Lagrangian Reduction and the Double Spherical Pendulum

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Abstract

This paper studies the stability and bifurcations of the relative equilibria of the double spherical pendulum, which has the circle as its symmetry group. This example as well as others with nonabelian symmetry groups, such as the rigid body, illustrate some useful general theory about Lagrangian reduction. In particular, we establish a satisfactory global theory of Lagrangian reduction that is consistent with the classical local Routh theory for systems with an abelian symmetry group.

1 Introduction

One of the goals of this paper is to study some dynamical features of the double spherical pendulum using techniques of geometric mechanics and bifurcation theory with symmetry. In doing this, we find that the energy momentum technique of Simo, Lewis and Marsden [1991] is useful for a stability analysis, but to get the linearized equations that enable one to detect bifurcations (such as the Hamiltonian–Hopf bifurcation), we use methods of Lagrangian reduction that are closely related to Routh's method; see Routh [1877, 1884]. A second goal of the paper is to develop the general theory of Lagrangian reduction. This paper develops the Lagrangian analogue of symplectic reduction on the Hamiltonian side; that is, there is a specified value of the momentum map that is chosen. In Marsden and Scheurle [1992], we develop the Lagrangian analogue of Poisson reduction and in so doing, the Euler-Lagrange-Poincaré equations play a central role.

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The double spherical pendulum is a mechanical system with an abelian symmetry group, but we extend the Routh method to the nonabelian case as well, even though it is normally regarded as being applicable only for abelian groups (see Arnold [1988], p. 86ff). We use a construction similar to the Dirac constraint method as one of the aids for the nonabelian case. The latter work is in fact related to the Lagrangian reduction and Clebsch variable techniques of Cendra, Ibort, and Marsden [1987]. The rigid body and the classical water molecule illustrate the use of nonabelian Lagrangian reduction, but only the rigid body as a simple nonabelian example will be discussed in this paper.

Our double pendulum example fits into the spirit of a number of interesting and similar analyses of mechanical systems with symmetry that have appeared recently in the literature. See Simo, Lewis, and Marsden [1991], Ballieul and Levi [1987, 1991] and Zombro and Holmes [1992] for further information and references.

All of the relative equilibria of the double spherical pendulum are found in the present paper. Amongst these are the special symmetric equilibria, such as the four states with both pendula pointing vertically, for example, with them both pointing straight down. This case requires special attention, and a beginning analysis is made for them here, but we do not attempt to make the analysis of this case complete. For the general equilibria, we are more complete, with the stabilities, both in terms of the energy-momentum method and spectral stability being found. Moreover, a Hamiltonian transcritical bifurcation of relative equilibria as dimensionless system parameters, depending on the mass and the length ratios, are varied is found. In addition, we find a (generic, or nonsemisimple) 1 : 1 resonance bifurcation—a so-called Hamiltonian Hopf bifurcation (see van der Meer [1985, 1990])—as these parameters and the angular momentum are varied. However, this bifurcation, while identified, is not explored in detail in the present paper (such as whether or not one has the "stable" or the "unstable" case).

For the symmetric equilibria, some bifurcation and stability information is obtained here and in Dellnitz, Marsden, Melbourne and Scheurle [1992]. Our suggested approach to this problem, which is only sketched, is that of blowing up singularities and regularization. However, no attempt is made to give a complete account, or to relate our method to that of singular reduction, such as found in Arms, Marsden, and Moncrief [1981], Arms, Cushman and Gotay [1991], Sjamaar, R. and E. Lerman [1991] and Cushman and Sjamaar [1991], and references therein, or to the methods for analyzing symmetric relative equilibria developed by Lewis [1992]. This would be of considerable interest, but is not the purpose of the present paper to explore. Not only this, but we do not investigate the Lagrangian reduction procedure near symmetric points. Again, we leave this for elsewhere.

It is interesting to note that the Lagrangian reduction procedure developed here is closely related in spirit, and in some details, to the structures that appear in the theory of nonholonomic constraints, as is given in, for example, Naimark and Fufaev [1972], Koiller [1992], and Bloch and Crouch [1992]. These connections will be explored elsewhere.

This topic of nonholonomic constraints is just one amongst several others that would be worth further study, such as the symmetric relative equilibria mentioned above and another is geometric phases in the Lagrangian setting, especially for motions near symmetric relative equilibria. Another interesting topic not addressed in this paper is the establishment and study of chaotic motions for the double spherical pendulum. We presume that this can be done using the Poincaré-Melnikov method adapted for systems with symmetry (Holmes and Marsden [1982a, b, 1983] and Wiggins [1988]). For a study along these lines for the double *planar* pendulum, see Burov [1986].

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2 Some Preliminaries

We recall a few facts about simple mechanical systems with symmetry. The general framework is that of a symplectic manifold (P, Ω) together with the symplectic action of a Lie group G on P, an *equivariant* momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ and a G-invariant Hamiltonian $H : P \to \mathbb{R}$. In this paper, we focus on the special case of a **simple mechanical system**, following terminology of Smale [1970]. That is, we choose $P = TQ$ or $P = T^*Q$, assume there is a Riemannian metric $\langle\langle\, , \rangle\rangle$ on Q, that G acts on Q by isometries (and so G acts on TQ by tangent lifts and on T^*Q by cotangent lifts) and that the Lagrangian is

$$L(q, v) = \frac{1}{2}\|v\|_q^2 - V(q), \tag{1}$$

or equivalently, the Hamiltonian is

$$H(q, p) = \frac{1}{2}\|p\|_q^2 + V(q), \tag{2}$$

where $\| \cdot \|_q$ denotes either the norm on T_qQ or the one induced on T_q^*Q, as is appropriate, and where V is a G-invariant potential.

We abuse notation slightly and write either (q, v) or v_q for a vector based at $q \in Q$ and $z = (q, p)$ or $z = p_q$ for a covector based at $q \in Q$. The pairing between T_q^*Q and T_qQ is written

$$\langle p_q, v_q \rangle, \quad \langle p, v \rangle \quad \text{or} \quad \langle (q, p), (q, v) \rangle. \tag{3}$$

Other natural pairings between spaces and their duals are also denoted $\langle\, , \rangle$.

The standard momentum map for simple mechanical G-systems is

$$\begin{aligned} \mathbf{J} : TQ \to \mathfrak{g}^*, &\quad \text{where} \quad \langle \mathbf{J}(q, v), \xi \rangle = \langle\langle v, \xi_Q(q) \rangle\rangle \\ \text{or} \quad \mathbf{J} : T^*Q \to \mathfrak{g}^*, &\quad \text{where} \quad \langle \mathbf{J}(q, p), \xi \rangle = \langle p, \xi_Q(q) \rangle \end{aligned} \tag{4}$$

where ξ_Q denotes the infinitesimal generator of $\xi \in \mathfrak{g}$ on Q. We use the same notation for $\mathbf{J}$ regarded as a map on either the cotangent or the tangent space; which is meant will be clear from the context.

Assume that G acts freely on Q so we can regard $Q \to Q/G$ as a principal G-bundle. [**Aside:** All one really needs is the action of G_μ on Q to be free and all the constructions can be done in terms of the bundle $Q \to Q/G_\mu$; here, G_μ is the isotropy subgroup for $\mu \in \mathfrak{g}^*$ for the coadjoint action of G on $\mathfrak{g}^*$. Recall that for abelian groups, $G = G_\mu$.]

For each $q \in Q$, let the **locked inertia tensor** be the map $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ defined by

$$\langle \mathbb{I}(q)\eta, \zeta \rangle = \langle\langle \eta_Q(q), \zeta_Q(q) \rangle\rangle. \tag{5}$$

Since the action is free, $\mathbb{I}(q)$ is indeed an inner product. The terminology comes from the fact that for coupled rigid or elastic systems, $\mathbb{I}(q)$ is the classical moment of inertia tensor of the corresponding rigid system. Most of the results of this paper hold in the infinite as well as the finite dimensional case. To expedite the exposition, we give many of the formulas in coordinates for the finite dimensional case. For instance,

$$\mathbb{I}_{ab} = g_{ij} A^i{}_a A^j{}_b, \tag{6}$$

where we write

$$[\xi_Q(q)]^i = A^i{}_a(q)\xi^a \tag{7}$$

relative to coordinates $q^i, i = 1, 2, \dots n$ on Q and a basis $e_a, a = 1, 2, \dots, m$ of $\mathfrak{g}$. In such a basis, the coordinates of $\xi \in \mathfrak{g}$ are defined by writing $\xi = \xi^a e_a$.

Define the map $\alpha : TQ \to \mathfrak{g}$ which assigns to each (q, v) the corresponding **angular velocity of the locked system**:

$$\alpha(q, v) = \mathbb{I}(q)^{-1}(\mathbf{J}(q, v)). \tag{8}$$

In coordinates,

$$\alpha^a = \mathbb{I}^{ab} g_{ij} A^i{}_b v^j \tag{9}$$

The map α is a connection, called the **mechanical connection** on the principal G-bundle $Q \to Q/G$. In other words, α is G-equivariant and satisfies $\alpha(\xi_Q(q)) = \xi$, both of which are readily verified. The horizontal space of the connection α is given by

$$\mathrm{hor}_q = \{(q, v) \mid \mathbf{J}(q, v) = 0\}; \tag{10}$$

i.e., the space orthogonal to the G-orbits. The vertical space consists of vectors that are mapped to zero under the projection $Q \to S = Q/G$; *i.e.,*

$$\mathrm{ver}_q = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}. \tag{11}$$

For each $\mu \in \mathfrak{g}^*$, define the 1-form α_μ on Q by

$$\langle \alpha_\mu(q), v \rangle = \langle \mu, \alpha(q, v) \rangle \tag{12}$$

i.e.,

$$(\alpha_\mu)_i = g_{ij} A^j{}_b \mu_a \mathbb{I}^{ab} \tag{13}$$

It follows from the idenitity $\alpha(\xi_Q(q)) = \xi$ that α_μ takes values in $\mathbf{J}^{-1}(\mu)$.

The horizontal-vertical decomposition of a vector $(q, v) \in T_qQ$ is given by

$$v = \mathrm{hor}_q v + \mathrm{ver}_q v \tag{14}$$

where

$$\mathrm{ver}_q v = [\alpha(q, v)]_Q(q) \quad \text{and} \quad \mathrm{hor}_q v = v - \mathrm{ver}_q v.$$

Notice that hor : $TQ \to \mathbf{J}^{-1}(0)$ and that, it may be regarded as a **velocity shift**.

The **amended potential** V_μ is defined by

$$V_\mu(q) = V(q) + \frac{1}{2}\langle \mu, \mathbb{I}(q)^{-1}\mu \rangle. \tag{15}$$

In coordinates,

$$V_\mu(q) = V(q) + \frac{1}{2}\mathbb{I}^{ab}(q)\mu_a\mu_b. \tag{16}$$

A **relative equilibrium** is a dynamic state that is also a one parameter group orbit. Various criteria characterizing relative equilibria are given in Simo, Lewis, and Marsden [1991], and we shall recall one of these, namely Smale's criterion, in the next section.

We shall need some facts about reduction and in particular, the cotangent bundle reduction theorem, so we recall these now.

For **symplectic reduction**, we begin with a symplectic manifold (P, Ω), a Lie group G acting by symplectic maps on P, an equivariant momentum map $\mathbf{J}$ for this action and a G-invariant Hamiltonian H on P. For $\mu \in \mathfrak{g}^*$, the isotropy subgroup G_μ leaves $\mathbf{J}^{-1}(\mu)$ invariant by equivariance. Assume for simplicity that μ is a regular value of $\mathbf{J}$, so that $\mathbf{J}^{-1}(\mu)$ is a smooth manifold and that G_μ acts freely and properly on $\mathbf{J}^{-1}(\mu)$, so that $\mathbf{J}^{-1}(\mu)/G_\mu =: P_\mu$ is a smooth manifold. Already in our example of the double spherical pendulum, we will encounter an interesting singular situation in which μ is not regular. We will indicate how we deal with this difficulty at that juncture.

Let $i_\mu : \mathbf{J}^{-1}(\mu) \to P$ denote the inclusion map and let $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ denote the projection. Note that

$$\dim P_\mu = \dim P - \dim G - \dim G_\mu. \tag{17}$$

Building on classical work of Jacobi, Liouville, Arnold and Smale, we have the Reduction Theorem of Marsden and Weinstein [1974] (see also Meyer [1973]):

There is a unique symplectic structure Ω_μ on P_μ *satisfying*

$$i_\mu^*\Omega = \pi_\mu^*\Omega_\mu. \tag{18}$$

Given a G-invariant Hamiltonian H on P, define the **reduced Hamiltonian** $H_\mu : P_\mu \to \mathbb{R}$ by $H = H_\mu \circ \pi_\mu$. The trajectories of X_H project to those of X_{H_μ}. An important problem is how to reconstruct trajectories of X_H from trajectories of X_{H_μ}. We do not address this here, but refer the reader to Marsden, Montgomery, and Ratiu [1990] and Marsden [1992] and remark that this reconstruction procedure naturally brings in the concept of geomertic phases.

One can also describe reduction in terms of orbits: $P_\mu \cong P_{\mathcal{O}}$ where

$$P_{\mathcal{O}} = \mathbf{J}^{-1}(\mathcal{O})/G$$

and $\mathcal{O} \subset \mathfrak{g}^*$ is the coadjoint orbit through μ. See Marsden [1981, 1992] for an exposition of this result of Marle, Kahzdan, Kostant, and Sternberg.

For cotangent bundles, a main result (due to Satzer, Marsden, and Kummer; see Abraham and Marsden [1978] and Kummer [1981]) says that *the reduction of a cotangent bundle T^*Q at $\mu \in \mathfrak{g}^*$ is a symplectic subbundle of $T^*(Q/G_\mu)$* or from the symplectic bundle point of view (due to Montgomery, Marsden and Ratiu [1984] and Montgomery [1986]) *is a bundle over $T^*(Q/G)$ with fiber the coadjoint orbit through μ*. Here, $S = Q/G$ is called **shape space**. From the Poisson bundle viewpoint, this reads: $(T^*Q)/G$ is a $\mathfrak{g}^*$-bundle over $T^*(Q/G)$, **or a Lie-Poisson bundle over the cotangent bundle of shape space.**

To see this, map $\mathbf{J}^{-1}(\mathcal{O}) \to \mathbf{J}^{-1}(0)$ by the map hor. This induces a map, denoted by $\mathrm{hor}_{\mathcal{O}}$, on the quotient spaces by equivariance:

$$\mathrm{hor}_{\mathcal{O}} : \mathbf{J}^{-1}(\mathcal{O})/G \to \mathbf{J}^{-1}(0)/G. \tag{19}$$

Reduction at zero is easy to describe: $\mathbf{J}^{-1}(0)/G$ is isomorphic with $T^*(Q/G)$ by the following identification: $\beta_q \in \mathbf{J}^{-1}(0)$ satisfies $\langle \beta_q, \xi_Q(q)\rangle = 0$ for all $\xi \in \mathfrak{g}$, so we can regard β_q as a one form on $T(Q/G)$.

As a set, the fiber of the map $\mathrm{hor}_{\mathcal{O}}$ is identified with $\mathcal{O}$. Therefore, we have realized $(T^*Q)_{\mathcal{O}}$ as a coadjoint orbit bundle over $T^*(Q/G)$.

The Poisson bracket structure of the **reduction bundle**

$$\mathrm{hor}_{\mathcal{O}} : (T^*Q)_{\mathcal{O}} \to T^*(Q/G)$$

is a synthesis of the Lie-Poisson structure, the cotangent structure, the magnetic and interaction terms, as has been investigated in Montgomery, Marsden and Ratiu [1984] and Montgomery [1986].

To obtain the symplectic structure, we restrict the map hor to $\mathbf{J}^{-1}(\mu)$ and quotient by G_μ to get a map of P_μ to $\mathbf{J}^{-1}(0)/G_\mu$. If $\mathbf{J}_\mu$ denotes the momentum map for G_μ, then $\mathbf{J}^{-1}(0)/G_\mu$ embeds in $\mathbf{J}_\mu^{-1}(0)/G_\mu \cong T^*(Q/G_\mu)$. *The resulting map* hor_μ *embeds P_μ into $T^*(Q/G_\mu)$*. This map is the one induced by the shifting map:

$$p_q \mapsto p_q - \alpha_\mu(q). \tag{20}$$

The symplectic form on P_μ is obtained by restricting the form on $T^*(Q/G_\mu)$ given by

$$\Omega_{\text{canonical}} + \mathbf{d}\alpha_\mu. \tag{21}$$

The two form $\mathbf{d}\alpha_\mu$ drops to a two form β_μ called the **magnetic term** on the quotient, so (**??**) defines the symplectic structure of P_μ. (The term "magnetic is used" because the same structure occurs in the dynamics of a pariticle moving in a maganetic field; see Marsden [1992] for more information). We also note that on $\mathbf{J}^{-1}(\mu)$ (and identifying vectors and covectors via the Legendre transformation, $[\alpha(v), \alpha(w)] = [\mu, \mu] = 0$, where $[\,,]$ is the Lie algebra bracket. *Therefore, the magnetic term β_μ may also be regarded as the form induced by the μ-component of the curvature.* One can compute the magnetic terms on the symplectic reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ in two ways: either by defining them as we have done, or by computing the connection for the action of the group G_μ and pulling the resulting magnetic two form back to $\mathbf{J}^{-1}(\mu)$.

Two limiting cases are noteworthy. The first (that one can associate with Arnold [1966]) is when $Q = G$ in which case $P_{\mathcal{O}} \cong \mathcal{O}$ and the base is trivial in the $P_{\mathcal{O}} \to T^*(Q/G)$ picture, while in the $P_\mu \to T^*(Q/G_\mu)$ picture, the fiber is trivial and the space is $Q/G_\mu \cong \mathcal{O}$. Here the description of the orbit symplectic structure induced by $\mathbf{d}\alpha_\mu$ coincides with that given by Kirillov [1976].

The other limiting case (that one can associate with (Routh [1877] and) Smale [1970]) is when $G = G_\mu$; for instance, this holds in the abelian case. Then

$$P_\mu = P_{\mathcal{O}} = T^*(Q/G)$$

with symplectic form $\Omega_{\text{canonical}} + \beta_\mu$.

We get a reduced Hamiltonian system on $P_\mu \cong P_{\mathcal{O}}$ obtained by restricting H to $\mathbf{J}^{-1}(\mu)$ or $\mathbf{J}^{-1}(\mathcal{O})$ and then passing to the quotient. This produces the reduced Hamiltonian function H_μ and thereby a Hamiltonian system on P_μ. The resulting vector field is the one obtained by restricting and projecting the Hamiltonian vector field X_H from P to P_μ. The resulting dynamical system X_{H_μ} on P_μ is called the **reduced Hamiltonian system**.

Let us compute H_μ in each of the pictures P_μ and $P_{\mathcal{O}}$. In either case the shift by the map hor is basic, so let us first compute the function on $\mathbf{J}^{-1}(0)$ given by

$$H_{\alpha_\mu}(q,p) = H(q, p + \alpha_\mu(q)). \tag{22}$$

Indeed,

$$\begin{aligned} H_{\alpha_\mu}(q,p) &= \frac{1}{2}\langle\langle p + \alpha_\mu, p + \alpha_\mu \rangle\rangle_q + V(q) \\ &= \frac{1}{2}\|p\|_q^2 + \langle\langle p, \alpha_\mu \rangle\rangle_q + \frac{1}{2}\|\alpha_\mu\|_q^2 + V(q). \end{aligned} \tag{23}$$

If $p = \mathbb{F}L \cdot v$, then $\langle\langle p, \alpha_\mu \rangle\rangle_q = \langle \alpha_\mu, v\rangle = \langle \mu, \alpha(q,v)\rangle = \langle \mu, \mathbb{I}(q)\mathbf{J}(p)\rangle = 0$ since $\mathbf{J}(p) = 0$. Thus, on $\mathbf{J}^{-1}(0)$,

$$H_{\alpha_\mu}(q,p) = \frac{1}{2}\|p\|_q^2 + V_\mu(q). \tag{24}$$

In $T^*(Q/G_\mu)$, we obtain H_μ by selecting a representative (q,p) of $T^*(Q/G_\mu)$ in $\mathbf{J}^{-1}(0) \subset T^*Q$, shifting it to $\mathbf{J}^{-1}(\mu)$ by $p \mapsto p + \alpha_\mu(q)$ and then evaluating H at this point. Thus, the above calculation (**??**) proves:

Proposition 2.1 *The reduced Hamiltonian H_μ is the function obtained by restricting to the symplectic subbundle $P_\mu \subset T^*(Q/G_\mu)$, the function*

$$H_\mu(q,p) = \frac{1}{2}\|p\|^2 + V_\mu(q) \tag{25}$$

defined on $T^*(Q/G_\mu)$ with the symplectic structure

$$\Omega_\mu = \Omega_{\rm can} + \beta_\mu \tag{26}$$

where β_μ is the two form on Q/G_μ obtained from $\mathbf{d}\alpha_\mu$ on Q by passing to the quotient. Here we use the quotient metric on Q/G_μ and identify V_μ with a function on Q/G_μ.

For example, if $Q = G$ and the symmetry group is G itself, then $P_\mu \subset T^*(Q/G_\mu)$ sits as the zero section. In fact P_μ is identified with $Q/G_\mu \cong G/G_\mu \cong \mathcal{O}_\mu$. In this example, the reduced symplectic form is "entirely magnetic".

To describe H_μ on $\mathbf{J}^{-1}(\mathcal{O})/G$ is easy abstractly; one just calculates H restricted to $\mathbf{J}^{-1}(\mathcal{O})$ and passes to the quotient. More concretely, we choose an element $[(q,p)] \in T^*(Q/G)$, where we identify the representative with an element of $\mathbf{J}^{-1}(0)$. We also choose an element $\nu \in \mathcal{O}$, a coadjoint orbit, and shift $(q,p) \mapsto (q, p+\alpha_\nu(q))$ to a point in $\mathbf{J}^{-1}(\mathcal{O})$. Thus, we get:

Proposition 2.2 *Regarding $P_\mu \cong P_{\mathcal{O}}$ as an $\mathcal{O}$-bundle over $T^*(Q/G)$, the reduced Hamiltonian is given by*

$$H_{\mathcal{O}}(q,p,\nu) = \frac{1}{2}\|p\|^2 + V_\nu(q)$$

where (q,p) is a representative in $\mathbf{J}^{-1}(0)$ of a point in $T^*(Q/G)$ and where $\nu \in \mathcal{O}$.

The symplectic structure in this second picture was described abstractly above. To describe it concretely in terms of $T^*(Q/G)$ and $\mathcal{O}$ in terms of Poisson bundles, see Montgomery, Marsden and Ratiu [1984].

3 Lagrangian Reduction and the Routhian

The general symplectic reduction procedure for Hamiltonian systems was recalled above. What is less known is how to reduce Lagrangian systems directly, although the abelian case was essentially known to Routh by around 1860; a modern account is given in Arnold [1988]. The procedure developed in this section is a geometrization and a generalization of the Routh procedure to the nonabelian case. It is a generally held belief that the Routh procedure "works" only in the abelian case. We are able to handle the general case by including conservative gyroscopic forces into the variational principle in the sense of Lagrange and d'Alembert. We also employ a Dirac constraint type of construction to include the cases in which the reduced space is not a tangent bundle (but it is a Dirac constraint set inside one). Some of the underlying ideas of this section are already found in Cendra, Ibort, and Marsden [1987]. The nonabelian case is illustrated by the rigid body below.

Given $\mu \in \mathfrak{g}^*$, define the **Routhian** $R^\mu : TQ \to \mathbb{R}$ as follows:

$$R^\mu(q,v) = L(q,v) - \langle \alpha(q,v), \mu \rangle \tag{27}$$

where α is the mechanical connection. This function is not the classical Routhian, but is closely related to it, as we shall see below. Notice that the Routhian has the form of a *Lagrangian with a gyroscopic term*; see Bloch, Krishnaprasad, Marsden, and Sanchez [1991] and Wang and Krishnaprasad [1992] for information on the use of gyroscopic systems in control theory.

A basic observation about the Routhian is that solutions of the Euler-Lagrange equations for L can be regarded as solutions of the Euler-Lagrange equations for the Routhian, with the addition of "magnetic forces". To understand this statement, define the **magnetic two form** β to be

$$\beta = \mathbf{d}\alpha_\mu, \tag{28}$$

a two form on Q. In coordinates,

$$\beta_{ij} = \frac{\partial \alpha_j}{\partial q^i} - \frac{\partial \alpha_i}{\partial q^j},$$

where we write $\alpha_\mu = \alpha_i dq^i$ and

$$\beta = \sum_{i<j} \beta_{ij} dq^i \wedge dq^j. \tag{29}$$

We say that $q(t)$ *satisfies the Euler-Lagrange equations for a Lagrangian $\mathcal{L}$ with the magnetic term β* provided that the associated variational principle in the sense of Lagrange and d'Alembert is satisfied:

$$\delta \int_a^b \mathcal{L}(q(t), \dot{q}) dt = \int_a^b \mathbf{i}_{\dot{q}} \beta \tag{30}$$

where the variations are over curves in Q with fixed endpoints and where $\mathbf{i}_{\dot{q}}$ is the interior product by $\dot{q}$. This condition is equivalent to the coordinate condition stating that the *Euler-Lagrange equations with gyroscopic forcing* are satisfied:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = \dot{q}^j \beta_{ij}. \tag{31}$$

Proposition 3.1 *A curve $q(t)$ in Q is a solution of the Euler-Lagrange equations for the Lagrangian L with momentum $\mathbf{J}(q, \dot{q}) = \mu$ iff it is a solution of the Euler-Lagrange equations for the Routhian R^μ with gyroscopic forcing given by β.*

Proof Let p denote the momentum conjugate to q for the Lagrangian L (so in coordinates, $p_i = g_{ij}\dot{q}^j$) and let $\mathfrak{p}$ be the corresponding conjugate momentum for the Routhian. Clearly, p and $\mathfrak{p}$ are related by the momentum shift $\mathfrak{p} = p - \alpha_\mu$. Thus by the chain rule, $\frac{d}{dt}\mathfrak{p} = \frac{d}{dt}p - T\alpha_\mu \cdot \dot{q}$, or in coordinates,

$$\frac{d}{dt}p_i = \frac{d}{dt}\mathfrak{p}_i - \frac{\partial \alpha_i}{\partial q^j}\dot{q}^j. \tag{32}$$

Likewise, $\mathbf{D}_q R^\mu = \mathbf{D}_q L - \mathbf{D}_q\langle \alpha(q,v), \mu\rangle$ or in coordinates,

$$\frac{\partial R^\mu}{\partial q^i} = \frac{\partial L}{\partial q^i} - \frac{\partial \alpha_j}{\partial q^i}\dot{q}^j \tag{33}$$

Subtracting these expressions, one finds (in coordinates, for convenience):

$$\begin{aligned} \frac{d}{dt}\frac{\partial R^\mu}{\partial \dot{q}^i} - \frac{\partial R^\mu}{\partial q^i} &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} + \left(\frac{\partial \alpha_j}{\partial q^i} - \frac{\partial \alpha_i}{\partial q^j}\right)\dot{q}^j \\ &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} + \beta_{ij}\dot{q}^j, \end{aligned} \tag{34}$$

which proves the result. ■.

Proposition 3.2 *For all* $(q,v) \in TQ$ *and* $\mu \in \mathfrak{g}^*$ *we have*

$$R^\mu = \frac{1}{2}\|\mathrm{hor}(q,v)\|^2 + \langle \mathbf{J}(q,v) - \mu, \xi\rangle - \left(V + \frac{1}{2}\langle \mathbb{I}(q)\xi, \xi\rangle\right) \tag{35}$$

where $\xi = \alpha(q,v)$.

Proof Use the definition $\mathrm{hor} = v - \xi_Q(q,v)$ and expand the square using the definition of $\mathbf{J}$. ■

Before describing the actual reduction procedure, we relate our Routhian with the classical one. If one has an abelian group G and can identify the symmetry group by a set of cyclic coordinates, then there is a simple formula which relates R^μ to the "classical" Routhian $R^\mu_{classical}$. In this case, we assume that G is the torus T^k (or a torus cross Euclidean space) and acts on Q by $q^\alpha \mapsto q^\alpha (\alpha = 1, \cdots, m)$ and $\theta^a \mapsto \theta^a + \varphi^a (a = 1, \cdots, k)$ with $\varphi^a \in [0, 2\pi)$, where $q^1, \cdots, q^m, \theta^1, \cdots, \theta^k$ are suitably chosen (local) coordinates on Q. Then G-invariance implies that the Lagrangian $L = L(q, \dot{q}, \dot{\theta})$ in (2.1) does not explicitly depend on the variables θ^a, *i.e.,* these variables are *cyclic*. Moreover, the infinitesimal generator ξ_Q of $\xi = (\xi^1, \cdots, \xi^k) \in \mathfrak{g}$ on Q is given by $\xi_Q = (0, \cdots, 0, \xi^1, \cdots, \xi^k)$, and the momentum map J has components given by $J_a = \partial L/\partial\dot{\theta}^a$, *i.e.,*

$$J_a(q, \dot{q}, \dot{\theta}) = g_{\alpha a}(q)\dot{q}^\alpha + g_{ba}(q)\dot{\theta}^b. \tag{36}$$

Thus, given $\mu \in \mathfrak{g}^*$, the **classical Routhian** is defined by (see, for example, Arnold [1988]).

$$R^\mu_{\mathrm{classical}}(q, \dot{q}) = [L(q, \dot{q}, \dot{\theta}) - \mu_a\dot{\theta}^a]_{|\dot{\theta}^a = \dot{\theta}^a(q,\dot{q})}, \tag{37}$$

where

$$\dot{\theta}^a(q, \dot{q}) = [\mu_c - g_{\alpha c}(q)\dot{q}^\alpha]\mathbb{I}^{ca}(q) \tag{38}$$

is the unique solution of $J_a(q, \dot{q}, \dot{\theta}) = \mu_a$ with respect to $\dot{\theta}^a$.

Proposition 3.3 $R^\mu_{\text{classical}} = R^\mu + \mu_c g_{\alpha a}\dot{q}^\alpha \mathbb{I}^{ca}$

Proof In the present coordinates we have

$$L = \frac{1}{2}g_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta + g_{\alpha a}(q)\dot{q}^\alpha\dot{\theta}^a + \frac{1}{2}g_{ab}(q)\dot{\theta}^a\dot{\theta}^b - V(q, \theta) \tag{39}$$

and

$$\alpha_\mu = \mu_a d\theta^a + g_{b\alpha}\mu_a\mathbb{I}^{ab}dq^\alpha. \tag{40}$$

By (2.14), (3.14) implies

$$\|\text{hor}_{(q,\theta)}(\dot{q}, \dot{\theta})\|^2 = g_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - g_{\alpha a}(q)g_{b\gamma}(q)\dot{q}^\alpha\dot{q}^\gamma\mathbb{I}^{ab}(q). \tag{41}$$

Using (2.16), (3.14), (3.16) and the identity $(\mathbb{I}^{ab}) = (g_{ab})^{-1}$, the proposition follows from the above definitions of R^μ and $R^\mu_{\text{classical}}$ by a straightforward algebraic computation. ■

Now we are ready to drop the variational principle (3.4) to the quotient space Q/G_μ, with $\mathcal{L} = R^\mu$. In this principle, the variation of the integral of R^μ is taken over curves satisfying the fixed endpoint condition; this variational principle therefore holds in particular if the curves are also constrained to satisfy the condition $\mathbf{J}(q, v) = \mu$. Then we find that the variation of the function R^μ restricted to the level set of $\mathbf{J}$ satisfies the variational condition. The restriction of R^μ to the level set equals

$$R^\mu = \frac{1}{2}\|\text{hor}(q, v)\|^2 - V_\mu \tag{42}$$

In this variational principle, the endpoint conditions can be relaxed to the condition that the ends lie on orbits rather than be fixed. This is because the kinetic part now just involves the horizontal part of the velocity, and so the endpoint conditions in the variational principle, which involve the contraction of the momentum $\mathfrak{p}$ with the variation of the configruration variable δq vanish if $\delta q = \zeta_Q(q)$ for some $\zeta \in \mathfrak{g}$, *i.e.,* if the variation is tangent to the orbit. The condition that (q, v) be in the μ level set of $\mathbf{J}$ means that the momentum $\mathfrak{p}$ vanishes when contracted with an infinitesimal generator on Q.

From the above displayed formula, we see that the function R^μ restricted to the level set defines a function on the quotient space $T(Q/G_\mu)$ – that is, it factors through the tangent of the projection map $\tau_\mu : Q \to Q/G_\mu$. The variational principle also drops, therefore, since the curves that join orbits correspond to those that have fixed endpoints on the base. Note, also, that the magnetic term defines a well-defined two form on the quotient as well, as is known from the Hamiltonian case, even though α_μ does not drop to the quotient in general. In terms of the coordinate representation for the special case of a torus action, and cyclic variables, this can be seen from the fact that α_μ depends on the θ-variables, whereas β does not, because the 1-form $\mu_a d\theta^a$ is closed. However, on the quotient we have the well defined magnetic two form

$$\beta = d(g_{b\alpha}\mu_a \mathbb{I}^{ab} dq^\alpha). \tag{43}$$

Here is what we have proved:

Proposition 3.4 *Suppose that $q(t)$ satisfies the Euler-Lagrange equations for L with $\mathbf{J}(q,\dot{q}) = \mu$, then the induced curve on Q/G_μ satisfies the* **reduced Lagrangian variational principle***, i.e., the variational principle of Lagrange-d'Alembert on Q/G_μ with magnetic term β and the Routhian dropped to $T(Q/G_\mu)$.*

In the special case of a torus action, i.e., with cyclic variables, as in Proposition **3.3***, this reduced variational principle is equivalent to the Euler-Lagrange equations for the classical Routhian which agrees with the classical procedure of Routh.*

A consequence of equation (**??**) and the preceeding proposition is the following result of Smale [1970]:

Proposition 3.5 *Relative equilibria are given by critical points of the amended potential*

Example The Rigid Body The rigid body is a nonabelian example with group $G = SO(3)$ and configuration space $Q = G$. Here, the reduced Lagrangian variational principle is a variational principle for curves on the momentum sphere — here $Q/G_\mu \cong S^2$. For it to be well defined, it is essential that one uses the variational principle in the sense of Lagrange and d'Alembert, and not in the naive sense of the Lagrange-Hamilton principle. In this case, one checks that the dropped Routhian is just (up to a sign) the kinetic energy of the body in body coordinates. The principle then says that the variation of the kinetic energy over curves with fixed points on the two sphere equals the integral of the magnetic term (in this case the magnetic term is a constant times the area element) contracted with the tangent to the curve. One can also check this by a direct verification. (If one wants a variational principle in the usual sense of the Lagrange-Hamilton variational principle, then one can do this by introduction of "Clebsch variables", as in Marsden and Weinstein [1983] and Cendra and Marsden [1987].)

The rigid body also shows that the reduced variational principle given by Proposition **3.4** in general is degenerate. This can be seen in two essentially equivalent ways; first, the projection of the constraint $\mathbf{J} = \mu$ can produce a nontrivial condition in $T(Q/G_\mu)$ — corresponding to the embedding as a symplectic subbundle of P_μ in $T^*(Q/G_\mu)$. For the case of the rigid body, the subbundle is the zero section, and the symplectic form is all magnetic (*i.e.,* all coadjoint orbit structure). The second way to view it is that the kinetic part of the induced Lagrangian is degenerate in the sense of Dirac, and so one has to cut it down to a smaller space to get well defined dynamics. In this case, one cuts down the metric corresponding to its degeneracy, and this is, coincidentally, the same cutting down as one gets by imposing the constraint coming from the image of $\mathbf{J} = \mu$ in the set $T(Q/G_\mu)$.

For the rigid body, and more generally, for T^*G the one form α_μ is independent of the Lagrangian, or Hamiltonian. It is in fact, the right invariant one form on G equaling μ at the identity, the same form used by Marsden and Weinstein [1974] in the identification of the reduced space. For the rigid body, the Routhian is computed to be $R^\mu = -\frac{1}{2}\mu^T \mathbb{I}^{-1}\mu$, where $\mathbb{I}$ is the *spatial* moment of intertia tensor (so that, up to sign, R^μ is the standard rigid body energy, and the variational principle becomes

$$\delta \int V_\mu(q)dt = \int \beta_\mu(\dot{q}, \delta q),$$

which is equivalent to the standard *first-order* Euler equations on $Q/G_\mu = S^2$. ♦

There is a well defined *reconstruction procedure* for these systems. One can horizonatally lift a curve in Q/G to a curve $\mathfrak{q}(t)$ in Q (which therefore has zero angular momentum) and then one rotates it by the group action by a time dependent group element solving the equation

$$\dot{g}(t) = g(t)\xi(t)$$

where $\xi(t) = \alpha(\mathfrak{q}(t))$, as is used in the development of geometric phases— Marsden, Montgomery, and Ratiu [1990]. We will discuss the geometry of the horizontal curve $\mathfrak{q}(t)$ elsewhere.

In the case of the rigid body, or more generally, for the case of T^*G the system obtained by the Lagrangian reduction procedure above is "already Hamiltonian" (in this case, the symplectic structure is "all magnetic").

In general, one arrives at the reduced Hamiltonian description on $P_\mu \subset T^*(Q/G_\mu)$ with the amended potential by performing a Legendre transform in the non-degenerate variables; *i.e.,* the fiber variables corresponding to the fibers of $P_\mu \subset T^*(Q/G_\mu)$. For example, for abelian groups, one would perform a Legendre transformation in all the variables.

If one prefers, one can get a reduced Lagrangian description in the angular velocity rather than the angular momentum variables. Here are some (still vague) ideas on how to do this. One keeps the relation $\xi = \alpha(q, v)$ unspecified till near the end. In this scenario, one starts by **enlarging** the space Q to $Q \times G$ (motivated by having a rotating frame in addition to the rotating structure (as in Krishnaprasad and Marsden [1987]) and one adds to the given Lagrangian, the rotational energy for the G variables using the locked inertia tensor to form the kinetic energy–the motion on G is thus dependent on that on Q. In this description, one has ξ as an independent velocity variable and μ is its Legendre transform. The Routhian is then seen already to be a Legendre transformation in the ξ and μ variables. One can delay making this Legendre transformation to the end, when the "locking device" that locks the motion on G to be that induced by the motion on Q by imposition of $\xi = \alpha(q, v)$ and $\xi = \mathbb{I}(q)^{-1}\mu$ or $\mathbf{J}(q, v) = \mu$.

Figure 1: The configuration space for the double spherical pendulum consists of two copies of the two sphere

4 The double spherical pendulum

Consider the mechanical system consisting of two coupled spherical pendulum moving without friction in a gravitational field. (See Figure 2.1).

Let the position vectors of each pendulum relative to their hinge points be denoted $\mathbf{q}_1$ and $\mathbf{q}_2$. These vectors are assumed to have fixed lengths l_1 and l_2 and the pendula masses are denoted m_1 and m_2. The configuration space is $Q = S^2_{l_1} \times S^2_{l_2}$, the product of spheres of radii l_1 and l_2 respectively. The Lagrangian is

$$\begin{aligned} L(\mathbf{q}_1, \mathbf{q}_2, \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2) &= \frac{1}{2} m_1 \|\dot{\mathbf{q}}_1\|^2 + \frac{1}{2} m_2 \|\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2\|^2 \\ & -m_1 g \mathbf{q}_1 \cdot \mathbf{k} - m_2 g (\mathbf{q}_1 + \mathbf{q}_2) \cdot \mathbf{k}. \end{aligned} \tag{44}$$

Here $\mathbf{q}_1 + \mathbf{q}_2$ represents the position of the second mass relative to an inertial frame, so (**??**) has the standard form of kinetic minus potential energy. We identify the velocity vectors $\dot{\mathbf{q}}_1$ and $\dot{\mathbf{q}}_2$ with vectors perpendicular to $\mathbf{q}_1$ and $\mathbf{q}_2$, respectively.

The conjugate momenta are

$$\mathbf{p}_1 = \frac{\partial L}{\partial \dot{\mathbf{q}}_1} = m_1 \dot{\mathbf{q}}_1 + m_2 (\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) \tag{45}$$

and

$$\mathbf{p}_2 = \frac{\partial L}{\partial \dot{\mathbf{q}}_2} = \mathbf{m}_2 (\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) \tag{46}$$

regarded as vectors in $\mathbb{R}^3$ that are only paired with vectors orthogonal to $\mathbf{q}_1$ and $\mathbf{q}_2$ respectively.

The Hamiltonian is therefore

$$\begin{aligned} H(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2) &= \frac{1}{2m_1}\|\mathbf{p}_1 - \mathbf{p}_2\|^2 + \frac{1}{2m_2}\|\mathbf{p}_2\|^2 \\ &\quad + m_1 g \mathbf{q}_1 \cdot \mathbf{k} + m_2 g(\mathbf{q}_1 + \mathbf{q}_2) \cdot \mathbf{k}. \end{aligned} \tag{47}$$

The equations of motion are given by the Euler-Lagrange equations for L or, equivalently by Hamilton's equations for H. To write these out explicitly, it is convenient to coordinatize the configuration space. We shall do this later.

As for the symmetry group, let $G = S^1$ act on Q by simultaneous rotation of the two pendula about the z-axis. If R_θ is the rotation by an angle θ, the action is

$$(\mathbf{q}_1, \mathbf{q}_2) \mapsto (R_\theta \mathbf{q}_1, R_\theta \mathbf{q}_2).$$

The infinitesimal generator corresponding to the rotation vector $\omega\mathbf{k}$, where $\omega \in \mathbb{R}$, is $\omega(\mathbf{k} \times \mathbf{q}_1, \mathbf{k} \times \mathbf{q}_2)$ and so the corresponding momentum map (conserved quantity) is the total angular momentum about the z axis, given by

$$\begin{aligned} \langle \mathbf{J}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2), \omega\mathbf{k}\rangle &= \omega[\mathbf{p}_1 \cdot (\mathbf{k} \times \mathbf{q}_1) + \mathbf{p}_2 \cdot (\mathbf{k} \times \mathbf{q}_2)] \\ &= \omega\mathbf{k} \cdot [\mathbf{q}_1 \times \mathbf{p}_1 + \mathbf{q}_2 + \mathbf{p}_2] \end{aligned}$$

i.e.,

$$\mathbf{J} = \mathbf{k} \cdot [\mathbf{q}_1 \times \mathbf{p}_1 + \mathbf{q}_2 \times \mathbf{p}_2]. \tag{48}$$

Note that from (**??**) and (**??**),

$$\begin{aligned} \mathbf{J} &= \mathbf{k} \cdot [m_1 \mathbf{q}_1 \times \dot{\mathbf{q}}_1 + m_2 \mathbf{q}_1 \times (\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) + m_2 \mathbf{q}_2 \times (\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2)] \\ &= \mathbf{k} \cdot [m_1(\mathbf{q}_1 \times \dot{\mathbf{q}}_1) + m_2(\mathbf{q}_1 + \mathbf{q}_2) \times (\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2)]. \end{aligned}$$

The locked inertia tensor $\mathbb{I}$ plays an important role in the general theory of relative equilibria and of the separation of internal and rotational modes. We refer to Simo, Lewis and Marsden [1991] for the general construction. For "simple" systems like this one, we use the fact that the locked inertia tensor is the moment of inertia of the system regarded as a rigid structure. Thus,

$$\mathbb{I}(\mathbf{q}_1, \mathbf{q}_2) = m_1\|\mathbf{q}_1^\perp\|^2 + m_2\|(\mathbf{q}_1 + \mathbf{q}_2)^\perp\|^2 \tag{49}$$

where $\|\mathbf{q}_1^\perp\|^2 = \|\mathbf{q}_1\|^2 - \|\mathbf{q}_1 \cdot \mathbf{k}\|^2$ is the square length of the projection of $\mathbf{q}_1$ onto the xy-plane. Note that $\mathbb{I}$ is the moment of inertia of the system about the $\mathbf{k}$-axis and in this example, it is a scalar function on configuration space.

Correspondingly, the amended potential is given by

$$V_\mu(\mathbf{q}_1, \mathbf{q}_2) = m_1 g \mathbf{q}_1 \cdot \mathbf{k} + m_2 g(\mathbf{q}_1 + \mathbf{q}_2) \cdot \mathbf{k} + \frac{1}{2}\frac{\mu^2}{m_1\|\mathbf{q}_1^\perp\|^2 + m_2\|(\mathbf{q}_1 + \mathbf{q}_2)^\perp\|^2}. \tag{50}$$

Here, the symplectically reduced space is $T^*(Q/S^1)$ which is 6 dimensional. It has a nontrivial magnetic term obtained by taking the differential of (**??**).

5 Relative Equilibria for the Double Spherical Pendulum

Relative equilibria of the double spherical pendulum are dynamic states that are in uniform rotation about the vertical axis. As we saw in §2, they are computed by finding the critical points of V_μ.

There are *four* obvious relative equilibria—the ones with $\mathbf{q}_1^\perp = 0$ and $\mathbf{q}_2^\perp = 0$, in which the individual pendula are pointing vertically upwards or vertically downwards. We begin with a search for solutions with each pendulum *pointing downwards*, and with $\mathbf{q}_1^\perp \neq 0$ and $\mathbf{q}_2^\perp \neq 0$. We will comment on the cases with one of the pendula pointing upwards below.

We express V_μ as a function of $\mathbf{q}_1^\perp$ and $\mathbf{q}_2^\perp$ by using the constraints, which, for downward pointing equilibria gives the third components:

$$q_1^3 = -\sqrt{l_1^2 - \|\mathbf{q}_1^\perp\|^2} \quad \text{and} \quad q_2^3 = -\sqrt{l_2^2 - \|\mathbf{q}_2^\perp\|^2}.$$

Thus,

$$V_\mu(\mathbf{q}_1^\perp, \mathbf{q}_2^\perp) = -(m_1+m_2)g\sqrt{l_1^2 - \|\mathbf{q}_1^\perp\|^2} - m_2 g\sqrt{l_2^2 - \|\mathbf{q}_2^\perp\|^2} + \frac{1}{2}\frac{\mu^2}{\mathbb{I}}. \tag{51}$$

Setting the derivatives of V_μ equal to zero gives

$$\left.\begin{aligned} (m_1+m_2)g\frac{\mathbf{q}_1^\perp}{\sqrt{l_1^2 - \|\mathbf{q}_1^\perp\|^2}} &= \frac{\mu^2}{I^2}[(m_1+m_2)\mathbf{q}_1^\perp + m_2\mathbf{q}_2^\perp] \\ m_2 g\frac{\mathbf{q}_2^\perp}{\sqrt{l_2^2 - \|\mathbf{q}_2^\perp\|^2}} &= \frac{\mu^2}{I^2}[m_2(\mathbf{q}_1^\perp + \mathbf{q}_2^\perp)] \end{aligned}\right\} \tag{52}$$

From (5.2) we see that the vectors $\mathbf{q}_1^\perp$ and $\mathbf{q}_2^\perp$ are parallel. Therefore, define a parameter α by

$$\mathbf{q}_2^\perp = \alpha\mathbf{q}_1^\perp \tag{53}$$

Also, let λ be defined by

$$\|\mathbf{q}_1^\perp\| = \lambda l_1. \tag{54}$$

Notice that α and λ determine the *shape* of the relative equilibrium. Also, define the **system parameters** r and m by

$$r = \frac{l_2}{l_1}, \qquad m = \frac{m_1+m_2}{m_2} \tag{55}$$

Then conditions (5.2) are equivalent to

$$\begin{aligned} \frac{mg}{l_1}\frac{1}{\sqrt{1-\lambda^2}} &= \frac{\mu^2}{\mathbb{I}^2}(m+\alpha) \\ \frac{g}{l_1}\frac{\alpha}{\sqrt{r^2-\alpha^2\lambda^2}} &= \frac{\mu^2}{\mathbb{I}^2}(1+\alpha) \end{aligned} \tag{56}$$

The restrictions on the parameters are as follows: First, from $\|\mathbf{q}_1^\perp\| \leq l_1$ and $\|\mathbf{q}_2^\perp\| \leq l_2$ we get

$$0 \leq \lambda \leq \min\{r/\alpha, 1\} \tag{57}$$

and next, from the equations (5.6) we get the restrictions

$$\alpha > 0 \quad \text{or} \quad -m < \alpha < -1 \tag{58}$$

The restrictions (5.8) are special to the downward pointing relative equilibria. There are equations similar to (5.6) (with plus and minus signs inserted at the appropriate points) and inequalities similar to (5.8) for relative equilibria with one or both of the pendula pointing upwards. One shows that, except the one with both pendula pointing straight upwards, there are no relative equilibria with *both* pendula pointing upwards and that the relative equilibria with one of the pendula pointing upwards fill out the remaining intervals on the α-axis, namely $\alpha < -m$ and $-1 < \alpha < 0$. Dividing the equations (5.6) to eliminate μ and using a little algebra then establishes the following result:

Theorem 5.1 *All of the relative equilibria of the double spherical pendulum are given by the four equilibria with the two pendula vertical and the points on the graph of*

$$\lambda^2 = \frac{L^2 - r^2}{L^2 - \alpha^2} \qquad \text{where} \qquad L(\alpha) = \left(1 + \frac{\alpha}{m}\right)\left(\frac{\alpha}{1+\alpha}\right). \tag{59}$$

subject to the restrictions (5.7). Relative equilibria with both pendula pointing downwards correspond to solutions satisfying the inequalities (5.8) and the remaining intervals on the α-axis correspond to solutions with one of the pendula pointing upwards.

From (5.6) we get either μ or ξ in terms of α. In Figures 5.1 and 5.2 we show the relative equilibria for two sample values of the system parameters. Note that there is a bifurcation of relative equilibria for fixed m and increasing r, and that it occurs within the range of restricted values of α and λ . Also note that there can be two or three relative equilibria for a given set of system parameters.

Note that there is just one branch with both pendula pointing downwards, emanating from the straight down state ($\lambda = 0$) and satisfying $-m < \alpha < -1$. Because of its spatial shape, we call this the **cowboy branch**. See Figure 5.3

The bifurcation of relative equilibria that happens between Figures 5.1 and 5.2 does so along the curve in the (m, r) plane given by

$$r = \frac{2m}{1+m}$$

as is readily seen. See Figure 5.4. For instance, for $m = 2$ one gets $r = 4/3$, in agreement with the figures. There is a Hamiltonian transcritical bifurcation when (m, r) lies on this curve where we think of λ as the bifurcation parameter, and α as the state. We can also think of μ as the bifurcation parameter, as it can be

Figure 2: The graphs of λ^2 versus α for $r = 1, m = 2$ and of $\lambda^2 = r^2/\alpha^2$.

Figure 3: The graph of λ^2 versus α for $r = 1.35$ and $m = 2$ and of $\lambda^2 = r^2/\alpha^2$

Figure 4: The shape of two relative equilibria of the double spherical pendulum.

Figure 5: The switch over curve in the (r, m) plane. To the left of the curve, the bifurcation branch emanating from the straight down state $\lambda = 0$ for negative α bends to the right, while to the right of the curve, it bends to the left.

expressed as a function of λ and α using (5.6) and (5.9). However, in that case, the branches extend to infinity, so using λ is more convenient, as it brings them into a finite region.

6 Stability of Relative Equilibria

According to the energy momentum method of Simo, Lewis, and Marsden [1991], to carry out the stability analysis for relative equilibria of the double spherical pendulum, one must compute $\delta^2 V_\mu$ on the subspace orthogonal to the G_μ-orbit. To do this, it is useful to introduce coordinates adapted to the problem and to work in Lagrangian representation. Specifically, let $\mathbf{q}_1^\perp$ and $\mathbf{q}_2^\perp$ be given polar coordinates (r_1, θ_1) and (r_2, θ_2) respectively. Then $\varphi = \theta_2 - \theta_1$ represents an S^1-invariant coordinate, the angle between the two vertical planes formed by the pendula. In these terms, one computes from our earlier expressions that the angular momentum is

$$\begin{aligned} J &= (m_1 + m_2) r_1^2 \dot{\theta}_1 + m_2 r_2^2 \dot{\theta}_2 \\ &= + m_2 r_1 r_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos\varphi + m_2 (r_1 \dot{r}_2 - r_2 \dot{r}_1) \sin\varphi \end{aligned} \tag{60}$$

and the Lagrangian is

$$\begin{aligned} L = \; & \frac{1}{2} m_1 (\dot{r}_1^2 + r_1^2 \dot{\theta}_1^2) + \frac{1}{2} m_2 \Big\{ \dot{r}_1^2 + r_1^2 \dot{\theta}_1^2 + \dot{r}_2^2 + r_2^2 \dot{\theta}_2^2 \\ & + 2(\dot{r}_1 \dot{r}_2 + r_1 r_2 \dot{\theta}_1 \dot{\theta}_2) \cos\varphi + 2(r_1 \dot{r}_2 \dot{\theta}_1 - r_2 \dot{r}_1 \dot{\theta}_2) \sin\varphi \Big\} \\ & + \frac{1}{2} m_1 \frac{r_1^2 \dot{r}_1^2}{l_1^2 - r_1^2} + \frac{1}{2} m_2 \left(\frac{r_1 \dot{r}_1}{\sqrt{l_1^2 - r_1^2}} + \frac{r_2 \dot{r}_2}{\sqrt{l_2^2 - r_2^2}} \right)^2 \\ & - m_1 g \sqrt{l_1^2 - r_1^2} - m_2 g \left(\sqrt{l_1^2 - r_1^2} + \sqrt{l_2^2 - r_2^2} \right). \end{aligned} \tag{61}$$

One also has, from (5.1),

$$\begin{aligned} V_\mu = \; & -m_1 g \sqrt{l_1^2 - r_1^2} - m_2 g \left(\sqrt{l_1^2 - r_1^2} + \sqrt{l_2^2 - r_2^2} \right) \\ & + \frac{1}{2} \frac{\mu^2}{m_1 r_1^2 + m_2 (r_1^2 + r_2^2 + 2 r_1 r_2 \cos\varphi)}. \end{aligned} \tag{62}$$

Notice that V_μ depends on the angles θ_1 and θ_2 only through $\varphi = \theta_2 - \theta_1$, as it should by S^1-invariance. Next one calculates the second variation at one of the relative equilibria found in §5. If we calculate it as a 3×3 matrix in the variables r_1, r_2, φ, then one checks that we will automatically be in a space orthogonal to the G_μ-orbits. One finds, after some computation, that

$$\delta^2 V_\mu = \begin{bmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & e \end{bmatrix} \tag{63}$$

where

$$
\begin{aligned}
a &= \frac{\mu^2(3(m+\alpha)^2-\alpha^2(m-1))}{\lambda^4 l_1^2 m_2(m+\alpha^2+2\alpha)^3}+\frac{g m_2 m}{l_1(1+\lambda^2)^{3/2}} \\
b &= (\operatorname{sign}\alpha)\frac{\mu^2}{\lambda^4 l_1^4 m_2}\frac{3(m+\alpha^2+2\alpha)+4\alpha(m-1)}{(m+\alpha^2+2\alpha)^3} \\
d &= \frac{\mu^2}{\lambda^4 l_1^4 m_2}\frac{3(\alpha+1)^2+1-m}{(m+\alpha^2+2\alpha)^3}+\frac{m_2 g}{l_1}\frac{r^2}{(r^2-\lambda^2\alpha^2)^{3/2}} \\
e &= \frac{\mu^2}{\lambda^2 l_1^2 m_2}\frac{\alpha}{(m+\alpha^2+2\alpha)^2}.
\end{aligned}
$$

Notice the zeros in (**??**); they are in fact a result of discrete symmetry, as in Harnad, Hurtubise, and Marsden [1992]. Without the help of these zeros (for example, if the calculation is done in arbitrary coordinates), the expression for $\delta^2 V_\mu$ might be intractible.

Based on this calculation one finds:

Proposition 6.1 *The signature of $\delta^2 V_\mu$ along the "straight out" branch of the double spherical pendulum (with $\alpha > 0$) is $(+,+,+)$ and so is (linearly and nonlinearly) stable. The signature along the cowboy branch is $(-,-,+)$ and along the remaining branches is $(-,+,+)$.*

The stability along the cowboy branch requires further analysis that we shall indicate below. The remaining branches are linearly unstable since the index along them is odd; cf. Oh [1987].

To get instability and bifurcation information along the cowboy branch, one needs to linearize the reduced equations and compute the corresponding eigenvalues. There are (at least) three methodologies that can be used for computing the reduced linearized equations:

i Compute the Euler-Lagrange equations from (**??**), drop them to $\mathbf{J}^{-1}(\mu)/G_\mu$ and linearize the resulting equations.

ii Obtain the linearized reduced equations using the normal (block diagonal) form of $\delta^2 H_\xi$ and that of the associated symplectic structure given in Hamiltonian form by Simo, Lewis, and Marsden [1991] or in Lagrangian form by Lewis [1991].

iii Perform Lagrangian reduction, (either by our intrinsic approach, or equivalently by the classical Routh procedure using the variables $(r_1, r_2, \theta_1, \theta_2)$ in which a variable complementary to the reduced variable φ, such as $\theta_1+\theta_2$, is cyclic, to obtain the Lagrangian structure of the reduced system and linearize it at a relative equilibrium.

For the double spherical pendulum, perhaps the first method is the quickest to get the answer, but of course the other methods provide insight and information about the structure of the system obtained.

7 The Reduced Linearized Equations for the Double Spherical Pendulum

The linearized system obtained by using one of the procedures above has the following standard form expected for abelian reduction:

$$M\ddot{q} + S\dot{q} + \Lambda q = 0. \tag{64}$$

In our case $q = (r_1, r_2, \varphi)$ and Λ is the matrix (**??**) given above. The mass matrix M is

$$M = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{12} & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix}$$

where

$$\begin{aligned} m_{11} &= \frac{m_1 + m_2}{1 - \lambda^2}, \quad m_{12} = (\operatorname{sign}\alpha) m_2 \left(1 + \frac{\alpha\lambda^2}{\sqrt{1-\lambda^2}\sqrt{r^2 - \alpha^2\lambda^2}}\right) \\ m_{22} &= m_2 \frac{r^2}{r^2 - \lambda^2\alpha^2}, \quad m_{33} = m_2 l_1^2 \lambda^2 (m-1) \frac{\alpha^2}{m + \alpha^2 + 2\alpha} \end{aligned}$$

and the gyroscopic matrix S (the magnetic term) is

$$S = \begin{bmatrix} 0 & 0 & s_{13} \\ 0 & 0 & s_{23} \\ -s_{13} & -s_{23} & 0 \end{bmatrix}$$

where

$$\begin{aligned} s_{13} &= \frac{\mu}{\lambda l_1} \frac{2\alpha^2(m-1)}{(m + \alpha^2 + 2\alpha)^2} \quad \text{and} \\ s_{23} &= -(\operatorname{sign}\alpha) \frac{\mu}{\lambda l_1} \frac{2\alpha(m-1)}{(m + \alpha^2 + 2\alpha)^2}. \end{aligned}$$

8 Bifurcations in the Double Spherical Pendulum

Above, we wrote the equations for the linearized solutions of the double spherical pendulum at a relative equilibrium in the form

$$M\ddot{q} + S\dot{q} + \Lambda q = 0 \tag{65}$$

for certain 3×3 matrices M, S and Λ. These equations have the Hamiltonian form $\dot{F} = \{F, H\}$ where $p = M\dot{q}$,

$$H = \frac{1}{2} p M^{-1} p + \frac{1}{2} q \Lambda q \tag{66}$$

and

$$\{F, K\} = \frac{\partial F}{\partial q^i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q^i}\frac{\partial F}{\partial p_i} - S_{ij}\frac{\partial F}{\partial p_i}\frac{\partial K}{\partial p_j} \tag{67}$$

i.e.,

$$\left.\begin{aligned} \dot{q} &= M^{-1}p \\ \dot{p} &= -S\dot{q} - \Lambda q = -SM^{-1}p - \Lambda q. \end{aligned}\right\} \tag{68}$$

The following is a standard useful observation:

Proposition 8.1 *The eigenvalues λ of the linear system (8.1) are given by the roots of*

$$\det[\lambda^2 M + \lambda S + \Lambda] = 0 \tag{69}$$

Proof Let (u, v) be an eigenvector of (8.4) with eigenvalue λ; then

$$M^{-1}v = \lambda u \quad \text{and} \quad -SM^{-1}v - \Lambda u = \lambda v$$

i.e., $-S\lambda u - \Lambda u = \lambda^2 M u$, so u is an eigenvector of $\lambda^2 M + \lambda S + \Lambda$. ■

For the double spherical pendulum, we call the eigenvalue γ (since λ is already used for something else in this example) and note that the polynomial

$$p(\gamma) = \det[\gamma^2 M + \gamma S + \Lambda] \tag{70}$$

is cubic in γ^2, as it must be, consistent with the symmetry of the spectrum of Hamiltonian systems. This polynomial can be readily analyzed for specific system parameter values. In particular, for $r = 1$ and $m = 2$, *one finds a Hamiltonian Hopf bifurcation along the cowboy branch as we go up the branch in Figure 5.1 with increasing λ starting at $\alpha = -\sqrt{2}$.* Since this bifurcation occurs in the reduced space, it amounts to a bifurcation from a periodic orbit in the original system (so periodic orbits that branch out give tori, etc.)

Notice that along the cowboy branch, in the region below the Hopf point, the relative equilibrium is energetically unstable, or formally unstable in the sense that the second variation of the effective Hamiltonian (or amended potential) is indefinite (it has index 2, while the spectrum of the linearized equations lies on the imaginary axis. One can guess that this means that the solution is very slowly unstable due to Arnold diffusion, but this is presumably a very delicate phenomenon. However, if one adds dissipation in the sense of friction in the internal variable φ, then the results of Bloch, Krishnaprasad, Marsden, and Ratiu [1991] show that the system becomes *linearly unstable*. Interestingly, this is consistent with experiments (Baillieul [1991] and Baillieul and Levi [1987, 1991]).

Another interesting feature is the fact that for certain system parameters, the Hamiltonian Hopf point can converge to the straight down *singular*(!) state with $\lambda = 0 = \mu$, or equivalently, $\mu = 0$. In this limit, the characteristic polynomial (8.6), and the linearized system (8.1) become degenerate. We also observe from (5.6) and (5.9) that in this limit, $\mu = O(\lambda^2)$. This degeneracy is due to the rotational symmetry of the limiting straight down state. To study this limit, one can blow up the singularity by rescaling the Lagrangian $\mathcal{L}_2(\lambda, r_1, r_2, \varphi)$ of the linearized equations at a relative equilibrium, as follows; let $r_1 = \lambda\mathfrak{r}_1, r_2 = \lambda\mathfrak{r}_2$ and set

$$\mathcal{L}_2^\lambda(\lambda, \mathfrak{r}_1, \mathfrak{r}_2, \varphi, \dot{\mathfrak{r}}_1, \dot{\mathfrak{r}}_2, \dot{\phi}) = \frac{1}{\lambda^2}\mathcal{L}_2(\lambda, \lambda\mathfrak{r}_1, \lambda\mathfrak{r}_2, \varphi, \lambda\dot{\mathfrak{r}}_1, \dot{\mathfrak{r}}_2, \dot{\phi})$$

or, equivalently in terms of μ,

$$\mathcal{L}_2^{\mu}(\mu, \mathfrak{r}_1, \mathfrak{r}_2, \varphi, \dot{\mathfrak{r}}_1, \dot{\mathfrak{r}}_2, \dot{\phi}) = \frac{1}{\mu}\mathcal{L}_2(\mu, \sqrt{\mu}\,\mathfrak{r}_1, \sqrt{\mu}\,\mathfrak{r}_2, \varphi, \sqrt{\mu}\,\dot{\mathfrak{r}}_1, \sqrt{\mu}\,\dot{\mathfrak{r}}_2, \dot{\phi})$$

In these new variables, the linearized equations, and the characteristic polynomial has a regular limit as $\lambda \to 0$. This limit can be studied over the (m, r) parameter plane. Corresponding to the cowboy branch, one finds both splitting (Hamiltonian Hopf) cases and passing cases of $1:1$ resonances of purely imaginary eigenvalues γ, when one of the parameters m and r is varied. In fact, a numerical study suggests that there is a whole curve of each of these resonance types in the (m, r) plane; see Dellnitz, Marsden, Melbourne, and Scheurle [1992]. The curve corresponding to the splitting case divides up the (m, r) plane into a region where, all along the cowboy branch as λ increases, we have linear instability, and a region where, along the cowboy branch, a Hamiltonian Hopf bifurcation occurs and the eigenvalues move from on the imaginary axis to off of it, as λ increases. The curve coresponding to the passing case lies in the region where the eigenvalues, for small λ, stay on the imaginary axis. We hope that a modification of the theory of Dellnitz, Melbourne and Marsden [1992] with the incorporation of antisymplectic symmetries (like reversibility), as well as symplectic ones, will be relevant for explaining this phenomenon, both for the symmetric straight down analysis, and the nearby solutions with μ close to zero. If successful, this analysis would also be relevant for many other situations involving singular reduction.

The passing cases in the straight down state noted above are analogues of the passing cases one sees in steady state bifurcation of Hamiltonian systems with symmetry, as in Golubitsky and Stewart [1987] and Lewis, Marsden, and Ratiu [1987]. One can of course expect interactions between steady state and resonance bifurcations in more complex systems, and this would be an interesting topic for future work.

In this paper, we dealt with the singularity at the straight down state in the zero angular momentum level set by directly blowing up the singularity; as we have mentioned in the introduction, it would be of interest to find out if this is related to the general work on singularities in phase spaces of Arms, Marsden, and Moncrief [1981], Arms, Cushman and Gotay [1991], Sjamaar, R. and E. Lerman [1991], and Cushman and Sjamaar [1991], for example.

We expect that the methods of this paper can be applied to a number of other situations as well. For example, the work of Lewis and Simo [1990] on pseudo-rigid bodies would be of interest to pursue, especially in connection with Hamiltonian bifurcations at symmetric relative equilibria.

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