# The Rotor and the Pendulum

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*In Honor of J.-M. Souriau*

**Abstract**

We show that Euler's equations for a free rigid body, and for a rigid body with a controlled feedback torque each reduce to the classical simple pendulum equation under an explicit cylindrical coordinate change of variables. These examples illustrate several ideas in Hamiltonian mechanics: Lie-Poisson reduction, cotangent bundle reduction, singular Lie-Poisson maps, deformations of Lie algebras, brackets on $\mathbb{R}^3$, simplifications obtained by utilizing the representation-dependence of Lie-Poisson reduction, and controlling instability by inducing global bifurcations among a set of equilibria using a control parameter.

## 1. Introduction

Even though the free rigid body is a classical and well understood system, some new and interesting features are still being uncovered. Notable amongst these is Montgomery's [1990] formula for the change in the geometric phase for the attitude of the body when the body angular momentum vector executes one period of its motion. In this paper we present a number of other results that also seem to be new. Perhaps the most interesting of these is the fact that the rigid body system in body angular momentum space (identified with $\mathbb{R}^3$) is filled with invariant elliptical cylinders on each of which the dynamics is, in elliptical cylindrical coordinates, *exactly* the dynamics of a standard simple pendulum.

Related to this is the variety of ways the rigid body equations can be written in Hamiltonian form as a Lie-Poisson system associated to a Lie algebra structure on $\mathbb{R}^3$. The standard choice is to use the Lie algebra $SO(3)$, but one can also use the Euclidean Lie algebra $SE(2)$ or the Lie algebra $SO(2,1)$. The deformation through these algebras, discussed abstractly in Weinstein [1983], is achieved explicitly in the rigid body simply by defining new Hamiltonians and Casimirs using linear combinations of the standard ones.

We make a similar analysis of the rigid body with the stabilizing torque feedback law introduced by Bloch and Marsden [1990] (see also Bloch, Krishnaprasad, Marsden, and Sanchez de Alvarez [1990]). In particular, this sort of analysis enables one to see how the stabilization is achieved from a geometric viewpoint. Some interesting global bifurcations accompany this stabilization. Combined with the ideas about geometric phases, this stabilization process may be useful for attitude control of rigid bodies.

## 2. The Free Rigid Body

Euler's classical equations for the rotational dynamics of a freely spinning rigid body are

$$\begin{aligned} I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3 \\ I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1 \\ I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2 \end{aligned} \qquad (EE\Omega)$$

where $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the *body angular velocity vector*, an overdot denotes time derivative, and $I_1 < I_2 < I_3$ denote the principal moments of inertia of the body. We can rewrite $(EE\Omega)$ as

$$\frac{d}{dt}\begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{I_2} - \frac{1}{I_3}\right)\Pi_2\Pi_3 \\ -\left(\frac{1}{I_1} - \frac{1}{I_3}\right)\Pi_1\Pi_3 \\ \left(\frac{1}{I_1} - \frac{1}{I_2}\right)\Pi_1\Pi_2 \end{bmatrix}, \qquad (EE)$$

where $(\Pi_1, \Pi_2, \Pi_3) = \vec{\Pi}$ denotes the *body angular momentum vector* given by $\Pi_i = I_i\Omega_i$, $i = 1, 2, 3$. Euler's equations are expressible in vector form as

$$\frac{d}{dt}\vec{\Pi} = \nabla H \times \nabla L,$$

where $H$ is the energy,

$$H = \frac{\Pi_1^2}{2I_1} + \frac{\Pi_2^2}{2I_2} + \frac{\Pi_3^2}{2I_3},$$

$$\nabla H = \left(\frac{\partial H}{\partial \Pi_1}, \frac{\partial H}{\partial \Pi_2}, \frac{\partial H}{\partial \Pi_3}\right) = \left(\frac{\Pi_1}{I_1}, \frac{\Pi_2}{I_2}, \frac{\Pi_3}{I_3}\right)$$

is the gradient of $H$ and $L$ is the square of the body angular momentum,

$$L = \frac{1}{2}\left(\Pi_1^2 + \Pi_2^2 + \Pi_3^2\right).$$

Hence, both $H$ and $L$ are conserved, and the rigid body motion itself takes place along the intersections of the level surfaces of the energy (ellipsoids) and the angular momentum (spheres) in $\mathbb{R}^3$. The centers of the energy ellipsoids and the angular momentum spheres coincide. This, along with the $(\mathbb{Z}_2)^3$ symmetry of the energy ellipsoids, implies that the two sets of level surfaces in $\mathbb{R}^3$ develop collinear gradients (*e.g.*, tangencies) at pairs of points which are diametrically opposite on an angular momentum sphere. At these points, collinearity of the gradients of $H$ and $L$ implies stationary rotations, *i.e.*, equilibria.

Euler's equations for the rigid body may also be written as

$$\frac{d}{dt}\vec{\Pi} = \nabla K \times \nabla N, \qquad (EE')$$

where $K$ and $N$ are linear combinations of energy and angular momentum of the form

$$\begin{pmatrix} K \\ N \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} H \\ L \end{pmatrix}, \qquad (SL2R)$$

with real constants $a, b, c$, and $d$ satisfying the unit determinant condition $ad-bc = 1$. Thus, the equations of rigid body motion are unchanged (so the trajectories of the motion in $\mathbb{R}^3$ remain unchanged) when the energy $H$ and angular momentum $L$ are replaced by the $SL(2,\mathbb{R})$-linear combinations $K$ and $N$. Notice that $K$ will be a quadratic form and that it can occur with *any signature except* $(0,0,0)$ *and, if the moments of inertia are distinct,* $(*,0,0)$ by suitably choosing $a$ and $b$.

For example, one may choose to eliminate one of the terms in each of $H$ and $L$, by choosing the linear combination given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{-1}{I_3} \\ -c & \frac{c}{I_1} \end{bmatrix}, \quad \text{with } c = \frac{1}{\left(\frac{1}{I_1} - \frac{1}{I_3}\right)},$$

so that
$$K = \frac{1}{2}\left(\frac{1}{I_1} - \frac{1}{I_3}\right)\Pi_1^2 + \frac{1}{2}\left(\frac{1}{I_2} - \frac{1}{I_3}\right)\Pi_2^2,$$

and
$$N = \frac{c}{2}\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\Pi_2^2 + \frac{1}{2}\Pi_3^2.$$

With this choice, the orbits for Euler's equations for rigid body dynamics are realized as motion along the intersections of two, orthogonally-oriented, *elliptic cylinders*, one elliptic cylinder a level surface of $K$, with its translation axis along $\Pi_3$ (where $K = 0$), and the other a level surface of $N$, with its translation axis along $\Pi_1$ (where $N = 0$).

For a general choice of $K$ and $N$, equilibria occur at points where the gradients of $K$ and $N$ are collinear. This can occur at points where the level sets are tangent (and the gradients both are non-zero), or at points where one of the gradients vanishes. In the elliptic cylinder case above, these two cases are points where the elliptic cylinders are tangent, and at points where the axis of one cylinder punctures normally through the surface of the other. The elliptic cylinders are tangent at one $\mathbb{Z}_2$-symmetric pair of points along the $\Pi_2$ axis, and the elliptic cylinders have normal axial punctures at two other $\mathbb{Z}_2$-symmetric pairs of points along the $\Pi_2$ and $\Pi_3$ axes.

The stability of the equilibria at the points of tangency is determined geometrically by whether the curvatures of the two surfaces have opposite sign (stable), or the same sign (unstable). When the two surfaces have curvatures of opposite sign near the tangent point, their intersections describe nested ellipses on each surface – this is the stable case. When the two surfaces have curvatures of the same sign near the tangent point, their intersections describe hyperbolas on each surface—this is the unstable case. In the elliptic cylinder example, the curvatures of the elliptic cylinders have the same sign at the points of tangency, so the pair of equilibria at the tangent points along the $\Pi_2$ axis are unstable. Equilibria at puncture points are stable; so, the two pairs of diametrically opposite equilibria at the puncture points along the $\Pi_2$ and $\Pi_3$ axes are stable. This geometric picture thus recovers the known equilibria and the stability properties of these equilibria for the rigid body, in a new parameterization in terms of intersections of two, orthogonally-oriented, elliptic cylinders in $\mathbb{R}^3$. The situation is shown in Figure 1.

Let us pursue the elliptic cylinders example further. We now change variables in the rigid body equations within a level surface of $K$. To simplify notation, we first define the three positive constants $k_i^2$, $i = 1, 2, 3$, by setting

$$K = \frac{\Pi_1^2}{2k_1^2} + \frac{\Pi_2^2}{2k_2^2} \quad \text{and} \quad N = \frac{\Pi_2^2}{2k_3^2} + \frac{1}{2}\Pi_3^2$$

and referring term by term to the formulae

$$K = \frac{1}{2}\left(\frac{1}{I_1} - \frac{1}{I_3}\right)\Pi_1^2 + \frac{1}{2}\left(\frac{1}{I_2} - \frac{1}{I_3}\right)\Pi_2^2$$

and
$$N = \frac{c}{2}\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\Pi_2^2 + \frac{1}{2}\Pi_3^2.$$

On the surface $K$ = constant, and setting $r = \sqrt{2K}$ = constant, define new variables $\theta$ and $p$ by

$$\Pi_1 = k_1 r\cos\theta, \quad \Pi_2 = k_2 r\sin\theta, \quad \Pi_3 = p.$$

In terms of these variables, the constants of the motion become

$$K = \frac{1}{2}r^2 \quad \text{and} \quad N = \frac{1}{2}p^2 + \left(\frac{k_2^2}{2k_3^2}r^2\right)\sin^2\theta.$$

Leave 23 picas of vertical space for Fig. 1

**Figure 1.** Intersection of the elliptic cylinders—level surfaces of $K$ and $N$; orbits change from "bound" to "running" as the level surface of $N$ passes through the critical value at which homoclinic orbits occur.

As we shall show in section 3, using a Poisson structure relevant to the equations of motion in the form $\frac{d}{dt}\vec{\Pi} = \nabla K \times \nabla N$, the variables $\theta$ and $p$ are, up to a scale factor, canonically conjugate, *i.e.*, the Poisson bracket of two functions of $\theta$ and $p$ are given in standard canonical form (up to a scale factor) as follows:

$$\{F, G\}_{\text{EllipCyl}} = \frac{1}{k_1 k_2}\left(\frac{\partial F}{\partial p}\frac{\partial G}{\partial \theta} - \frac{\partial F}{\partial \theta}\frac{\partial G}{\partial p}\right).$$

In particular,

$$\{p, \theta\}_{\text{EllipCyl}} = \frac{1}{k_1 k_2}.$$

The quantity $N$ is the Hamiltonian in these variables—note that $N$ has the form of kinetic plus potential energy—and the equations of motion express themselves in Hamiltonian form in terms of the canonical Poisson bracket. Namely,

$$\frac{d}{dt}\theta = \{N, \theta\}_{\text{EllipCyl}} = \frac{1}{k_1 k_2}\frac{\partial N}{\partial p} = \frac{1}{k_1 k_2}p,$$

$$\frac{d}{dt}p = \{N,p\}_{\rm EllipCyl} = \frac{-1}{k_1k_2}\frac{\partial N}{\partial\theta} = \frac{-1}{k_1k_2}\frac{k_2^2}{k_3^2}r^2\sin\theta\cos\theta.$$

Combining these equations of motion gives

$$\frac{d^2}{dt^2}\theta = \frac{-r^2}{2k_1^2k_3^2}\sin 2\theta,$$

or, in terms of the original rigid body parameters,

$$\frac{d^2}{dt^2}\theta = -K\left(\frac{1}{I_1}-\frac{1}{I_2}\right)\left(\frac{1}{I_1}-\frac{1}{I_3}\right)\sin 2\theta.$$

Thus, we have proved that

**Theorem 1.** *Rigid body motion reduces to pendulum motion on level surfaces of K.*

Another way of saying this is as follows: regard rigid body angular momentum space as the union of the level surfaces of $K$, so the dynamics of the rigid body is recovered by looking at the dynamics on each of these level surfaces. On each level surface, the dynamics is equivalent to a simple pendulum. In this sense, we have proved that:

**Corollary.** *The dynamics of a rigid body in three dimensional body angular momentum space is a union of two dimensional simple pendula phase portraits.*

**Remarks.** By restricting to a nonzero level surface of $K$, the pair of rigid body equilibria along the $\Pi_3$ axis are excluded. (This pair of equilibria can be included by permuting the indices of the moments of inertia.) The other two pairs of equilibria, along the $\Pi_1$ and $\Pi_2$ axes, lie in the $p = 0$ plane at $\theta = 0$, $\pi/2$, $\pi$, and $3\pi/2$. Since $K$ is positive, the stability of each equilibrium point is determined by the relative sizes of the principle moments of inertia, which affect the overall sign of the right-hand-side of the pendulum equation. The well-known results about stability of equilibrium rotations along the least and greatest principle axes, and instability around the intermediate axis, are immediately recovered from this overall sign, combined with the stability properties of the pendulum equilibria. For $K > 0$ and $I_1 < I_2 < I_3$, this overall sign is negative, so the equilibria at $\theta = 0$ and $\pi$ (along the $\Pi_1$ axis) are stable, while those at $\theta = \pi/2$ and $3\pi/2$ (along the $\Pi_2$ axis) are unstable. The factor of 2 in the argument of the sine in the pendulum equation is explained by the $\mathbb{Z}_2$ symmetry of the level surfaces of $K$ (or, just as well, by their invariance under $\theta \mapsto \theta + \pi$). Under this discrete symmetry operation, the equilibria at $\theta = 0$ and $\pi/2$ exchange with their counterparts at $\theta = \pi$ and $3\pi/2$, respectively, while the elliptical level surface of $K$ is left invariant. By construction, the Hamiltonian $N$ in the reduced variables $\theta$ and $p$ is also invariant under this discrete symmetry.

### 3. Alternative Poisson Structures for the Rigid Body

The standard ***rigid body Poisson bracket*** on two functions $F_1$ and $F_2$ of $\vec{\Pi}$ is given by the *minus Lie Poisson bracket* for $\mathfrak{so}(3)^*$:

$$\{F_1, F_2\} = -\vec{\Pi} \cdot (\nabla F_1 \times \nabla F_2). \qquad (RBB)$$

In case Euler's equations are rewritten as

$$\frac{d}{dt}\vec{\Pi} = \nabla K \times \nabla N, \qquad (EE')$$

where $K$ and $N$ are given as above by an $SL(2,\mathbb{R})$ matrix

$$\begin{pmatrix} K \\ N \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} H \\ L \end{pmatrix}, \qquad (SL2R)$$

one checks that the equations $(EE')$ are Hamiltonian with energy $N$ and the Poisson bracket

$$\{F_1, F_2\} = -\nabla K \cdot (\nabla F_1 \times \nabla F_2). \qquad (PBK)$$

One verifies also that the bracket $(PBK)$ -essentially a "Nambu" bracket (see Nambu [1973]), indeed defines a Poisson structure on $\mathbb{R}^3$. Clearly the function $K$ is a Casimir for this bracket; *i.e.*, $\{K, F\} = 0$ for any function $F$. One can now directly verify the formula $\{F, G\}_{\mathrm{EllipCyl}}$ for the Poisson bracket on level sets of the function $K$ in the elliptic cylinder case by a straightforward calculation.

We shall see shortly *that the bracket $(PBK)$ is in fact a Lie Poisson bracket.* Let $\mathbf{K}$ be the symmetric $3 \times 3$ matrix associated with the quadratic form $K$; *i.e.*, $K(\mathbf{v}) = \frac{1}{2}\mathbf{v}^t \cdot \mathbf{K} \cdot \mathbf{v}$, where $^t$ denotes transpose. Thus, the gradient of $K$ is $\nabla K(\vec{\Pi}) \cdot \mathbf{v} = \mathbf{d}K(\vec{\Pi}) \cdot \mathbf{v} = \vec{\Pi} \cdot \mathbf{K}\mathbf{v}$, and so $(PBK)$ may be written

$$\{F_1, F_2\}_K(\vec{\Pi}) = -\vec{\Pi} \cdot \mathbf{K}(\nabla F_1 \times \nabla F_2). \qquad (PBK')$$

Note that these formulas are valid even if the matrix $\mathbf{K}$ is singular.

Define the following bracket on $\mathbb{R}^3$:

$$[\mathbf{u}, \mathbf{v}]_K = \mathbf{K} \cdot (\mathbf{u} \times \mathbf{v}). \qquad (LAK)$$

This defines a Lie algebra structure on $\mathbb{R}^3$, as is straightforward to verify.

For $\mathbf{K}$ nonsingular, the Lie algebra structure $(LAK)$ can be explicitly identified with that of the orthogonal group of $\mathbf{K}$ as follows. The orthogonal group $O(K)$ is the group of linear transformations of $\mathbb{R}^3$ that leave the quadratic form $K$ invariant. It consists of the set of $3 \times 3$ matrices $\mathbf{A}$ that satisfy the condition $\mathbf{A}^t\mathbf{K}\mathbf{A} = \mathbf{K}$. The corresponding Lie algebra $\mathfrak{o}(K)$ is the Lie algebra of $3\times3$ matrices S that satisfy the condition $\mathbf{S}^t\mathbf{K} + \mathbf{K}\mathbf{S} = \mathbf{0}$, *i.e.*, $\mathbf{KS}$ is a skew matrix, and with the standard commutator bracket of matrices. Since $\mathbf{K}$ is nonsingular, the equation

$$\mathrm{S}\mathbf{v} = \mathbf{s} \times \mathrm{K}\mathbf{v} \qquad (LAI)$$

defines an isomorphism between $\mathrm{S} \in \mathfrak{o}(\mathrm{K})$ and $\mathbf{s} \in \mathbb{R}^3$. The following is a straightforward verification:

**Lemma 1.**

1. *If* $\mathbf{K}$ *is nonsingular, the isomorphism* $(LAI)$ *is a Lie algebra isomorphism between the Lie algebra* $\mathfrak{o}(K)$ *with the commutator bracket and* $\mathbb{R}^3$ *with the bracket* $(LAK)$*; i.e.,* $[\mathrm{S}_1, \mathrm{S}_2] = \mathrm{K}(\mathbf{s}_1 \times \mathbf{s}_2)$.
2. *If* $\mathbf{K}$ *is singular, then the Lie algebra structure is that of the euclidean Lie algebra* $\mathfrak{se}(2)$ *if* $\mathrm{K}$ *has signature* $(+,+,0)$*, and that of the Heisenberg algebra if* $\mathbf{K}$ *has signature* $(+,0,0)$.

**Lemma 2.** *The Poisson bracket* $\{F_1, F_2\}_K(\vec{\Pi}) = -\vec{\Pi}\cdot\mathbf{K}(\nabla F_1 \times \nabla F_2)$ *on* $\mathbb{R}^3$ *is the minus Lie Poisson bracket for the Lie algebra* $\mathbb{R}^3_K$*, defined to be* $\mathbb{R}^3$ *with the Lie algebra bracket* $[\mathbf{u}, \mathbf{v}]_K = \mathbf{K}\cdot(\mathbf{u} \times \mathbf{v})$.

To see this, recall that the general minus Lie-Poisson bracket formula (see, for example, Marsden and Weinstein [1983]) is given on $\mathfrak{g}^*$, the dual of a general Lie algebra $\mathfrak{g}$ by

$$\{F_1, F_2\} = -\left\langle \mu, \left[\frac{\partial F_1}{\partial \mu}, \frac{\partial F_2}{\partial \mu}\right]\right\rangle.$$

In our case, the Lie algebra is $\mathbb{R}^3_K$ and the dual space is identified with $\mathbb{R}^3$ via the standard dot product. The functional derivative $\frac{\partial F}{\partial \mu}$ is thus the ordinary gradient, and $\mu$ gets replaced by $\vec{\Pi}$, so Lemma 2 follows.

We can gain insight into why $K$ is a Casimir for the bracket $\{F_1, F_2\}_K$ by noting that the coadjoint action of the Lie algebra $\mathbb{R}^3_K$ on the dual space is given by taking the dual of the adjoint action:

$$\langle ad^*_{\mathbf{s}}\vec{\Pi}, \mathbf{r}\rangle = \langle \vec{\Pi}, ad_{\mathbf{s}}\mathbf{r}\rangle = \langle \vec{\Pi}, \mathrm{K}\cdot(\mathbf{s} \times \mathbf{r})\rangle = \langle \mathrm{K}\vec{\Pi} \times \mathbf{s}, \mathbf{r}\rangle$$

and so
$$ad^*_{\mathbf{s}}\vec{\Pi} = (\mathrm{K}\vec{\Pi}) \times \mathbf{s}.$$

Vectors of this form are tangent to the coadjoint orbit through $\vec{\Pi}$. Notice that the differential of $K$ vanishes on vectors of this form; so $K$ is a Casimir and therefore $K$ is constant on (connected components of) orbits. (This corresponds to the fact, observed earlier, that $K$ is a Casimir for the bracket $\{F,H\}_K$.) In the nonsingular case, *the coadjoint orbit through* $\vec{\Pi}$ *is, up to connected components, exactly the level set of* $K$ *through* $\vec{\Pi}$. The orbit is given algebraically as follows: write $\vec{\Pi} = \mathbf{K}\mathbf{v} \times \mathbf{w}$; then the orbit through $\vec{\Pi}$ consists of vectors of the form $Ad^*_{\mathbf{A}}\vec{\Pi} = \mathbf{A}^t\mathbf{K}\mathbf{v} \times \mathbf{A}^t\mathbf{w}$ as $\mathbf{A}$ ranges over $O(K)$. In the case of the euclidean Lie algebra, which corresponds to the case of the elliptic cylinders, the regular coadjoint orbits are given by cotangent bundles to circles, with the canonical symplectic structure (up to a factor, depending on the radius of the circle). Again, we see by a dimension count that the orbits are the level sets of $K$. We have proved the following:

**Lemma 3.** *The (connected components of the) coadjoint orbits for* $\mathbb{R}^3_K \cong \mathbb{R}^3$ *are the level sets of* $K$ *if* $\mathbf{K}$ *is nonsingular, or if* $\mathbf{K}$ *has signature* $(+,+,0)$. *Tangent vectors to the coadjoint orbit through* $\vec{\Pi}$ *are given by vectors in* $\mathbb{R}^3$ *of the form* $ad^*_{\mathbf{s}}\vec{\Pi} = (\mathbf{K}\vec{\Pi}) \times \mathbf{s}$ *and the symplectic structure on the orbit is given by* $\Omega\big((\mathbf{K}\vec{\Pi}) \times \mathbf{s}_1, (\mathbf{K}\vec{\Pi}) \times \mathbf{s}_2\big) = -\mathbf{K} \cdot (\mathbf{s}_1 \times \mathbf{s}_2)$.

The rigid body can, correspondingly, be regarded as a left invariant system on the group $O(K)$ or $SE(2)$. The special case of $SE(2)$ is the one in which the orbits are cotangent bundles. The fact that one gets a cotangent bundle in this situation is a special case of the cotangent bundle reduction theorem using the semidirect product reduction theorem; see Marsden, Ratiu, and Weinstein [1984]. For the Euclidean group it says that the coadjoint orbits of the Euclidean group of the plane are given by reducing the cotangent bundle of the rotation group of the plane by the trivial group, giving the cotangent bundle of a circle with its canonical symplectic structure up to a factor. This is the abstract explanation of why, in the elliptic cylinder case above, the variables $\theta$ and $p$ were, up to a factor, canonically conjugate. This general theory is also consistent with the fact that the Hamiltonian $N$ is of the form kinetic plus potential energy. In fact, in the cotangent bundle reduction theorem, one always gets a Hamiltonian of this form, with the potential being changed by the addition of an amendment to give the *amended potential.* In the case of the pendulum equation, the original Hamiltonian is purely kinetic energy and so the potential term in $N$, namely $\left(\frac{k^2}{2k_3^2} r^2\right)\sin^2\theta$, is entirely amendment. See Abraham and Marsden [1978] for the general theory.

We summarize some of our findings as follows:

**Theorem 2.** *Euler's equations for a free rigid body are Lie Poisson with the Hamiltonian* $N$ *for the Lie algebra* $\mathbb{R}^3_K$ *where the underlying Lie group is the orthogonal group of* $K$ *if the quadratic form is nondegenerate, and is the Euclidean group of the plane if* $K$ *has signature* $(+,+,0)$. *In particular, all the groups* $SO(3)$, $SO(2,1)$, *and* $SE(2)$ *occur as the parameters* $a$, $b$, $c$, *and* $d$ *are varied. (If the body is a Lagrange body (with two moments of inertia equal), then the Heisenberg group occurs as well.)*

**Remark.** The same richness of Hamiltonian structure was found in the Maxwell-Bloch system in David and Holm [1990]. As in the case of the rigid body, the $\mathbb{R}^3$ motion for the Maxwell-Bloch system may also be realized as motion along the intersections of two orthogonally oriented cylinders. However, in this case, one cylinder is parabolic in cross section, while the other is circular. Upon passing to parabolic cylindrical coordinates, the Maxwell-Bloch system reduces to the ideal Duffing equation, while in circular cylindrical coordinates, the pendulum equation results. The $SL(2, R)$ matrix transformation $(SL2R)$ in the Maxwell-Bloch case provides a parametrized array of (offset) ellipsoids, hyperboloids, and cylinders, along whose intersections the $\mathbb{R}^3$ motion takes place.

## 4. Rigid Body with Controlled Feedback Torque

In this section we illustrate how to control the stability properties of equilibria for a dynamical system by using a control parameter that induces global bifurcations. The example is the free rigid body with a single torque about its major axis introduced by Bloch and Marsden [1990]. A similar analysis for other systems, such as that in Bloch, Krishnaprasad, Marsden, and Sánchez de Alvarez [1990] coming from a rigid body system with internal rotors is possible as well.

Euler's equations for the free rigid body with a single torque, $u$, about its major axis are given by

$$\frac{d}{dt}\begin{bmatrix}\Pi_1\\ \Pi_2\\ \Pi_3\end{bmatrix} = \begin{bmatrix}\left(\frac{1}{I_2}-\frac{1}{I_3}\right)\Pi_2\Pi_3\\ \left(\frac{1}{I_1}-\frac{1}{I_3}\right)\Pi_1\Pi_3\\ \left(\frac{1}{I_1}-\frac{1}{I_2}\right)\Pi_1\Pi_2+u\end{bmatrix}.$$

Following Bloch and Marsden [1990], we employ the feedback rule

$$u = -k\Pi_1\Pi_2,$$

where $k$ is the feedback gain parameter. We refer to the system with this feedback as the ***controlled system***. The equations of the controlled system are expressible in vector form as

$$\frac{d}{dt}\Pi = \nabla H' \times \nabla L' \qquad (CEE)$$

where $H'$ is the controlled-system energy,

$$H' = \frac{\Pi_1^2}{2I_1} + \frac{\Pi_2^2}{2I_2} + \frac{\Pi_3^2}{2\gamma I_3},$$

$L'$ is the square of the controlled-system angular momentum,

$$L' = \frac{1}{2}(g\Pi_1^2 + g\Pi_2^2 + \Pi_3^2),$$

and the parameter $g$ is defined by

$$g = 1 - \frac{k}{\left(\frac{1}{I_1} - \frac{1}{I_2}\right)}.$$

Note that the parameter $g$ in these equations contains elements of both the rigid body mass distribution, and its internal feedback torque along the 3-axis.

The vector-cross-product form of the controlled system $(CEE)$ ensures that both of the quantities $H'$ and $L'$ are conserved, and that the motion of the controlled system takes place in $\mathbb{R}^3$ along the intersections of the level surfaces of the two conserved quantities. The level surfaces $H'$ and $L'$ are clearly of either elliptic type, or hyperbolic type, depending upon the sign of $g$. (The case $g = 0$ is degenerate and will be discussed later. The rigid body case treated earlier is recovered for $g = 1$.) Regardless of the sign of $g$, the centers of the level surfaces of the quadratic functions $H'$ and $L'$ coincide at the origin of coordinates in $\mathbb{R}^3$, so points on these level surfaces where the gradients of $H'$ and $L'$ are collinear (*i.e.*, equilibrium points) will occur in opposing ($\mathbb{Z}_2$-symmetric, $\mathbf{x} \mapsto -\mathbf{x}$) pairs. In particular, tangencies for which the $H'$ and $L'$ level surfaces have curvatures of the same sign produce unstable equilibria, while tangencies of $H'$ and $L'$ having opposite sign curvatures produce stable equilibria. From the expressions for $H'$ and $L'$ above, it is clear that the sign of $g$ will play a key role in the stability properties of the equilibria.

Euler's equations for the controlled system may be re-expressed as

$$\frac{d}{dt}\vec{\Pi} = \nabla K' \times \nabla N',$$

where $K'$ and $N'$ are linear combinations of the controlled-system energy and angular momentum given by

$$\begin{pmatrix} K' \\ N' \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} H' \\ L' \end{pmatrix},$$

with real constants $a$, $b$, $c$, and $d$ satisfying the unit determinant condition $ad - bc = 1$. The controlled system is unchanged (and so the trajectories of its motion in $\mathbb{R}^3$ remain unchanged) when its energy $H'$ and angular momentum $L'$ are replaced by the $SL(2,\mathbb{R})$ linear combinations $K'$ and $N'$. By analogy to the rigid body discussed in the previous sections, one may choose to eliminate one term in each of $H'$ and $L'$ by choosing the linear combination given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{-1}{gI_3} \\ -c & \frac{c}{gI_3} \end{bmatrix}, \quad \text{with } c = \frac{g}{\left(\frac{1}{I_1} - \frac{1}{I_3}\right)},$$

so that
$$K' = \frac{1}{2}\left(\frac{1}{I_1} - \frac{1}{I_3}\right)\Pi_1^2 + \frac{1}{2}\left(\frac{1}{I_2} - \frac{1}{I_3}\right)\Pi_2^2,$$

and
$$N' = \frac{c}{2}\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\Pi_2^2 + \frac{1}{2}\Pi_3^2.$$

With this choice, the controlled system may be regarded as describing motion taking place along the intersection of a level surface of $K'$, an elliptic cylinder with its translation axis along $\Pi_3$ (where $K' = 0$), and a level surface of $N'$, which (depending on the sign of $g$) is either an elliptic cylinder ($g > 0$), or a hyperbolic cylinder ($g < 0$); in either case with its translation axis along $\Pi_1$ (where $N' = 0$). For $g = 0$, the level surfaces of $N'$ are horizontal planes, $\Pi_3^2 = \text{constant}$.

We restrict the controlled system to a level surface of $K'$ by the same change of variables as for the rigid body. In terms of the three positive constants $k_i^2$, $i = 1, 2, 3$, defined in the rigid-body case, we have

$$K' = \frac{\Pi_1^2}{2k_1^2} + \frac{\Pi_2^2}{2k_2^2} \quad \text{and} \quad N' = \frac{g\Pi_2^2}{2k_3^2} + \frac{1}{2}\Pi_3^2.$$

In terms of the elliptic polar coordinates defined by

$$\Pi_1 = k_1 r\cos\theta,\ \Pi_2 = k_2 r\sin\theta,\ \Pi_3 = p,$$

and with $r = \sqrt{2K'} = \text{constant}$, the conserved quantities become

$$K' = \frac{1}{2}r^2, \quad \text{and} \quad N' = \frac{1}{2}p^2 + g\left(\frac{k_2^2}{k_3^2}r^2\right)\sin^2\theta.$$

Transforming to the canonically conjugate variables $\theta$ and $p$ now gives

$$\frac{d}{dt}\theta = \{N', \theta\} = \frac{1}{k_1k_2}\frac{\partial N'}{\partial p} = \frac{1}{k_1k_2}p,$$

$$\frac{d}{dt}p = \{N', p\} = \frac{-1}{k_1k_2}\frac{\partial N'}{\partial \theta} = \frac{-g}{k_1k_2}\frac{k_2^2}{k_3^2}r^2\sin\theta\cos\theta.$$

Combining these equations of motion gives

$$\frac{d^2}{dt^2}\theta = \frac{-gr^2}{2k_1^2k_3^2}\sin 2\theta,$$

or, in terms of the original rigid body parameters,

$$\frac{d^2}{dt^2}\theta = -gK'\left(\frac{1}{I_1}-\frac{1}{I_2}\right)\left(\frac{1}{I_1}-\frac{1}{I_3}\right)\sin 2\theta.$$

Thus, the controlled system also reduces to the pendulum equation on level surfaces of $K'$. The stability of the equilibria of the controlled system now depends upon the sign of $g$, as well as the relative sizes of the principle moments of inertia of the body. In particular, negative $g$ stabilizes the equilibrium along the intermediate axis, while *destabilizing* the other two equilibria. This occurs because the introduction of the control parameter $g$ causes a global bifurcation of the rigid body system in which the equilibria exchange stability.

Leave 16 picas of vertical space for Fig. 2

**Figure 2.** The change in phase portrait of the controlled rigid body as the parameter $g$ passes through zero.

The general theory of reduction discussed in section 3 of course applies just as well to this example, and puts into a different light the fact observed by Bloch and Marsden [1990], that the controlled equations are Lie Poisson for the group $SO(2,1)$ in the stabilized situation.

## 5. Conclusions

In this paper we have shown how the Euler equations for a rigid body (even possibly including a stabilizing control torque) is Hamiltonian simultaneously with respect to a variety of Lie Poisson structures on $\mathbb{R}^3$. These Lie-Poisson structures are those for the three dimensional groups $SO(3)$, $SO(2,1)$, and the Euclidean group of the plane. For the euclidean group of the plane, the coadjoint orbits are elliptic cylinders with a canonical cotangent structure. In these canonical variables, which are given explicitly in elliptic cylindrical coordinates, the rigid body equations become transformed to the equations of a simple planar pendulum. This is analogous to the situation of David and Holm [1990] in which both the pendulum and the Duffing equation occur.

The geometry of the intersections of the level surfaces can be used to understand the stability of the system dynamics and how the feedback parameter affects the controlled system. There is a bifurcation and a change of stability as this parameter passes through zero (see Figure 2).

One potential application of these ideas is for reorienting satellites by controlled switching from one stable equilibrium to the opposite one (rotation by $\pi$, to the $\mathbb{Z}_2$-partner equilibrium) by momentarily destabilizing it, then restabilizing the partner equilibrium using the $g$ control parameter. The angle of switch is a geometric phase and can be calculated using the phase method of Montgomery; see Marsden, Montgomery, and Ratiu [1990]. The switch itself is accomplished by passing from an equilibrium to its $\mathbb{Z}_2$ partner—it is a *particular* geometrical phase associated to the $\mathbb{Z}_2$ symmetry of the level surfaces of the Casimir, $K'$. The rate at which the switching process takes place scales with the magnitude of the control parameter $g$, thereby allowing fine precision control through the adjustment of $g$ near zero.

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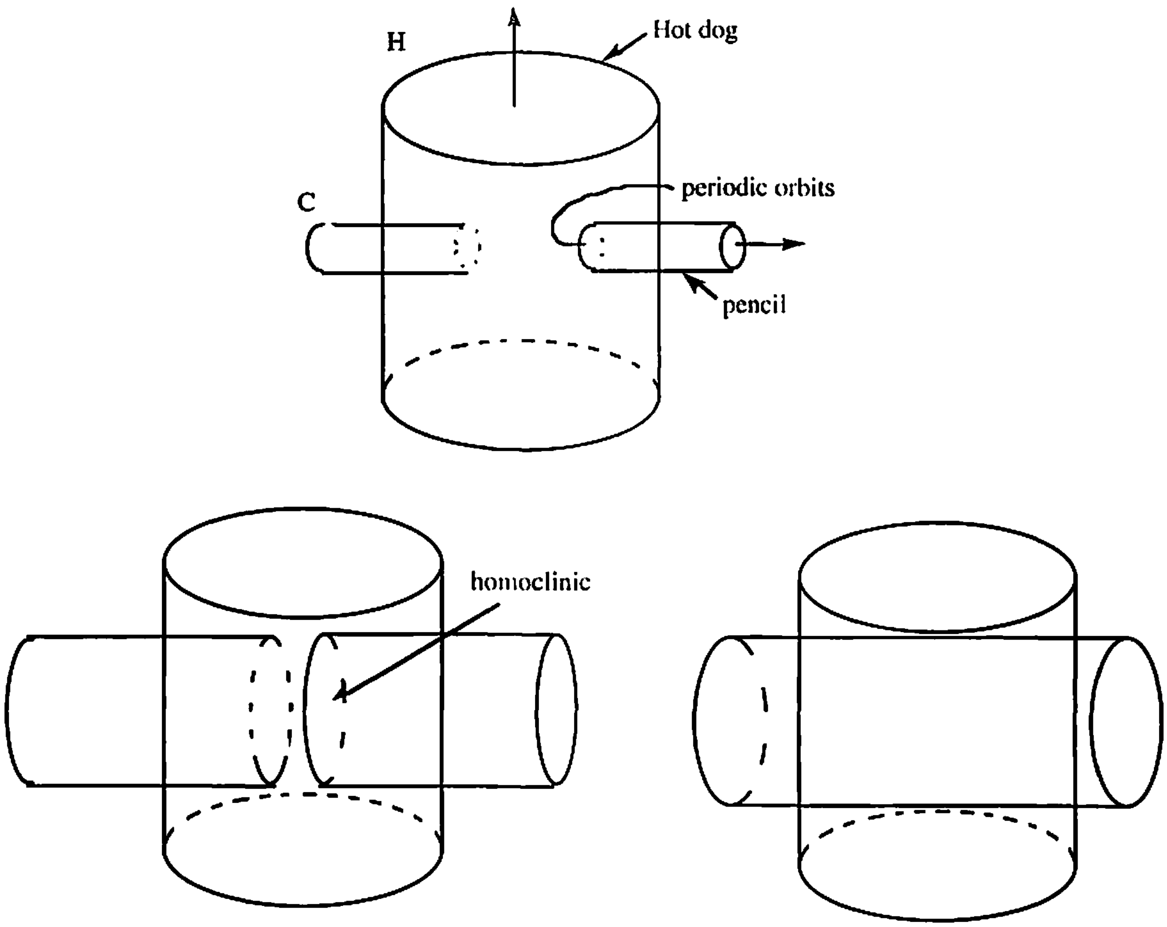
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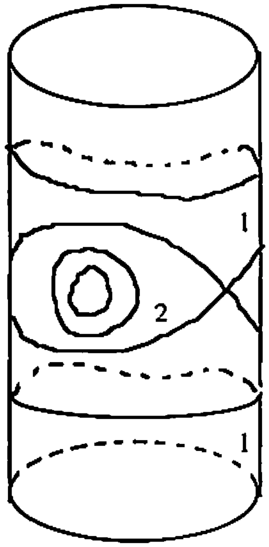
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