}$$

If, say, l, is negative, then we replace $\overline{z_1}^l$ by $z_1^{|l|}$. Note that $-J \leq Z \leq J$ and that these invariants are related by

$$X^2 + Y^2 = k^{-|l|}l^{-|k|}(J+Z)^{|l|}(J-Z)^{|k|}. \tag{1.22}$$

In the case of the $1:1$ resonance ($k=1, l=1$) the invariants (X, Y, Z) comprise the components of the momentum map of the standard $SU(2)$ action on $\mathbf{C}^2$; this action is relevant in this case, since it is the symmetry group of J.

For the $1:1$ resonance with J fixed equation (1.22) defines a sphere. For the $1:-1$ resonance, one gets a hyperboloid. For other values of k and l one can get objects with "pinches" and this is important in many resonance problems.

In performing Poisson reduction, one normally constructs the quotient space $\mathbf{C}^2/S^1$ and calculates its induced Poisson bracket. However, except for the case of $k=1$ and $l=1$, the action, while locally free (apart from the origin), is not free, and so one has to be careful about singularities in the quotient space. For example, for $k=1$ and $l=2$, the action of the group element $e^{i\pi}$ leaves points of the form $(0, z_2) \in \mathbf{C}^2$ invariant.

For each real number m, define the map $\phi_m : \mathbf{R}^3 \to \mathbf{R}$ by

$$\phi_m = X^2 + Y^2 - k^{-|l|}l^{-|k|}(m+Z)^{|l|}(m-Z)^{|k|}, \tag{1.23}$$

so that the relation (1.22) between the variables X, Y, Z, J becomes $\phi_J(X, Y, Z) = 0$.

The quotient $\mathbf{C}^2/S^1$ is identifiable with $\mathbf{R}^3$ coordinatized by (X, Y, Z) and carries the quotient Poisson structure given as follows. Let F and G be given functions of X, Y, Z and let (X, Y, Z) lie on the set $\phi_m(X, Y, Z) = 0$. Then

$$\{F, G\}(X, Y, Z) = \nabla\phi_m \cdot (\nabla F \times \nabla G). \tag{1.24}$$

This result is proved as follows. Define $f = F \circ \pi$ where π is the map sending $(z_1, z_2) \mapsto (X, Y, Z)$. The Poisson bracket on $\mathbf{C}^2$ associated to the symplectic structure (1.19) is given by

$$\{f, g\} = -\mathrm{Im}\left\langle \nabla_{z_1} f, \nabla_{z_1} g \right\rangle - \mathrm{Im}\left\langle \nabla_{z_2} f, \nabla_{z_2} g \right\rangle, \tag{1.25}$$

where the gradient uses the real inner product. The bracket (1.24) follows by computing $\{f, g\}$ using the chain rule. We expect that this "pinched sphere"

picture will be a useful general tool for resonance problems (many of these results are due to Kummer [1990]).

The symplectic leaves in the above Poisson structure are given by the symplectic reduced spaces, namely by the sets $\phi_m = 0$ corresponding to the symplectic reduced spaces $J^{-1}(m)/S^1$. The bracket above is the Poisson bracket associated with these leaves. The leaves $\phi_m = 0 \subset \mathbf{R}^3$ are, in general, "pinched spheres". If h is a Hamiltonian on $\mathbf{C}^2$ that is invariant under the action of S^1, it induces a function H on $\mathbf{R}^3$ and the reduced equations on the pinched sphere $\phi_m = 0$ are given by the (Euler-like) equations

$$\dot{V} = \nabla H \times \nabla \phi_m, \tag{1.26}$$

where $V = (X, Y, Z)$. In Knobloch, Mahalov, and Marsden [1993], the above equations are studied in detail, along with bifurcation studies and their physical implications.

2 The Liapunov-Schmidt Procedure for Hamiltonian Systems

As we have mentioned, the second main approache is to use solution specific techniques. In particular, the search for periodic solutions by means of the Liapunov-Schmidt procedure has been very successful in the Hopf and other bifurcations. This method is developed for systems with symmetry in Golubitsky and Schaeffer [1985], Chapter 8, §2. We show how to add to this theory, the extra conditions of being Hamiltonian. We first phrase this in terms of systems with a first integral and indicate how it can be used to give a proof of the Liapunov center theorem. Below, we shall show how it can be used in resonance problems. Another application of the methods here to the periodic orbit structure of the Hamiltonian Hopf bifurcation is given in Dellnitz and Golubitsky [1993].

We first consider the abstract Liapunov-Schmidt theory for systems with a first integral.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $F : \mathcal{X} \to \mathcal{Y}$ be a C^k map ($k \geq 1$) whose zeros we seek. Assume that $F(0) = 0$. Let Γ be a group acting linearly on both $\mathcal{X}$ and $\mathcal{Y}$ and suppose that F is Γ-equivariant. Also, assume there is a *first integral* in the sense that there is another Banach space $\mathcal{W}$ on which Γ also acts, and an equivariant map

$$\Phi : \mathcal{X} \times \mathcal{Y} \to \mathcal{W}$$

of class C^l (for $l \geq 1$), such that

$$\Phi(x, F(x)) = 0 \tag{2.27}$$

for all $x \in \mathcal{X}$ (or in a neighborhood of zero, since the constructions here are local).

Let $\mathcal{K} = \ker dF(0)$, where d is the Frechet derivative, and assume there is a Γ-invariant closed complement $\mathcal{M}$ so that $\mathcal{X} = \mathcal{K} \oplus \mathcal{M}$ and similarly, let $\mathcal{R} = \text{range}\, dF(0)$, assume it is closed and has a Γ-invariant closed complement N, so $\mathcal{Y} = \mathcal{R} \oplus \mathcal{N}$. Let $\mathrm{P} : \mathcal{Y} \to \mathcal{R}$ denote the projection with kernel $\mathcal{N}$ and (locally) define a C^k map $w : \mathcal{K} \to \mathcal{N}$ with $w(0) = 0$ by solving

$$\mathrm{P}F(k, w(k)) = 0 \tag{2.28}$$

for w using the implicit function theorem. By uniqueness of the solution $w(k)$ and invariance of the splitting under Γ, one finds that w is Γ-equivariant. One also finds that

$$dw(0) = 0. \tag{2.29}$$

Higher derivatives of w at zero are determined by implicit differentiation.

To see how the first integral (2.27) affects the bifurcation equation $g(k) = 0$, where

$$g(k) := (I - \mathrm{P})F(k, w(k)) = 0, \tag{2.30}$$

define $\varphi : \mathcal{K} \times \mathcal{N} \to \mathcal{W}$ by

$$\varphi(k, n) = \Phi((k, w(k)), n). \tag{2.31}$$

Observe that φ is Γ-equivariant because w and Φ are. From (2.28) and (2.30),

$$\begin{aligned} \varphi(k, g(k)) &= \Phi((k, w(k)), (I - \mathrm{P})F(k, w(k))) \\ &= \Phi((k, w(k)), F(k, w(k))) \end{aligned}$$

and so $\varphi(k, g(k)) = 0$ by (2.27). We interpret

$$\varphi(k, g(k)) = 0 \tag{2.32}$$

as the first integral for the bifurcation equation (2.30). This imposes conditions on the bifurcation equation similar to conditions imposed by equivariance.

The Liapunov Center Theorem below follows from the *proof* of the Liapunov center theorem due to Duistermaat (see Abraham and Marsden [1978], 496–499). The proof here is based on the Liapunov-Schmidt procedure (Vanderbauwhede [1982], Golubitsky, Krupa and Lim [1991]).

Start with a C^k system $\dot{x} = f(x)$ on a manifold $M, k \geq 1$, and assume

(H1) x_0 is a fixed point: $f(x_0) = 0$. Let $A = f'(x_0)$ be the linearization at x_0.

(H2) $\pm\omega_0 i$ are simple eigenvalues of A and $k\omega_0 i$ is not an eigenvalue for $k = 0, 2, 3, \ldots$. Let V denote the eigenspace associated to the eigenvalues $\pm\omega_0 i$.

(H3) f has a C^{k+1} first integral $H : M \to \mathbf{R}$; *i.e.*, $dH(x) \cdot f(x) = 0$ with the properties:

(a) $dH(x_0)|V = 0$

(b) $d^2H(x_0)|V$ is positive (or negative) definite (For Hamiltonian systems, this follows automatically, as in Abraham and Marsden [1978], §5.6.)

Theorem 2.1 *Under conditions* (H1)–(H3), *there is a one-parameter family of periodic orbits in a neighborhood of x_0 with periods close to $2\pi/\omega_0$.*

These periodic orbits define an invariant C^k manifold of dimension two that is tangent to V at x_0.

Proof Rescaling time, we can assume that $\omega_0 = 1$. Passing to a coordinate chart in M, we can assume $M = \mathbf{R}^n$ since the theorem, while intrinsic for manifolds, is local.

Let $\mathcal{C}^1_{2\pi}$ be the space of C^1 maps of S^1 to $\mathbf{R}^n$ and let $\mathcal{C}^0_{2\pi}$ be the corresponding C^0 maps. Define

$$F : \mathcal{C}^1_{2\pi} \times \mathbf{R} \to \mathcal{C}^0_{2\pi}$$

by

$$F(u, \tau) = (1 + \tau)\frac{du}{ds} - f(u). \tag{2.33}$$

The map F is C^k by the so-called "Ω-lemma" (see, for example, Abraham, Marsden and Ratiu [1988], §2.4.

Here we let S^1 act on $\mathcal{Y} = \mathcal{C}^0_{2\pi}$ by $\theta \cdot (v)(s) = v(s + \theta)$ and on $\mathcal{C}^1_{2\pi}$ by the same action. Then S^1 acts on $\mathcal{X} = \mathcal{C}^1_{2\pi} \times \mathbf{R}$ by acting trivially on the second factor. Note that F is S^1 equivariant.

The derivative of F at $u = x_0$ is given by

$$Lv \equiv dF(x_0, 0) \cdot v = v' - Av.$$

By (H2),

$$\ker L = \operatorname{span}\{\operatorname{Re}(e^{is}v), \operatorname{Im}(e^{is}v)\}$$

where $Av = iv$. Thus, we can identify the kernel of dF with $\mathbf{C}$; explicitly, $z \to \operatorname{Re}(ze^{is}v)$. Using the Fredholm alternative, let $\mathcal{M} = \operatorname{range} L^*$ be the orthogonal complement of $\mathcal{K} = \ker L$, and let $\mathcal{N} = \ker L^*$ be the orthogonal complement to $\mathcal{R} = \operatorname{range} L$.

The Liapunov-Schmidt procedure with an integral may now be applied. Define the complements to the kernel and the range using the adjoint of the linearization, as in the Fredholm alternative. The adjoint depends on an S^1-invariant inner product. To simplify our procedures, we choose the L^2 inner product induced by one that, on V, coincides with $d^2H(0)|V$, (or its negative, should $d^2H(0)|V$ be negative definite).

Thus, the Liapunov-Schmidt procedure gives a bifurcation equation

$$g : \mathbf{C} \times \mathbf{R} \to \mathbf{C}$$

and, being S^1 equivariant, it has the form (see Golubitsky and Schaeffer [1985], p. 344)

$$g(z,\tau) = p(|z|^2,\tau)z + q(|z|^2,\tau)iz. \tag{2.34}$$

In the reversible case, one has $p = 0$ identically, so that one just needs to solve the equations $q = 0$. Here, we to show that $q = 0$ *implies* $p = 0$.

The calculation in Golubitsky and Schaeffer [1985], p. 344–349 shows that

$$p(0,0) = 0, q(0,0) = 0, p_\tau(0,0) = 0 \quad \text{and} \quad q_\tau(0,0) = -1.$$

Thus, we can solve $q = 0$ for $\tau = \tau(|z|^2)$. We claim that this automatically defines the required manifold of periodic orbits. To show this, we need to show that $\tau = \tau(|z|^2)$ implies $p = 0$ for z small. To do this we use the first integral. Define

$$\Phi : \mathcal{X} \times \mathcal{Y} \to \mathbf{R}$$

by

$$\Phi((u,\tau),v) = \int_0^{2\pi} dH(u(s)) \cdot v(s) ds.$$

Note that

$$\begin{aligned} \Phi((u,\tau),F(u,\tau)) &= \int_0^{2\pi} dH(u(s)) \cdot \left[(1+\tau)\frac{du}{ds} - f(u(s))\right] ds \\ &= \int_0^{2\pi} (1+\tau)\frac{d}{ds}H(u(s)) - dH(u(s)) \cdot f(u(s)) ds, \end{aligned}$$

which vanishes as H is a first integral of f. Thus (2.27) holds, so we may conclude that the bifurcation equation satisfies (2.32). In other words, we have

$$\varphi((z,\tau), g(z,\tau)) = 0,$$

i.e.,

$$\Phi(((z,\tau) + w(z,\tau)), g(z,\tau)) = 0,$$

i.e.,

$$\int_0^{2\pi} dH(u_z(s)) \cdot g(z,\tau) ds = 0.$$

Here, $u_z(s)$ is the element of $\mathcal{X} = C^1_{2\pi}$ of the form $\mathrm{Re}(ze^{is}v) + O(z^2)$ and $g(z)$ stands for $\zeta = p(|z|^2,\tau)z + q(|z|^2,\tau)iz$ regarded as an element of $\mathcal{Y}$ by

$$\zeta \mapsto \mathrm{Re}(\zeta e^{is} v^*) \tag{2.35}$$

where v^* is the eigenvector for A^*, *i.e.*,, $A^* v^* = iv^*$.

Choose $\tau = \tau(|z|^2)$ so $q = 0$ as above. Thus, we get

$$0 = p(|z|^2\tau(|z|^2)) \int_0^{2\pi} dH(\mathrm{Re}(ze^{is}v + O(|z|^2) \cdot \mathrm{Re}(ze^{is}v^*) ds. \tag{2.36}$$

Now consider $z = r$ real and let

$$\Psi(r) = \int_0^{2\pi} dH(\mathrm{Re}(re^{is}v + O(|z|^2) \cdot \mathrm{Re}(e^{is}v^*)ds.$$

Note that $\Psi(0) = 0$; a simple calculation then gives

$$\Psi'(0) = \int_0^{2\pi} d^2H(0) \cdot (\mathrm{Re}(e^{is}v), \mathrm{Re}(e^{is}v^*))ds.$$

By the choice of inner product,

$$\Psi'(0) = \int_0^{2\pi} \langle v, v^* \rangle ds = 2\pi$$

and so $\Psi(r) > 0$ for $r > 0$ and small. Thus, from (2.36), $p(|z|^2, \tau(|z|^2)) = 0$ for z small.

The smoothness of the manifold of periodic orbits follows directly from the smoothness of $\tau(|z|^2)$ which is guaranteed by the implicit function theorem. QED

The above proof of the Liapunov center theorem automatically produces an analogous equivariant theorem, whose hypotheses we now explain.

Let $\Gamma \subset O(n)$ be a given group acting in the usual way on $\mathbf{R}^n$. Assume the vector field $f : \mathbf{R}^n \to \mathbf{R}^n$ is Γ-equivariant and that

(E1) $f(x_0) = 0$ *where* x_0 *is fixed by* Γ.

As before let $A = df(x_0)$ and observe that Γ-equivariance of f implies that A commutes with Γ. It follows that eigenspaces of A are Γ-invariant. In particular, eigenvalues of high multiplicity may be expected. Our second assumption is the following.

(E2a) $\pm\omega_0 i$ *are eigenvalues of* A *and* $k\omega_0 i$ *is not an eigenvalue for* $k = 0, 2, 3 \ldots$. *Let* V *denote the generalized eigenspace associated to the eigenvalues* $\pm\omega_0 i$.

We replace the previous assumption that the eigenvalues $\pm\omega_0 i$ are simple by Γ-simple. This notion is the Γ-equivariant version of simple eigenvalues. See Golubitsky, Stewart and Schaeffer [1988].

Definition 2.2 *The action of the group* Γ *on* V *is said to be* Γ-***simple*** *if either*

(a) $V = W \oplus W$ *and* Γ *acts absolutely irreducibly on* W, *or*

(b) Γ *acts irreducibly, but not absolutely irreducibly, on* V.

Our next assumption is the following:

(E2b) Γ *acts* Γ*-simply on* V.

There is a natural action of S^1 on V induced by the one-parameter group $\exp(sA^T)$. Since A commutes with Γ, so does this action of S^1. Thus, we have a well defined action of $\Gamma \times S^1$ on V. We now determine families of periodic solutions by looking for their symmetries.

(E3a) *Let* $\Sigma \subset \Gamma \times S^1$ *be an isotropy subgroup of the action of* $\Gamma \times S^1$ *on* V. *Assume that* $\dim \mathrm{Fix}_V(\Sigma) = 2$.

(E3b) f *has a* C^{k+1} *first integral* $H : \mathbf{R}^{2m} \to \mathbf{R}$ *such that*

i $dH(x_0)|\mathrm{Fix}(\Sigma) = 0$

ii $d^2H(x_0)|\mathrm{Fix}(\Sigma)$ *is positive or negative definite.*

Theorem 2.3 *Under conditions* (E1)–(E3), *there is a one-parameter family of periodic orbits in a neighborhood of* x_0 *with* Σ *symmetry and periods close to* $2\pi/\omega_0$. *These periodic orbits form an invariant* C^k *manifold of dimension two that is tangent to* $\mathrm{Fix}(\Sigma)$ *at* x_0.

The only remaining point for discussion is what is meant by a symmetry of a periodic solution. An element $\sigma = (\gamma, \theta) \in \Gamma \times S^1$ is called a ***symmetry*** of a 2π-periodic function $x(t)$ if $\sigma x(t) = x(t - \theta)$. That is, the periodic orbit is preserved by γ and uniqueness of solutions to ODE's guarantees the existence of a unique θ.

The proof proceeds by noting that the same Liapunov-Schmidt reduction technique as above can be used, and that the reduced map g is $\Gamma \times S^1$-equivariant with the S^1 action induced by phase shift being identical to the S^1 action induced by A. Equivariance implies that $g : \mathrm{Fix}(\Sigma) \times \mathbf{R} \to \mathrm{Fix}(\Sigma)$ has the form (2.34). Using (E3a) we can identify $\mathrm{Fix}(\Sigma)$ with $\mathbf{C}$. The proof now proceeds as before.

The machinery of finding two-dimensional fixed-point subspaces in Γ-equivariant Hopf bifurcation can now be applied to finding periodic solutions in systems with first integrals. For example, in $O(2)$-equivariant systems with $\dim V = 4$, Theorem **2.3** produces both rotating and standing waves. We also note that Golubitsky and Stewart [1993] show that there are surprising group-theoretic conditions restricting those Γ that can produce two-dimensional fixed-point subspaces. Finally, note that the preceding result may be applied to the reduction of a system by the continuous part of a symplectic group action at a regular value of the momentum map, but with the discrete symmetry remaining. We refer to Marsden [1992] for the general set up.

3 Resonances

Nest, we show that the Hamiltonian resonance bifurcation equations can be reduced to those of the same form as in the reversible case. We do this by using the general Liapunov Schmidt procedure for systems with an integral, using the stronger condition of being Hamiltonian (or, if one prefers, the condition of being symplectic) as the first integral.

To do this, assume that we have a symplectic form Ω on $\mathbf{R}^n$. Passing to Darboux coordinates, we may assume that $n = 2r$ and Ω is in canonical form; that is, Ω is the skew symmetric bilinear form with the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

where 1 denotes the $r \times r$ identity.Assume that the given vector field f is Hamiltonian relative to Ω. That is, we have the identity

$$\Omega(f(u), v) = dH(u) \cdot v$$

for all $v \in \mathbf{R}^{2r}$. Define the map

$$\Phi : C^1_{2\pi} \times \mathbf{R} \times C^0_{2\pi} \to [C^1_{2\pi}]^*$$

by

$$\Phi(u, \tau, v) \cdot U = \int_0^{2\pi} \left\{ \Omega(v - (1+\tau)\frac{du}{ds}, U) + dH(u) \cdot U \right\} ds.$$

The condition that

$$\Phi(u, \tau, F(u, \tau)) = 0$$

now is a restatement of the condition that the vector field be Hamiltonian with Hamiltonian function H. We also observe that this condition can be rephrased as saying that the map F regarded as a (parameter dependent) vector field on the space $C^1_{2\pi}$ is Hamiltonian with respect to the weak symplectic form given by integration of the given symplectic form around loops, and with the Hamiltonian function given by

$$\mathcal{H}(u, \tau) = \int_0^{2\pi} \left\{ \frac{1}{2}\Omega(1+\tau)\frac{du}{ds}, u) - H(u) \right\} ds.$$

We also note that this point of view was used by Weinstein in one of the approaches to the Weinstein-Moser theorem (Weinstein [1978]).

Now we set up the bifurcation equation for the $k : l$ resonance in a form suitable for applying the symplectic integral condition. We make some preliminary assumptions:

(RH1) x_0 is a fixed point: $f(x_0) = 0$. Let $A = df(x_0)$ be the linearization at x_0 and let k and l be distinct positive integers.

(RH2) $\pm k\omega_0 i$ and $\pm l\omega_0 i$ are simple eigenvalues of A and $p\omega_0 i$ is not an eigenvalue for p a nonnegative integer other than k or l. Let v_k and v_l be the eigenvectors of A for the eigenvalues $k\omega_0 i$ and $l\omega_0 i$ respectively.

We can assume that ω_0 is one by a temporal rescaling, so we assume that this is done. Now we set up the Liapunov Schmidt procedure for the map F that was defined in the previous section. Let $\mathcal{X} = C^1_{2\pi} \times \mathbf{R}$ and $\mathcal{Y} = C^0_{2\pi}$ (together with bifurcation parameters to be included later). Let S^1 act on the first two components of $\mathcal{X}$ by $\theta \cdot (u)(s) = u(s+\theta)$ and similarly on $\mathcal{Y}$. Note that F is S^1 equivariant.

The derivative of F at $u = x_0, \tau = 0$ is given by

$$\bar{L}v \equiv dF(x_0, 0) \cdot v = \dot{v} - Av$$

By (RH2), $\ker dF(x_0, 0)$ is spanned by $\mathrm{Re}(e^{iks}v_k), \mathrm{Im}(e^{iks}v_k), \mathrm{Re}(e^{ils}v_l)$, and $\mathrm{Im}(e^{ils}v_l)$. Thus, we can identify the kernel of dF with $\mathbf{C}^2$; explicitly,

$$(z_1, z_2) \to (\mathrm{Re}(z_1 e^{iks}v_k), \mathrm{Re}(z_2 e^{ils}v_l)).$$

Using the Fredholm alternative, let $\mathcal{M} = \mathrm{range}\, L^*$ be the orthogonal complement of $\mathcal{K} = \ker L$, and let $\mathcal{N} = \ker L^*$ be the orthogonal complement to $\mathcal{R} = \mathrm{range}\, L$.

The Liapunov-Schmidt procedure with the symplectic integral, as described above, may now be applied. This procedure is done relative to a choice of an inner product that is Γ-invariant. The bifurcation equation for $\bar{F}$ has the form

$$g : \mathbf{C}^2 \times \mathbf{R} \times \text{parameter space} \to \mathbf{C}^2$$

where the second two components of g are obtained from the first two by differentiation, and, being S^1 equivariant, the first two components have the following form (but the case $k = 1$ is a little different)

$$P_1 z_1 + Q_1 \bar{z_1}^{l-1} z_2^k = 0 \tag{3.37}$$
$$P_2 z_2 + Q_2 z_1^l \bar{z_2}^{k-1} = 0 \tag{3.38}$$

For reversible systems, the real parts of these equations vanish, and the imaginary parts can be examined by using singularity theory. (Consult Golubitsky, Stewart, and Marsden [1993] for details). Interestingly, as the following remarks show, this same analysis applies to the Hamiltonian case. As above, the general condition imposed on the bifurcation equation is given by

$$\Phi((k + w(k), g(k)) = 0$$

where $k \in \mathcal{K}$, where $w(k)$ is the implicit function determined by the Liapunov-Schmidt procedure, and where g is the bifurcation map. As above, we use the

Golubitsky, M., I. Stewart and D. Schaeffer [1988] *Singularities and Groups in Bifurcation Theory*, Vol. II, Springer-Verlag, AMS Series, **69**.

Golubitsky, M. and I. Stewart [1993] An algebraic criterion for symmetric Hopf bifurcation, *Proc. Roy. Soc. Lond. A* (to appear).

Golubitsky, M., I. Stewart and J.E. Marsden [1993] The $k : l$ resonance for reversible systems (in preparation).

Guckenheimer, J. and A. Mahalov [1992] Instability induced by symmetry reduction, *Phys. Rev. Lett.* **68**, 2257–2260.

Knobloch, E., Mahalov and J.E. Marsden [1993], Normal forms for three-dimensional parametric instabilities in ideal hydrodynamics, (in preparation).

Kummer, M. [1990], On resonant classical Hamiltonians with n frequencies, *J. Diff. Eq.*, **83**, 220–243.

Lewis, D., J.E. Marsden and T.S. Ratiu [1987] Stability and bifurcation of a rotating liquid drop, *J. Math. Phys.* **28**, 2508–2515.

Marsden, J.E. [1992], *Lectures on Mechanics* London Mathematical Society Lecture note series, **174**, Cambridge University Press.

Marsden, J.E. and J. Scheurle [1993] Lagrangian reduction and the double spherical pendulum, *ZAMP* **44**, 17–43.

Oh, Y.G., N. Sreenath, P.S. Krishnaprasad and J.E. Marsden [1989] The Dynamics of Coupled Planar Rigid Bodies Part 2: Bifurcations, Periodic Solutions, and Chaos, *Dynamics and Diff. Eq'ns.* **1**, 269–298.

Sanders, J.A. and F. Verhulst [1985] *Averaging Methods in Nonlinear Dynamical Systems.* Springer-Verlag.

Vanderbauwhede, A. [1982] *Local Bifurcation and Symmetry* Research Notes in Mathematics, Pitman, Boston, 1982.

Weinstein, A. [1978] Bifurcations and Hamilton's principle, *Math. Zeit.* **159**, 235–248.

fact that $w(k)$ vanishes with its first derivative; *i.e.*, it is of higher order, and so one needs to compute the quantity

$$\Phi((k, g(k)).$$

This condition separates into two conditions, one for each of the two components of (3.37). Here we use the fact that the first integral condition is a *vector* condition, and, correspondingly, that U can be chosen arbitrarily. In fact, we choose, respectively, U to be the two components of $f(u)-(1+\tau)\frac{du}{ds}$ projected to the kernel. Then, the condition (modulo higher order terms) becomes the same as the conditions we found in the case of the Liapunov center theorem, with the difference now being that we find them separately for the two components of the bifurcation equation. Thus, the argument already given above for the Liapunov Center theorem shows that the real parts of each of the bifurcation equations vanishes when the imaginary part does. Thus, the analysis for the reversible case applies here as well.

References

Abraham, R. and J.E. Marsden [1978] *Foundations of Mechanics.* Second Edition, Addison-Wesley Publishing Co., Reading, Mass..

Abraham, R., J.E. Marsden and T.S. Ratiu [1988] *Manifolds, Tensor Analysis, and Applications.* Second Edition, Springer-Verlag, New York.

Golubitsky, M. and M. Dellnitz [1993] A Liapunov-Schmidt approach to the Hamiltonian Hopf bifurcation (in preparation).

Dellnitz, M., I. Melbourne and J.E. Marsden [1992] Generic bifurcation of Hamiltonian vector fields with symmetry, *Nonlinearity* **5**, 979–996.

Duistermaat, H. [1983] *Bifurcations of periodic solutions near equilibrium points of Hamiltonian systems.* Springer-Verlag Lecture Notes in Mathematics, **1057**, 57–105.

Golubitsky, M., M. Krupa and C. Lim [1991] Time reversibility and particle sedimentation, *SIAM J. Appl. Math.*, **51**, 49–72.

Golubitsky, M. and J.E. Marsden [1993] The Liapunov-Schmidt procedure for systems with a first integral (preprint).

Golubitsky, M. and D. Schaeffer [1985] *Singularities and Groups in Bifurcation Theory*, Vol. 1, Springer-Verlag, AMS Series, **51**.

Golubitsky, M. and I. Stewart [1987] Generic bifurcation of Hamiltonian systems with symmetry, *Physica D* **24**, 391–405.