A Hamiltonian-dissipative decomposition of normal forms of vector fields

D. Lewis* and J. Marsden†

1. The Elphick-Iooss normal form

We consider dynamical systems in two variables with nilpotent linearization at the origin. We show that the behavior of the equilibria of such systems is determined by a modified Hamiltonian function which is constructed from an appropriate normal form for the vector field. In particular, the equilibria of the dynamical system correspond to critical points of the modified Hamiltonian and the local behavior of the vector field near an equilibrium is determined by the second variation of the modified Hamiltonian and its time derivative.

The normal form used here is the one described by Elphick et al. [1987]. This normal form is determined using an inner product $(\ |\ )$ on the space of vector fields with homogeneous polynomial entries. The inner product behaves well with respect to the adjoint operator, allowing an explicit characterization of the kernel of the adjoint in terms of vector fields commuting with the linear group action associated to L_0^*, where L_0^* is the adjoint (with respect to the Euclidean inner product) of the linearization L_0 of the equations of motion at the origin. After the normal form is obtained, the inner product is used to split the vector field in normal form into its Hamiltonian and dissipative components. (The word dissipative is used here in a generalized sense; the dissipation can be positive or negative and is defined precisely below.) The Hamiltonian subspace of the equivariant vector fields is determined by using the canonical symplectic structure on $\mathbf{R}^2$ to compute the Hamiltonian vector fields with L_0^* invariant Hamiltonians. The subspace of dissipative vector fields is defined to be the orthogonal complement to the Hamiltonian subspace with respect to $(\ |\ )$. For illustration, we consider the nilpotent operator

$$L_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

Here, the space of equivariant Hamiltonian vector fields is determined by Hamiltonians $H(x,y) = \frac{1}{2}y^2 + U(x)$ generating vector fields of the form $(y, -U_x(x))$. The dissipative subspace consists of L_0^* invariant functions (i.e. functions of x alone) multiplying the L_0^* equivariant vector field (x, y). Thus the Hamiltonian vector fields turn out to have the normal form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -U_x \end{aligned} \tag{2}$$

*Mathematical Sciences Institute, Cornell University, Ithaca NY 14853. Research supported in part by the Army Research Office.

†Department of Mathematics, Cornell University, Ithaca NY 14853 and Department of Mathematics, University of California, Berkeley, Berkeley CA 94720. Research partially supported by DOE contract DE-AT03-88ER-12097 and the Mathematical Sciences Institute at Cornell University.

(where U is thought of as a polynomial expression in x) and the dissipative vector fields have the form

$$\begin{aligned} \dot{x} &= a(x)x \\ \dot{y} &= a(x)y. \end{aligned} \tag{3}$$

Adding (2) and (3) gives the Hamiltonian-dissipative decomposition

$$\begin{aligned} \dot{x} &= y + a(x)x \\ \dot{y} &= -U_x + a(x)y, \end{aligned} \tag{4}$$

which is the *Elphick-Iooss normal form associated to* L_0.

The decomposition techniques described above are not restricted to vector fields with nilpotent linearization. An analogous situation arises for the harmonic oscillator, with Hamiltonian $H(x,y) = \frac{1}{2}(x^2 + y^2)$, which leads to the normal form for the Hopf bifurcation. The linearization of this normal form is *not* nilpotent. For a discussion of the Hopf bifurcation as the unfolding of the Hamiltonian vector field associated to H, see Schmidt [1976].

Let us now recall some of the general formalism behind the results. The inner product $(\ |\ )$ is defined as follows. First define an inner product $\langle\ |\ \rangle$ on the space P_n of homogeneous polynomials of degree n on $\mathbf{R}^m$ by

$$\langle P|Q\rangle = P(\partial)Q(\mathbf{x})|_{\mathbf{x}=0},$$

where $\partial_j = \partial\ /\partial x^j$. For example,

$$\langle (x^i)^\alpha | (x^j)^\beta \rangle = \alpha!\delta_{\alpha\beta}\delta^{ij}.$$

This inner product has the property that

$$\langle QR|P\rangle = R(\partial)Q(\partial)P(\mathbf{x})|_{\mathbf{x}=0} = \langle R|Q(\partial)P\rangle. \tag{5}$$

The inner product on the space of scalar polynomials can be extended to polynomial vector fields on $\mathbf{R}^m$, i.e., polynomials with values in $\mathbf{R}^m$, by defining

$$(V|W) = \sum_{j=1}^{m} \langle V^j | W^j \rangle.$$

This inner product behaves nicely with respect to pullback by linear maps. Let $A : \mathbf{R}^m \to \mathbf{R}^m$ be a linear transformation. We first note that

$$\langle P \circ A|Q\rangle = \langle P|Q \circ A^*\rangle,$$

where $(P \circ A)\mathbf{x} = P(A\mathbf{x})$ and A^* is the transpose of A with respect to the usual Euclidean inner product on $\mathbf{R}^m$. This can be seen by applying the chain rule to show that $\partial_{\mathbf{x}} = A\partial_{\mathbf{y}}$, where $\mathbf{y} = A^*\mathbf{x}$. It follows from the extension of the inner product $\langle\ |\ \rangle$ to vector valued polynomials that $(\ |\ )$ satisfies $(AV|W) = (V|A^*W)$. Hence

$$(A^{-1}V \circ A|W) = (V|(A^*)^{-1}W \circ A^*), \tag{6}$$

i.e. *the adjoint* (with respect to the inner product $(\ |\ )$ on P_n) *of the pullback map is the pullback map of the adjoint* (with respect to the Euclidean inner product on $\mathbb{R}^m$).

We consider now the extended adjoint operator $\mathcal{A}^{(n)} : P_n \to P_n$ given by

$$\mathcal{A}^{(n)}V = \left.\frac{d}{dt}\right|_{t=0} \left(e^{-L_0 t} V \circ e^{L_0 t}\right),$$

where L_0 is the linear part of the vector field to be put in normal form. We make use of (6) to obtain a simple description of the decomposition

$$P_n = \mathrm{Im}\,\mathcal{A}^{(n)} \oplus \left(\mathrm{Ker}\,\mathcal{A}^{(n)}\right)^*.$$

Taking $A = e^{L_0 t}$ and applying (6), we see that

$$\begin{aligned}
(\mathcal{A}^{(n)}V|W) &= \left(\left.\frac{d}{dt}\right|_{t=0} \left(e^{-L_0 t} V \circ e^{L_0 t}\right) |W\right) \\
&= \left(V|\left.\frac{d}{dt}\right|_{t=0} \left(\left(e^{-L_0 t}\right)^* W \circ \left(e^{L_0 t}\right)^*\right)\right) \\
&= \left(V|\left.\frac{d}{dt}\right|_{t=0} \left(e^{-L_0^* t} W \circ e^{L_0^* t}\right)\right) \\
&= (V|\mathcal{A}_*^{(n)}W),
\end{aligned}$$

where $(\mathcal{A}_*^{(n)}W)[X] = [W[X], L_0^* X]$. Hence $\left(\mathrm{Ker}\,\mathcal{A}^{(n)}\right)^* = \mathrm{Ker}\,\mathcal{A}_*^{(n)}$ and so we have the decomposition

$$P_n = \mathrm{Im}\,\mathcal{A}^{(n)} \oplus \mathrm{Ker}\,\mathcal{A}_*^{(n)},$$

i.e. all vector fields orthogonal to the image of $\mathcal{A}^{(n)}$ commute with L_0^*. We say that a polynomial $\Sigma_n V_n$ is in *normal form* (with respect to the given inner product) if V_n is orthogonal to $\mathrm{Im}\,\mathcal{A}^{(n)}$. In particular, the terms appearing in the normal form commute with L_0^*. This definition arises as follows: Assume that the equations take the form

$$\dot{\mathbf{x}} = L_0\mathbf{x} + \mathbf{g}(\mathbf{x}), \tag{7}$$

where the vector-valued polynomial $\mathbf{g}$ is of order $n \geq 2$. One seeks a vector-valued polynomial $\mathbf{p}$ of order n such that the nonlinear term $\mathbf{g}$ is removed to order n by a change of coordinates of the form $\mathbf{x} = \mathbf{y} + \mathbf{p}(\mathbf{y})$. Application of the chain rule yields

$$\begin{aligned}
\dot{\mathbf{p}} &= D\mathbf{p}(\mathbf{y}) \cdot \dot{\mathbf{y}} \\
&= D\mathbf{p}(\mathbf{y}) \cdot L_0\mathbf{y} + \text{higher order terms;}
\end{aligned}$$

thus (7) is satisfied at order n if and only if

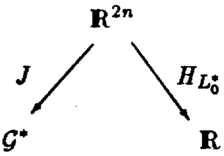
$$D\mathbf{p}(\mathbf{y}) \cdot L_0\mathbf{y} = L_0\mathbf{p}(\mathbf{y}) + \mathbf{g}(\mathbf{y}),$$

i.e., if

$$\begin{aligned}
\mathbf{g}(\mathbf{y}) &= -L_0\mathbf{p}(\mathbf{y}) + D\mathbf{p}(\mathbf{y}) \cdot L_0\mathbf{y} \\
&= -\left(\mathcal{A}^{(n)}\mathbf{p}\right)(\mathbf{y}).
\end{aligned}$$

Hence, only $\mathbf{g} \in \mathrm{Im}\,\mathcal{A}^{(n)}$ can be removed by this method. (See Guckenheimer and Holmes [1983] for a discussion of normal forms.)

In some cases (in particular, the nilpotent case (1)), the invariant functions are functions of a momentum map associated with an appropriate dual pair. (See Marsden and Weinstein [1983] and Weinstein [1984].) More specifically, assume the system associated to L_0^* is completely integrable via some Lie group G in the following sense: the action of the Hamiltonian vector field $X_{H_{L_0^*}}$ on the level sets of the momentum map $J : \mathbf{R}^{2n} \to \mathcal{G}^*$ is transitive. Then *functions which are invariant under the action induced by $X_{H_{L_0^*}}$ 'collectivize' with respect to J; i.e. if a function $\mathbf{R}^{2n} \to \mathbf{R}$ is invariant under the action of $X_{H_{L_0^*}}$, then there exists a function $f : \mathcal{G}^* \to \mathbf{R}$ such that $F = f \circ J$.*



A dual pair

In the nilpotent case (1), we take $\mathbf{R}$ to be our Lie group G acting on $\mathbf{R}^2$ by translations in y, i.e. $\xi : (x,y) \mapsto (x, y+\xi)$. This action has infinitesimal generator $\xi_{\mathbf{R}^2}(x,y) = (0,\xi)$ and momentum map $J(x,y) = J(x,y)\cdot 1 = x$. Hence the level sets of J are simply vertical lines. The dynamics generated by L_0^* are Hamiltonian with respect to $H_{L_0^*} = \frac{1}{2}x^2$. This Hamiltonian has the Hamiltonian vector field $X_{H_{L_0^*}} = (0,-x)$, which induces the action $\xi : (x,y) \mapsto (x, y-\xi x)$ of $\mathbf{R}$ on $\mathbf{R}^2$. This action is transitive on the level sets of J (except for $J^{-1}(0)$). Hence this system is Hamiltonian and completely integrable with respect to G and the fact that the invariant functions are functions of x alone is predicted by the collectivization result noted above. Another interesting example we hope to look at is the dual pair associated to 1:1 resonance; see Golubitsky and Stewart [1987].

If there is an additional Lie group $\tilde{G}$ which acts on both $\mathbf{R}^{2n}$ and $\mathcal{G}^*$, and if the momentum map J associated to the group G is equivariant with respect to the action of $\tilde{G}$, *then $\tilde{G}$ invariance collectivizes in the following sense: if $F = f \circ J$ is $\tilde{G}$ invariant, then f is $\tilde{G}$ invariant.* We shall use this fact in §4.

Remark: An alternative normal form for the nilpotent system is given by

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -\nabla p + f(u)v. \end{aligned} \tag{8}$$

If we make the change of coordinates $x = u$ and $y = v - ax$, where the function $a(x)$ satisfies

$$2a + a_x x = f(x),$$

then we see that the normal form (8) corresponds to the Elphick-Iooss normal form

$$\begin{aligned} \dot{x} &= y + ax \\ \dot{y} &= -\nabla U + ay \end{aligned} \tag{9}$$

for $U(x) = p(x) - \int^x a^2 x\, dx$.

An alternate Hamiltonian-dissipative decomposition, which treats differential polynomials in one independent and one dependent variable, characterizes the dissipative component of the system as a series of differentials of Euler-Lagrange expressions. A description of this decomposition, including an interpretation of the dissipative terms as determining a generalized Rayleigh dissipation function, is presented in Olver and Shakiban [1988]. The relationship between the Olver-Shakiban decomposition and the decomposition presented here will be the subject of future investigation.

1.1 Sample phase portraits

We now present a few samples of vector fields on $\mathbf{R}^2$ which have the same Hamiltonian component, but distinct dissipative components. We choose as our underlying Hamiltonian the function

$$H(x,y) = \tfrac{1}{2}y^2 + .1x - .05x^2 - \tfrac{1}{3}x^3,$$

which we view as an unfolding of the nilpotent Hamiltonian discussed in the previous section. The phase portrait for this Hamiltonian system, given in figure 1, yields the 'fish' which characteristically appears in unfoldings of the double zero eigenvalue problem. This system has a center and a saddle point lying on the x-axis. The following three figures (2–4) illustrate the effect of the addition of various order one dissipative components to the vector field. In figure two, the center at the 'eye of the fish' has become a source; the saddle point remains a saddle. In addition, a new equilibrium, a second source, has appeared. In figure three, the 'eye' is now a sink; the other two equilibria, a saddle and a source, are similar to those in figure two. Figure four is qualitatively similar to figure three, in the sense that it possesses a sink, saddle and source, but the saddle and source are sufficiently near one another that they are hardly visible in the graph. To illustrate this system in greater detail, we focus on the region of phase space near the lower 'fin' and vary the constant c in the dissipation function $a(x) = c + x$ in figures 5a – d. As the constant c increases, the saddle and the source move towards each other; when $c = .1344035$, they collide and disappear. (In figures 5b and 5c, the location of the saddle is indicated by a cross; the location of the source is marked by a triangle.)

In all three dissipative cases the behavior along the upper 'fin' is fairly consistent with that of the underlying Hamiltonian system. It is to be expected that the neutral center at the 'eye' should become either a source or a sink when a nonconservative term is added to the vector field. The most striking distinctions in the phase portraits occur in the region containing the lower 'fin' of the original system. The appearance of the new fixed point could not be predicted from a local analysis. One consistent feature in the dissipative flows is the existence of a curve, roughly determined by the unstable eigenvector of the saddle and the most strongly repelling direction of the source, which predominantly determines the behavior of the flow in the lower righthand quadrant. Such a curve appears in almost all dissipative variants of the Hamiltonian flow given above. The equilibria and eigenvalues for the phase portraits displayed in figures one through five are given in the following table.

Figure	Dissipation	Equilibria	Eigenvalues
1	$a(x) = 0$	$(-0.370156, 0)$ $(0.270156, 0)$	$\pm 0.800195i$ $(0.800195, -0.800195)$
2	$a(x) = x^2$	$(-0.360501, 0.0468512)$ $(0.272494, -0.0202335)$ $(1, -1)$	$0.259923 \pm 0.798682i$ $(0.948178, -0.651166)$ $(3.04881, 0.951191)$
3	$a(x) = x^3$	$(-0.368706, -0.0184809)$ $(0.270321, -0.00533972)$ $(1, -1)$	$-0.125309 \pm 0.79956i$ $(0.849601, -0.750835)$ $(3.6619, 1.3381)$
4	$a(x) = .1344 + x$	$(-0.345222, -0.0727806)$ $(0.536518, -0.359959)$ $(0.539904, -0.36406)$	$-0.383434 \pm 0.795884i$ $(1.87994, -0.00158842)$ $(1.88692, 0.00158862)$
5a	$a(x) = 0$	(-0.370156,0) $(0.270156, 0)$	$\pm 0.800195i$ $(0.800195, -0.800195)$
5b	$a(x) = .13 + x$	$(-0.344397, -0.0738376)$ $(0.482062, -0.295052)$ $(0.602334, -0.44111)$	$-0.386595 \pm 0.7956i$ $(1.76258, -0.0563945)$ $(2.01036, 0.0566391)$
5c	$a(x) = .134 + x$	$(-0.345148, -0.0728771)$ $(0.520387, -0.340535)$ $(0.55676, -0.384588)$	$-0.383722 \pm 0.795859i$ $(1.84621, -0.0170522)$ $(1.92121, 0.0170752)$
5d	$a(x) = .14 + x$	$(-0.346259, -0.0714189)$	$-0.379388 \pm 0.796218i$

Table 1. Eigenvalue information for figures 1 – 5.

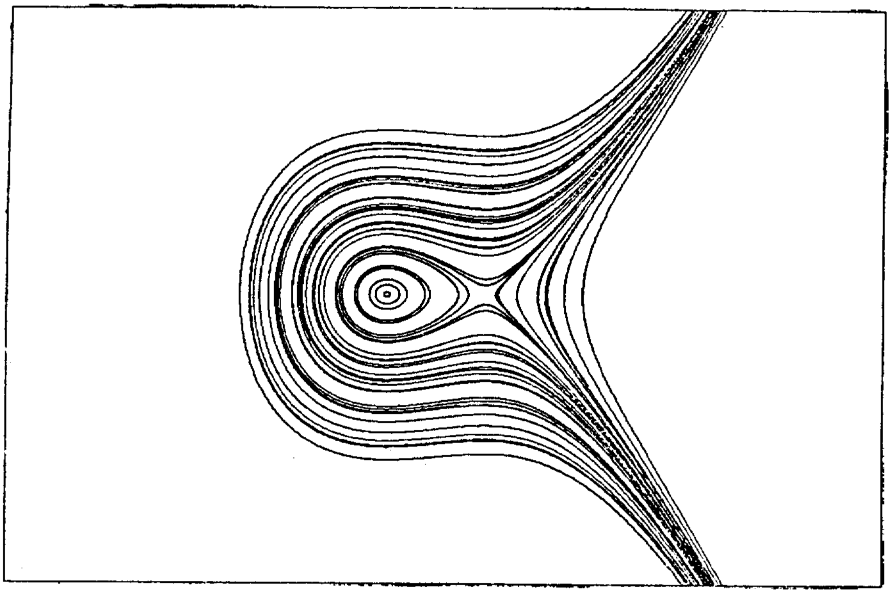


fig. 1

$$H(x,y) = \tfrac{1}{2}y^2 + .1x - .05x^2 - \tfrac{1}{3}x^3$$

$$\dot{x} = y$$

$$\dot{y} = -.1 + .1x + x^2$$

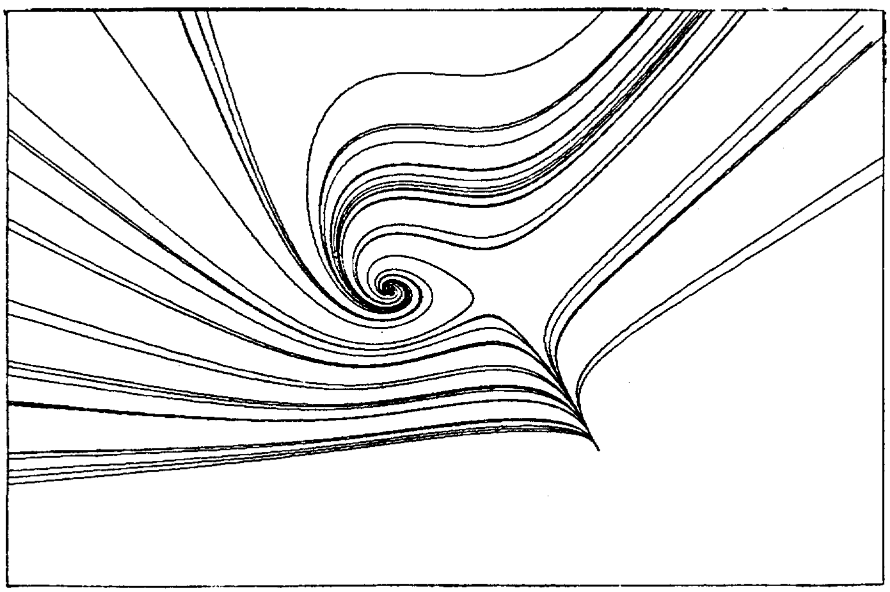


fig. 2

$$a(x) = x^2$$

$$\dot{x} = y + x^3$$

$$\dot{y} = -.1 + .1x + x^2 + x^2y$$

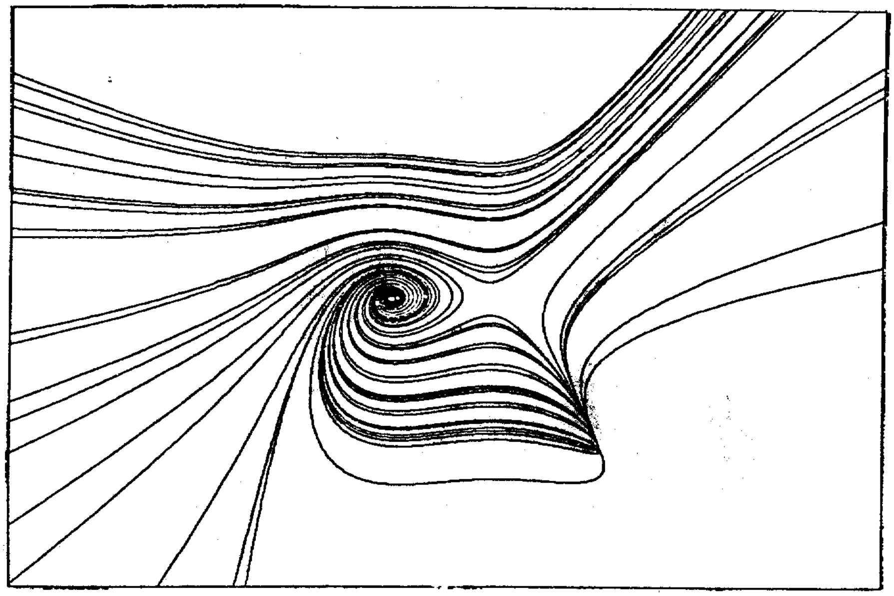


fig. 3

$$\begin{aligned} a(x) &= x^3 \\ \dot{x} &= y + x^4 \\ \dot{y} &= -.1 + .1x + x^2 + x^3 y \end{aligned}$$

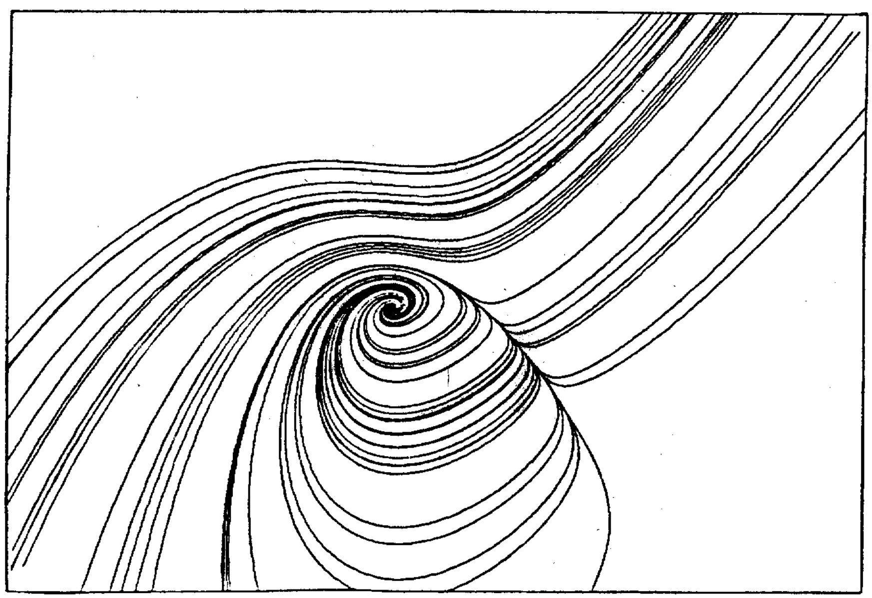


fig. 4

$$\begin{aligned} a(x) &= .1344 + x \\ \dot{x} &= y + (.1344 + x)x \\ \dot{y} &= -.1 + .1x + x^2 + (.1344 + x)y \end{aligned}$$

fig. 5a

$$H(x,y) = \tfrac{1}{2}y^2 + .1x - .05x^2 - \tfrac{1}{3}x^3$$
$$\dot{x} = y$$
$$\dot{y} = -.1 + .1x + x^2$$

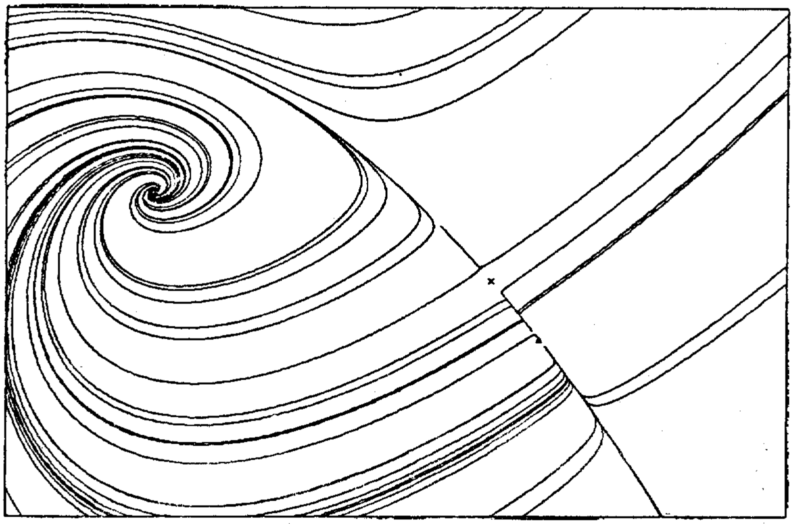


fig. 5b

$$a(x) = .13 + x$$
$$\dot{x} = y + (.13 + x)x$$
$$\dot{y} = -.1 + .1x + x^2 + (.13 + x)y$$

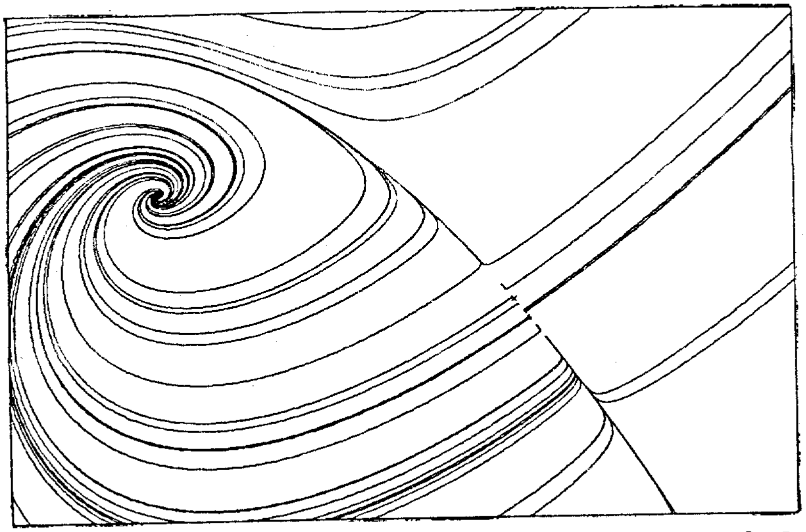


fig. 5c

$$a(x) = .134 + x$$
$$\dot{x} = y + (.134 + x)x$$
$$\dot{y} = -.1 + .1x + x^2 + (.134 + x)y$$

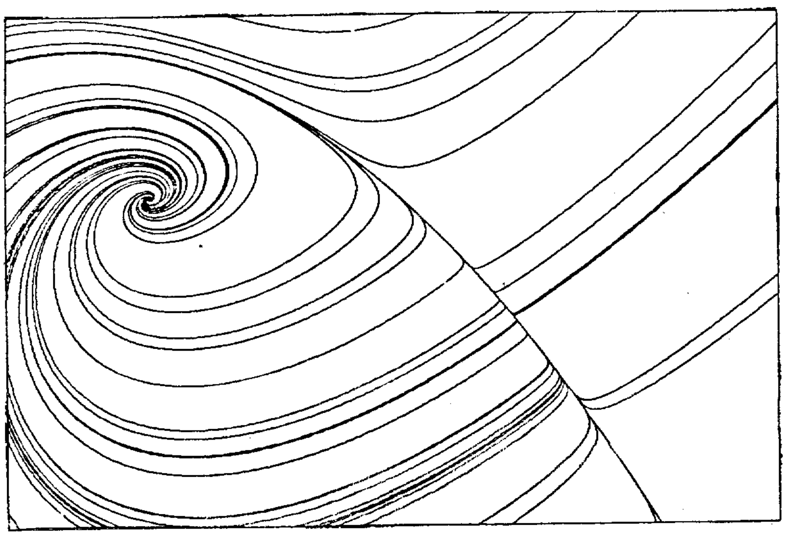


fig. 5d

$$a(x) = .14 + x$$
$$\dot{x} = y + (.14 + x)x$$
$$\dot{y} = -.1 + .1x + x^2 + (.14 + x)y$$

2. THE MODIFIED HAMILTONIAN

The following remarks show that *the equilibria of a two dimensional system given in normal form with respect to the nilpotent operator (1) may be analysed by studying the critical points of a function* H_a determined by the normal form. In particular, *a saddle point of the function* H_a *is a saddle point of the equations of motion and a local minimum of* H_a *is a sink (source) of the equations of motion if* $\dot{H}_a \leq (\geq) 0$ *in a neighborhood of the critical point.* (H_a has no local maxima.)

Given the dissipative system in normal form

$$\begin{aligned} \dot{x} &= y + a(x)x \\ \dot{y} &= -U_x(x) + a(x)y \end{aligned}$$

we compute the linearized equations of motion

$$\begin{aligned} \dot{\delta x} &= \delta y + (a_x x + a)\delta x \\ \dot{\delta y} &= (-U_{xx} + a_x y)\delta x + a\delta y \end{aligned}$$

with matrix

$$L = \begin{pmatrix} a + a_x x & 1 \\ -(U_{xx} + aa_x x) & a \end{pmatrix}$$

at equilibrium. (Note: We write a for the function $a(x)$, a_x for the derivative da/dx, etc.)

We compute

$$\det L = a^2 + 2aa_x x + U_{xx} = (a^2 x + U_x)_x$$

and

$$\operatorname{tr} L = 2a + a_x x.$$

Define the *modified Hamiltonian function*

$$H_a(x,y) = \tfrac{1}{2}y^2 + U(x) + a(x)xy + f(x), \tag{10}$$

where

$$f(x) = \tfrac{1}{2}a^2x^2 + \int^x a^2 x \, dx.$$

H_a has partial derivatives

$$\begin{aligned} \frac{\partial H_a}{\partial x} &= U_x + (a + a_x x)y + f_x \\ &= U_x + (a + a_x x)y + (2a + a_x x)ax \\ &= (U_x - ay) + (2a + a_x x)(y + ax) \\ \frac{\partial H_a}{\partial y} &= y + ax. \end{aligned}$$

It follows that

$$\begin{aligned} 0 = \dot{x} &= y + ax \\ 0 = \dot{y} &= -U_x + ay \end{aligned} \iff \nabla H_a(x,y) = 0,$$

i.e. $\nabla H_a(\mathbf{x}_e) = 0$ if and only if $\mathbf{x}_e$ is an equilibrium. Alternatively, one can see that critical points of H_a correspond to equilibria by writing the equations of motion in the form

$$\begin{aligned} \dot{x} &= \frac{\partial H_a}{\partial y} \\ \dot{y} &= -\frac{\partial H_a}{\partial x} + (2a + a_x x)\,\dot{x}. \end{aligned}$$

In this formulation, the function $(2a + a_x x)\,\dot{x}$, which is linear in the velocity, plays the role of a traditional dissipative term. This can be viewed in terms of a Rayleigh dissipation function $R(x)$ given by

$$\begin{aligned} R(x) &= \int^x (2a + a_x x)\, dx \\ &= ax + \int^x a\, dx. \end{aligned}$$

H_a has second partial derivatives

$$\begin{aligned} \frac{\partial^2 H_a}{\partial x^2} &= U_{xx} + (2a + a_{xx} x) y + f_{xx} \\ \frac{\partial^2 H_a}{\partial x \partial y} &= a + a_x x \\ \frac{\partial^2 H_a}{\partial y^2} &= 1, \end{aligned}$$

with values

$$\begin{aligned} \frac{\partial^2 H_a}{\partial x^2}(\mathbf{x}_e) &= U_{xx} + 4a a_x x + a_x^2 x^2 + 2a^2 \\ \frac{\partial^2 H_a}{\partial x \partial y}(\mathbf{x}_e) &= a + a_x x \\ \frac{\partial^2 H_a}{\partial y^2}(\mathbf{x}_e) &= 1 \end{aligned}$$

at equilibrium. The Jacobian determinant at equilibrium is

$$\begin{aligned} \det D^2 H_a(\mathbf{x}_e) &= U_{xx} + a^2 + 2a a_x x \\ &= (U_x + a^2 x)_x \\ &= \det L. \end{aligned} \tag{11}$$

We note here that

$$\frac{\partial^2 H_a}{\partial x^2}(\mathbf{x}_e) = ((ax)_x)^2 + \det D^2 H_a(\mathbf{x}_e),$$

hence $D^2 H_a(\mathbf{x}_e)$ must be either positive definite (in the case $\det D^2 H_a(\mathbf{x}_e) > 0$) or indefinite (in the case $\det D^2 H_a(\mathbf{x}_e) < 0$). It follows from (11) that if $\det D^2 H_a(\mathbf{x}_e) > 0$, then $\det L > 0$, so that the real parts of the eigenvalues of the linearized equations must be of the same sign, i.e. the equilibrium must be a source, sink or center. On the other hand, if $\det D^2 H_a(\mathbf{x}_e) < 0$, then $\det L < 0$ and so the eigenvalues of L must be real and have opposite sign, i.e. the equilibrium must be a saddle point. In the case of positive determinant, an additional computation is required to determine whether the equilibrium is a source or a sink. The sign of the the real part of the eigenvalues is determined by the sign of the trace of L. We shall show that this sign equals the sign of the trace of the Jacobian of the time derivative of the modified Hamiltonian; this sign also determines the sign of $\dot{H}_a$ itself in a neighborhood of the equilibrium. In particular, an equilibrium $\mathbf{x}_e$ is a sink if and only if it is a strict local minimum of the modified Hamiltonian H_a and $\dot{H}_a \leq 0$ on a neighborhood of $\mathbf{x}_e$. If $\mathbf{x}_e$ is a source, then H_a again has a local minimum at $\mathbf{x}_e$, but $\dot{H}_a \geq 0$ on a neighborhood of $\mathbf{x}_e$.

We now compute the derivatives of $\dot{H}_a$:

$$\begin{aligned} \dot{H}_a &= \frac{-\partial H_a}{\partial x}\dot{x} + \frac{\partial H_a}{\partial y}\dot{y} \\ &= (y + ax)\left((2a + a_x x)y + f_x\right) \\ &= (y + ax)^2(2a + a_x x). \end{aligned} \tag{12}$$

Using the Rayleigh dissipation function $R(x) = \int(2a + a_x x)dx$, we can write

$$\dot{H}_a = R_x \dot{x}^2.$$

At equilibrium

$$\dot{H}_a(\mathbf{x}_e) = \frac{\partial H_a}{\partial x}(\mathbf{x}_e)\frac{\partial H_a}{\partial y}(\mathbf{x}_e) = 0. \tag{13}$$

It is clear from the factorization (12) that

$$\frac{\partial \dot{H}_a}{\partial x}(\mathbf{x}_e) = \frac{\partial \dot{H}_a}{\partial y}(\mathbf{x}_e) = 0.$$

In fact, any point (x, y) such that $\dot{x} = 0$ is a critical point of $\dot{H}_a$.

$\dot{H}_a(\mathbf{x}_e)$ has second partial derivatives

$$\begin{aligned} \frac{\partial^2 \dot{H}_a}{\partial x^2}(\mathbf{x}_e) &= 2(a + a_x x)^2(2a + a_x x) \\ \frac{\partial^2 \dot{H}_a}{\partial x \partial y}(\mathbf{x}_e) &= 2(a + a_x x)(2a + a_x x) \\ \frac{\partial^2 \dot{H}_a}{\partial y^2}(\mathbf{x}_e) &= 2(2a + a_x x). \end{aligned}$$

It follows that $\det D^2\dot{H}_a(\mathbf{x}_e) = 0$,

$$\operatorname{tr} D^2\dot{H}_a(\mathbf{x}_e) = 2(2a + a_x x)\left(1 + (a + a_x x)^2\right),$$

and hence $D^2\dot{H}_a(\mathbf{x}_e)$ has eigenvalues 0 and $\operatorname{tr} D^2\dot{H}_a(\mathbf{x}_e)$. We note that the eigenvector associated to the 0 eigenvalue of $D^2\dot{H}_a(\mathbf{x}_e)$ is $(1, -(ax)_x)$ and the eigenvector associated to the $\operatorname{tr} D^2\dot{H}_a(\mathbf{x}_e)$ eigenvalue is $((ax)_x, 1)$. In particular, the eigenvector with eigenvalue 0 is tangent to the curve $y = -ax$ determined by the conditions $\dot{H}_a = 0$ (equivalently $\dot{x} = 0$) and the other eigenvector is perpendicular to this curve.

We see that

$$\begin{aligned}\operatorname{tr} D^2 \dot{H}_a(\mathbf{x}_e) &= 2(2a + a_x x)\left(1 + (a + a_x x)^2\right) \\ &= 2\operatorname{tr} L\left(1 + (a + a_x x)^2\right).\end{aligned}$$

Since the term within the parentheses is always positive, the sign of $\operatorname{tr} L$ (and hence the sign of the eigenvalues of L in the case that $\det L > 0$) is determined by the sign of $\operatorname{tr} D^2 \dot{H}_a(\mathbf{x}_e)$. Clearly, we also have $\partial^2 \dot{H}_a/\partial y^2(\mathbf{x}_e) = 2(2a + a_x x) = 2\operatorname{tr} L$, so one can work with this partial derivative, rather than the trace, if it is more convenient.

The modified Hamiltonian H_a provides information about periodic orbits, as well as equilibria. It is clear that a periodic orbit cannot be entirely contained within a region where $\dot{H}_a$ has fixed sign. If an orbit moves down across the level sets of H_a while in a region where $\dot{H}_a < 0$, then it must pass through a region where $\dot{H}_a > 0$ if it is to return to the level set on which it began. From the factorization (12) of $\dot{H}_a$ we note that the regions of fixed sign for $\dot{H}_a$ are bounded by the vertical lines $(2a + a_x x)^{-1}(0)$. ($\dot{H}_a = 0$ along the curve $y = -ax$, but does not change sign when crossing this curve.) Hence any periodic orbit must cross one of the lines $(2a + a_x x)^{-1}(0)$. According to the Poincaré index theorem, the region enclosed by a periodic orbit must contain an equilibrium point, which we know must lie along the curve $y = -ax$. Thus, a periodic orbit must enclose segments of both the $y = -ax$ and the $(2a + a_x x)^{-1}(0)$ branches of the set $\dot{H}_a = 0$.

The functions H_a and $\dot{H}_a$ can also be used to detect the occurrence of a Hopf bifurcation. (See Marsden and McCracken [1976] or Guckenheimer and Holmes [1983], for instance.) From the characteristic equation $\lambda^2 + \operatorname{tr} L\,\lambda + \det L = 0$ for the eigenvalues of L, observe that L has purely imaginary eigenvalues if and only if $\operatorname{tr} L = 0$ and $\det L > 0$. As discussed above, conditions on the determinant and trace of L can be expressed in terms of $\det D^2 H_a$ and $\operatorname{tr} D^2 \dot{H}_a$. In particular, L has purely imaginary eigenvalues if $\det D^2 H_a > 0$, i.e. if H_a has a local minimum at $\mathbf{x}_e$, and if $\operatorname{tr} D^2 \dot{H}_a = 0$. These conditions may be expressed geometrically as follows: Consider the factorization

$$\dot{H}_a = (y + ax)^2(2a + a_x x)$$

and note that if we plot the set $\dot{H}_a^{-1}(0)$, then an equilibrium has purely imaginary eigenvalues if and only if it is a local minimum of H_a lying at the intersection of the curve $y = -ax$ and the vertical line(s) $(2a + a_x x)^{-1}(0)$.

The non-zero speed condition for Hopf bifurcation can be checked by computing the derivative of $\operatorname{tr} D^2 \dot{H}_a$ (or $\partial^2 \dot{H}_a/\partial y^2$) with respect to the bifurcation parameter μ at $\mathbf{x}_e$. We note that

$$\begin{aligned}\left(\operatorname{tr} D^2 \dot{H}_a(\mathbf{x}_e)\right)_\mu &= \left(2\operatorname{tr} L\left(1 + (a + a_x x)^2\right)\right)_\mu \\ &= 2\operatorname{tr} L_\mu\left(1 + (a + a_x x)^2\right),\end{aligned}$$

since, by assumption, $\operatorname{tr} L = 0$ at criticality. Hence $\operatorname{sign}\left(\operatorname{tr} D^2 \dot{H}_a(\mathbf{x}_e)\right)_\mu = \operatorname{sign}(\operatorname{tr} L_\mu)$.

In summary, much of the qualitative behavior of a system of the form (10) can be determined by plotting the level sets of the function H_a and locating the regions on which $\dot{H}_a$ is positive, negative or zero:

- *Equilibria occur at critical points of H_a. These must lie along the curve $y = -ax$ making up one or more branches of the set $\dot{H}_a^{-1}(0)$.*

- *Local minima of H_a correspond to sinks, sources, or nodes of the equations of motion. Ignoring degeneracies, the equilibrium is attracting (repelling) if $\dot{H}_a$ is non-positive (non-negative) in a neighborhood of the equilibrium.*
- *Saddle points of the equations of motion correspond to saddle points of H_a.*
- *Periodic orbits must cross the vertical lines $(2a + a_x x)^{-1}(0)$ which bound the regions on which $\dot{H}_a$ has fixed sign. In particular, a Hopf bifurcation can occur only at a singularity in $\dot{H}_a^{-1}(0)$, i.e. where the curve $y = -ax$ intersects the line $(2a+a_x x)^{-1}(0)$.*

Remark: The second time derivative of H_a contains a factor of $\dot{x}$. In addition, the higher order time derivatives of H_a seem to factor nicely at points where $\dot{x} = 0$. We have

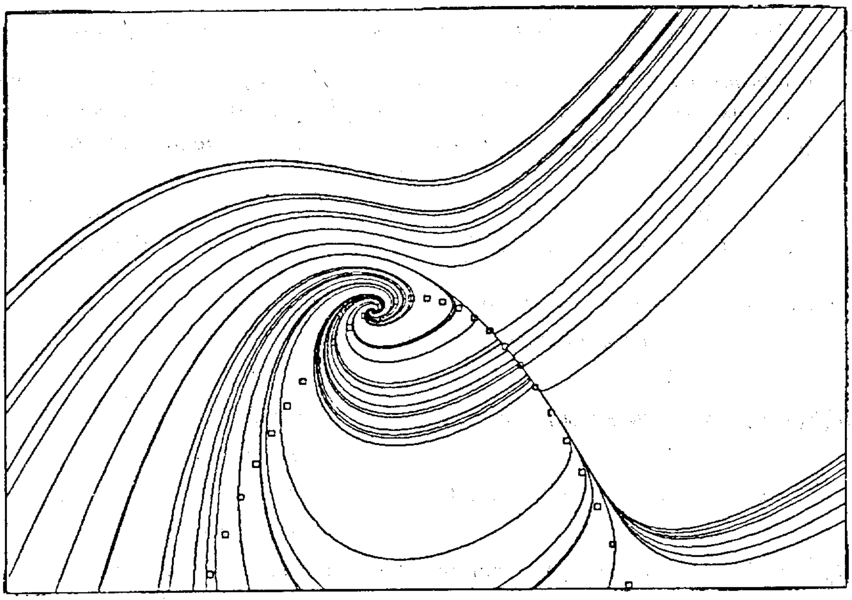
$$\begin{aligned}
\ddot{H}_a(x,y) &= (y + ax)(\text{a complicated function}) \\
&= \dot{x}(\text{a complicated function}) \\
&= 0 && \text{if } \dot{x} = 0 \\
H_a^{(3)}(x,y) &= 2(2a + a_x x)(a^2 x + U_x)^2 && \text{if } \dot{x} = 0 \\
&= 0 && \text{if } \dot{y} = \dot{x} = 0 \\
H_a^{(4)}(x,y) &= 6(2a + a_x x)^2(a^2 x + U_x)^2 && \text{if } \dot{x} = 0 \\
&= 0 && \text{if } \dot{y} = \dot{x} = 0 \\
H_a^{(5)}(x,y) &= 2(a^2 x + U_x)^2(\text{a complicated function}) && \text{if } \dot{x} = 0 \\
&= 0 && \text{if } \dot{y} = \dot{x} = 0.
\end{aligned}$$

This behavior may or may not continue indefinitely; these facts could be useful for the analysis of degenerate singularities.

2.1 Examples

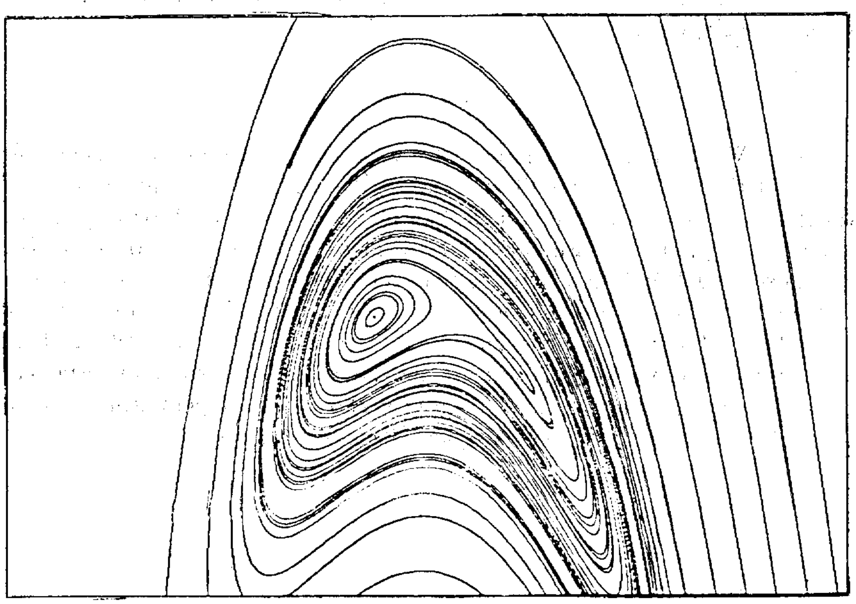
The following figures show the phase portraits of several dissipative nilpotent vector fields and the level sets of the associated modified Hamiltonians. The curves made up of small boxes indicate the curve $y = -ax$; the vertical line indicated by small boxes in figures 8a and 9a is the line $(2a + a_x x)^{-1}(0)$. Figure 6 shows a member of the one parameter family considered in figures 4 and 5; it can be seen that the two fixed points which eventually collide and disappear lie in an 'arm' of the level sets of the modified Hamiltonian. This 'arm' shrinks as the parameter c is increased, forcing the two equilibria together. Figures 7 and 8 show the effect of a quadratic and, respectively, cubic dissipative term on the underlying Hamiltonian $\frac{1}{2}y^2 - \frac{1}{3}x^3$. Finally, figure 9 shows a flow possessing a periodic orbit, which straddles the regions of increasing and decreasing energy.

$$\begin{aligned} H(x,y) &= \tfrac{1}{2}y^2 + .1x - .05x^2 - \tfrac{1}{3}x^3 \\ a(x) &= .13 + x \\ H_a(x,y) &= \tfrac{1}{2}y^2 + .1x - .0331x^2 - .1167x^3 + .75x^4 + (.13 + x)xy \end{aligned}$$



$$\begin{aligned} \dot{x} &= y + (.13 + x)x \\ \dot{y} &= -.1 + .1x + x^2 + (.13 + x)y \end{aligned}$$

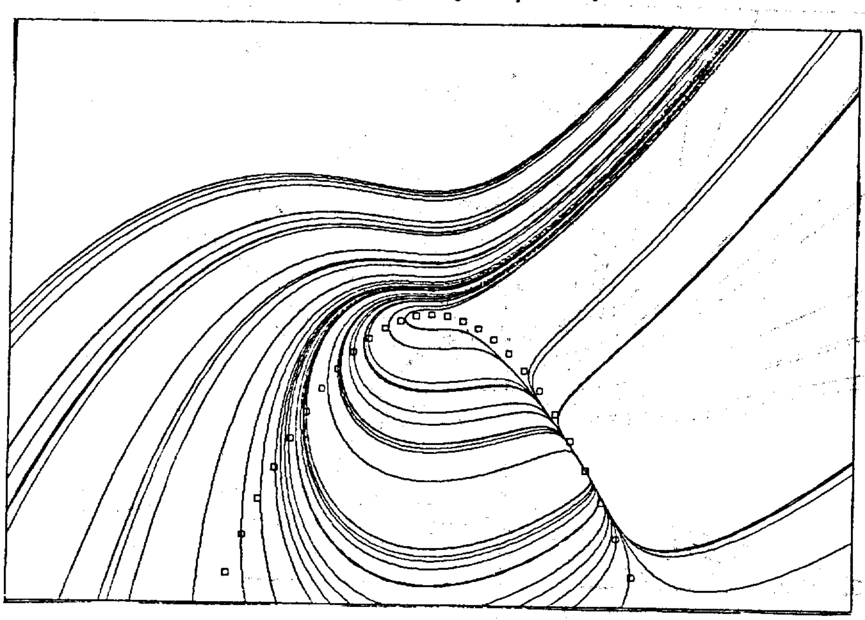
fig. 6a



$$\begin{aligned} \dot{x} &= y + (.13 + x)x \\ \dot{y} &= -.1 + .0662x + .35x^2 - 3x^3 + (.13 + 2x)y \end{aligned}$$

fig. 6b

$$
\begin{aligned}
H(x,y) &= \tfrac{1}{2}y^2 - \tfrac{1}{3}x^3 \\
a(x) &= x \\
H_a(x,y) &= \tfrac{1}{2}y^2 - \tfrac{1}{3}x^3 - \tfrac{3}{4}x^4 + x^2 y
\end{aligned}
$$



$$
\begin{aligned}
\dot{x} &= y + x^2 \\
\dot{y} &= x^2 + xy
\end{aligned}
$$

fig. 7a

fig. 7b

$$
\begin{aligned}
\dot{x} &= y + x^2 \\
\dot{y} &= x^2 - 3x^3 + 2xy
\end{aligned}
$$

$$H(x,y) = \tfrac{1}{2}y^2 - \tfrac{1}{3}x^3$$
$$a(x) = x^2$$
$$H_a(x,y) = \tfrac{1}{2}y^2 - \tfrac{1}{3}x^3 + \tfrac{2}{3}x^6 + x^3y$$

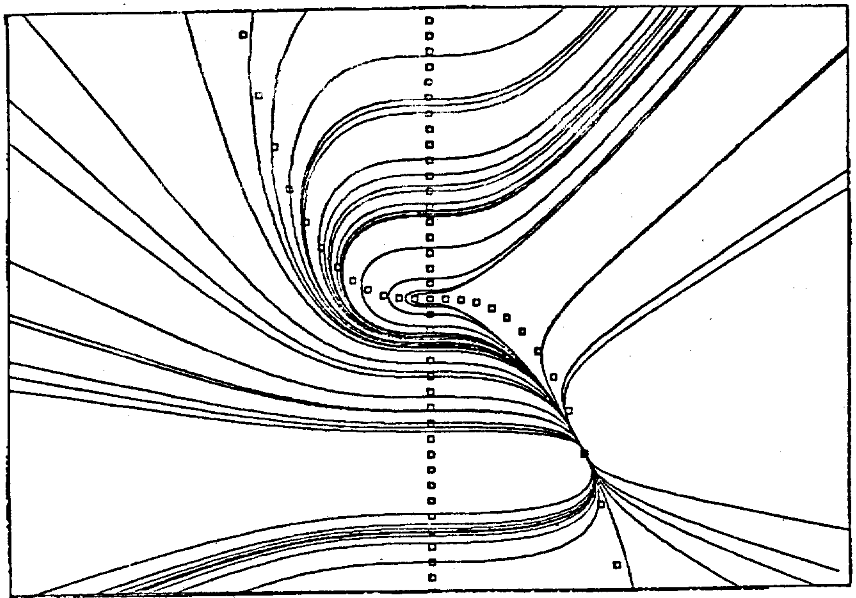


fig. 8a

$$\dot{x} = y + x^3$$
$$\dot{y} = x^2 + x^2y$$

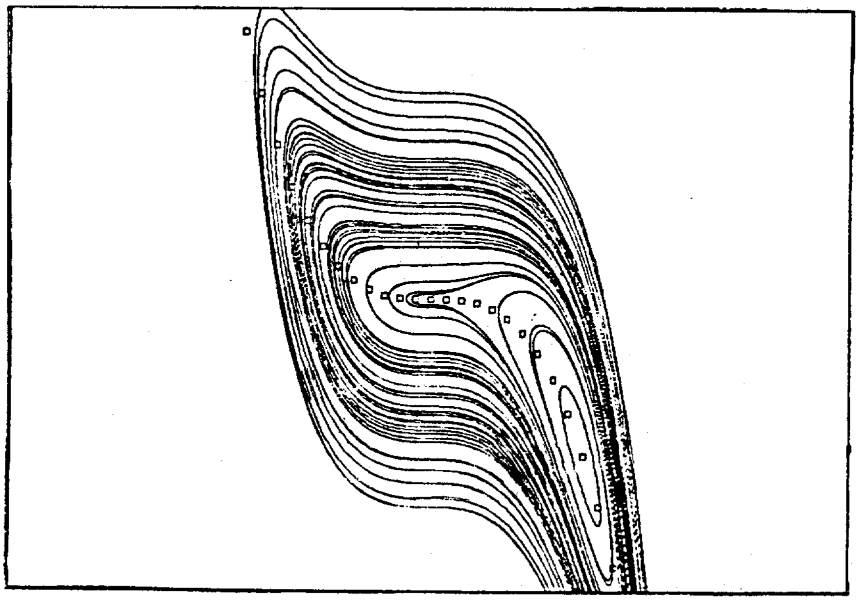
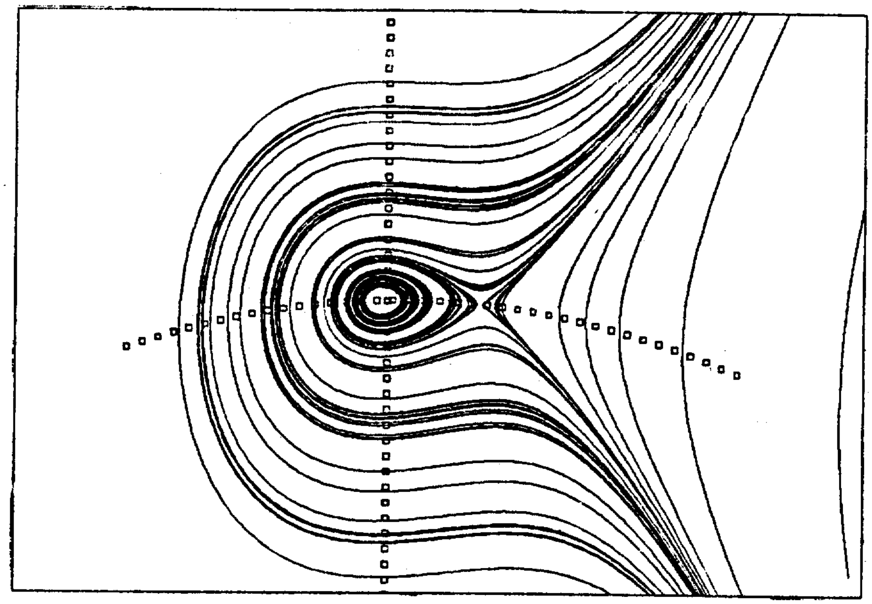


fig. 8b

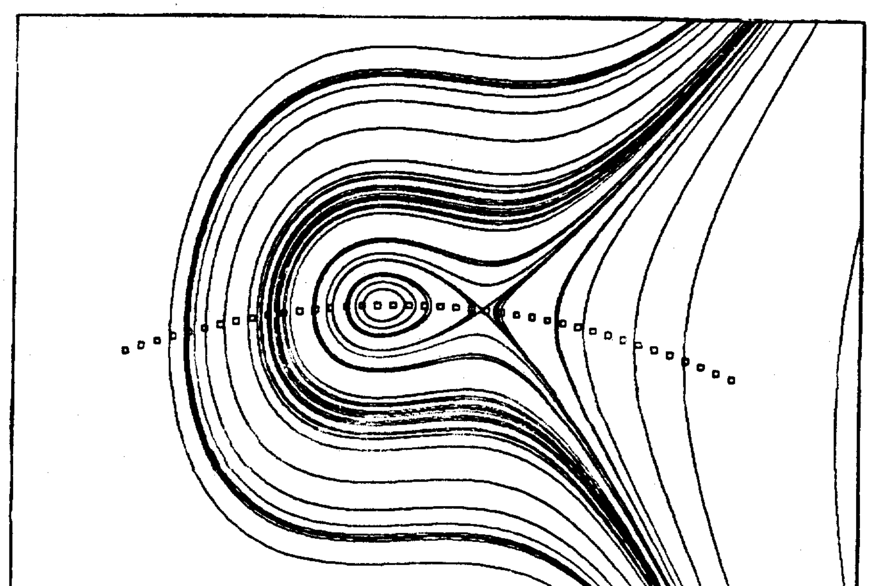
$$\dot{x} = y + x^3$$
$$\dot{y} = x^2 + 4x^5 + 3x^2y$$

$$
\begin{aligned}
H(x,y) &= \tfrac{1}{2}y^2 + .1x - .05x^2 - \tfrac{1}{3}x^3 \\
a(x) &= .05 + .1x \\
H_a(x,y) &= \tfrac{1}{2}y^2 + .1x - .0475x^2 - .325x^3 + .0075x^4 + (.05 + .1x)xy
\end{aligned}
$$



$$
\begin{aligned}
\dot{x} &= y + (.05 + .1x)x \\
\dot{y} &= -.1 + .1x + x^2 + (.13 + x)y
\end{aligned}
$$

fig. 9a



$$
\begin{aligned}
\dot{x} &= y + (.05 + .1x)x \\
\dot{y} &= -.1 + .095x + .975x^2 - .03x^3 + (.05 + .2x)y
\end{aligned}
$$

fig. 9b

3. A FOUR DIMENSIONAL EXAMPLE

3.1 The normal form and modified Hamiltonian

Consider the class of 4-dimensional dynamical systems with the nilpotent linearization

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

at the origin. The action of $\mathbf{R}$ on $\mathbf{R}^4$ associated to L_0^* is

$$t : (x, y, u, v) \mapsto (x, y + tx, u, v + tu).$$

The invariant functions are functions of x, u, and $uy - xv$. (There is a momentum map and dual pair interpretation of these invariant functions, just as in the planar case.) A basis for the space of equivariant vector fields is given by:

$$\begin{aligned} \mathbf{e}_1 &= (x, y, 0, 0) \\ \mathbf{e}_2 &= (0, 0, u, v) \\ \mathbf{e}_3 &= (u, v, 0, 0) \\ \mathbf{e}_4 &= (0, 0, x, y) \\ \mathbf{e}_5 &= (0, 1, 0, 0) \\ \mathbf{e}_6 &= (0, 0, 0, 1) \end{aligned}$$

where the vectors $\mathbf{e}_5$ and $\mathbf{e}_6$ are to be multiplied by functions of x and u alone (to avoid redundancies).

The Hamiltonian vector fields in normal form are of the form

$$X_H(x, y, u, v) = (y + U_y, -U_x, v + U_v, -U_u),$$

with Hamiltonian $H = \frac{1}{2}(y^2 + v^2) + U$, where U is an invariant function. The space of 'dissipative' equivariant vector fields is computed using the identity (5), which implies that a vector field $\mathbf{d} = (d^1, d^2, d^3, d^4)$ is perpendicular to (the higher order terms of) the Hamiltonian vector fields if

$$yd^1 - xd^2 + vd^3 - ud^4 = 0.$$

Thus, *the 'dissipative' vector fields are spanned by*

$$\begin{aligned} \mathbf{d}_1 &= (x, y, 0, 0) \\ \mathbf{d}_2 &= (0, 0, u, v) \\ \mathbf{d}_3 &= (u, 0, 0, y) \\ \mathbf{d}_4 &= (0, v, x, 0) \\ \mathbf{d}_5 &= (0, u, 0, -x). \end{aligned} \tag{14}$$

We now consider systems combining both Hamiltonian and 'dissipative' components. A special subclass of such systems consists of systems of the form

$$\begin{aligned} \dot{x} &= y + \alpha(x)x \\ \dot{y} &= -U_x(x,u) + \alpha(x)y \\ \dot{u} &= v + \beta(u)u \\ \dot{v} &= -U_u(x,u) + \beta(u)v. \end{aligned} \tag{15}$$

Define the modified Hamiltonian

$$H_a(x,y,u,v) = \tfrac{1}{2}\left(y^2+v^2\right) + U + \alpha xy + \beta uv$$
$$+\tfrac{1}{2}\left(\alpha^2x^2+\beta^2u^2\right) + \int^x \alpha^2 x\,dx + \int^u \beta^2 u\,du.$$

H_a has partial derivatives

$$\begin{aligned} \frac{\partial H_a}{\partial x} &= U_x - \alpha y + (2\alpha + \alpha_x x)(y+\alpha x) \\ &= -\dot{y} + (2\alpha + \alpha_x x)\,\dot{x} \\ \frac{\partial H_a}{\partial y} &= y + \alpha x \\ &= \dot{x} \\ \frac{\partial H_a}{\partial u} &= U_u - \beta v + (2\beta + \beta_u u)(v + \beta u) \\ &= -\dot{v} + (2\beta + \beta_u u)\,\dot{u} \\ \frac{\partial H_a}{\partial v} &= v + \beta u \\ &= \dot{u}. \end{aligned}$$

Hence

$$\begin{aligned} \dot{x} &= \frac{\partial H_a}{\partial y} \\ \dot{y} &= -\frac{\partial H_a}{\partial x} + \dot{R}_1 \\ \dot{u} &= \frac{\partial H_a}{\partial v} \\ \dot{v} &= -\frac{\partial H_a}{\partial u} + \dot{R}_2, \end{aligned}$$

where

$$R_1(x) = \int^x (2\alpha + \alpha_x x)\,dx \quad \text{and} \quad R_2(u) = \int^u (2\beta + \beta_u u)\,du$$

act as Rayleigh dissipation functions. *Thus, again equilibria are exactly the critical points of* H_a.

We compute that H_a has time derivative

$$\begin{aligned} \dot{H}_a &= (2a + a_x x)(y + ax)^2 + (2b + b_u u)(v + bu)^2 \\ &= (2a + a_x x)\dot{x}^2 + (2b + b_u u)\dot{u}^2. \end{aligned}$$

Remark: The 'angular momentum' term $xv - uy$ has time derivative

$$\begin{aligned}(xv-uy)^{\cdot} &= U_x u + U_y v - U_u x - U_v y + (a(x)+b(u))(xv-uy) \\ &= U_1 u - U_2 x + (a(x)+b(u))(xv-uy).\end{aligned}$$

3.2 Eigenvalues of L

We define the following quantities:

$$\begin{aligned}
\tau_1 &= R^1_x = 2\alpha + \alpha_x x \\
\tau_2 &= R^2_u = 2\beta + \beta_u u \\
\tau_3 &= \left(\alpha^2 x + U_x\right)_x \\
\tau_4 &= \left(\beta^2 u + U_u\right)_u \\
a_1 &= -\operatorname{tr} L = -(\tau_1 + \tau_2) \\
a_2 &= \tau_3 + \tau_4 + \tau_1\tau_2 \\
a_3 &= -(\tau_2\tau_3 + \tau_1\tau_4) \\
a_4 &= \det L = \tau_3\tau_4 - (U_{xu})^2 \\
\Lambda_1 &= a_3 - a_1 a_2 = \tau_1\tau_3 + \tau_2\tau_4 + \tau_1^2\tau_2 + \tau_1\tau_2^2 \\
\Lambda_2 &= a_3\Lambda_1 + (\operatorname{tr} L)^2 \det L = a_3(a_3 - a_1 a_2) + a_1^2 a_4 \\
&= -(U_{xu})^2(\tau_1+\tau_2)^2 - \tau_1\tau_2\left((\tau_3-\tau_4)^2 + a_1 a_3\right)
\end{aligned}$$

Using these quantities, we can use the following formulas to compute quantities R_i, $i = 1, \ldots, 4$ with the property that the signs of the real parts of the eigenvalues of L are the opposite of the signs of the R_i. (We obtained these formulas from Armbruster [1988].) We compute that

$$R_1 = -\frac{1}{\operatorname{tr}} L, \quad R_2 = -\frac{\operatorname{tr} L^2}{\Lambda_1}, \quad R_3 = \frac{\Lambda_1^2}{\operatorname{tr} L\,\Lambda_2}, \quad R_4 = \frac{\Lambda_2}{\det L\,\Lambda_1}. \tag{16}$$

Hence, the signs of eigenvalues are given by the signs of $\operatorname{tr} L$, Λ_1, $-\Lambda_2 \operatorname{tr} L$ and $-\Lambda_1\Lambda_2 \det L$. Thus, we have the following table

det L	tr L	Λ_1	Λ_2	Eigenvalues
+	+	+	+	$++--$
+	+	+	−	$++++$
+	+	−	+	$+--+$
+	+	−	−	$+-+-$
+	−	+	+	$-++-$
+	−	+	−	$-+-+$
+	−	−	+	$--++$
+	−	−	−	$----$
−	+	+	+	$++-+$
−	+	+	−	$+++-$
−	+	−	+	$+---$
−	+	−	−	$+-++$
−	−	+	+	$-+++$
−	−	+	−	$-+--$
−	−	−	+	$--+-$
−	−	−	−	$---+$

If D^2H_a is positive definite, then both τ_3 and τ_4 are positive; computing the necessary entries of the table, we find that if both τ_1 and τ_2 are positive (negative), i.e. if the derivatives of the Rayleigh functions are both positive (negative), then all eigenvalues of L have negative (positive) real parts. This provides an extension of the result for a two dimensional system stating that *a local minimum of H_a is locally attracting or repelling depending on the sign of the derivative of the Rayleigh function at the equilibrium.*

We now show that if *D^2H_a is indefinite, then the equilibrium must have at least one eigenvalue with positive real part.* If all eigenvalues have negative real part, then $\det L > 0$, $\operatorname{tr} L < 0$, $\Lambda_1 < 0$, and $\Lambda_2 < 0$. If D^2H_a is to be indefinite, τ_3 must be negative; τ_4 must also be negative if the positive determinant condition is to be satisfied. Since $\operatorname{tr} L < 0$, we can write $\tau_1 = -\tau_2 - \delta_1$ for some positive constant δ_1. We also write $\tau_3 = \tau_4 - \delta_2$ for some constant δ_2. The condition $\tau_3 < 0$ implies $\delta_2 > \tau_4$. Expressing Λ_1 and Λ_2 in terms of τ_2, τ_4, δ_1, and δ_2, we obtain

$$\Lambda_1 = \delta_1\delta_2 + \delta_2\tau_2 - \delta_1\tau_4 + \delta_1^2\tau_2 + \delta_1\tau_2^2$$

and

$$\Lambda_2 = -\delta_1^2 (U_{xu})^2 + \tau_2(\tau_2 + \delta_1)\left(\delta_2^2 + \delta_1\delta_2\tau_2 + \delta_1^2\tau_4\right).$$

If we are to satisfy both $\det L > 0$ and $\Lambda_2 < 0$, we must have

$$\tau_4(\tau_4 - \delta_2) \geq (U_{xu})^2 \geq \frac{\tau_2(\tau_2 + \delta_1)\left(\delta_2^2 + \delta_1\delta_2\tau_2 + \delta_1^2\tau_4\right)}{\delta_1^2}. \tag{17}$$

In particular, we must be able to satisfy

$$\tau_4\delta_1^2(\tau_4 - \delta_2) - \tau_2(\tau_2 + \delta_1)\left(\delta_2^2 + \delta_1\delta_2\tau_2 + \delta_1^2\tau_4\right) = -(\delta_1\tau_4 + \delta_2\tau_2)\Lambda_1 > 0. \tag{18}$$

If we wish to satisfy both this condition and the necessary condition $\Lambda_1 < 0$, we must have $\delta_1\tau_4 + \delta_2\tau_2 > 0$. This holds if and only if

$$\delta_1^* := -\frac{\delta_2\tau_2}{\tau_4} > \delta_1 > 0.$$

Since $\tau_4 < 0$, this implies that unless $\delta_2\tau_2 > 0$, the equilibrium cannot be stable.

We now consider the necessary conditions for $\Lambda_1 < 0$ and test to see if these can be satisfied under the preceding assumptions. We compute that Λ_1 changes sign at

$$\delta_1^{\pm} := \frac{\tau_4 - \tau_2^2 - \delta_2}{2\tau_2} \pm \sqrt{\left(\frac{\tau_4 - \tau_2^2 - \delta_2}{2\tau_2}\right)^2 - \delta_2}.$$

In addition, we note that when $\delta_1 = 0$, then $\Lambda_1 = \delta_2\tau_2 > 0$.

We consider first the case when both τ_2 and δ_2 are positive. Then, since $\tau_4 - \tau_2^2 - \delta_2 < 0$ and $\delta_2 > 0$, both δ_1^+ and δ_1^- are negative. Hence, for all positive values of d_1 the sign of Λ_1 equals the sign of Λ_1 for $\delta_1 = 0$, which is positive. Thus there must be at least one unstable direction if τ_2 and δ_2 are positive.

The final possibility is the case where both τ_2 and δ_2 are negative. In this case, $\delta_1^+ > 0$ and $\delta_1^- < 0$. Hence, to satisfy $\Lambda_1 < 0$, we must have $\delta_1 > \delta_1^+$. At the same time, if we are to satisfy (18), we must have $\delta_1 < \delta_1^*$. It follows that an allowable range of values for δ_1 exists only if $\delta_1^* > \delta_1^+$. We compute that $\delta_1^* = \delta_1^+$ for $\delta_2 = 0$ and $\delta_2 = \tau_4 + \tau_4^2/(\tau_4 - \tau_2^2)$. Since $\tau_4 < 0$, it follows that $\delta_1^* - \delta_1^+$ has constant sign on the permissible range

$$0 > \delta_2 > \tau_4 > \tau_4 + \frac{\tau_4^2}{\tau_4 - \tau_2^2}.$$

When $\delta_2 = \tau_4$, then

$$\delta_1^* - \delta_1^+ = \frac{-\tau_2 - \sqrt{\tau_2^2 - 4\tau_4}}{2} < 0. \tag{19}$$

Hence there is no possible value of δ_1 for which all of the necessary conditions can be satisfied. This proves the claim.

4. S^1 SYMMETRIC SYSTEMS

Now consider a system on $\mathbf{R}^4$ with the same nilpotent linearization as in §3, but now possessing S^1 symmetry; i.e. the equations of motion are equivariant under the group action

$$\theta : (x, y, u, v) \mapsto (\cos\theta\, x + \sin\theta\, u, \cos\theta\, y + \sin\theta\, v, -\sin\theta\, x + \cos\theta\, u, -\sin\theta\, y + \cos\theta\, v).$$

Functions which commute with both the S^1 action and the action induced by L are functions of $x^2 + u^2$ and $xv - uy$. The equivariant vector fields consist of gradients of functions of the invariant functions and multiples of the vector fields (x, y, u, v) and $(0, v, 0, -x)$ by such functions. We consider first the case in which the dissipative vector field is a multiple of $(0, v, 0, -x)$ alone. Thus the permissible vector fields in normal form are of the form

$$\begin{aligned} \dot{x} &= y + U_y \\ \dot{y} &= -U_x + ud \\ \dot{u} &= v + U_v \\ \dot{v} &= -U_u - xd, \end{aligned}$$

where U and d are functions of $x^2 + u^2$ and $xv - uy$.

In addition to the trivial solution $(x, y, u, v) = \mathbf{0}$, this system has a nontrivial steady state solution at any point (x, y, u, v) satisfying

$$\begin{aligned} y &= -U_y = uU_1 \\ v &= -U_v = -xU_1 \\ U_1 &= \frac{U_2^2}{2} \\ d &= 0, \end{aligned}$$

where U_1 and U_2 denote the derivatives of U with respect to the first and second components. Note that the first two equalities imply that $xy + uv = 0$.

Let $f : \mathbf{R} \to \mathbf{R}$ be the function determined by the relation

$$f(x^2 + u^2) = xV(x,u) - vY(x,u),$$

where $Y(x,u)$ and $V(x,u)$ are the solutions of $\dot{x} = \dot{u} = 0$. (This function can be computed explicitly for U_1 quadratic as described above.) Define the modified Hamiltonian

$$H_a = \tfrac{1}{2}(y^2 + v^2) + p(x^2 + u^2, xv - uy) + \int_0^{xv-uy} d(x^2 + u^2, s)ds - \int^{x^2+u^2} \int_0^{f(t)} d_1(t,s)ds\, dt.$$

H_a has partial derivatives

$$\begin{aligned} \frac{\partial H_a}{\partial x} &= 2x \int_{f(x^2+u^2)}^{xv-uy} d_1(u^2 + x^2, s)ds + vd(u^2 + x^2, vx - uy) \\ &\quad + vU_2(u^2 + x^2, vx - uy) + 2xU_1(u^2 + x^2, vx - uy) \\ \frac{\partial H_a}{\partial y} &= y - ud(u^2 + x^2, vx - uy) - uU_2(u^2 + x^2, vx - uy) \\ \frac{\partial H_a}{\partial u} &= 2u \int_{f(x^2+u^2)}^{xv-uy} d_1(u^2 + x^2, s)ds - yd(u^2 + x^2, vx - uy) \\ &\quad - yU_2(u^2 + x^2, vx - uy) + 2uU_1(u^2 + x^2, vx - uy) \\ \frac{\partial H_a}{\partial v} &= v + xd(u^2 + x^2, vx - uy) + xU_2(u^2 + x^2, vx - uy). \end{aligned}$$

As before, equilibria are critical points of H_a. In the special case that the potential U is a function of $x^2 + u^2$ alone, we have the second order system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -U_x + ud \\ \dot{u} &= v \\ \dot{v} &= -U_u - xd, \end{aligned}$$

with nontrivial solutions at points where $y = v = U_1 = d = 0$. In this case, the modified Hamiltonian takes on the simpler form

$$H_a = \tfrac{1}{2}(y^2 + v^2) + U(x^2 + u^2) + \int_0^{xv-uy} d(x^2 + u^2, s)ds.$$

If we consider the combination of both radial and 'angular' dissipation, then the equations of motion are of the form

$$\begin{aligned}
\dot{x} &= y + U_y + ax \\
\dot{y} &= -U_x + ud + ay \\
\dot{u} &= v + U_v + au \\
\dot{v} &= -U_u - xd + av,
\end{aligned}$$

where a is a function of $x^2 + u^2$ and $xv - uy$. This system has nontrivial equilibria when

$$\begin{aligned}
y &= -\frac{du + 2a^2 x}{2a} \\
v &= \frac{dx - 2a^2 u}{2a} \\
U_1 &= \frac{d^2 - 4a^4}{8a^2} \\
U_2 &= -\frac{d}{2a}.
\end{aligned}$$

Alternatively, the conditions for the existence of nontrivial equilibria may be expressed as

$$\begin{aligned}
y &= U_2 u \pm x\mu \\
v &= -U_2 x \pm u\mu \\
d &= \pm 2U_2\mu \\
a &= \mp\mu
\end{aligned}$$

where $\mu = \sqrt{U_2^2 - 2U_1}$.

4.1 $O(2)$ symmetric systems

As a further simplification, we can specialize to the case of a system with $O(2)$ symmetry, which eliminates the $xv - uy$ terms which were possible in the S^1 case. The $O(2)$ symmetric case has nontrivial equilibria when

$$\begin{aligned}
U' &= -\tfrac{1}{2}a^2 \\
y &= -ax \\
v &= -au.
\end{aligned}$$

Letting $\ell = x^2 + u^2$, the modified Hamiltonian takes the form

$$H_a = \tfrac{1}{2}\left(y^2 + v^2\right) + U + a(xy + uv) + \tfrac{1}{2}a^2\ell + \int^{\ell} a(s)^2 ds.$$

The linearized system has characteristic equation

$$0 = \lambda^4 - 2(2a + a'\ell)\lambda^3 + 4(a^2 + (a^2 + U')'\ell)\lambda^2 - 8a(\tfrac{1}{2}a^2 + U')'\ell\lambda$$

and eigenvalues

$$0, 2a, \text{ and } (a\ell)' \pm \tfrac{1}{2}\sqrt{((a\ell)')^2 - 4(\tfrac{1}{2}a^2 + U')'\ell}. \qquad (20)$$

It is clear from (20) that the equilibrium will be stable only if the following conditions hold:
(i) $a > 0$

(ii) $(a\ell)' < 0$

(iii) $\left(\frac{1}{2}a^2 + U'\right)' > 0$.

Combining condition (iii) with the condition $\frac{1}{2}a^2 + U' = 0$ required for equilibrium, we see that the function $\frac{1}{2}a^2 + U'$ must change sign from negative to positive as ℓ increases through its value at equilibrium. Similarly, we note that $-a\ell = xY(x,u) + uV(x,u)$ must be a positive increasing function at a stable equilibrium point.

The second variation of the Hamiltonian has characteristic equation

$$\begin{aligned} 0 \;=\; & \lambda^4 - 2\left(1 + a^2 + 2(\tfrac{1}{2}a^2 + U')'\ell + 2(a\ell)'a'\ell\right)\lambda^3 \\ & + \left((1+a^2)^2 + 4(2+a^2)(\tfrac{1}{2}a^2 + U')'\ell + 4a'(1+a^2)(a\ell)'\ell\right)\lambda^2 \\ & -4(1+a^2)(\tfrac{1}{2}a^2 + U')'\ell\lambda \end{aligned}$$

and eigenvalues

$$0, 1 + a^2, \tfrac{1}{2}\left(\mu \pm \sqrt{\mu^2 - 16(\tfrac{1}{2}a^2 + U')'\ell}\right),$$

where

$$\mu = 1 + (a + 2a'\ell)^2 + 4(\tfrac{1}{2}a^2 + U')'\ell.$$

If the equilibrium is stable, then $(\frac{1}{2}a^2 + U')' > 0$, which implies $\mu > 0$ and all eigenvalues of D^2H_a (except the zero eigenvalue, of course) have positive real part. Thus, to some extent, the analysis in the planar case can be expected to generalize to the higher dimensional case. In future work, we intend to turn attention to the analysis of specific systems, such as pseudo-rigid bodies (see Lewis and Simo [1989]).

Acknowledgements. This paper was directly inspired by some comments of Marty Golubitsky. We also thank Dieter Armbruster, John Guckenheimer, Phil Holmes, Peter Olver, and Jan Sanders for helpful comments.

REFERENCES

Armbruster D. [1988] Private communication.

Elphick C., Tirapegui E., Brachet M.E., Coullet P., and Iooss G. [1987] A simple global characterization for normal forms of singular vector fields. *Physica* **29D**. 95–127.

Golubitsky M. and Stewart I. [1987] Generic bifurcation of Hamiltonian systems with symmetry *Physica* **24D**. 391–405.

Guckenheimer J. and Holmes P. [1983] **Nonlinear oscillations, dynamical systems and bifurcation of vector fields.** Springer Applied Math Series.

Lewis D. and Simo J.C. [1989] Nonlinear stability of pseudo-rigid bodies. (preprint)

Marsden J.E. and McCracken M. [1976] **The Hopf bifurcation and its applications.** Springer Applied Math Series **19**.

Marsden J.E. and Weinstein A. [1983] Coadjoint orbits, vortices and Clebsch variables for incompressible fluids. *Physica* **7 D**. 305–323.

Olver P.J. and Shakiban C. [1988] Dissipative decomposition of ordinary differential equations. *Proc. Roy. Soc. Edin.* **109A**. 297–317.

Schmidt D. [1976] Hopf's bifurcation theorem and the center theorem of Liapunov. **The Hopf bifurcation and its applications.** Springer Applied Math Series **19**. 95–103.

Weinstein A. [1984] The local structure of Poisson manifolds. *J. Diff. Geom.* **18**. 523–557.