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Covariant Poisson Brackets for Classical Fields

J. E. MARSDEN AND R. MONTGOMERY

Department of Mathematics, University of California, Berkeley, California 94720

P. J. MORRISON

Department of Physics and Institute for Fusion Studies, University of Texas, Austin, Texas 78712

AND

W. B. THOMPSON

Department of Physics, University of California, San Diego, La Jolla, California 92093

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Poisson brackets that are spacetime covariant are presented for a variety of relativistic field theories. These theories include electromagnetism, general relativity, and general relativistic fluids and plasmas in Eulerian representation. The examples presented suggest the development of a general theory; the beginnings of such a theory are presented. Our covariant bracket formalism provides a general setting for, amongst other things, clarifying the transition from the covariant formalism to the dynamical 3 + 1 Hamiltonian formalism of Dirac and Arnowitt, Deser, and Misner. We illustrate the relevant procedures with electromagnetism.

1. INTRODUCTION

The purpose of this paper is to show how to write the equations of some specific general relativistic field theories in covariant Poisson bracket form. Our approach is to proceed from explicit examples to some speculations on the structure of the underlying mathematical theory. For each of the examples, the field equations will be shown to be equivalent to equations of the form

$$\{F, S\} = 0, \tag{1.1}$$

where F is an arbitrary function of the fields and S is an action integral. The theories considered fall into two categories:

A. *Pure fields*, typified by gauge fields, where F and S in (1.1) are functions of the basic field variables ϕ^A and their conjugate momenta π^μ_A.

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B. *Media fields*, such as those describing relativistic fluids and plasmas in Eulerian representation, where F and S are just functions of the basic fields (without the addition of conjugate momenta).

In either case, the formalism has these features:

1. It involves an integration over both space and time.

2. Equation (1.1) satisfies the usual properties of Poisson brackets, such as Jacobi's identity, so the space of fields forms a Poisson manifold (see Dirac [9]).

3. S is the action (Lagrangian) suitably expressed as a function of the field variables.

For pure fields, the bracket contains a spacetime vector field V^μ which would correspond to the choice of a slicing, were a 3 + 1 Dirac–ADM decomposition performed (see Fischer and Marsden [11] and Isenberg and Nester [19] for reviews). For media fields, the bracket is a covariant extension of brackets of Lie–Poisson type that are now common for 3 + 1 relativistic and non-relativistic media fields (Iwinski and Turski [20], Morrison [37], Marsden and Weinstein [34], etc.).

We reiterate: the aim of this paper is purely observational. We simply observe that many relativistic field theories can be written in covariant Poisson bracket form. The basic mathematical underpinnings of the present work are not claimed to be worked out. To complete the basic theory, one should tie up the present results with the multisymplectic approach (see for example, Kijowski and Tulczyjew [25]).

It is anticipated that the covariant Poisson bracket formalism will be useful for calculation. In this direction, Kaufman and Holm [22] have used a covariant single particle bracket (due to Ignatiev) with success. The present formalism would be interesting to pursue along these and other lines.

To motivate some of our results, let us first consider the simple case of particle mechanics. We recall that the canonical Hamilton equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{1.2}$$

can be written as $\delta S = 0$, where the action

$$S[\gamma] = \int (p_i \dot{q}^i - H(q, p))\, dt \tag{1.3}$$

is regarded as a functional on Γ, the space of paths $\gamma(t) = (q(t), p(t))$ in phase space with appropriate boundary conditions (see, e.g., Arnold [2, p. 243]). Let us rewrite this variational principle in terms of a Poisson bracket on Γ. For functionals F and G of paths γ, set

$$\{F, G\}(\gamma) = \int \left(\frac{\delta F}{\delta q^i} \frac{\delta G}{\delta p_i} - \frac{\delta G}{\delta q^i} \frac{\delta F}{\delta p_i} \right) dt, \tag{1.4}$$

where the functional derivatives are defined by

$$\frac{d}{ds}F(\gamma+s\,\delta\gamma)|_{s=0}=\int\left(\frac{\delta F}{\delta\gamma}\delta\gamma\right)dt=\int\left(\frac{\delta F}{\delta q^i}\delta q^i+\frac{\delta F}{\delta p_i}\delta p_i\right)dt \tag{1.5}$$

for variations $\delta\gamma$ vanishing at the endpoints of γ. It is straightforward to check that $\delta S[\gamma]=0$; i.e., γ solves Hamilton's equation if and only if

$$\{F, S\}(\gamma)=0 \tag{1.6}$$

for all functionals F. It is this variational principle for Hamilton's equations that we shall generalize and apply to field theory. The covariant theory does not, of course, single out a time direction; rather space and time occur on equal footing, as will be seen below.

For general covariant brackets, the operation $F\mapsto\{F, S\}$ may be viewed as a variation of the action S along a direction in function space determined by F. In this sense, (1.1) can be viewed as a reformulation of the conventional variational approach to field theory. However, it is also a generalization and unifying principle, for conventional field theories treat electromagnetism and fluids, for example, in a rather different way.

Field theories of the traditional Euler–Lagrange form have been analyzed by symplectic methods in a fairly well-developed way. See, for example, Chernoff and Marsden [7], Dedecker [8], Abraham and Marsden [1], Kijowski and Tulczyjew [25] and Gimmsy [14] and references therein. However, a number of important field theories do not fit this mold, just as rigid body equations in body representation, i.e., in terms of the body angular momenta, admit a simple Poisson description, but not a symplectic one (since, for example, there are three equations); see for example, Sudarshan and Mukunda [41] and Holmes and Marsden [18] for these descriptions. There are also Poisson bracket formulations of the equations of fluids and plasmas in Eulerian description, as is now well known. For reviews, see Morrison [38], Holm and Kupershmidt [17], Marsden, Weinstein, Ratiu, Schmid and Spencer [35], Marsden and Morrison [29], and Marsden [28]. We shall show, however, that the relativistic version of these theories (either interacting with gravity or with a fixed background) do admit a simple covariant Poisson bracket description. Thus, by means of (1.1), we obtain a unifying principle for media as well as pure fields. For relativistic media field theories written in either Eulerian or Lagrangian description but in $3+1$ dynamical formulation, a Poisson bracket formulation is known; see Bialynicki-Birula and Iwinski [6], Iwinski and Turski [20], Bialynicki-Birula, Hubbard and Turski [5], Kunzle and Nester [26], Tulczyjew [43], Holm and Kupershmidt [17], Bao, Marsden and Walton [3], and Holm [15]. We note that the idea of covariant *canonical* variables has been around for awhile; it is presented, for example, in Barut [4].

Because of its generality, Eq. (1.1) is a natural starting point for obtaining $3+1$ reductions that result in Hamiltonian formalisms. To see how this works for our motivating particle mechanics example, we suppose that the F of (1.6) has the form

$$F = \int n(t)\, \mathscr{F}(p, q)\, dt, \tag{1.7}$$

where $n(t)$ is an arbitrary function of time and $\mathscr{F}$ is an arbitrary function of the q's and p's. Upon inserting (1.7) and (1.3) into (1.6) we obtain

$$\{F, S\}(\gamma) = \int n(t)(\dot{\mathscr{F}} - \{\mathscr{F}, H\}^{(3)})\, dt = 0, \tag{1.8}$$

where $\{\mathscr{F}, H\}^{(3)}$ is the conventional Poisson bracket. Since $n(t)$ is arbitrary,

$$\dot{\mathscr{F}} = \{\mathscr{F}, H\}^{(3)}. \tag{1.9}$$

This is, of course, equivalent to (1.2).

Starting with a symplectic formulation of classical field theory for pure fields, Gimmsy [14] shows how to obtain the $3+1$ adjoint Hamiltonian form of ADM and Fischer and Marsden [11, 12] (a field theoretic generalization of (1.9) with arbitrary spacetime slicings). The results here give an alternative setting for the same procedures. We illustrate this for electromagnetism in Section 2. The formulation can also be shown to yield the $3+1$ brackets of Bao, Marsden and Walton [3] for general relativistic fluids. In addition, the incorporation of covariant momentum maps should be possible for these covariant Poisson structures, as well as a covariant version of the reduction procedure (Marsden and Weinstein [33]). The latter would enable one, for example, to pass directly from a covariant Hamiltonian description of a relativistic fluid or plasma in material representation to one in spacetime representation (see Holm [15] for some results in this direction).

The plan of the paper is as follows. We will first present Maxwell's equations, the relativistic Maxwell–Vlasov system, general relativity and general relativistic fluids as examples. The covariant Poisson bracket form is exhibited explicitly in each case and the $3+1$ transition for electromagnetism is given. (Other examples are similar; the authors have treated additional cases, such as the Einstein–Maxwell, Yang–Mills or relativistic Liouville equations. For the non-relativistic Liouville equation, see Marsden, Morrison and Weinstein [30]). We conclude with some remarks on how these results suggest a general formulation of classical field theory.

2. Electromagnetism

To begin, we deal with Maxwell's equations on Minkowsi spacetime. For the usual Euler–Lagrange variational principles, see, for example, Jackson [21]. (The results are generalizable to arbitrary background spacetimes and to general gauge fields.) Let A denote the four vector potential, thought of as a one-form on Minkowski space. Let

$$F = dA, \qquad \text{i.e.,} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.1}$$

be the electromagnetic field tensor, where $\partial_\mu = \partial/\partial x^\mu$; $\mu = 0, 1, 2, 3$; and x^0, x^1, x^2, x^3 are the usual Minkowski coordinates.

The standard Lagrangian for the theory with an external current density J^μ is

$$L[A] = \int \mathscr{L} := \int (-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu) \, d^4x \tag{2.2}$$

where indices are raised and lowered using the Minkowski metric. In order to define a Legendre transformation, we introduce the covariant momentum variables, $\pi^{\mu\nu}$ as follows:

$$\pi^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu}. \tag{2.3}$$

The primary constraint manifold is defined to be the image of the map defined by (2.3), where $\mathscr{L}$ is regarded as defined on the space of A_μ and $\partial_\nu A_\mu$'s. This image space is the space of pairs of fields $(A_\mu, \pi^{\mu\nu})$ with $\pi^{\mu\nu}$ skew symmetric, and is our basic covariant phase space.

If F is a functional of A and π, one defines the functional derivatives as usual, being cautious about the constraint on $\pi^{\mu\nu}$ (just as one must be cautious about the div $B = 0$ constraint in the MHD and Maxwell–Vlasov equations). Namely, $\delta F/\delta \pi^{\mu\nu}$ is a *skew* tensor satisfying

$$\frac{d}{ds} F(\pi^{\mu\nu} + s\delta\pi^{\mu\nu})|_{s=0} = \int \frac{\delta F}{\delta \pi^{\mu\nu}} \delta\pi^{\mu\nu} d^4x$$

for $\delta\pi^{\mu\nu}$ a skew symmetric perturbation.

The covariant Poisson bracket of two functions F and G of A_μ and $\pi^{\mu\nu}$ is defined by

$$\{F, G\}_V (A, \pi) = \int \left(\frac{\delta F}{\delta A_\mu} \frac{\delta G}{\delta \pi^{\mu\nu}} - \frac{\delta G}{\delta A_\mu} \frac{\delta F}{\delta \pi^{\mu\nu}} \right) V^\nu d^4x, \tag{2.4}$$

where V^ν is an arbitrary vector field on spacetime, and functional derivatives are defined as usual. (The vector field V is related to the passage to dynamical equations—see Remark 3 in Section 6.) The bracket (2.4) can be written as

$$\{F, G\}_V (A, \pi) = \int \{F, G\}_\nu V^\nu d^4x$$

where (2.5)

$$\{F, G\}_\nu = \frac{\delta F}{\delta A_\mu} \frac{\delta G}{\delta \pi^{\mu\nu}} - \frac{\delta G}{\delta A_\mu} \frac{\delta F}{\delta \pi^{\mu\nu}}$$

is the associated density.

Let S be defined by the covariant analogue of (1.3), namely

$$S[A, \pi] = \int [\pi^{\mu\nu} A_{\mu,\nu} - H(A, \pi)]\, d^4x, \tag{2.6}$$

where

$$\begin{aligned} H(A, \pi) &= \tfrac{1}{4}\pi_{\mu\nu}\pi^{\mu\nu} + A_\mu J^\mu \\ &= \pi^{\mu\nu} A_{\mu,\nu} - \mathscr{L}. \end{aligned} \tag{2.7}$$

We claim that Maxwell's equations are equivalent to

$$\{F, S\}_V (A, \pi) = 0 \tag{2.8}$$

for all V and F. The statement (2.8) is clearly equivalent to

$$\frac{\delta S}{\delta \pi^{\mu\nu}} = 0 \quad \text{and} \quad \frac{\delta S}{\delta A_\mu} = 0, \tag{2.9}$$

i.e., to

$$\pi^{\mu\nu} = -(\partial^\mu A^\nu - \partial^\nu A^\mu) \quad \text{and} \quad \pi^{\mu\nu}{}_{,\nu} = -J^\mu, \tag{2.10}$$

which, together with $F = dA$, are the Maxwell equations. (We remark that the choice $\hat{S}(A, \pi) = \int [\frac{1}{2}\pi^{\mu\nu} F_{\mu\nu} - H(A, \pi)]\, d^4x$ would have yielded skew symmetry of π as one of the consequences of (2.8), but (2.6) seems to be a more useful version for the general theory; in fact one has, in general, a fair amount of freedom in the choice of S. We have followed an analogue of the form (1.3). In other cases, for example gravity, we do not follow such an analogue, but rather directly transcribe the Lagrangian into phase space variables.)

Let us now see how this relates to the standard 3 + 1 canonical theory in which A and $-E$ are conjugate variables (see the earlier references or Marsden and Weinstein [34], for example). We choose coordinates so the spacetime vector field V is

$$V = \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}. \tag{2.11}$$

To get a closed system, we choose $J = 0$ and rewrite S from (2.6) as

$$S = \int \left\{ \int [\pi^{i0} A_{i,0} - \mathscr{H}]\, d^3x \right\} dt, \tag{2.12}$$

where Latin indices run over 1, 2, 3 and where

$$\mathscr{H} = \tfrac{1}{2}(\pi_{ij}\pi^{ij} + \pi_{i0}\pi^{i0}). \tag{2.13}$$

Choose

$$F[\phi, \pi] = \int n(t)\, \bar{F}[A_i, \pi^{i0}]\, dt, \tag{2.14}$$

where $\bar{F}$ is a function of the $3+1$ variables A_i and π^{i0}. Clearly

$$\frac{\delta F}{\delta A_i} = n(t) \frac{\delta \bar{F}}{\delta A_i} \wedge dt$$
$$\frac{\delta F}{\delta \pi^{i\mu}} = \delta_{\mu 0} n(t) \frac{\delta \bar{F}}{\delta \pi^{i0}}. \tag{2.15}$$

Hence

$$\begin{aligned}\{F, S\}_{\partial/\partial x^0} &= \int \left[\int \left\{ \frac{\delta \bar{F}}{\delta A_i} \left(A_{i,0} - \frac{\partial \mathscr{H}}{\partial \pi^{i0}} \right) \right.\right. \\ &\quad \left.\left. + \left(\pi^{i0}{}_{,0} + \frac{\partial \mathscr{H}}{\partial A_i} \right) \frac{\delta \bar{F}}{\delta \pi^{i0}} \right\} d^3x \right] n(t)\, dt \\ &= \int [\dot{\bar{F}} - \{\bar{F}, \bar{H}\}^{(3)}]\, n(t)\, dt,\end{aligned} \tag{2.16}$$

where

$$\bar{H} = \int \mathscr{H} d^3x \tag{2.17}$$

and $\{,\}^{(3)}$ is the usual canonical Poisson bracket for functionals of the canonically conjugate variables A_i, $\pi^i = \pi^{i0}$. Since $n(t)$ is arbitrary (2.16) yields

$$\dot{\bar{F}} = \{\bar{F}, \bar{H}\}^{(3)}. \tag{2.18}$$

In deriving (2.18) we regard $\bar{H}$ as a function of just A_i and π^i. To do this, we re-insert the relation

$$\pi^{ij} = -(\partial^i A^j - \partial^j A^i).$$

This is an indication of a general and well-known feature of 3+1'ing: one must do more than simply replace the π^μ_A with $\pi_A = \pi^0_A$—the left out momenta π^i_A must be regarded as functions of the ϕ^A and π_A through the Legendre transformation.

If we set $\pi^i = -E^i$, $\pi^{ij} = \varepsilon^{ijk} B_k$, we thus recover the usual canonical formalism for electromagnetism. (The formalism with E and B as the basic variables—the Pauli–Born–Infeld bracket—requires a reduction by the gauge group of electromagnetism. See Marsden and Weinstein [34]).

3. The Relativistic Maxwell–Vlasov Equations

A special relativistic particle moves in an external electromagnetic field $F = dA$ according to the Lorentz force law

$$\frac{dx^\mu}{d\tau} = u^\mu; \qquad \frac{du^\mu}{d\tau} = \frac{e}{m} F^{\mu\nu} u_\nu, \tag{3.1}$$

where τ is the particle's proper time, e is its charge and m its rest mass. Declare

$$p_\mu = m u_\mu + \frac{e}{c} A_\mu \tag{3.2}$$

to be canonically conjugate to x^μ and set

$$H = \frac{m}{2} u^\mu u_\mu = \frac{1}{2m}\left(p^\mu - \frac{e}{c} A^\mu\right)\left(p_\mu - \frac{e}{c} A_\mu\right). \tag{3.3}$$

Thus (3.1) are equivalent to Hamilton's equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}; \qquad \frac{dp_\mu}{d\tau} = \frac{-\partial H}{\partial x^\mu} = \frac{e}{c} u^\nu \frac{\partial A_\nu}{\partial x^\mu}. \tag{3.4}$$

A relativistic plasma density $f(x, p)\, d^4x d^4p$ is constant along its particles' world lines:

$$\frac{df}{d\tau} = \frac{\partial f}{\partial x^\mu} u^\mu + \frac{e}{c} \frac{\partial f}{\partial p_\mu} u^\nu \frac{\partial A_\nu}{\partial x^\mu} = 0. \tag{3.5}$$

We may rewrite this as

$$\{f, H\}_{xp} = 0, \qquad \text{where } \{f, g\}_{xp} = \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial g}{\partial x^\mu}\frac{\partial f}{\partial p_\mu}. \tag{3.6}$$

The basic field for the Vlasov theory is the plasma phase space density function. As in Iwinski and Turski [20] and in the non-relativistic case (Morrison [37] and Marsden and Weinstein [34]) we define the bracket of two functionals F, G of f to be of Lie–Poisson form:

$$\{F, G\}(f) = \int f \left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\}_{xp} d^4x d^4p. \tag{3.7}$$

Let

$$S[f] = \int f(x, p)\, H(x, p)\, d^4x d^4p \tag{3.8}$$

so $\delta S/\delta f = H$. An integration by parts shows that the covariant bracket equation

$$\{F, S\}(f) = 0 \tag{3.9}$$

is equivalent to the relativistic Vlasov equation (3.5) [or (3.6)]. (As in Kaufman and Holm [22, p. 278), one must suitably restrict the fields and functionals so these integrals converge.)

The basic fields for the relativistic Maxwell–Vlasov equations are triples $(A_\mu, \pi^{\mu\nu}, f)$. The bracket of two functions of (A, π, f) is just the sum of (2.3) and (3.7):

$$\{F, G\}_V (A, \pi, f) = \int \left(\frac{\delta F}{\delta A_\mu}\frac{\delta G}{\delta \pi^{\mu\nu}} - \frac{\delta G}{\delta A_\mu}\frac{\delta F}{\delta \pi^{\mu\nu}}\right) V^\nu d^4x$$
$$+ \int f \left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\}_{xp} d^4x\, d^4p. \tag{3.10}$$

Let

$$S[A, \pi, f] = \int \left(\pi^{\mu\nu} A_{\mu,\nu} - \frac{1}{4}\pi_{\mu\nu}\pi^{\mu\nu}\right) d^4x$$
$$+ \int f(x, p)\frac{1}{2m}\left(p_\mu - \frac{e}{c}A_\mu\right)\left(p^\mu - \frac{e}{c}A^\mu\right) d^4x d^4p. \tag{3.11}$$

The field equations are

$$\{F, S\}_V (A, \pi, f) = 0 \tag{3.12}$$

for all F and all V. These are obviously equivalent to

$$\frac{\delta S}{\delta \pi^{\mu\nu}} = 0 \quad \text{and} \quad \frac{\delta S}{\delta A_\mu} = 0$$

and (3.13)

$$\int f \left\{\frac{\delta F}{\delta f}, \frac{\delta S}{\delta f}\right\}_{xp} d^4x d^4p = 0.$$

These are, in turn, equivalent to the relativistic Maxwell–Vlasov equations

$$\frac{\partial f}{\partial x^\mu} u^\mu + \frac{e}{c}\frac{\partial f}{\partial p_\mu} u^\nu \frac{\partial A_\nu}{\partial x^\mu} = 0$$

$$\partial_\mu F^{\mu\nu} = \frac{e}{c}\int u^\nu f(x, p)\, d^4p \tag{3.14}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Remarks. 1. Here we have not mentioned the obvious physical constraint that f vanishes unless $u^\alpha u_\alpha = -1$. This can be treated a posteriori since it can be shown that if $f(x, p)$ is a solution of the relativistic Vlasov equation (3.5) defined on all of xp space, then $g(x, p) = f(x, p)\, \delta(u^\alpha u_\alpha + 1)$, where $u_\alpha = (1/m)(p_\alpha - eA_\alpha/c)$, also is a solution. Alternatively, this constraint can be treated by restricting to density functions concentrated on the mass shell at the outset.

2. The bracket (3.7) is literally the Lie-Poisson bracket for the group of canonical transformations on xp space, the cotangent bundle of spacetime. Thus, this part of the bracket can be regarded as the reduction from canonical coordinates in Lagrangian representation by the particle relabeling group. In Lagrangian representation, the bracket has a form similar to (2.3); the vector field V^ν should disappear during reduction because one relabels by world lines, not by points (x, p). This is part of a general covariant reduction process which is planned for future development.

3. Another reduction process that we plan to pusue is the elimination of the gauge freedom for electromagnetism via reduction. This should re-express the bracket in terms of $F^{\mu\nu}$ and f alone and build in the div E constraint. When expressed dynamically, this should reproduce the known bracket for relativistic plasmas (Iwinski and Turki [20] and Bialynicki-Birula, Hubbard and Turski [5]), and should coincide with the non-relativistic bracket (Morrison [37], Marsden and Weinstein [34]).

4. General Relativity

The basic field variables we use for general relativity are the contravaiant symmetric two-tensor $g^{\alpha\beta}$ representing the dual metric and the "conjugate momenta" $\pi^\mu_{\alpha\beta}$ which are symmetric in α and β. We shall identify $\pi^\mu_{\alpha\beta}$ with the affine connection; this is standard, although not strictly true from the point of view of the Legendre transformation because of second derivatives of $g_{\alpha\beta}$ in the Lagrangian density (see Misner, Thorne and Wheeler [36, Chap. 21], Kijowski and Szczyrba [24], and Szczyrba [42]).

The Poisson brackets are of the same form as (2.3), namely

$$\{F, G\}_V (g, \pi) = \int \left(\frac{\delta F}{\delta g^{\alpha\beta}} \frac{\delta G}{\delta \pi^\mu_{\alpha\beta}} - \frac{\delta G}{\delta g^{\alpha\beta}} \frac{\delta F}{\delta \pi^\mu_{\alpha\beta}} \right) V^\mu d^4x. \tag{4.1}$$

Here functional derivatives are defined so that those with respect to $g^{\alpha\beta}$ are tensors:

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} F(g + \lambda \delta g) = \int \frac{\partial F}{\partial g^{\alpha\beta}} \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$$

whereas those with respect to $\pi^{\mu}_{\alpha\beta}$ are tensor densities:

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} F(\pi+\lambda\delta\pi)=\int \frac{\delta F}{\delta\pi^{\gamma}_{\alpha\beta}}\,\delta\pi^{\gamma}_{\alpha\beta}\,d^4x.$$

The action is the usual one written in terms of $g^{\alpha\beta}$ and $\pi^{\mu}_{\alpha\beta}$:

$$S[g,\pi]=\int g^{\alpha\beta}R_{\alpha\beta}(\pi)\sqrt{-g}\,d^4x-8\pi\int L^*\sqrt{-g}\,d^4x, \tag{4.2}$$

where $\sqrt{-g}\,d^4x$ is the volume element on spacetime,

$$R_{\alpha\beta}=\partial_{\lambda}\pi^{\lambda}_{\alpha\beta}-\partial_{\beta}\pi^{\lambda}_{\alpha\lambda}-\pi^{\lambda}_{\nu\beta}\pi^{\nu}_{\alpha\lambda}+\pi^{\lambda}_{\nu\lambda}\pi^{\nu}_{\alpha\beta} \tag{4.3}$$

and where

$$\frac{\delta}{\delta g^{\alpha\beta}}\int L^*\sqrt{-g}\,d^4x=T_{\alpha\beta}$$

is an (externally imposed) stress–energy tensor.

The covariant bracket equations are

$$\{F,S\}_V=0 \qquad \text{for all} \quad F,V:$$

i.e.,

$$\frac{\delta S}{\delta g^{\alpha\beta}}=0 \qquad \text{and} \qquad \frac{\delta S}{\delta\pi^{\mu}_{\alpha\beta}}=0. \tag{4.4}$$

The first equation yields the field equations

$$G_{\alpha\beta}=8\pi T_{\alpha\beta}, \tag{4.5}$$

where

$$G_{\alpha\beta}=R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R^{\lambda}{}_{\lambda}$$

while the second equation can be shown to imply

$$\begin{aligned}(g^{\alpha\beta}\sqrt{-g})_{;\mu}:=\partial_{\mu}(g^{\alpha\beta}\sqrt{-g})+\pi^{\alpha}_{\mu\nu}g^{\nu\beta}\sqrt{-g}\\ +\pi^{\beta}_{\mu\nu}g^{\nu\alpha}\sqrt{-g}-\pi^{\nu}_{\mu\nu}g^{\alpha\beta}\sqrt{-g}=0,\end{aligned} \tag{4.6}$$

which implies that π is the Levi–Civita connection:

$$\pi^{\mu}_{\alpha\beta}=\tfrac{1}{2}g^{\mu\nu}(\partial_{\alpha}g_{\nu\beta}+\partial_{\beta}g_{\nu\alpha}-\partial_{\nu}g_{\alpha\beta}). \tag{4.7}$$

(See Misner, Thorne, and Wheeler [36], Chap. 21, Sect. 2].)

We may think of (4.2) as depending parametrically on a set of matter and radiation fields ϕ^A through an additional Lagrangian L^*. To couple these fields to the gravitational fields $(g^{\alpha\beta}, \pi^{\gamma}_{\alpha\beta})$ we need a covariant bracket for the ϕ. Then the equations

$$\{F(\phi), S\}_\nu = 0$$

ought to be equivalent to

$$\nabla \cdot T = 0$$

and the field equations for the ϕ. This suggestion is followed in the next section.

Our treatment of general relativity is of course a reformulation of the standard Palatini variational principle. One interesting feature of our bracket formulation is that it allows an interesting coupling with media fields, as we shall see.

On the negative side, our choice of how to write S in (4.2) is somewhat ad hoc following the Palatini formalism and not as close to the form (1.3) as one might like. Also, the correspondence between the covariant bracket (4.1) and the canonical Dirac–ADM bracket through a $3+1$ process is suggestive, but it has not been worked out. Difficulties of this or equivalent sorts are common to all the canonical or symplectic treatments of general relativity we know of.

5. General Relativistic Fluids

We consider a perfect adiabatic fluid coupled to gravity; see Misner, Thorne, and Wheeler [36, Chap. 22] for background. One can similarly treat, we presume, plasmas coupled to general relativity (the Maxwell–Einstein–Vlasov system) or charged general relativistic fluids or general relativistic MHD.

The basic fluid quantities are the following *scalar* fields:

ρ = fluid mass-energy per unit rest three volume

n = baryon number density per unit rest three volume

σ = entropy per unit rest three volume

p = pressure in a rest frame

s = entropy per baryon

μ = relativistic inertial mass per unit rest three volume.

We have the relations

$$\sigma = ns \qquad \text{and} \qquad u = p + \rho. \tag{5.1}$$

The equation of state has the form

$$\rho = \rho(n, \sigma) \tag{5.2}$$

and the pressure is determined by the Legendre transform

$$p = n\frac{\partial\rho}{\partial n} + \sigma\frac{\partial\rho}{\partial\sigma} - \rho. \tag{5.3}$$

The basic fluid variable are taken to be

$$n, \sigma, \text{ and } M_\alpha = \mu u_\alpha.$$

Here u^α is the four velocity of the fluid, which satisfies $u^\alpha u_\alpha = -1$, i.e., $M^\alpha M_\alpha = \mu^2$. This constraint is to be imposed *after* functional derivatives, i.e., variations are taken. Here indices are raised using the Lorentz dual metric $g^{\alpha\beta}$. The constraint $u^\alpha u_\alpha = -1$ can either be imposed directly, as we do, or be viewed as a constraint in the sense of Dirac associated to the gauge symmetry of curve reparametrizations. (The latter requires some work on covariant momentum maps—see Section 6 below.)

The fluid brackets are taken to be Lie–Poisson with a structure similar to that in the non-relativistic case (Morrison and Greene [39], Dzyaloshinskii and Volovick [10]):

$$\begin{aligned}\{F, G\}(M, n, \sigma) = \int d^4x\,\sqrt{-g}\Big[&M_\alpha\left(\frac{\delta G}{\delta M_\beta}\partial_\beta\frac{\delta F}{\delta M_\alpha} - \frac{\delta F}{\delta M_\beta}\partial_\beta\frac{\delta G}{\delta M_\alpha}\right)\\ &- n\left(\frac{\delta G}{\delta M_\alpha}\partial_\alpha\frac{\delta F}{\delta n} - \frac{\delta F}{\delta M_\alpha}\partial_\alpha\frac{\delta G}{\delta n}\right) - \sigma\left(\frac{\delta G}{\delta M_\alpha}\partial_\alpha\frac{\delta F}{\delta\sigma} - \frac{\delta F}{\delta M_\alpha}\partial_\alpha\frac{\delta G}{\delta\sigma}\right)\Big].\end{aligned} \tag{5.4}$$

The Lie algebra underlying this Lie–Poisson bracket is a semi-direct product of vector fields and (densities × densities), similar to the nonrelativistic case (see Marsden [27], Holm and Kupershmidt [16] and Marsden *et al.* [35]). Here, functional derivatives are defined to be vectors or scalars, *not* densities:

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} F(M + \lambda\delta M) = \int \frac{\delta F}{\delta M_\alpha}\,\delta M_\alpha\sqrt{-g}\,d^4x$$

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} F(n + \lambda\delta n) = \int \frac{\delta F}{\delta n}\,\delta n\sqrt{-g}\,d^4x$$

and

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} F(\sigma + \lambda\delta\sigma) = \int \frac{\delta F}{\delta\sigma}\,\delta\sigma\sqrt{-g}\,d^4x.$$

We note that the two minus signs in (5.4) are in apparent disagreement with the non-relativistic and 3 + 1 version of the theory (see the above references and Bao, Marsden and Walton [3, Eq. (1C.13]). However, when the covariant theory is decomposed into its $3+1$ parts, this discrepancy should disappear (for example, when a bracket of vector fields on spacetime is decomposed, the result looks like a semi-direct product bracket, but with a relative sign switch due to the signature $(+++-)$ of the spacetime metric; cf. Fischer and Marsden [12, Appendix II].)

For the coupled system we use the variables

$$(g^{\alpha\beta}, \pi^{\mu}_{\alpha\beta}, M_\alpha, n, \sigma)$$

and use the bracket (5.4) plus (4.1). For the action we take

$$S[g, \pi, M, n, \sigma) = \int g^{\alpha\beta} R_{\alpha\beta}(\pi) \sqrt{-g}\, d^4x$$

$$- 8\pi \int \left(\frac{-1}{2\mu} g^{\alpha\beta} M_\alpha M_\beta + V(n, \sigma) \right) \sqrt{-g}\, d^4x, \qquad (5.5)$$

where $R_{\alpha\beta}$ is given by (4.3) and

$$V(n, \sigma) = \tfrac{1}{2}[\, p(n, \sigma) - \rho(n, \sigma)]. \qquad (5.6)$$

We note that the fluid term in (5.5), when evaluated on the constraint set $M^\alpha M_\alpha = \mu^2$ is proportional to the integral of the pressure. The covariant bracket equations are

$$\{F, S\}_{V} = 0 \qquad (5.7)$$

for all F and V. Choosing $F = F(g, \pi)$ gives

$$\pi^{\mu}_{\alpha\beta} = \text{Levi-Civita connection of } g$$

and

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

where

$$T_{\alpha\beta} = \mu u_\alpha u_\beta + p g_{\alpha\beta}, \qquad (5.8)$$

as in the previous section. In getting $\delta S_{\text{fluid}}/\delta g^{\alpha\beta} = 8\pi T_{\alpha\beta}$, we used the constraint $g^{\alpha\beta} M_\alpha M_\beta = -\mu^2$ after taking the variation. Choosing $F = F(n)$ and $F = F(\sigma)$ gives

$$(nu^\alpha)_{;\alpha} = 0 \quad \text{and} \quad (\sigma u^\alpha)_{;\alpha} = 0, \qquad (5.9)$$

i.e., conservation of baryon number and entropy. (The apparent discrepancy in (5.8) by a factor of 2 is discussed in Bao, Marsden and Walton [3].) Finally, choosing $F = F(M_\alpha)$ gives $\nabla \cdot T = 0$, which, of course, also follows from (5.8) and the Bianchi identity.

6. General Canonical Field Theories

We sketch here a framework in which the canonical brackets (2.5) and (4.1) can be constructed and in which the Euler–Lagrange equations for a pure field are equivalent to the covariant bracket equations. As we have remarked this also covers, in principle, fluids and plasmas by reduction of this structure from Lagrangian (material) representation to Eulerian (spatial) representation. (In $3+1$) form, the connection between these is discussed in Holm [15].) General relativity, as usual, is anomalous: it is formally similar, but does not quite fit the scheme presented here.

Our fields are assumed to be sections of a vector bundle $\pi: Y \to X$ over a base manifold X (we take X to be spacetime—but for plasmas it is T^*(spacetime) or the mass hyperboloid therein). We suspect that most of what we describe also works for a general fiber bundle, but we have restricted to the vector bundle case for simplicity. The fields are described in local coordinates by $\phi^A(x^\mu)$, where A is a multi-index for field components and x^μ are spacetime coordinates. Let $\mathscr{L}$ be a given Lagrangian density defined on $J^1(Y)$, the first jet bundle of Y. Recall that the fiber $J^1_x(Y)$ of $J^1(Y)$ over a point $y \in T_x$ is

$$J^1_y(Y) = Y_x \otimes T^*_x X = \mathscr{L}(T_x X, Y_x). \tag{6.1}$$

The Lagrangian density of a field ϕ is locally given by $\mathscr{L}(\phi^A, \partial_\mu \phi^A)$. The field equations are the usual Euler–Lagrange equations for $\mathscr{L}$,

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^A)} = \frac{\partial \mathscr{L}}{\partial \phi^A}, \tag{6.2}$$

and we set

$$\pi^\mu_A = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^A)}. \tag{6.3}$$

We now describe (6.3) intrinsically (cf. Kijowski and Tulczyjew [25]). Let $\Lambda^4 X$ be the bundle of four forms (densities) over X so

$$\mathscr{L}: J^1(Y) \to \Lambda^4(X). \tag{6.4}$$

Let P be the bundle over X whose fiber at x is

$$P_x = (Y_x \otimes T^*_x X)^* \otimes \Lambda^4_x X \approx T_x X \otimes Y^*_x \otimes \Lambda^4_x X. \tag{6.5}$$

Describe P by local coordinates (ϕ^A, π^μ_A). The Legendre transformation is the fiber derivative of $\mathscr{L}$:.

$$F\mathscr{L}: J^1(Y) \to P$$

given locally by

$$(x^\mu, \phi^A, \partial_\mu \phi^A) \mapsto (x^\mu, \phi^A, \pi^\mu_A),$$

where π^μ_A is given by (6.3).

Let F and G be functionals of sections of P. Then we have a Poisson bracket

$$\{F, G\}_v (\phi, \pi) = \int_X \left(\frac{\delta F}{\delta \phi^A} \frac{\delta G}{\delta \pi^\mu_A} - \frac{\delta G}{\delta \phi^A} \frac{\delta F}{\delta \pi^\mu_A} \right) V^\mu d^4 x, \tag{6.6}$$

where $\delta F/\delta \phi^A$ is a section of $Y^* \otimes \Lambda^4 X \to X$ and $\delta G/\delta \pi^\mu_A$ is a section of $Y \otimes T^*X \to X$. They pair together to give a section of $T^*X \otimes \Lambda^4 X \to X$ which can be contracted with the vector field V and the resulting four form integrated over X. For each fixed V, the bracket makes the sections of P into a Poisson manifold.

The primary constraint set C is the image of the Legendre transformation. We will assume that it is a vector subbundle of P. This will be the case, for example, if $\mathscr{L}$ is quadratic in $\partial_\mu \phi^A$ and if its "kinetic matrix" $\partial^2 \mathscr{L} / \partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)$ has constant rank.

Let l: $P \to P$ be a smooth vector bundle projection with im $l = C$. For example, in electromagnetism, l would project any tensor density onto its skew-symmetric part. Then l^*: $Y \otimes T^*X \to Y \otimes T^*X$. Set $C^* = \text{im}\, l^*$, a subbundle of $Y \otimes T^*x$. This is a bundle dual to C, so that functional derivatives with respect to the constrained covariant momenta naturally take values in C^*:

$$D_\pi F \cdot \delta\pi = \int_X \frac{\delta F}{\delta \pi^\mu_A} \cdot \delta \pi^\mu_A,$$

where F is a functional of sections of C, $\delta\pi$ is a variation in C and $\delta F/\delta \pi^\mu_A$ takes values in C^*; thus $(\delta F/\delta \pi^\mu_A) \cdot \delta \pi^\mu_A$ is a density on X.

One may now define brackets of functionals on $\Gamma(C)$ (sections of C) by the same formula as before (6.6), however, where the $\delta/\delta \pi^\mu_A$ are interpreted as sections of C^*. These brackets satisfy all the conditions for Poisson brackets. The only non-obvious condition is the Jacobi identity. To check this, we extend functionals F on $\Gamma(C)$ to functionals $\tilde{F} = l^* F$ on $\Gamma(P)$ as follows:

$$\tilde{F}(\phi, \pi) = F(\phi, l\pi).$$

For a general extension $\hat{F}$, we have

$$l^* \frac{\delta \hat{F}}{\delta \pi} = \frac{\delta F}{\delta \pi}$$

by the fiber linearity of l. However, for our extension,

$$l^* \frac{\delta \tilde{F}}{\delta \pi} = \frac{\delta \tilde{F}}{\delta \pi} = \frac{\delta F}{\delta \pi},$$

since $l_\perp^*(\delta\tilde{F}/\delta\pi)=0$, where $l_\perp = \mathrm{id} - l$. Note that the bracket of two such extensions is again such an extension and hence

$$\{\tilde{F}, \tilde{G}\} = \{F, G\}^{\sim}. \tag{6.7}$$

Now Jacobi's identity follows since it holds for the extended functions.

Remarks. 1. Equation (6.7) is expressed by saying that $(\phi, \pi) \mapsto (\phi, l\pi)$ is a Poisson map. Thus the constrained brackets are the pull back of the full brackets by the projection $\Gamma(P) \to \Gamma(C)$. Note that the *injection* $\Gamma(C) \to \Gamma(P)$ is *not* a Poisson map for $C \neq P$. In fact,

$$\{F, G\} - \{\hat{F}, \hat{G}\} = \int \frac{\delta \hat{F}}{\delta \phi} l_\perp^* \frac{\delta \hat{G}}{\delta \pi} - \frac{\delta \hat{G}}{\delta \phi} l_\perp^* \frac{\delta \hat{F}}{\delta \pi},$$

for arbitrary extensions.

2. For general fiber bundles, or if C is not a vector bundle, the results just described require extension. This development should be done in conjunction with examples such as general relativistic fluids written in the Lagrangian (material) picture.

The "Hamiltonian density" is uniquely defined on C by

$$H(\phi, \pi) = \pi^\mu_A \partial_\mu \phi^A - \mathscr{L}(\phi^A, \partial_\mu \phi^A). \tag{6.7}$$

At first, the right-hand side is defined on $J^1(Y) \times C$. However, the partial derivative with respect to $\partial_\mu \phi^A$ is zero, so we get a well-defined density on C. Set

$$S[\phi, \pi] = \int_X [\pi^\mu_A \partial_\mu \phi^A - H(\phi, \pi)]\, d^4x \tag{6.8}$$

and note that

$$\frac{\delta S}{\delta \phi^A} = 0 \quad \text{and} \quad \frac{\delta S}{\delta \pi^\mu_A} = 0, \tag{6.9}$$

i.e.,

$$\{F, S\}_V = 0 \qquad \text{for all } F, V \tag{6.10}$$

reproduce respectively

$$\frac{\partial}{\partial x^\mu} \pi^\mu_A = -\frac{\delta H}{\delta \phi^A} \quad \text{and} \quad \partial_\mu \phi^A = \frac{\delta H}{\delta \pi^\mu_A}, \tag{6.11}$$

which are equivalent to the Euler–Lagrange equations (6.2).

Remarks. 1. In the above setting, only canonical brackets are described. Non-canonical brackets, such as those for fluids and plasmas, are expected to come from canonical brackets in Lagrangian representation as in the non-relativistic case by a covariant version of the reduction process. See Marsden, Ratiu and Weinstein [31, 32].

2. We conjecture that covariant momentum maps associated with a group action G on P should be defined to be maps $J: P \to \mathfrak{g}^* \otimes TX \otimes \Lambda^4 P$ where $\mathfrak{g}$ is the Lie algebra of G. These should be consistent with the covariant momentum maps defined in Gimmsy [14] and should include standard Noether identities.

As in Gimmsy [14], one can presumably show that for an appropriately covariant localized theory, J vanishes on solutions of the field equations and that these conditions $J = 0$ correspond to first class constraints in the sense of Dirac [9]. The momentum maps should play a key role in the reduction process, as in the nonrelativistic case (Marsden *et al.* [35].

3. As already noted, the mathematical development of a systematic $3+1$ analysis is incomplete. This requires further development of the theory along the lines of Remarks 1 and 2. Once this is done, the $3+1$ analysis should proceed as in Gimmsy [14]. In particular, the $3+1$ procedure applied to the covariant brackets and field equations should directly yield the dynamical Poisson brackets and the evolution equations in bracket form (which is equivalent to the adjoint form of Fischer and Marsden [11, 12]). As we saw in Section 2, the vector V in the bracket (6.6) plays an important role in the $3+1$ process. It corresponds to the arbitrariness in the choice of the direction of time and to the lapse and shift which appear in the dynamical formulation. Forming the variables ϕ^A and $\pi_A = \pi^\mu_A V_\mu$ is a first step in constructing conjugate variables for the $3+1$ formalism. Subsequently, one must also eliminate the so-called "atlas fields" (such as the temporal component of A in electromagnetism), as in Gimmsy [14].

4. The results of this paper also need to be studied with a view towards understanding limits and averaging (see, for example, Weinstein [44] and Similon, Kaufman and Holm [40]). For example, we presume that the fluid bracket (5.4) can be derived from the plasma bracket (3.10) in the cold plasma limit (Gibbons, Holm and Kuperschmidt [13]) and that, as in Marsden *et al.* [35], taking moments via reduction gives a Poisson map between these structures.

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