Reprinted from

M.W. Hirsch, J.E. Marsden, and M. Shub
Editors

From Topology to Computation:
Proceedings of the Smalefest

Springer-Verlag
New York Berlin Heidelberg London Paris
Tokyo Hong Kong Barcelona Budapest

45
Steve Smale and Geometric Mechanics

JERROLD E. MARSDEN*

1. Some Historical Comments

In the period 1960–1965, geometric mechanics was "in the air." Some key papers were available, such as Arnold's work on KAM theory and a little had made it into textbooks, such as Mackey's book on quantum mechanics and Sternberg's book on differential geometry. In this period, Steve was working on his dynamical systems program. His survey article (Smale [1967]) contained important remarks on how geometric mechanics (specifically Hamiltonian systems on symplectic manifolds) fits into the larger dynamical systems framework. In 1966 at Princeton, Abraham ran a seminar using a preprint of the survey article and it was through this paper that I first encountered Smale's work. After he visited the seminar, the importance of what he was doing was obvious; also, it became evident that there was great power in asking simple, penetrating, and sometimes even seemingly naive questions. I should add that in the mathematical physics seminar at Princeton that I also had the good fortune of attending, Eugene Wigner had a remarkably similar aura.

Smale's dynamical systems work suggested developing similar ideas like structural stability, dynamic bifurcations, and genericity in the context of mechanics. Structural stability aspects were developed by Abraham, Buchner, Robinson, Robbin, and others. The generic bifurcations of equilibria, relative equilibria, Hamiltonian–Krein–Hopf bifurcations that can occur in Hamiltonian systems has been studied by Williamson, Arnold, Meyer, van der Meer, Duistermaat, Cushman, Golubitsky, Stewart, and others. See Abraham and Marsden [1978], Marsden [1992] and Delliniz, Melbourne and Marsden [1992] for further information and references.

In 1966–1967, two important personal events occured. First, Abraham gave his lectures on mechanics at Princeton from which our book *Foundations*

* Research partially supported by NSF Grant DMS-8922704 and DOE Contract DE-FGO3-88ER25064.

of Mechanics arose. Second, Arnold's [1966] paper on rigid-body mechanics and ideal fluid mechanics appeared. The latter paper influenced me profoundly; it showed how the dynamics of these systems could be interpreted as geodesic flow on $SO(3)$ with a left-invariant metric and on $\mathrm{Diff}_{\mathrm{vol}}(\Omega)$—the volume-preserving diffeomorphism groups of, say, a region Ω in $\mathbb{R}^3$—with the right-invariant metric defined by the kinetic energy of the fluid. This paper, together with the emerging work of Kostant and Souriau on the role of symmetry groups and the momentum map, laid important ideas latent in the traditional approach to mechanics, in clear and concise geometric terms.

Smale gave a course of lectures on mechanics in the fall of 1968 at Berkeley, the semester I arrived. This led to his two-part paper Topology and Mechanics (Smale [1970]).

In the same period, I completed a paper with Ebin (Ebin and Marsden [1970]) in which we put Arnold's work on fluid mechanics in the context of Sobolev (H^s) manifolds and showed the remarkable fact that Arnold's geodesic flow on H^s-$\mathrm{Diff}_{\mathrm{vol}}(\Omega)$ (the volume-preserving diffeomorphisms of Ω to itself of Sobolev class H^s) comes from a *smooth* geodesic spray. This fact is remarkable because it allows one to solve the initial value problem using only Picard iteration and ordinary differential equations theory on TH^s-$\mathrm{Diff}_{\mathrm{vol}}(\Omega)$ and one would not expect this since the Euler equations for fluid mechanics are rather nasty PDEs, not ODEs! This led to a number of interesting analytic and numerical developments in fluid mechanics. Around this same time, I began work with Fischer on the Hamiltonian structure of general relativity (see, for example, Fischer and Marsden [1972]). At this stage I had only a rough idea how these two topics might be related to Smale's work.

Already around 1971, with so much happening in geometric mechanics, Abraham and I started work on the second edition of *Foundations of Mechanics* with the help of Ratiu and Cushman. Doing so prompted thoughts about how all of these ingredients might fit together. Especially interesting was the question of how Smale's papers might be linked with Arnold's. Smale used symmetry ideas in the context of tangent and cotangent bundles of configuration spaces with Hamiltonians of the form kinetic plus potential energy; that is, he dealt with *simple mechanical systems*. The examples and some of the theory were concerned with *abelian symmetry* groups. In this context, Smale's work contained some of the essential ideas of what we now call *reduction theory*. It was natural to attempt to put Smale's ideas and those of Arnold in the more general and unifying context of symplectic manifolds. Doing so led to the paper with Weinstein (Marsden and Weinstein [1974]) that was completed in early 1972. Some of these ideas were found independently by Meyer [1973] whose paper appears in the proceedings of the 1971 conference organized by Peixoto that was, in effect, a large conference on Steve's work on dynamical systems as a whole.

This effort led to the now fairly well-developed area of reduction theory. There are expositions of this subject available in the 10 or so texts and monographs that are currently devoted to geometric mechanics (Abraham and Marsden [1978], Guillemin and Sternberg [1984], and Arnold [1989] are examples). We will come back to a description of one of the reduction theorems later (the cotangent bundle reduction theorem) and give an indication of why this approach made so much fall beautifully into place. Briefiy, if one starts with a cotangent bundle T^*Q, and a Lie group G acting on Q, then the quotient $(T^*Q)/G$ is a bundle over $T^*(Q/G)$ with fiber $\mathfrak{g}^*$, the dual of the Lie algebra of G One has the following structure of the Poisson reduced space $(T^*Q)/G$ (its symplectic leaves are the symplectic reduced spaces of Marsden and Weinstein [1974]).

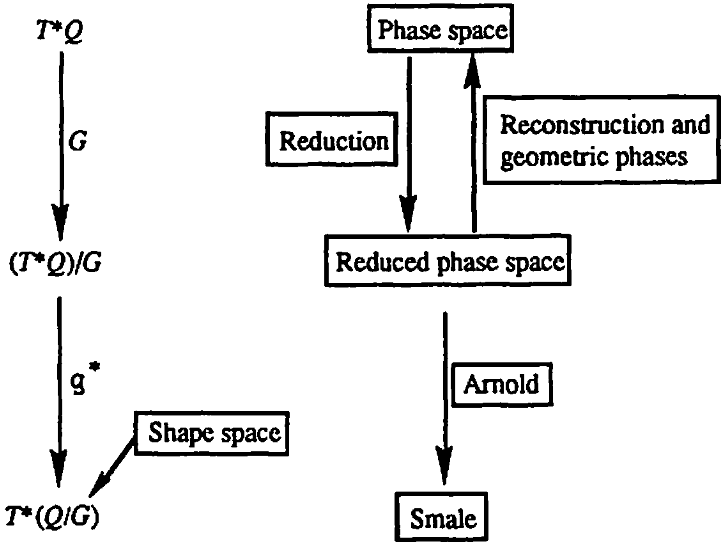

FIGURE 1

Thus, one can say—perhaps with only a slight danger of oversimplification—that reduction theory synthesises the work of Smale, Arnold (and their predecesors of course) into a bundle, with Smale as the base and Arnold as the fiber. This bundle has interesting topology and carries mechanical connections (with associated Chern classes and Hannay–Berry phases) and has interesting singularities (Arms, Marsden, and Moncrief, Guillemin and Sternberg, Atiyah, and others). We will describe some of these features later.

2. Highlights from Topology and Mechanics

One of Smale's main goals was to use topology, especially Morse theory, to estimate the number of relative equilibria in a given simple mechanical system with symmetry, such as the n-body problem, and to study the associated bifurcations as the energy and momentum are varied. This approach proved to be quite successful and has been carried on by Palmore, Cushman, Fomenko, and others to give more detailed information in the n-body problem and basic information in other problems like vortex dynamics, rigid-body mechanics, and other integrable systems.

One should also mention that this paper of Smale did a lot for the subject itself. The paper attracted worldwide attention and brought many excellent young people into the field. The idea of using topology and geometry in a classical subject to bring new insights and fresh ideas must have been quite appealing.

Smale's strategy was to study the topology of the level sets of the energy-momentum map $H \times \mathbf{J}: P \to \mathbb{R} \times \mathfrak{g}^*$ on a given phase space P with a given Hamiltonian H and a symplectic group action having a momentum mapping $\mathbf{J}: P \to \mathfrak{g}^*$. He lays out a program for studying bifurcations in the level sets of the energy-momentum mapping as the level value changes. In doing so, he sets out the basic equivariance properties of momentum maps, apparently independent of the other people normally credited with introducing the momentum map, namely, Lie, Kostant, and Souriau. (An interesting historical note is that Lie had most of the essential ideas, including—according to Weinstein—the fact that the momentum map is a Poisson map, its equivariance, the symplectic structure on the coadjoint orbits, and more, all back in 1890!) Smale also made the important observation that *a point $z \in P$ is a regular point of* $\mathbf{J}$ *iff the symmetry* (*isotropy*) *group of z is discrete.* This idea surfaces again in the study of the solution space of the Einstein or Yang–Mills equations, as we shall see below.

One of the most important objects that Smale introduced was the *amended potential* V_μ that plays a vital role in current developments in stability and bifurcation of relative equilibria. The amended potential is a geometric generalization of the classical construction of the "effective potential" in the (planar) two-body problem, which is obtained by adding to the given potential $V(r)$, the centrifugal potential at angular momentum value μ; in this simple case,

$$V_\mu = V + \frac{\mu^2}{2r^2}.$$

In this situation, reduction corresponds to the elimination of the angular variable θ (division by the group $G = S^1$) and the replacement of the potential V by the potential V_μ.

As we shall see later, in special situations, such as the abelian case, the reduction of a simple mechanical system is again a simple mechanical system, but the general situation, even for groups like the rotation group, is more complicated. The fact that the abelian reduction of a simple mechanical system is again a simple mechanical system is essentially contained in Smale's paper, but it has a long history with a surprising amount contained in the work of Routh around 1860 in his books on mechanics. See, for example, Routh [1877]. In fact, Routh's work contains results that get rediscovered from time to time in the modern literature, but that is another story.

For problems like rigid-body mechanics and fluid mechanics, one has to deal with Lie–Poisson structures on $\mathfrak{g}^*$ (the "Arnold fiber") and magnetic terms on $T^*(Q/G)$ (the "Smale base"). The magnetic terms modify the canonical symplectic structure on $T^*(Q/G)$ with the addition of the μ-component of the curvature of a connection on $Q \to Q/G$ called the *mechanical connection*. This connection, defined below, is given implicitly in Smale's paper (it is introduced in §6 of his paper).

Smale studies relative equilibria by applying critical point theory to V_μ. The function V_μ contains much of the information of $E \times \mathbf{J}$ through the general fact proved by Smale that a point of Q is the configuration of a relative equilibrium if and only if it is a critical point of V_μ. (Some of these concepts are recalled in the next section.)

Smale's examples deal with the abelian case (when the "Arnold fiber" has trivial Poisson structure); in this situation, V_μ defines a function on Q/G, and on this quotient space, one expects the critical points to be generically nondegenerate. In the general case (such as a rotating rigid body with internal structure), one has to carefully synthesize the analysis of Arnold and Smale to get the sharpest information.

This basic theory, and some simple but very informative examples, comprise part I of "Topology and Mechanics." Part II is concerned with the planar n-body problem in which $G = S^1$ is the planar rotation group and Q is $\mathbb{R}^{2n}$, minus collision points. The study of relative equilibria is done by determining the global topology of the level sets of $E \times \mathbf{J}$ and their quotients by S^1—Theorems A and B of the paper. Theorem C relates this to critical points of V_μ and Theorem D determines the bifurcation set. Theorem E relates the topology of the rduced phase space to that of the configuration space. Corollaries give more details for $n = 2$ and $n = 3$. An interesting consequence of these results is Moulton's theorem stating that there are $n!/2$ classes of *colinear* relative equilibria. The fact that one is looking for colinear relative equilibria enables one to reduce the problem to one of finding critical points of a function on *real* projective $n - 2$ space minus collisions. In fact, a combinational argument shows that this space has $n!/2$ components and the corresponding function has a single nondegenerate maximum on each component. There have, of course, been many important contributions to the n-body problem since 1970, such as those of Palmore, McGehee, Mather, Meyer, and others. The book of Meyer and Hall [1991] can be consulted for some of the relevant literature.

3. A Glimpse at Reduction Theory

In this section we will focus on *some* of the ways Smale's paper is connected with *some* of the current research in geometric mechanics. No attempt is made at thoroughness here—the focus is on selected topics of personal interest only! In particular, we focus on some aspects of reduction theory, one of the most fruitful outgrowths of Smale's and Arnold's work. Of course, others also deserve much credit for setting the foundations, especially Lie, Kostant, Kirillov, and Souriau.

Let P be a symplectic manifold and G be a group acting symplectically on P. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ its dual. Let G act on $\mathfrak{g}^*$ by the coadjoint action and let $\mathbf{J}: P \to \mathfrak{g}^*$ be an equivariant momentum map; that is, $\mathbf{J}$ is equivariant and $\mathbf{J}$ generates the group action in the sense that for each $\xi \in \mathfrak{g}$,

$$X_{\langle \mathbf{J}, \xi \rangle} = \xi_P,$$

where X_f is the Hamiltonian vector field determined by the function f and ξ_P is the infinitesimal generator of the action on P.

If G_μ is the isotropy subgroup of $\mu \in \mathfrak{g}^*$, the *reduced space* at μ is

$$P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu.$$

Equivariance guarantees that G_μ acts on $\mathbf{J}^{-1}(\mu)$, so the quotient makes sense. If μ is a (weakly) regular value and the quotient is nonsingular, the *reduction theorem* states that P_μ is a symplectic manifold and that G-invariant Hamiltonian systems on P decend to Hamiltonian systems on P.

Given a G-invariant hamiltonian H on P, a *relative equilibrium* (in the terminology of Poincaré) is a point in P whose dynamic orbit equals a one-parameter group orbit. Relative equilibria correspond to critical points of $H \times \mathbf{J}$ and to (dynamically) fixed points on P_μ.

There are three interrelated special cases. First, if $P = T^*G$, then the reduced space at μ is the coadjoint orbit through μ and its reduced symplectic structure is that of Kirillov, Kostant, and Souriau. Second, if $\mu = 0$ and $P = T^*Q$ (with the canonical cotangent structure), then $P_0 = T^*(Q/G)$ with the canonical symplectic structure. Finally, if G is abelian (or $G = G_\mu$), then $P_\mu = T^*(Q/G)$ although the structure on $T^*(Q/G)$ need not be canonical.

We need to also recall that the coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$ are the symplectic leaves in the *Lie–Poisson structure*

$$\{F, K\}(\mu) = \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu}\right]\right\rangle,$$

where one uses "$-$" for left actions and "$+$" for right actions. Here $\delta F/\delta \mu \in \mathfrak{g}$ is the *generalized functional derivative* defined by

$$\left\langle \frac{\delta F}{\delta \mu}, \nu \right\rangle = \mathbf{D}F(\mu) \cdot \nu$$

for all $\nu \in \mathfrak{g}^*$. As is well-known [or follows using Poisson reduction in the form $(T^*G)/G \cong \mathfrak{g}^*$], this bracket makes $\mathfrak{g}^*$ into a Poisson manifold. This term "Lie–Poisson" structure was coined by Marsden and Weinstein [1983] since this expression occurs *explicitly* in Lie's work around 1890.

If $\mathcal{O}$ is the coadjoint orbit through μ, then following Marle [1976] and

Kazhdan, Kostant, and Sternberg [1978], one finds a symplectic identification

$$P_\mu \cong \mathbf{J}^{-1}(\mathcal{O})/G \subset P/G$$

which shows that P_μ may be identified with a symplectic leaf in the Poisson reduced space P/G. An account of this, along with some additional information is given in Marsden [1981] . Reduction theory has been applied to a large number of interesting situations—the literature is too vast to survey here. We just mention a few: see Cushman and Rod [1982] for a penetrating application to resonances, Deprit [1983] for a solution of the problem of Jacobi's elimination of the node in the n-body problem, Bobenko et al. [1989] for integrable systems and a group-theoretic resolution of the integrability of the Kowalewski top, Marsden and Weinstein [1982, 1983] and Marsden, Ratiu and Weinstein [1984] for applications in fluid and plasma dynamics, and David, Holm, and Tratnik [1990] for applications to polarization lasers. Consult Guillemin and Sternberg [1984] for some applications involving representation theory.

Perhaps the most interesting case is the one considered by Smale: $P = T^*Q$ with the canonical symplectic structure. We assume G acts on Q and hence by cotangent lift on T^*Q. We also assume we are dealing with a Hamiltonian of the form kinetic plus potential energy, where the metric g on Q is G-invariant and where the potential $V: Q \to \mathbb{R}$ is G-invariant.

The *cotangent bundle reduction theorem* states that *the reduced space P_μ is a bundle over $T^*(Q/G)$ with fiber $\mathcal{O}$, the orbit through μ.* The corresponding Poisson statement is that the space $(T^*Q)/G$ is a $\mathfrak{g}^*$-bundle over $T^*(Q/G)$. The corresponding description of the symplectic or Poisson structure is a nontrivial synthesis of the Lie–Poisson structure on $\mathfrak{g}^*$ and the canonical (plus magnetic) structure on $T^*(Q/G)$. This was worked out in Montgomery, Marsden, and Ratiu [1984] motivated by the cases of the Hamiltonian structure for the interaction of a fluid or plasma with an electromagnetic field and the work of Sternberg and Weinstein on the geometry of Wong's equations that describe the motion of a particle in a Yang–Mills field (see Montgomery [1984] and references therein).

The proof of the cotangent bundle reduction theorem (see, for example, Marsden [1992] for a recent account) utilizes two crucial ideas, each of which is in Smale's paper (one of them implicitly). The first is the *mechanical connection* and the second is the associated *momentum shift*.

The *locked inertia tensor* is the map $\mathbb{I}: Q \to \mathfrak{g}^* \otimes \mathfrak{g}^* \cong L(\mathfrak{g}, \mathfrak{g}^*)$ defined by

$$\langle \mathbb{I}(q)\xi, \eta \rangle = \langle\langle \xi_Q(q), \eta_Q(q) \rangle\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing and $\langle\langle \cdot, \cdot \rangle\rangle$ is the metric pairing. If the action is locally free, then $\mathbb{I}(q)$ is a positive definite symmetric tensor. Define $\alpha: TQ \to \mathfrak{g}$ by

$$\alpha(v_q) = \mathbb{I}(q)^{-1}\mathbf{J}(\mathbb{F}L(v_q)),$$

where $v_q \in T_qQ$ and $\mathbb{F}L: TQ \to T^*Q$ is the Legendre transformation determined by the metric. The above formula for α is equivalent to the one given by Smale.

The first person to note and exploit the fact that $\alpha: TQ \to \mathfrak{g}$ is a G-connection on the bundle $Q \to Q/G$ seems to have been Kummer [1981]. The μ-component of the curvature of α is added to the canonical symplectic structure on $T^*(Q/G)$ in the reduction process. This was observed, without the language of connections, and the phrase "magnetic term" coined by Abraham and Marsden [1978].

The picture of $Q \to Q/G$ as a G-bundle carrying the connection α is the beginning of the story of the "gauge theory of deformable bodies" and the remarkable work of Wilczek, Shapere, and Montgomery on the link between optimal control and the motion of a colored particle moving in the Yang–Mills field α. See Shapere and Wilczek [1989], Montgomery [1990] and references therein.

The μ-component of α defines a one-form $\alpha_\mu: Q \to T^*Q$. One of the properties of a connection translates to

$$\alpha_\mu \in \mathbf{J}^{-1}(\mu),$$

which is the way Smale thought of α. The mechanical connection was explicitly used by Smale to describe the amended potential as the composition of the Hamiltonian with α_μ.

The momentum shift $T^*Q \to T^*Q$ taking a covector p_q at q to the covector $p_q - \alpha_\mu(q)$ therefore maps $\mathbf{J}(\mu)$ to $\mathbf{J}^{-1}(0)$. This, in effect, replaces reduction at μ by reduction at 0, which gives $T^*(Q/G)$. It is by this means that the cotangent bundle reduction theorem is proved.

Reduction has its counterpart, *reconstruction*, which is part of the theory. This concerns how one constructs dynamic trajectories in $\mathbf{J}^{-1}(\mu) \subset P$ given the reduced dynamic trajectory in P_μ. It turns out that α induces a connection on the G_μ-bundle $J^{-1}(\mu) \to P_\mu$, and the horizontal lift and holonomy of this connection play a basic role in reconstruction and in the interpretation of geometric phases (Hannay–Berry phases). See Marsden, Montgomery, and Ratiu [1990] for details. In particular, in this reference one will find a beautiful formula of Montgomery for the phase shift of a rigid body—when a rigid body undergoes a periodic motion in its reduced (body angular-momentum space), then the actual body does not return to its original position, but undergoes a rotation about the constant spatial angular momentum vector through an angle given by

$$\Delta\theta = -\Lambda + \frac{2ET}{\|\mu\|},$$

where Λ is the solid angle on the sphere of radius $\|\mu\|$ enclosed by the trajectory in body angular-momentum space, E is its energy, and T is the period. The first term, *the geometric phase*, is the holonomy of the canonical one form regarded as an S^1 connection on the Hopf bundle $\mathbf{J}^{-1}(\mu) \to S^2$. This type of formula proves to be useful in a number of problems, such as the control of the attitude of a rigid body with internal rotors; see Bloch, Krishnaprasad, Marsden, and Sánchez de Alvarez [1992].

Geometric phases come up in a variety of other problems as well, and the ideas of reduction and reconstuction can be useful for understanding them. Some of these are described in Marsden, Montgomery, and Ratiu [1990] and Montgomery [1990]. Many other applications involve integrable systems—one of these is the phase shift that one sees when two solitons interact. This is described in Alber and Marsden [1992]. Others that seem likely are phenomena like Stokes' drift in fluid mechanics.

4. Other Directions

Here we describe a few other recent research directions, each of which has a specific link with Smale's paper.

4.1. General Methodology

Most academics get judged on specific contributions—in mathematics, one is ideally judged on specific theorems. In many circumstances, this is a sound procedure, but what often turns out to be more valuable for science as a whole is the point of view or pedagogical approach that is developed. Smale (along with Poincaré, Arnold, Atiyah, Singer, and a few others) gave us the valuable and influential point of view of dynamical systems and, more generally, of *global or geometric analysis.* This view has profoundly influenced a whole generation of workers and has had a pervasive effect, often taken for granted. It has also indirectly influenced areas Steve never worked in. For instance, global analysis ideas have proved useful in nonlinear elasticity, even to the point of designing better numerical codes. The book of Marsden and Hughes [1983] is typical of many works showing this impact.

4.2. Stability of Relative Equilibria

The context of the cotangent bundle reduction theorem provides a setting for another synthesis of the works of Arnold [1966] and Smale [1970]. This concerns explicit (computable) criteria for the dynamic stability of relative equilibria. The main recent works on this point are Simo, Posbergh, and Marsden [1990] and Simo, Lewis, and Marsden [1991].

The Lie–Poisson reduction methods of Arnold are built around the fact, mentioned above, that $(T^*G)/G \cong \mathfrak{g}^*$. In this case, Arnold worked out explicit stability criteria and applied them to rigid bodies and fluids. Working in the context of fluids, to overcome technical difficulties with the PDEs involved, he developed what is now known as the *Arnold method* or the *energy-Casimir method* in Arnold [1969] and related references. This technique was developed by a number of authors and was applied to a variety of fluid and plasma problems; Holm et al. [1985] contains a fairly complete survey and bibliography to that date.

In the general case, Smale's work suggests that one should test for stability by looking at the second variation $\delta^2 V_\mu$ at a critical point in Q. However, should one view it on Q/G_μ or on $\mathfrak{g}^* \times Q/G$? How does the Arnold stability criterion fit in? This is an interesting point because the philosophies of the two approaches are rather different. From the point of view of Smale, things are considerably simpler in the abelian case and here the criterion is clear—test $\delta^2 V_\mu$ for positive definiteness on Q/G. A more general suggestion is to test $\delta^2 V_\mu$ for definiteness on Q/G_μ (this criterion is an exercise in *Foundations of Mechanics* but is implicit in Smale's paper). Note that if $Q = G$, then Q/G_μ is the orbit through μ, so one can expect $\delta^2 V_\mu$ to correspond to the Arnold criterion on a coadjoint orbit. However, Arnold's philosophy was different: the energy-Casimir method is more tractable if one relaxes the restriction to orbits in the spirit of the Lagrange multiplier theorem. Namely, we add to the Hamiltonian a function that Poisson commutes with every other function, that is, with a *Casimir function*. (As an aside, we note for amusement only that some would like to call such a Casimir function a "Casimirian," so it would sound just like a Hamiltonian or a Lagrangian. Unfortunately, English is neither logical nor perfect—we also do not call a "Green's function" a "Greenian," even though it probably is more correct to do that.)

At this point in the history of geometric mechanics, it was not clear whether the energy-Casimir method of Arnold or the second variation method suggested by Smale's work was the more appropriate. A motivation for looking more deeply into this problem came from nonlinear elasticity. Here, the complexity of the orbits of $\mathfrak{g}^*$ means that Casimir functions are difficult or impossible to find. Arnold already realized this for *three*-dimensional ideal flow (where the only known Casimir is the helicity) and this fact surely was a discouragement for the method. Abarbanel and Holm [1987] made some progress on this problem by working directly in material representation, before reduction. (It would be interesting to return to this and related questions in plasma physics studied by Morrison [1987] in the light of the block diagonalization work described below, and the work in progress of Bloch, Krishnaprasad, Marsden and Ratiu on *instability* criteria with the addition of dissipation obtained using Chetaev's method.)

All of these factors led to the development of the *energy-momentum method* (or block-diagonalization, or reduced energy-momentum method). The key to this method is the development of a *synthesis* of the Arnold and Smale methods. One splits the space of variations of (a concrete realization of) Q/G_μ into variations in G/G_μ and variations in Q/G. With the appropriate splitting, one gets the block-diagonal structure

$$\delta^2 V_\mu = \begin{bmatrix} \text{Arnold form} & 0 \\ 0 & \text{Smale form} \end{bmatrix},$$

where the Smale form means $\delta^2 V_\mu$ computed on Q/G. This method turns out to be an extremely powerful one when applied to specific systems such as spinning satellites with flexible appendages.

Perhaps even more interesting is the structure of the linearized *dynamics* near a relative equilibrium. That is, both the augmented Hamiltonian $H_\xi = H - \langle \mathbf{J}, \xi \rangle$ *and* the symplectic structure can be *simultaneously* brought into the following normal form:

$$\delta^2 H_\xi = \begin{bmatrix} \text{Arnold form} & 0 & 0 \\ 0 & \text{Smale form} & 0 \\ 0 & 0 & \text{Kinetic energy} > 0 \end{bmatrix}$$

and

$$\text{Symplectic Form} = \begin{bmatrix} \text{Coadjoint orbit form} & * & 0 \\ -* & \text{Magnetic (coriolis)} & I \\ 0 & -I & 0 \end{bmatrix}$$

where the columns represent the *coadjoint orbit variable* (G/G_μ), the *shape variables* (Q/G), and the *shape momenta*, respectively. The term $*$ is an *interaction term* between the group variables and the shape variables. The magnetic term is the curvature of the μ-component of the mechanical connection, as we described earlier.

For $G = SO(3)$, this form captures all the essential features in a well-organized way: centrifugal forces in V_μ, coriolis forces in the magnetic term, and the interaction between internal and rotational modes. In fact, in this case, the splitting of variables solves an important problem in mechanics: *how to efficiently separate rotational and internal modes near a relative equilibrium.*

4.3. Bifurcation and Symmetry Breaking

Smale realized, as pointed out earlier, that the symmetry group of a point in phase space determines how degenerate it is for the momentum map. Correspondingly, one expects, from the work of Golubitsky and co-workers, that these symmetry groups will play a vital role in the bifurcation theory of relative equilibria and its connections with dynamic stability theory. The beginnings of this theory has started and it will be tightly tied with the normal form methods of Subsection 4.2. Smale concentrated on the topology of the level sets of $H \times \mathbf{J}$ and their associated bifurcations as the level sets vary. However, in many problems one also wants to vary other system parameters as well.

A simple example will perhaps help here. Consider the dynamics of a particle moving without friction in a rotating circular hoop, as in Fig. 2.

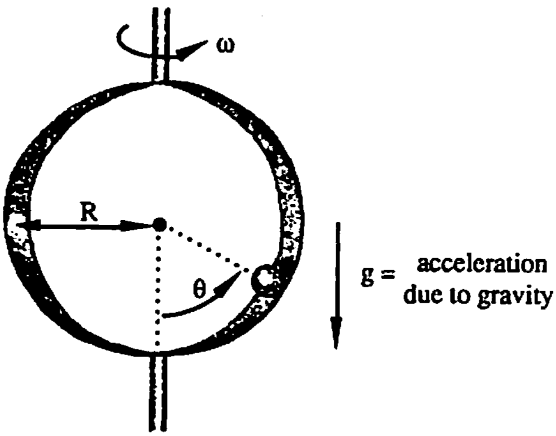

FIGURE 2. A ball in a rotating hoop.

As the angular velocity ω of the hoop increases past $\sqrt{g/R}$, a Hamiltonian pitchfork bifurcation occurs near the central equilibrium point, as in Fig. 3.

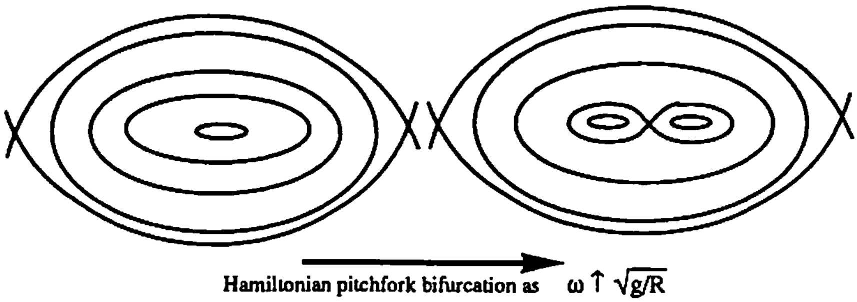

FIGURE 3. The Hamiltonian bifurcation for the ball in the rotating hoop.

The stability of the central point, which has $\mathbb{Z}_2$ symmetry, gets transferred to the bifurcating solutions, for which the $\mathbb{Z}_2$ symmetry is lost.

Related ideas appear in the work of Golubitsky and Stewart [1987] and in the study of a rotating planar liquid drop (with a free boundary held with a surface tension τ) in Lewis, Marsden, and Ratiu [1987] and Lewis [1989]. In the latter, a circular drop loses its circular symmetry to a drop with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry as the angular momentum of the drop is increased (although the stability analysis near the bifurcation is somewhat delicate). There are also interesting stability and bifurcation results in the dynamics of vortex patches, especially those of Wan in a series of papers starting with Wan and Pulvirente [1984].

4.4. Discrete Symmetries

In mechanics, the time-honored discrete symmetry is reversibility—the *anti-symplectic* involution $(q, p) \to (q, -p)$. However, there are many interesting discrete symmetries that are *symplectic*—the spatial $\mathbb{Z}_2$ symmetry of the ball in the hoop in Fig. 2 being a simple example. In bifurcation theory with symmetry, the Golubitsky school shows that discrete symmetries (and their corresponding fixed point sets, etc.) play an important role in the theory. A similar thing is true in the Hamiltonian case. For instance, discrete *and* continuous symmetries play a key role in the wonderful work of Bobenko, Reyman, and Semenov-Tian-Shansky [1989] that puts the integrability of the Kowalewski top into a reduction-theoretic framework. These ideas have been put into a general framework of *discrete reduction* by Harnard, Hurtubise, and Marsden [1991]. It would be of interest to go back to Smale's program with these discrete symmetry ideas to see their effect.

4.5. Singularity Structures in Solution Spaces

We already noted that Smale observed that singular points of J are points with symmetry. This is a simple but a profound observation with far-reaching implications. Abstractly, it turns out that level sets of J typically have *quadratic* singularities at its singular (= symmetric) points, as was shown by Arms, Marsden and Moncrief [1981]. In the abelian case, the images of these symmetric points are the vertices, edges, and faces of the convex polyhedron $J(P)$ in the Atiyah–Guillemin–Sternberg–Kirwan convexity theory. (See Atiyah [1982] and Guillemin and Sternberg [1984].)

These ideas apply in a remarkable way to solution spaces of relativistic field theories, such as Einstein's equations of general relativity and the Yang–Mills equations. Here the theories have symmetry groups and, appropriately interpreted, corresponding momentum maps. The relativistic field equations split into two parts—Hamiltonian hyperbolic evolution equations and elliptic constraint equations. The solution space structure is determined by the elliptic constraint equations, which, in turn, say nothing other than the momentum map vanishes.

A fairly long story of both geometry and analysis is needed to really establish this, but the result is easy to understand in the terms we have given: The *solution space has a quadratic singularity precisely at those field points that have symmetry.* For further details, see Fischer, Marsden, and Moncrief [1980] and Arms, Marsden and Moncrief [1982].

Whereas these results were motivated by perturbation theory of classical solutions (gravitational waves as solutions of the linearized Einstein equations, etc.), there is some evidence that these singularities have quantum implications. For example, there appears to be evidence that, in the Yang–Mills case, wave functions tend to concentrate near singular points (see, for example, Emerich and Romer [1990]). It would be of interest to explore these ideas further using the theory developed by Sjamaar [1990] and Sjamaar and Lerman [1991].

4.6. Mechanical Integrators

With Steve's more recent interests in computation, it might be appropriate to note that there is quite a bit of activity in developing numerical codes that respect the underlying structure of a mechanical system with symmetry. For example, one can develop codes that preserve exactly the energy-momentum map $H \times \mathbf{J}$ *or* that preserve the symplectic structure and $\mathbf{J}$ (it turns out that one *cannot* do *all* of these; see Ge and Marsden [1988]). There are too many references to adequately survey here, but the one just cited, Channell and Soovel [1990], references therein, and recent works of Feng, Krishnaprasad, and Simo, will give one a start.

To obtain an integrator preserving $\mathbf{J}$ is related to finding an algorithm $F_{\Delta t}: P \to P$ that is consistent with the symmetry. To get one that preserves H, one can base the analysis on a discretization of the variational principle, and to get one preserving the symplectic structure, one can discretize the Hamilton–Jacobi equation.

One of the interesting things about these integrators is that they seem to perform better than conventional ones (such as Runga–Kutta schemes) in long-term integrations where chaotic dynamics becomes important.

4.7. Homoclinic Chaos

Of course, Smale is noted for the famous "horseshoe" that is associated to homoclinic tangles. The technical way that this is handled is via what is usually called the *Birkhoff–Smale theorem*, which associates an invariant Cantor set having a well-understood symbolic dynamics to a homoclinic tangle. This phenomena had its origins in the work of Poincaré on the three-body problem and led to the *Poincaré–Melnikov–Arnold* technique for explicitly finding homoclinic tangles in specific systems. (See Wiggins [1988] for a thorough account of these topics.) We note that the method has proved effective in establishing homoclinic chaos for PDEs by using infinite-dimensional versions of the Poincaré–Melnikov–Arnold theorem and the Birkhoff–Smale theorem; see Holmes and Marsden [1981].

It is also interesting to note that horseshoes and reduction fit nicely together and this is needed when one proves that various systems with symmetry (such as rigid bodies with attachments) have homoclinic chaos (see Holmes and Marsden [1983]). Here ones sees that Smale's work on chaos in dynamical systems and his work on symmetry and mechanics fit together in a mutually suportive way. This is, of course, just one example of the many connections running through different parts of Smale's work when it is viewed on a global scale.

Epilogue

In this paper, I have sketched only some of Smale's involvement with mechanics. He was also interested in problems such as rotating fluid masses and provided much good advice about such problems. He was also interested in elementary particles, and using topology to help classify them—see Abraham [1960]. This subject is of course now in vogue with people like Witten, who is a good example of someone who has a blend of analysis, geometry, and topology in the Smale spirit.

A curious twist in Smale's work involves his recent work and the work of others on linear programming and computational complexity described elsewhere in this volume. We are now witnessing the beginnings of deep links between this work and mechanics by people like Deift, Brockett, Bloch, Flaschka, Ratiu, and others. For example, efficient ways of diagonalizing martices can be done by following the dynamics of integrable Hamiltonian systems (for instance, of Toda type) on appropriate spaces of matrices. This is one of many nice illustrations of the conference theme "unity and diversity" that runs through Smale's work and the approach he takes to his topics. Whereas there is a broad *diversity* in the subject matter, there is a *deep unity*, not only the obvious one of using global analysis methodology throughout his work, but nonobvious ones, like the preceeding link between computational techniques and mechanics, that repeats in unexpected yet beautiful ways in each of the subjects that he treated.

References

Abarbanel, H.D.I., and D.D. Holm [1987] Nonlinear stability analysis of inviscid flows in three dimensions: incompressible fluids and barotropic fluids. *Phys. Fluids* **30**, 3369–3382.

Abraham, R. [1960] Lectures of Smale on Differential Topology. Mimeographed notes, Columbia University.

Abraham, R., and J. Marsden [1978] *Foundations of Mechanics.* Addison-Wesley Publishing Co., Reading, Ma.

Alber, M., and J.E. Marsden [1992] On geometric phases and the phase shift of solitons, *Comm. Math. Phys.* **149**, 217–240.

Arms, J.M. [1981] The structure of the solution set for the Yang–Mills equations. *Math. Proc. Camb. Phil Soc.* **90**, 361–372.

Arms, J.M., J.E. Marsden, and V. Moncrief [1981] Symmetry and bifurcations of momentum mappings. *Commun. Math. Phys.* **78**, 455–478.

Arms, J.M., J.E. Marsden, and V. Moncrief [1982] The structure of the space solutions of Einstein's equations: II. Several Killings fields and the Einstein–Yang–Mills equations. *Ann. Phys.* **144**, 81–106.

Arnold, V.I. [1966] Sur la géometrie differentielle des groupes de Lie de dimenson infinie et ses applications à l'hydrodynamique des fluids parfaits. *Ann. Inst. Fourier, Grenoble* **16**, 319–361.

Arnold, V.I. [1969] On an a priori estimate in the theory of hydrodynamical stability. *Amer. Math. Soc. Transl.* **79**, 267–269.

Arnold, V.I. [1989] *Mathematical Methods of Classical Mechanics. Second Edition.* Springer-Verlag, New York.

Atiyah, M. [1982] Convexity and commuting Hamiltonians. *Bull. Lond. Math. Soc.* **14**, 1–15.

Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden, and G. Sánchez de Alvarez [1992] Stabilization of rigid body dynamics by internal and external torques, *Automatica* **28**, 745–756.

Bobenko, A.I., A.G. Reyman, and M.A. Semenov-Tian-Shansky [1989] The Kowalewski Top 99 years later: A Lax pair, generalizations and explicit solutions. *Commun. Math. Phys.* **122**, 321–354.

Channell, P., and C. Scovel [1990] Symplectic integration of Hamiltonian systems. *Nonlinearity* **3**, 231–259.

Cushman, R., and D. Rod [1982] Reduction of the semi-simple 1:1 resonance. *Physica D* **6**, 105–112.

David, D., D.D. Holm, and M. Tratnik [1990] Hamiltonian chaos in nonlinear optical polarization dynamics. *Phys. Rept.* **187**, 281–370.

Dellnitz, M., I. Melbourne and J.E. Marsden [1992] Generic bifurcation of Hamiltonian vector fields with symmetry, *Nonlinearity* **5**, 979–996.

Deprit, A. [1983] Elimination of the nodes in problems of N bodies. *Celestial Mech.* **30**, 181–195.

Ebin, D.G., and J.E. Marsden [1970] Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. Math.* **92**, 102–163.

Emmrich, C., and H. Römer [1990] Orbifolds as configuration spaces of systems with gauge symmetries. *Commun. Math. Phys.* **129**, 69–94.

Fischer, A.E., and J.E. Marsden [1972] The Einstein equations of evolution—a geometric approach. *J. Math. Phys.* **13**, 546–68.

Fischer, A.E., J.E. Marsden, and V. Moncrief [1980] The structure of the space of solutions of Einstein's equations, I: One Killing field. *Ann. Inst. H. Poincaré* **33**, 147–194.

Ge, Zhong, and J.E. Marsden [1988] Lie–Poisson integrators and Lie–Poisson Hamiltonian–Jacobi theory. *Phys. Lett. A* **133**, 134–139.

Golubitsky, M., and I. Stewart [1987] Generic bifurcation of Hamiltonian systems with symmetry. *Physica* **24D**, 391–405.

Guillemin, V., and S. Sternberg [1984] *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge.

Harnad, J., J. Hurtubise, and J. Marsden [1991] Reduction of Hamiltonian systems with discrete symmetry, preprint.

Holm, D.D., J.E. Marsden, T. Ratiu, and A. Weinstein [1985] Nonlinear stability of fluid and plasma equilibria. *Phys. Rept.* **123**, 1–116.

Holmes, P.J., and J.E. Marsden [1981] A partial differential equation with infinitely many periodic orbits: chaotic oscillations of a forced beam. *Arch. Rat. Mech. Anal.* **76**, 135–166.

Holmes, P.J., and J.E. Marsden [1983] Horseshoes and Arnold diffusion for Hamiltonian systems on Lie groups. *Indiana Univ. Math. J.* **32**, 273–310.

Kazhdan, D., B. Kostant, and S. Sternberg [1978] Hamiltonian group actions and dynamical systems of Calogero type. *Commun. Pure Appl. Math.* **31**, 481–508.

Kummer, M. [1981] On the construction of the reduced phase space of a Hamiltonian system with symmetry. *Indiana Univ. Math. J.* **30**, 281–291.

Lewis, D., [1989] Nonlinear stability of a rotating planar liquid drop. *Arch. Rat. Mech. Anal.* **106**, 287–333.

Lewis, D., J.E. Marsden, and T.S. Ratiu [1987] Stability and bifurcation of a rotating liquid drop. *J. Math. Phys.* **28**, 2508–2515.

Lewis, D., J.E. Marsden, T. Ratiu, and J.C. Simo [1990] Normalizing connections and the energy-momentum method. *Proceedings of the CRM Conference on Hamiltonian systems, Transformation Groups, and Spectral Transform Methods*, edited by Harnad and Marsden. CRM Press, Montreal, pp. 207–227.

Marle, C.M. [1976] Symplectic manifolds, dynamical groups and Hamiltonian mechanics. *Differential Geometry and Relativity*, edited by M. Cahen and M. Flato. D. Reidel, Boston.

Marsden, J.E. [1981] *Lectures on Geometric Methods in Mathematical Physics*. SIAM, Philadelphia, PA.

Marsden, J.E. [1992] *Lectures on Mechanics*, London Math. Soc. Lecture Notes **174**, Cambridge Univ. Press.

Marsden, J.E., and T.J.R. Hughes [1983] *Mathematical Foundations of Elasticity*. Prentice-Hall, Redwood City, CA.

Marsden, J.E., R. Montgomery, and T. Ratiu [1990] Reduction, symmetry, and phases in mechanics. *Mem. AMS* **436**.

Marsden, J.E., and T.S. Ratiu [1986] Reduction of Poisson manifolds. *Lett. Math. Phys.* **11**, 161–170.

Marsden, J.E., T. Ratiu, and A. Weinstein [1984] Semi-direct products and reduction in mechanics. *Trans. Amer. Math. Soc.* **281**, 147–177.

Marsden, J.E., and A. Weinstein [1974] Reduction of symplectic manifolds with symmetry. *Rep. Math. Phys.* **5**, 121–130.

Marsden, J.E., and A. Weinstein [1982] The Hamiltonian structure of the Maxwell–Vlasov equations. *Physica D* **4**, 394–406.

Marsden, J.E., and A. Weinstein [1983] Coadjoint orbits, vortices and Clebsch variables for incompressible fluids. *Physica D* **7**, 305–323.

Meyer, K.R. [1973] Symmetries and integrals in mechanics. *Dynamical Systems*, edited by M. Peixoto Academic Press, New York, pp. 259–273.

Meyer, K.R., and G. Hall [1991] *Hamiltonian Mechanics and the n-body Problem*. Springer-Verlag, New York.

Montaldi, J.A., R.M. Roberts, and I.N. Stewart [1988] Periodic solutions near equilibria of symmetric Hamiltonian systems. *Phil. Trans. R. Soc. Lond. A* **325**, 237–293.

Montgomery, R. [1984] Canonical formulations of a particle in a Yang–Mills field. *Lett. Math. Phys.* **8**, 59–67.

Montgomery, R. [1990] Isoholonomic problems and some applications. *Commun. Math Phys.* **128**, 565–592.

Montgomery, R., J. Marsden, and T. Ratiu [1984] Gauged Lie–Poisson structures. *Contrib. Math. AMS* **28**, 101–114.

Morrison, P.J. [1987] Variational principle and stability of nonmonotone Vlasov–Poisson equilibria. *Z. Naturforsch.* **42a**, 1115–1123.

Oh, Y.G., N. Sreenath, P.S. Krishnaprasad and J.E. Marsden [1989] The dynamics of coupled planar rigid bodies. Part 2: Bifurcations, periodic solutions, and chaos. *Dynam. Diff. Eqs.* **1**, 269–298.

Patrick, G., [1989] The dynamics of two coupled rigid bodies in three space. *Contrib. Math. AMS* **97**, 315–336.

Routh, E.J., [1877] Stability of a given state of motion. Reprinted in *Stability of Motion*, edited by A.T. Fuller. Halsted Press, New York, 1975.

Shapere, A., and F. Wilczeck [1989] Geometry of self-propulsion at low Reynolds number. *J. Fluid Mech.* **198**, 557–585.

Simo, J.C., D. Lewis, and J.E. Marsden [1991] Stability of relative equilibria I: The reduced energy momentum method. *Arch. Rat. Mech. Anal*, **115**, 15–59.

Simo, J.C., T.A. Posbergh, and J.E. Marsden [1990] Stability of coupled rigid body and geometrically exact rods: block diagonalization and the energy-momentum method. *Physics Rept*, **193**, 280–360.

Sjamaar, R. [1990] Singular orbit spaces in Riemannian and symplectic geometry. Thesis, Utrecht.

Sjamaar, R., and E. Lerman [1991] Stratified symplectic spaces and Reduction, *Ann. of Math.* **34**, 375–422.

Smale, S [1967] Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73**, 747–817.

Smale, S [1970] Topology and mechanics. *Invent. Math.* **10**, 305–331; **11**, 45–64.

Smale, S [1980] *The Mathematics of Time*. Springer-Verlag, New York.

Wan, Y.H., and M. Pulvirente [1984] Nonlinear stability of circular vortex patches. *Commun. Math. Phys.* **99**, 435–450.

Wiggins, S. [1988] *Global Bifurcations and Chaos*. Springer-Verlag, and Applied Mathematical Sciences, New York.