HORIZONTAL LIN CONSTRAINTS, CLEBSCH POTENTIALS AND VARIATIONAL PRINCIPLES ON PRINCIPAL FIBER BUNDLES

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1. INTRODUCTION.

The geometrical description of Lin constraints and variational principles in terms of Clebsch variables proposed recently by Cendra & Marsden [1] can be generalized in a way including among its wide range of applications those systems defined not only on configuration spaces which are products of Lie groups and vector spaces, but with configuration spaces being non-trivial principal fiber bundles with structural group G.. This generalization aims to cope with problems such as for example, fluids with free boundaries or Yang-Mills fields.

Let L be a Lagrangian defined on the tangent bundle of a Lie group G and assume that L is invariant under the natural action of the group G on TG lifted from the left (or right) translations of G on itself. This permits the definition of a Lagrangian $L_{\mathfrak{g}}$ in body (or space) coordinates, that is, in the Lie algebra of G, which will be denoted in what follows by $\mathfrak{g}$. One discovers that it is impossible to get the equations of motion in $\mathfrak{g}$ induced from those defined in TG by L using the naive variational principle associated to $L_{\mathfrak{g}}$. The reason for this phenomena is that variational principles do not behave naturally with respect to the geometrical process involved in reduction. The first method proposed historically to overcome this difficulty consisted in introducing a "Clebsch representation" for the system $L_{\mathfrak{g}}$ in $\mathfrak{g}$, thereby extending the reduced space and introducing a new

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Lagrangian providing the correct equations of motion. It is plausible that if we use the same idea in the context of configuration spaces which are non-trivial principal fiber bundles, we would have to describe separately the variations along the vertical ('group') directions and the 'horizontal' (base) directions. That means that we have to choose a connection A in order to separate group degrees of freedom from the base dynamics. Obviously such a choice will have non-trivial global consequences in the description of the reduced dynamics in the same way as the equations of motion for a particle moving in a Yang-Mills field depend on the connection we choose in the gauge fiber bundle of the system (see [2] and references therein).

So far, the main point is how we should incorporate this idea to the extended Lagrangian picture. This is done by changing the Lin constraint used in the trivial bundle case [1] by using a horizontal version of it, in such a way that it decouples the vertical and the horizontal degrees of freedom. This will be thouroughly described in § 3. § 2 will be devoted to the description of some notation and ideas about principal and associated fiber bundles, and finally in § 4 we will show the explicit form of the equations of motion for invariant Lagrangians.

2. CLEBSCH VARIABLES AND ASSOCIATED BUNDLES.

Let P be a principal fiber bundle with base space B, structural group G (acting on the right) and projection π. Let ρ be a linear action of G on a vector space V, i.e. $\rho: G \to GL(V)$ is a group homomorphism. We define the bundle $P \times_G V$ with fiber V and base B associated with P and the action of G on V by $(P \times V)/G$ where $(p,a)g = (pg, g^{-1}a)$ for every $(p,a) \in P \times V$ and $g \in G$. We will denote the equivalence class of (p,a) as $[p,a]$ or even pa for short, and define $\pi_V: P \times_G V \to B$ by $\pi_V(p,a) = \pi(p)$. Denoting by pr_1 the projection of $P \times V$ into the first factor and by Π the canonical projection from $P \times V$ into $P \times_G V$, we get the following conmutative diagram, showing us that pr_1 is a principal bundle isomorphism along the map π_V:

$$\begin{array}{ccc} P \times V & \xrightarrow{pr_1} & P \\ \Pi \downarrow & & \downarrow \pi \\ P \times_G V & \xrightarrow[\pi_V]{} & B \end{array}$$

Moreover, for each $p \in P$ the map i_p sending V into the fiber of $P \times_G V$ over the point $\pi(p)$ by $a \to pa$ is a linear isomorphism of vector spaces and satisfies $i_p = i_{pg} \circ \rho(g^{-1})$. The tangent map $Ti_p(a): T_a V (=V) \to T_{pa} \pi_V^{-1}(\pi(p))(=V)$ is a linear map denoted simply by p, i.e. $Ti_p(a)(\dot{a}) = p\dot{a}$. Likewise, we can define a map $i_a: P \to P \times_G V$ for each $a \in V$ by $i_a(p) = pa$. Notice that if the image of a fiber of P in $P \times_G V$ is an embedding, the tangent map

$Ti_a(p){:}T_pP \to T_{pa}P\times_G V$ is an injective linear mapping denoted in what follows by a, i.e. $Ti_a(p)(\dot p) = a(\dot p)$ for every $\dot p \in T_pP$. Finally if we are given two curves $(p(t),a(t))$ in $P\times V$, the tangent vector to the curve $p(t)a(t)$ at the point pa is given by $T_a i_p(\dot a) + T_p i_a(\dot p) = a\dot p + p\dot a$.

Given a connection A on P, a parallel transport operation is induced in any of its associated bundles by the projection under Π of the equivariant distribution of parallel planes H defined by A in TP, $H_p = \mathrm{Ker}A_p$. For every $p\in P$ we have the decomposition $T_pP = H_p \oplus V_p$, where V_p is the vertical subspace of TP at p, so a horizontal subspace H_V at the point pa in $P\times_G V$ is spanned by tangent vectors to curves of the form $p(t)a(t)$ with $p(t)$ horizontal, i.e. $A(\dot p)=0$. The vertical subbundle of the bundle $T(P\times_G V)$ is $\mathrm{Ker}T\pi_V$, that means that the vertical subspace at the point pa is just $T_{pa}\pi_V(pa){=}V$. Using that identification we can realize the horizontal distribution H_V as the Kernel of the V-valued 1-form A_V, which is defined as follows:

$$A_V(pa)(X) = X - X^h = X^v$$

where X^h is the horizontal projection of X, using the decomposition $T_{pa}(P\times_G V) = V\oplus H_V(pa)$. It is easy to see that for a fixed p, the curve $pa(t)$ is vertical, and that the horizontal component of $p(t)a$ for fixed a is $p^h(t)a$, ($p^h(t)$ denotes the horizontal lifting of $\pi p(t)$). For a given pair of curves $(p(t),a(t))$ on $P\times V$ the horizontal component of the tangent vector $\dot{\overline{pa}}$ at pa is $a\dot p^h$, and then $A_V(pa)(\dot{\overline{pa}}) = a(\dot p - \dot p^h) + p\dot a = aA_p(\dot p) + p\dot a$

3. HORIZONTAL LIN CONSTRAINTS AND VARIATIONAL PRINCIPLES.

As in the previous section let P be a principal fiber bundle, and $\rho{:}G\to GL(V)$ a linear representation of G on the vector space V. We assume that the following technical conditions are satisfied:

i. There is a G-invariant open set $U\subset V$.

ii. G is embedded into its orbit Ga for each $a\in U$.

Notice that the last condition means that G can be thought as a subset of V, because for each choice of a point $a_0\in U$, we can identify G with the orbit of G through a_0.

To properly handle the notions about variational principles in nontrivial principal fiber bundles we shall introduce some notation about the different spaces of curves which appear in the statement of the main result. We denote the space of curves in P with fixed origin $p_0\in P$ by $\Omega_{p_0}(P)$ and the space of curves with fixed endpoints p_0, p_1, by $\Omega_{p_0,p_1}(P)$. Likewise the space of curves in V with origin a_0 will be denoted by $\Omega_{a_0}(V)$ and the space of curves with fixed endpoints a_0,a_1 will be denoted by $\Omega_{a_0,a_1}(V)$. Curves in $\Omega_{p_0}(P)$ with the endpoint p_1 lying over a fixed base point $x_1\in B$ will be denoted by $\Omega_{p_0,x_1}(P)$.

It is clear that given a curve $p(t)$ there is a unique decomposition $p(t)=p^h(t).g^p(t)$, where $g^p(0)=e$ and $p^h(t)$ is horizontal with respect to a fixed connection A. Given a curve $p(t)\in\Omega_{p_o x_1}(P)$ and $a_o\in U$, there exists a unique curve denoted $a^p(t)$ in $\Omega_{a_o}(V)$ such that $p(t)a(t)$ is horizontal and $pa(0)=p_o a_o$, (this curve is defined by $a(t)=(g^p)^{-1}(t)a_o$ and notice that if $a_o\in U$, then the whole curve $a(t)$ is contained in U). Horizontal Lin constraints arise from the splitting of 'vertical' and 'horizontal' variations of curves via a connection to get a correct variational principle using a Clebsch representation. Fix a point p_o and a point p_1 in the fiber $\pi^{-1}(x_1)$, which represent the endpoints for the variational problem on P. Define the set of curves $(p(t),a(t))$ with $p(t)\in\Omega_{p_o x_1}(P)$ and $a(t)$ having a fixed starting point a_o, but a variable endpoint $a_1(t)$ varying in such a way that if $p(t)a(t)$ is horizontal, it must necessarily happen that $p(t_1)=p_1$. The curves $p(t)a(t)$ belonging to this set are in one-to-one correspondence with the curves in $\Omega_{p_o p_1}(P)$, which is just what we need to stablish a good variational principle for the reduced system.

The previous remarks and comentaries can be summarized in the following technical lemma.

Lemma. *Let P be a principal fiber bundle as before and $P\times_G V$ the associated vector bundle with respect to the linear action ρ of G on V. Let p_o, p_1 be a pair of points of P with basepoints $x_o x_1$ respectively and $a_o\in U$. Then there exists a unique map a_1 from $\Omega_{p_o}(P)$ into $U\subset V$ satisfying the following property: If $p(t)\in\Omega_{p_o x_1}(P)$, $a(t)\in\Omega_{a_o a_1(p)}(U)$ and $p(t)a(t)$ is horizontal, then necessarily $p(t_1)=p_1$.*

In particular, denoting by $\Omega_{p_o x_1 a_o a_1(p)}(P\times V)$ the set of curves $(p(t),a(t))$ such that $p(t_0)=p_0$, $\pi(p(t_1))=x_1$, $a(t_0)=a_0$ and $a(t_1)=a_1(p)$, and denoting by $\Omega^H_{p_o x_1 a_o a_1(p)}(P\times V)$ the subset of curves that are horizontal in $P\times_G V$, we see that the map $\Phi:\Omega^H_{p_o x_1 a_o a_1(p)}(P\times V)\to\Omega_{p_o p_1}(P)$ defined by $\Phi(p,a)=p$ is onto.

The proof is straightforward noticing that $a_1(p)$ defined by the formula $a_1(p)=i_{p_1}^{-1}(p^h(t_1)a_o)$ satisfies the required property. Notice that $a_1(p)=g_1^{-1}(p)a_o$ where $g_1(p)$ is defined by the equation $p(t_1)=p_1 g_1(p)$. That implies, because of the G-invariance of U, that $a_1(p)\in U$.

Since $A_V(\frac{d}{dt}(p(t)a(t)))=0$ is equivalent to $p(t)a(t)$ being horizontal, it follows that $\Omega^H_{p_o x_1 a_o a_1(p)}(P\times V)$ is the subset of $\Omega_{p_o x_1 a_o a_1(p)}(P\times V)$ defined by the constraint $A_V(\frac{d}{dt}(pa))=0$. Now we will introduce the constraint defined in this way in the variational principle using a Lagrange multiplier. Using the 'Clebsch representation space' $V\times V^*$ we will allow arbitrary variations of the curves in V^* and so the new term in the Lagrangian will look like

$$\langle A_V \frac{d}{dt}(p(t)a(t)), p(t)b(t)\rangle$$

with $(p(t),a(t),b(t))\in\Omega_{p_o,x_1,a_o,a_1(p)}(P\times V)\times\Omega(V^*)$.

Finally we include some comentaries and definitions involving G-invariant Lagrangians on P. L is G-invariant if $L(p,\dot p)=L(pg,\dot p g)$ with $(p,\dot p)\in TP$ and $g\in G$. The action of G on V induces an action ρ^{*-1} on V^*. This action permits us to define the associated bundle $P\times_G(V\times V^*)$ over B with fiber $V\times V^*$ with respect to the action (ρ,ρ^{*-1}). This bundle has a natural pairing $\langle,\rangle$ given by $\langle pa, p\alpha\rangle=\langle a,\alpha\rangle$, for every $(a,\alpha)\in V\times V^*$. We define the Lagrangian L^V on $T(P\times V\times V^*)$ by the formula:

$$L^V(p,\dot p;a,\dot a,\alpha,\dot\alpha)=L(p,\dot p)+\langle A_V(\frac{d}{dt}(pa)), p\alpha\rangle$$

or using the formulas previously obtained for A_V we get:

$$L^V(p,\dot p;a,\dot a,\alpha,\dot\alpha)=L(p,\dot p)+\langle p\dot a+aA_p(\dot p), p\alpha\rangle=L(p,\dot p)+\langle A_p(\dot p), J(a,\alpha)\rangle+\theta_{(a,\alpha)}(\dot a,\dot\alpha)$$

where $J:V\times V^*\to g^*$ is the momemtum map associated to the action (ρ,ρ^{*-1}) and θ is the canonical 1-form on $V\times V^*$.

From the Lagrange multiplier theorem we have the following

Theorem. *Fixing $p_o,p_1\in P$ and $a_o\in U$, the following assertions are equivalent:*

i. $p(t)\in\Omega_{p_o,p_1}(P)$ is a critical point of the functional $S:\Omega_{p_o,p_1}(P)\to$ R defined by

$$S[p]=\int_0^1 L(p,\dot p)dt$$

ii. $(p(t),a(t),\alpha(t))\in\Omega_{p_o,x_1,a_o,a_1(p)}(P\times V)\times\Omega(V^*)$ is a critical point of the functional $S^V:\Omega_{p_o,a_o,a_1(p)}(P\times V)\times\Omega(V^*)\to$ R defined by

$$S^V[p,a,\alpha]=\int_0^1 L^V(p,\dot p;a,\dot a,\alpha,\dot\alpha)dt$$

for some $(a(t),\alpha(t))\in\Omega_{a_o,a_1(p)}(V)\times\Omega(V^*)$

iii. If L is invariant these are also equivalent to the curve $(x(t),v(t),a(t),\alpha(t))\in\Omega_{x_o,x_1}(B)\times\Omega(g)\times\Omega_{a_o,a_1}(V)\times\Omega(V^*)$, where $a_1=i_{p_1}^{-1}(x^h(t_1)a_o$ is a critical point of the functional $\bar S^V$ induced from S^V defined by

$$\bar S^V[p,a,\alpha]=\int_0^1\left[L(p,\dot p)+\langle A(p,\dot p), J(a,\alpha)\rangle+\theta_{(a,\alpha)}(\dot a,\dot\alpha)\right]dt$$

All the terms in the definition of $\bar S^V$ are obviously G-invariant, and the curve $(p,\dot p)$ has a unique expression as $(x,\dot x,v)$ fixing p_o,p_1. This is easily proved recalling that from the decomposition $p(t)=p^h(t)g^p(t)$, we have

$$A(p,\dot p)=A(p^h g^p,\dot p^h g^p+p^h\dot g^p)=A(p^h g^p, p^h g^p(g^p)^{-1}\dot g^p)=v$$

where $v=(g^p)^{-1}\dot{g}^p$. Since for given $p_o \in P$ and $p(t) \in \Omega_{p_o}(P)$, the decomposition above is unique, we have the one-to-one map $\Omega_{p_o}(P) \to \Omega^H_{p_o}(P) \times \Omega(g)$ defined by $p \to (p^h, A(p,\dot{p}))$, whose inverse is obtained as follows: Given $v \in \Omega(g)$ find $g \in \Omega_e(G)$ such that $\dot{g} = g v$ (this amounts to solve a time dependent ordinary differential equation on G), then, set $p = p^h g$. Finally, the map $\Omega_{p_o p_1}(P) \to \Omega_{x_o x_1}(B) \times \Omega(g)$ given by $p \to (x, v)$ with $x = \pi(p)$ is a one-to-one correspondence proving that each curve $(p, \dot{p})$ has a unique reprsentation as $(x, \dot{x}, v)$.

4. EQUATIONS OF MOTION.

The next step in our discussion will be to derive the equations of motion for an invariant Lagrangian L using the variational principle provided by the previous theorem.

Consider first vertical variations of the curve p, i.e. let $p_\lambda(t)$ be $p(t)g_\lambda(t)$ with $g_\lambda \in \Omega_e(G)$. Notice that this variation is compatible with the set $\Omega_{p_o x_1 a_o a_1(p)}(P \times V)$. After some computations we get the following equation:

$$\frac{\partial L}{\partial v}(p, \dot{p}^h, v) = J(a, \alpha) \tag{1}$$

where $J(a,\alpha)$ is, as above, the momentum map.

If we consider horizontal variations of p, we will get as usual Euler-Lagrange's equations of motion:

$$\left[\left[\frac{\partial L}{\partial p} - \frac{d}{dt}\frac{\partial L}{\partial \dot{p}}\right](p, \dot{p})\right](\delta p^h) = 0 \tag{2}$$

where δp^h is an arbitrary horizontal vector at p. Writing equation 2 in local coordinates $(x, \dot{x}, v)$ we got

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial v} F(\dot{x}, .) \tag{2'}$$

where F is the curvature of the connection A. In other words, coordinates $(x, \dot{x}, v)$ correspond to the description of $(TP \times TV)/G$ as a fiber bundle over the adjoint bundle of P (with base TB and fiber g) with fiber TV.

Arbitrary variations on the curves $\alpha(t)$ in V^* lead to

$$\dot{a} - v_V(a) = 0 \tag{3}$$

Finally, variations of the curves $a(t)$ compatible with the set of curves we are dealing with, gives us

$$\dot{\alpha} - v_{V^*}(\alpha) = 0$$

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