FORMAL STABILITY OF LIQUID DROPS WITH SURFACE TENSION

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Abstract

A plane circular liquid drop with radius r, surface tension τ and rotating with angular frequency Ω is shown to be formally stable, in the sense of a positive definite second variation of a combination of conserved quantities, if $\frac{3\tau}{r^3} > (\frac{\Omega}{2})^2$. The proof is based on the Energy-Casimir method and the Hamiltonian structure of dynamic free boundary problems.

0. Introduction

Since the pioneering work of Arnold [1966a,b,c] on the Hamiltonian formulation of incompressible fluid dynamics and nonlinear stability of certain equilibrium planar flows, the Energy-Casimir method has been applied to a number of fluid and plasma stability problems. This method generalizes the classical δW method primarily in its ability to deal with non-static flows; this is accomplished by the use of conserved quantities other than energy, such as angular momentum and generalized enstrophy. The reader is referred to the articles in Marsden [1984], Holm, Marsden, Ratiu and Weinstein [1985], Abarbanel, Holm, Marsden and Ratiu [1985], Holm, Marsden and Ratiu [1985], and Wan and Pulvirente [1985] for recent applications and additional references.

The general method, called the **Energy-Casimir method**, proceeds as follows: First we find a conserved quantity C such that $H + C$, where H is the energy, has a critical point at the equilibrium to be studied. Then the second variation $\delta^2(H + C)$ is calculated and tested for definiteness at the equilibrium. If it is definite, one refers to the equilibrium as being **formally stable**. Formal stability implies linearized stability; although many authors have claimed that it also implies nonlinear stability, it is known by example (Ball and Marsden [1984]) that additional estimates are required to justify this assertion. These are often provided by convexity estimates, as given in the aforementioned references,

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or by Sobolev type estimates, which can suffice for some semilinear equations (Marsden and Hughes [1983]) or in some one-dimensional problems such as KdV soliton stability (Benjamin [1972], Bona [1975]).

In this paper we find conditions which insure formal stability of a planar circular liquid drop of radius r, with surface tension τ and rotating with angular velocity Ω. The surface of the drop is a free boundary. The conserved quantities used are angular momentum and generalized enstrophy. The second variation is shown to be positive definite if $\frac{3\tau}{r^3} > \left(\frac{\Omega}{2}\right)^2$. In particular, one has linearized stability in the H^1 norm on fluid variations and the H^1 norm on boundary variations under these circumstances. (In future work the questions of global existence of smooth solutions near this equilibrium solution and (rigorous) nonlinear stability will be addressed.) Formal stability for the spherical drop in three dimensions and circular shear flow in an annulus are also discussed.

The paper is organized as follows. In section one the Hamiltonian structure for our free boundary problem (see equations 1.7) is recalled. The Poisson brackets are derived by the usual procedure of reduction, as in Marsden and Weinstein [1982,1983] and Marsden, Ratiu and Weinstein [1984a,b]. These results are reviewed from Lewis, Marsden, Montgomery and Ratiu [1985]. While they are not absolutely necessary for the stability results, they provide a useful setting. In the second section the first and second variation calculations are carried out and formal stability is deduced.

The two-dimensional results presented here are closely related to those given in Sedenko and Iudovich [1978], although we obtain a less restrictive condition relating the surface tension coefficient and the angular velocity than the one given in their paper. (We cannot check their calculations since many steps are obscure or omitted; our final answers differ.) We feel, however, that our approach has the advantage of fitting into the general framework of stability analysis outlined in Holm, Marsden, Ratiu and Weinstein [1985]. Sedenko and Iudovich, following work of Arnold [1965] for fixed boundary fluids, consider relative equilibrium restricted to the "Helmholtz layer" of equivorticial flows; these layers are essentially the symplectic leaves of the Poisson manifold $\mathcal{N}$ defined below. In our argument, rather than explicitly restricting our variations to a specific layer or leaf, we introduce the generalized enstrophy functions and angular momentum as Lagrange multipliers and allow our variations to range over all tangent vectors to the space $\mathcal{N}$. Formal stability of two dimensional free boundary problems have also been considered by Artale and Salusti [1984], who consider rotational gravity waves without surface tension.

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1. Poisson Bracket and Equations of Motion

The dynamic variables we consider are the free boundary Σ and the spatial velocity field $\mathbf{v}$, a divergence free vector field on the region D_Σ bounded by Σ. The surface Σ is an element of the set $\mathcal{S}$ of closed curves (respectively surfaces) in $\mathbf{R}^2$ (respectively $\mathbf{R}^3$) diffeomorphic to the boundary of a reference region D and enclosing the same area (respectively volume) as D. We let $\mathcal{N}$ denote the space of all such pairs $(\Sigma, \mathbf{v})$.

The Poisson bracket will be defined for functions $F, G : \mathcal{N} \to \mathbf{R}$ which possess **functional derivatives**, defined as follows:

i) $\frac{\delta F}{\delta \mathbf{v}}$ is a divergence free vector field on D_Σ such that

$$D_{\mathbf{v}}F(\Sigma,\mathbf{v})\cdot\delta\mathbf{v} = \int_{D_\Sigma} \langle \tfrac{\delta F}{\delta \mathbf{v}}, \delta\mathbf{v}\rangle dA, \tag{1.1}$$

where the (Fréchet) derivative $D_{\mathbf{v}}F$ is computed with Σ fixed.

ii) $\frac{\delta F}{\delta \varphi}$ is the function on Σ with zero integral given by

$$\tfrac{\delta F}{\delta \varphi} = \left\langle \tfrac{\delta F}{\delta \mathbf{v}}, \nu \right\rangle, \tag{1.2}$$

where ν is the unit normal to Σ. (The symbol φ represents the potential for the gradient part of $\mathbf{v}$ in the Helmholtz, or Hodge, decomposition.)

iii) $\frac{\delta F}{\delta \Sigma}$ is a function on Σ determined up to an additive constant as follows. A variation $\delta\Sigma$ of Σ is identified with a function on Σ representing the infinitesmal variation of Σ in its normal direction. It follows from the incompressibility assumption that $\delta\Sigma$ has zero integral. The zero integral condition is dual (with respect to the L_2 pairing on Σ) to the additive constant ambiguity of $\frac{\delta F}{\delta \Sigma}$. We can smoothly extend $\mathbf{v}$ to a neighborhood of Σ, making it possible to fix $\mathbf{v}$ while varying Σ. Thus we can define the partial derivative $D_\Sigma F(\Sigma,\mathbf{v})$, which may be shown to be independent of the extension of $\mathbf{v}$ as long as F is C^1 as $\mathbf{v}$ varies in the C^1 topology. We then let $\frac{\delta F}{\delta \Sigma}$ be the function determined up to an additive constant by

$$\int_\Sigma \tfrac{\delta F}{\delta \Sigma}\delta\Sigma\; ds = D_\Sigma F(\Sigma,\mathbf{v})\cdot\delta\Sigma. \tag{1.3}$$

As an example, we compute the functional derivative with respect to Σ of a function of the form $F(\Sigma) = \int_\Sigma f(\Sigma)\, ds$ for some smooth function f of $\mathbf{x}$ defined in a neighborhood of a given Σ. Let Σ_ϵ be a curve in $\mathcal{S}$ with tangent vector $\delta\Sigma$ at Σ and let η_ϵ be a curve in $Emb(\partial D, \mathbf{R}^2)$, the manifold of embeddings of ∂D into $\mathbf{R}^2$, such that $\frac{d}{d\epsilon}|_{\epsilon=0}\eta_\epsilon = [(\delta\Sigma)\nu]\circ\eta_0$. Let $f_\epsilon : \partial D \mapsto \mathbf{R}$ be given by $f_\epsilon(X) := f(\eta_\epsilon(X))$ for $X \in D$ and let $ds_\epsilon := \eta_\epsilon{}^* ds$. Define $D_\Sigma F(\Sigma)\cdot\delta\Sigma := \frac{d}{d\epsilon}|_{\epsilon=0}\int_{\partial D} f_\epsilon ds_\epsilon$. The functional derivative $\frac{\delta F}{\delta \Sigma}$, if it exists, is the function modulo constants such that $\int_\Sigma \frac{\delta F}{\delta \Sigma}\cdot\delta\Sigma\; ds = D_\Sigma F(\Sigma)\cdot\delta\Sigma$. We calculate

$$\begin{aligned} D_\Sigma F(\Sigma)\cdot\delta\Sigma &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_{\partial D} f_\epsilon(X) ds_\epsilon \\ &= \int_{\partial D} [df(\eta_0(X))\cdot\delta\Sigma\nu(\eta_0(X))ds_0 + (f\kappa\delta\Sigma)(\eta_0(X))ds_0] \\ &= \int_\Sigma (\tfrac{\partial f}{\partial \nu} + f\kappa)(\mathbf{x})\cdot\delta\Sigma(\mathbf{x})ds \end{aligned}$$

which follows from the change of variables formula and the formula for the first variation of arc length. Thus, in this case,

$$\tfrac{\delta F}{\delta \Sigma} = \tfrac{\partial f}{\partial \nu} + \kappa f. \tag{1.4}$$

We now define the Poisson bracket on $\mathcal{N}$ as follows. For functions F and G mapping $\mathcal{N}$ to $\mathbf{R}$ and possessing functional derivatives as defined above, set

$$\{F,G\} = \int_{D_\Sigma} \langle \omega, \tfrac{\delta F}{\delta \mathbf{v}} \times \tfrac{\delta G}{\delta \mathbf{v}} \rangle \, dA + \int_\Sigma \left(\tfrac{\delta F}{\delta \Sigma} \tfrac{\delta G}{\delta \varphi} - \tfrac{\delta G}{\delta \Sigma} \tfrac{\delta F}{\delta \varphi} \right) ds, \tag{1.5}$$

where $\omega = \operatorname{curl} \mathbf{v}$. For irrotational (potential) flow $\omega = 0$, and so this bracket reduces to the canonical bracket found by Zakharov [1968].

This Poisson bracket on $\mathcal{N}$ is derived from the canonical cotangent bracket on $T^*\mathcal{C}$, where, in the two-dimensional case, $\mathcal{C} = Emb_{vol}(D, \mathbf{R}^2)$ is the manifold of volume-preserving embeddings of a two-dimensional reference manifold D into $\mathbf{R}^2$, by reduction by the group $\mathcal{G} = Diff_{vol}(D)$, the group of volume-preserving diffeomorphisms of D (i.e. the group of particle relabelling transformations). Elements of $T^*\mathcal{C}$ are pairs (η, μ) where $\eta : D \to \mathbf{R}^2$ is an element of $\mathcal{C}$ and μ, the momentum density, is a divergence free one form over η ; i.e. to each reference point $X \in D$, μ assigns a one form on $\mathbf{R}^2$ based at the spatial point $x = \eta(X)$. We map $T^*\mathcal{C}$ onto $\mathcal{N}$ by the map $\Pi_{\mathcal{N}} : T^*\mathcal{C} \to \mathcal{N}$ which takes (η, μ) to $(\Sigma, \mathbf{v})$ such that $\Sigma = \partial(\eta(D))$ and $\langle \mathbf{v}(x), \mathbf{w}(x) \rangle = \mu(X) \cdot \mathbf{w}(x)$, for all vector fields $\mathbf{w}$ on D_Σ, where $x = \eta(X)$ and $\langle \, , \, \rangle$ is the Euclidean inner product. The map $\Pi_{\mathcal{N}}$ is invariant under the right action of $\mathcal{G}$ and so induces a bijection $\bar{\Pi}_{\mathcal{N}} : T^*\mathcal{C}/\mathcal{G} \to \mathcal{N}$ which is a diffeomorphism in the appropriate topologies. Thus $\mathcal{N}$ inherits a Poisson structure determined by the relation

$$\{F, G\} \circ \Pi_{\mathcal{N}} = \{F \circ \Pi_{\mathcal{N}}, G \circ \Pi_{\mathcal{N}}\}_{T^*\mathcal{C}}.$$

One computes the resulting bracket to be (1.5).

Remark. In some cases it may be necessary to use a more general Poisson bracket than that described above. While considerably more complicated, the generalized bracket has the advantage that it is defined for a larger class of functions. Of concern to us at present are the generalized enstrophy functions $C(\Sigma, \mathbf{v}) = \int_{D_\Sigma} \Phi(\omega) dA$, where ω is the vorticity, which we will use in the following stability analysis. These functions do not have functional derivatives of the form previously described.

We say that a function F on $\mathcal{N}$ has **generalised functional derivatives** if there exist

i) $\frac{\delta F}{\delta \Sigma}(\Sigma, \mathbf{v})$ a function on Σ determined up to a constant,

ii) $\frac{\delta \hat{F}}{\delta \mathbf{v}}(\Sigma, \mathbf{v})$ a divergence free vector field on D_Σ, and

iii) $\frac{\delta \check{F}}{\delta \mathbf{v}}(\Sigma, \mathbf{v})$ a vector field on Σ

such that

$$DF(\Sigma, \mathbf{v}) \cdot (\delta\Sigma, \delta\mathbf{v}) = \int_{D_\Sigma} \left\langle \tfrac{\delta \hat{F}}{\delta \mathbf{v}}, \delta \mathbf{v} \right\rangle dA + \int_\Sigma \left(\tfrac{\delta F}{\delta \Sigma} \, \delta\Sigma + \left\langle \tfrac{\delta \check{F}}{\delta \mathbf{v}}, \delta \mathbf{v} \right\rangle \right) ds$$

for all variations $(\delta\Sigma, \delta\mathbf{v})$. The functional derivatives $\frac{\delta \hat{F}}{\delta \mathbf{v}}$ and $\frac{\delta \check{F}}{\delta \mathbf{v}}$ are determined only up to the addition of a harmonic function, as may be seen by applying the divergence theorem.

The generalized bracket on $\mathcal{N}$ is

$$\begin{aligned}\{F,G\} = &\int_{D_\Sigma} \left\langle \omega, \tfrac{\delta \hat{F}}{\delta \mathbf{v}} \times \tfrac{\delta \hat{G}}{\delta \mathbf{v}} \right\rangle dA \\ &+ \int_\Sigma \Big(\left\langle \omega, \tfrac{\delta \hat{F}}{\delta \mathbf{v}} \times \tfrac{\delta \check{G}}{\delta \mathbf{v}} + \tfrac{\delta \check{F}}{\delta \mathbf{v}} \times \tfrac{\delta \hat{G}}{\delta \mathbf{v}} \right\rangle \\ &\quad + \left\langle \tfrac{\delta F}{\delta \Sigma}\nu, \tfrac{\delta \hat{G}}{\delta \mathbf{v}} \right\rangle + \left\langle \nabla p_F, \tfrac{\delta \check{G}}{\delta \mathbf{v}} \right\rangle - \left\langle \tfrac{\delta G}{\delta \Sigma}\nu, \tfrac{\delta \hat{F}}{\delta \mathbf{v}} \right\rangle - \left\langle \nabla p_G, \tfrac{\delta \check{F}}{\delta \mathbf{v}} \right\rangle \Big) ds, \end{aligned} \tag{1.6}$$

where p_F, a "pressure" associated with $\frac{\delta \hat{F}}{\delta \mathbf{v}}$, is the solution of the Dirichlet problem: $\Delta p_F = -\operatorname{div}((\nabla \mathbf{v}) \cdot \frac{\delta \hat{F}}{\delta \mathbf{v}}), p_F|\Sigma = \frac{\delta F}{\delta \Sigma} - \langle (\nabla \mathbf{v}) \cdot \frac{\delta \hat{F}}{\delta \mathbf{v}}, \nu \rangle$ and $(\nabla \mathbf{v}) \cdot \frac{\delta \hat{F}}{\delta \mathbf{v}}$ is determined by the relation $\langle \mathbf{u}, (\nabla \mathbf{v}) \cdot \frac{\delta \hat{F}}{\delta \mathbf{v}} \rangle = \langle (\mathbf{u} \cdot \nabla)\mathbf{v}, \frac{\delta \hat{F}}{\delta \mathbf{v}} \rangle$ for all vector fields $\mathbf{u}$ on D_Σ. Due to the non-uniqueness of the functional derivatives the generalized bracket is not well-defined for all pairs F, G with functional derivatives as given above; if, however, we require that either $\frac{\delta \check{F}}{\delta \mathbf{v}}$ or $\frac{\delta \check{G}}{\delta \mathbf{v}}$ equals zero, then $\{F, G\}$ is uniquely defined. One can check that the generalized enstrophy functions have functional derivatives

$$\begin{aligned} \tfrac{\delta C}{\delta \Sigma} &= \Phi(\omega), \\ \tfrac{\delta \hat{C}}{\delta \mathbf{v}} &= \operatorname{curl}\, (\Phi'(\omega)\hat{\mathbf{z}}) \\ \text{and} \qquad \tfrac{\delta \check{C}}{\delta \mathbf{v}} &= \Phi'(\omega)\hat{\mathbf{z}} \times \nu \end{aligned}$$

and that they are Casimirs in the sense that $\{C, F\} = 0$ for **any** function F on $\mathcal{N}$ with functional derivatives such that $\frac{\delta \check{F}}{\delta \mathbf{v}} = 0$. (We will not use the generalized bracket for any further calculations in this paper.)

We now consider the equations of motion for a planar liquid drop consisting of an incompressible, inviscid fluid with a free boundary and forces of surface tension on the boundary and show that for the appropriate Hamiltonian H and the Poisson bracket (1.5) defined above, these equations are equivalent to the relation $\dot{F} = \{F, H\}$ for all functions F on $\mathcal{N}$ possessing functional derivatives. The equations of motion for an ideal fluid with a free boundary Σ are

$$\begin{aligned} &\tfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p, \\ &\tfrac{\partial \Sigma}{\partial t} = \langle \mathbf{v}, \nu \rangle, \\ &\operatorname{div} \mathbf{v} = 0 \qquad \text{and} \qquad p|\Sigma = \tau\kappa, \end{aligned} \tag{1.7}$$

where κ is the mean curvature of the surface Σ and τ is the surface tension coefficient, which is a numerical constant. Using notation for the two dimensional case, we take our Hamiltonian to be

$$H(\Sigma, \mathbf{v}) = \int_{D_\Sigma} \tfrac{1}{2}|\mathbf{v}|^2 dA + \tau \int_\Sigma ds. \tag{1.8}$$

The functional derivatives of H are computed to be

$$\begin{aligned} \tfrac{\delta H}{\delta \mathbf{v}} &= \mathbf{v}, \\ \tfrac{\delta H}{\delta \varphi} &= \left\langle \tfrac{\delta H}{\delta \mathbf{v}}, \nu \right\rangle = \langle \mathbf{v}, \nu \rangle, \\ \text{and, using (1.4),} \qquad \tfrac{\delta H}{\delta \Sigma} &= \tfrac{1}{2}|\mathbf{v}|^2 + \tau\kappa, \end{aligned} \tag{1.9}$$

where $\frac{\delta H}{\delta \Sigma}$ is taken modulo constants. Thus for arbitrary F possessing functional derivatives, (1.5) gives

$$\begin{aligned}\{F,H\} &= \int_{D_\Sigma} \left\langle \omega, \tfrac{\delta F}{\delta \mathbf{v}} \times \mathbf{v} \right\rangle dA + \int_\Sigma \left(\left\langle \tfrac{\delta F}{\delta \Sigma}\mathbf{v}, \nu \right\rangle - \left[\tfrac{1}{2}|\mathbf{v}|^2 + \tau\kappa\right] \tfrac{\delta F}{\delta \varphi} \right) ds \\ &= \int_{D_\Sigma} \left\langle \tfrac{\delta F}{\delta \mathbf{v}}, \mathbf{v} \times \omega - \nabla\left(\tfrac{1}{2}|\mathbf{v}|^2\right) \right\rangle dA + \int_\Sigma \left(\tfrac{\delta F}{\delta \Sigma} \langle \mathbf{v}, \nu \rangle - \left\langle \tfrac{\delta F}{\delta \mathbf{v}}, \tau\kappa\nu \right\rangle \right) ds \\ &= \int_{D_\Sigma} \left\langle \tfrac{\delta F}{\delta \mathbf{v}}, -(\mathbf{v}\cdot\nabla)\mathbf{v} \right\rangle dA + \int_\Sigma \left(\tfrac{\delta F}{\delta \Sigma} \langle \mathbf{v}, \nu \rangle - \left\langle \tfrac{\delta F}{\delta \mathbf{v}}, \tau\kappa\nu \right\rangle \right) ds.\end{aligned}$$

If (1.7) holds, then we find

$$\begin{aligned}\dot F &= \int_{D_\Sigma} \left\langle \tfrac{\delta F}{\delta \mathbf{v}}, \tfrac{\partial \mathbf{v}}{\partial t} \right\rangle dA + \int_\Sigma \tfrac{\delta F}{\delta \Sigma} \tfrac{\partial \Sigma}{\partial t}\, ds \\ &= \int_{D_\Sigma} \left\langle \tfrac{\delta F}{\delta \mathbf{v}}, -(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla p \right\rangle + \int_\Sigma \tfrac{\delta F}{\delta \Sigma} \langle \mathbf{v}, \nu \rangle\, ds \\ &= \int_{D_\Sigma} \left\langle \tfrac{\delta F}{\delta \mathbf{v}}, -(\mathbf{v}\cdot\nabla)\mathbf{v} \right\rangle dA + \int_\Sigma \left(\tfrac{\delta F}{\delta \Sigma} \langle \mathbf{v}, \nu \rangle - \left\langle \tfrac{\delta F}{\delta \mathbf{v}}, \tau\kappa\nu \right\rangle \right) ds,\end{aligned}$$

so $\dot F = \{F,H\}$. Conversely, $\dot F = \{F,H\}$ implies (1.7) by this same calculation. If F has only generalized functional derivatives, $\dot F = \{F,H\}$ is still equivalent to (1.7), but now we use the bracket (1.8).

2. Stability of Two-dimensional Circular Flow

We consider the stability of the planar incompressible fluid flow such that the boundary Σ_e is a circle of radius r and the fluid is rigidly rotating with angular velocity Ω. For this equilibrium solution of the equations of motion, we shall find a conserved quantity C such that $H_C := H + C$ has a critical point at the equilibrium and then test for definiteness of its second variation. In infinite-dimensional systems, such as fluid flow, we have already noted that definiteness of the second variation is not sufficient to guarantee nonlinear stability, but it does imply stability under the linearized dynamics.

One class of conserved quantities consists of the Casimirs of the Poisson manifold $\mathcal{N}$, i.e. functions C on $\mathcal{N}$ satisfying $\{C,F\} = 0$ for all functions F for which the bracket is defined. We will make use of Casimirs of the form $C_1(\Sigma,\mathbf{v}) = \int_{D_\Sigma} \Phi(\omega)dA$, where Φ is a C^2 function on $\mathbf{R}^2$ and $\omega = \langle \text{curl } \mathbf{v}, \hat{\mathbf{z}} \rangle$. We will also include the angular momentum $C_2(\Sigma,\mathbf{v}) = \int_{D_\Sigma} \langle \mathbf{v}\times\mathbf{x}, \hat{\mathbf{z}} \rangle dA$. C_2 is the momentum map associated to the left action of the rotation group $SO(2)$ on $\mathcal{N}$. The conservation of C_2 is a consequence of the invariance of the Hamiltonian H under the $SO(2)$ action, which implies $\dot C_2 = \{C_2, H\} = 0$. The inclusion of C_2 in the modified Hamiltonian H_C allows us, roughly speaking, to view the fluid from a rotating frame with arbitrary angular velocity. Mathematically, C_2 enables us to cancel an otherwise troublesome cross term in the second variation of H. In the course of the calculation we shall also fix a translational frame.

Thus, we take our total conserved quantity to be

$$H_C(\Sigma,\mathbf{v}) = \int_{D_\Sigma} (\tfrac{1}{2}|\mathbf{v}|^2 + \mu < \mathbf{v}\times\mathbf{x}, \hat{\mathbf{z}} > + \Phi(\omega))dA + \tau\int_\Sigma ds, \tag{2.1}$$

where μ is a constant, as yet undetermined. Using elementary vector identities, we can rewrite (2.1) as

$$H_C(\Sigma,\mathbf{v}) = \int_{D_\Sigma} (\tfrac{1}{2}|\tilde{\mathbf{v}}|^2 - \tfrac{1}{2}\mu^2|\mathbf{x}| + \Phi(\omega))dA + \tau\int_\Sigma ds, \tag{2.2}$$

where $\tilde{\mathbf{v}} := \mathbf{v} - \mu\hat{\mathbf{z}}\times\mathbf{x}$. This rephrasing corresponds to viewing the fluid from a frame rotating with constant angular velocity μ; $\tilde{\mathbf{v}}$ is the fluid velocity in the rotating frame.

The first variation of H_C is computed to be

$$\begin{aligned} &DH_C(\Sigma,\mathbf{v})\cdot(\delta\Sigma,\delta\mathbf{v}) \\ &= \int_{D_\Sigma} (\langle\tilde{\mathbf{v}},\delta\mathbf{v}\rangle + \Phi'(\omega)\cdot\langle \text{curl } \delta\mathbf{v}, \hat{\mathbf{z}}\rangle)\, dA + \int_\Sigma \left(\tfrac{1}{2}|\tilde{\mathbf{v}}|^2 - \tfrac{1}{2}\mu^2|\mathbf{x}|^2 + \tau\kappa + \Phi(\omega)\right)\delta\Sigma\, ds. \end{aligned} \tag{2.3}$$

We now consider the case where Σ_e is a circle of radius r and $\mathbf{v}_e = \frac{\Omega}{2}\hat{\mathbf{z}}\times\mathbf{x}$ for some constant Ω, i.e. the equilibrium flow is rigid rotation with angular velocity Ω. The circle Σ_e has constant mean curvature $\kappa = \frac{1}{r}$. We require DH_C to vanish at this equilibrium. Since $\omega_e = \langle\text{curl } \mathbf{v}_e, \hat{\mathbf{z}}\rangle = \Omega$, DH_C depends on Φ only through the constants $\Phi(\Omega)$ and $\Phi'(\Omega)$. If we set $\mu = \frac{\Omega}{2}$, corresponding to choosing a frame moving with the rigidly rotating fluid, then $\tilde{\mathbf{v}}_e = \mathbf{0}$, so

$$\begin{aligned} &DH_C(\Sigma_e,\mathbf{v}_e)\cdot(\delta\Sigma,\delta\mathbf{v}) \\ &\quad = \int_{D_\Sigma} \Phi'(\Omega)\cdot\langle\text{curl } \delta\mathbf{v},\hat{\mathbf{z}}\rangle\, dA + \left(-\tfrac{1}{2}\left(\frac{\Omega}{2}\right)^2 r^2 + \frac{\tau}{r} + \Phi(\Omega)\right)\int_\Sigma \delta\Sigma\, ds \\ &\quad = \int_{D_\Sigma} \Phi'(\Omega)\cdot\langle\text{curl } \delta\mathbf{v},\hat{\mathbf{z}}\rangle\, dA, \end{aligned}$$

since $\delta\Sigma$ satisfies $\int_\Sigma \delta\Sigma\, ds = 0$. Thus $DH_C(\Sigma_e,\mathbf{v}_e) = 0$ iff $\Phi'(\Omega) = 0$. For convenience we choose Φ to be such that $\Phi(\Omega) = 0$, $\Phi'(\Omega) = 0$ and $\Phi''(\Omega) = 1$. (We choose a non-zero value for $\Phi''(\Omega)$ since it will improve our a priori estimates, as will be discussed below.)

The second variation of H_C at a general point $(\Sigma,\mathbf{v})$ is calculated to be

$$\begin{aligned} D^2H_C(\Sigma,\mathbf{v})\cdot(\delta\Sigma,\delta\mathbf{v})^2 &= \int_{D_\Sigma} \left(|\delta\mathbf{v}|^2 + \Phi''(\omega)\cdot|\text{curl } \delta\mathbf{v}|^2\right) dA \\ &+ \int_\Sigma \Big[2\left(\langle\tilde{\mathbf{v}},\delta\mathbf{v}\rangle + \Phi'(\omega)\cdot\langle\text{curl } \delta\mathbf{v},\hat{\mathbf{z}}\rangle\right)\delta\Sigma \\ &\quad + \left(\tfrac{1}{2}|\tilde{\mathbf{v}}|^2 - \tfrac{1}{2}\mu^2|\mathbf{x}|^2 + \tau\kappa + \Phi(\omega)\right)\left(\delta^2\Sigma + \kappa\delta\Sigma^2\right) \\ &\quad + \tfrac{\partial}{\partial\nu}\left(\tfrac{1}{2}|\tilde{\mathbf{v}}|^2 - \tfrac{1}{2}\mu^2|\mathbf{x}|^2 + \Phi(\omega)\right)\delta\Sigma^2 - \tau(\Delta\delta\Sigma)\delta\Sigma - \tau\kappa^2\delta\Sigma^2\Big] ds, \end{aligned} \tag{2.4}$$

where Δ is the Laplacian on Σ and $\delta^2\Sigma$ is the variation of $\delta\Sigma$ with respect to Σ (see the earlier comments on the computation of functional derivatives with respect to Σ). The presence of the terms involving $\delta^2\Sigma$ is due to the constraints on the variations of Σ arising from the fact that the manifold S of boundary curves is not a linear space; for fixed Σ the space of $\mathbf{v}$'s on Σ is linear, so no such $\delta^2\mathbf{v}$ term arises. The only non-obvious term in the second variation (2.4) is the derivative with respect to Σ of the boundary term of the first variation. This derivative is computed in the following manner. Write the last term of (2.3) as follows:

$$\int_\Sigma \left(\tfrac{1}{2}|\bar{\mathbf{v}}|^2 - \tfrac{1}{2}\mu^2|\mathbf{x}|^2 + \tau\kappa + \Phi(\omega)\right)\delta\Sigma\, ds$$
$$= \int_\Sigma \left(\tfrac{1}{2}|\bar{\mathbf{v}}|^2 - \tfrac{1}{2}\mu^2|\mathbf{x}|^2 + \Phi(\omega)\right)\delta\Sigma\, ds + \tau\int_\Sigma \kappa\delta\Sigma\, ds.$$

Using equation (1.4) and the definition given in the general computation of $\frac{\delta F}{\delta\Sigma}$ for $F(\Sigma)$ of the form $\int_\Sigma f(\Sigma,\mathbf{x})ds$, we see that the first term of the preceeding expression has derivative with respect to Σ given by

$$\int_\Sigma \Big[\frac{\partial}{\partial\nu}\frac{1}{2}\left(|\bar{\mathbf{v}}|^2 - \mu^2|\mathbf{x}|^2 + \Phi(\omega)\right)\delta\Sigma^2 + \tfrac{1}{2}\left(|\bar{\mathbf{v}}|^2 - \mu^2|\mathbf{x}|^2 + \Phi(\omega)\right)\delta^2\Sigma$$
$$+\tfrac{1}{2}\left(|\bar{\mathbf{v}}|^2 - \mu^2|\mathbf{x}|^2 + \Phi(\omega)\right)\kappa\delta\Sigma^2\Big]ds.$$

The Σ variation of the second integral is clearly the second variation of the arc length of Σ with respect to $\delta\Sigma$, which may be computed to be

$$\tau\int_\Sigma [-(\Delta\delta\Sigma)\delta\Sigma + \kappa\delta^2\Sigma]ds.$$

Adding these two terms and regrouping gives expression (2.4).

For the circular flow described above the second variation reduces to

$$D^2H_C(\Sigma_e,\mathbf{v}_e)\cdot(\delta\Sigma,\delta\mathbf{v})^2 = \int_{D_\Sigma}\left(|\delta\mathbf{v}|^2 + |\mathrm{curl}\ \delta\mathbf{v}|^2\right)dA$$
$$+\int_\Sigma\left[\left(-\tfrac{1}{2}\left(\frac{\Omega}{2}\right)^2 r^2 + \frac{\tau}{r}\right)(\delta^2\Sigma + \kappa\delta\Sigma^2) - \left(\frac{\Omega}{2}\right)^2 r\delta\Sigma^2 - \tau(\Delta\delta\Sigma)\delta\Sigma - \frac{\tau}{r^2}\delta\Sigma^2\right]ds$$
$$= \int_{D_\Sigma}\left(|\delta\mathbf{v}|^2 + |\mathrm{curl}\ \delta\mathbf{v}|^2\right)dA - \int_\Sigma\left[\left(\frac{\Omega}{2}\right)^2 r\delta\Sigma^2 - \tau(\Delta\delta\Sigma)\delta\Sigma - \frac{\tau}{r^2}\delta\Sigma^2\right]ds, \tag{2.5}$$

since the integral $\int_\Sigma(\delta^2\Sigma + \kappa\delta\Sigma^2)\,ds$ is the variation with respect to $\delta\Sigma$ of $\int_\Sigma \delta\Sigma\ ds$, which is identically zero due to our restriction to area preserving variations. It follows that $D^2H_C(\Sigma_e,\mathbf{v}_e)$ is positive-definite iff

$$\tau\int_\Sigma\left(-\frac{1}{r^2}\delta\Sigma^2 - (\Delta\delta\Sigma)\delta\Sigma\right)ds > \left(\frac{\Omega}{2}\right)^2 r\int_\Sigma \delta\Sigma^2 ds \tag{2.6}$$

for all area preserving variations $\delta\Sigma$.

We can simplify the expression of this condition by estimating $-(\Delta\delta\Sigma)\delta\Sigma$ using eigenvalues of the negative of the Laplacian on the circle of radius r. The eigenfunctions are $\delta\Sigma_{k,\phi}(\theta) := \cos\ k(\theta-\phi)$ with eigenvalues $\lambda_{k,\phi} = \left(\frac{k}{r}\right)^2$ for all positive integers k and $\phi \in [0, 2\pi)$. It is clear that the left side of (2.6) equals zero when $\delta\Sigma = \delta\Sigma_{1,\phi} = \cos(\theta-\phi)$. This eigenfunction corresponds to an infinitesmal translation in the ϕ direction, as $\cos(\theta-\phi)$ is the linearization of the normal perturbation $\Delta\Sigma_{\epsilon,\phi} = \epsilon\cos(\theta-\phi) + \sqrt{r^2 - \epsilon^2\sin^2(\theta-\phi)} - r$ associated to a displacement of length ϵ in the ϕ direction. If we wish to consider our system modulo position, regarding two configurations as equivalent if one can be obtained from the other by a Euclidean motion, then we can simply ignore the perturbations generated by the lowest eigenfunctions $\delta\Sigma_{1,\phi}$ and test for the definiteness of D^2H_C only with respect to perturbations which actually distort the drop shape. In this case, taking $\lambda_{2,\phi} = \frac{4}{r^2}$ as the lowest admissible eigenvalue, it follows from (2.6) that D^2H_C is positive-definite iff

$$\frac{3\tau}{r^3} > \left(\frac{\Omega}{2}\right)^2.$$

Remarks. 1. This procedure of ignoring Euclidean motions is equivalent to evaluating the definiteness of the second variation on the quotient space of fields $(\Sigma, \mathbf{v})$ modulo Euclidean motions; in other words, it is precisely establishing formal stability of our solution viewed as a **relative equilibrium** in the sense of Poincaré; see Marsden and Weinstein [1974] or Abraham and Marsden [1978] for the abstract theory.

2.The interior integral in the second variation (2.5) is equivalent to the square of the H^1 norm of $\delta\mathbf{v}$; had we chosen $\Phi''(\omega) = 0$ rather than $\Phi''(\omega) = 1$ this term would have equaled the square of the L^2 norm of $\delta\mathbf{v}$ instead. We expect that, as in the proof of global existence of two dimensional flows (Kato [1967]), this term will be useful in our investigation of nonlinear stability. A key difficulty will be to determine if the stability estimates are sufficient to prevent the breaking of small surface waves. For the somewhat related problem of vortex patches (without surface tension) it is known that surface waves can break; nevertheless one still has stability (Wan and Pulvirente [1985]).

3. The formal stability analysis outlined above for a circular liquid drop in $\mathbf{R}^2$ may also be applied to a spherical drop in $\mathbf{R}^3$ rotating about, for example, the $\hat{\mathbf{z}}$ axis. The generalized enstrophy functions $\Phi(\omega)$ are not conserved in the three-dimensional case and are therefore dropped from the Hamiltonian H_C; otherwise, the analysis procedes as in the two-dimensional case, with the following numerical differences: our curvature conventions are such that the mean curvature κ of the sphere equals $\frac{2}{r}$, the second variation of area is given by $\tau\int_\Sigma[-(\Delta\delta\Sigma)\delta\Sigma + \frac{\kappa}{2}\delta^2\Sigma]ds$ and the first and second eigenvalues of the Laplacian on the sphere are, respectively, $\frac{2}{r^2}$ and $\frac{6}{r^2}$. Thus, in the case of the two-sphere, D^2H_C is positive-definite iff

$$\frac{4\tau}{r^3} > \left(\frac{\Omega}{2}\right)^2.$$

4. The stability criteria found by Sedenko and Iudovich [1978] for circular shear flow in an annulus may be obtained by the methods described above; in fact, we find a less restrictive relationship between τ and ω_e than is given in their paper. We consider the equilibrium flow $\mathbf{v}_e = \frac{\omega_e(|\mathbf{x}|)}{2}\hat{\mathbf{z}} \times \mathbf{x}$ in the annulus $r_0 \leq |\mathbf{x}| \leq r_\Sigma$ with fixed inner boundary Σ_0 of radius r_0 and free outer boundary Σ of radius r_Σ; $\omega_e : [r_0, r_\Sigma] \mapsto \mathbf{R}$ is a C^1 function with no critical points. We add the conserved quantity $\lambda \int_{\Sigma_0} \mathbf{v} \cdot dl$, where λ is an as yet undetermined constant, to the modified Hamiltonian (2.1). Taking the first variation of H_C as in (2.3) and integrating the $\Phi'(\omega)$ term by parts, we find that

$$
\begin{aligned}
&DH_C(\Sigma, \mathbf{v}) \cdot (\delta\Sigma, \delta\mathbf{v}) \\
&\quad = \int_{D_\Sigma} \langle \tilde{\mathbf{v}} + \operatorname{curl}(\Phi'(\omega)\hat{\mathbf{z}}), \delta\mathbf{v}\rangle \, dA - \int_\Sigma \Phi'(\omega)\delta\mathbf{v} \cdot dl - \int_{\Sigma_0} \Phi'(\omega)\delta\mathbf{v} \cdot dl + \lambda \int_{\Sigma_0} \delta\mathbf{v} \cdot dl \\
&\qquad + \int_\Sigma \left(\tfrac{1}{2}|\tilde{\mathbf{v}}|^2 - \tfrac{1}{2}\mu^2|\mathbf{x}|^2 + \tau\kappa + \Phi(\omega)\right) \delta\Sigma \, ds.
\end{aligned}
$$

Using the techniques outlined in Holm, Marsden, Ratiu and Weinstein [1985], we find a function Φ of ω such that $\tilde{\mathbf{v}}_e = \operatorname{curl}(\Phi'(\omega_e)\hat{\mathbf{z}})$ and $\Phi'(\omega_e(r_\Sigma)) = 0$ (it is essential for this step that ω_e have no critical points). Letting $\lambda = \Phi'(\omega_e(r_0))$ and $\mu = \frac{\omega_e(r_\Sigma)}{2}$, we obtain $DH_C(\Sigma_e, \mathbf{v}_e) = 0$. The condition $\tilde{\mathbf{v}}_e = \operatorname{curl}(\Phi'(\omega_e)\hat{\mathbf{z}})$ implies

$$\Phi''(\omega_e) = \frac{|\mathbf{x}|(\omega_e(|\mathbf{x}|) - \omega_e(r_\Sigma))}{2\omega_e'(|\mathbf{x}|)}.$$

The second variation at the equilibrium point is computed to be

$$
\begin{aligned}
&D^2H(\Sigma_e, \mathbf{v}_e) \cdot (\delta\Sigma, \delta\mathbf{v})^2 \\
&= \int_{D_\Sigma} (|\delta\mathbf{v}|^2 + \Phi''(\omega_e) \cdot |\operatorname{curl} \delta\mathbf{v}|^2) dA + \int_\Sigma \left[\left(\frac{\omega_e}{2}\right)^2 |\mathbf{x}|\delta\Sigma^2 + \tau(\Delta\delta\Sigma)\delta\Sigma + \frac{\tau}{|\mathbf{x}|^2}\delta\Sigma^2\right] ds.
\end{aligned}
$$

It follows that the flow is formally stable iff

$$\frac{|\mathbf{x}|(\omega_e(|\mathbf{x}|) - \omega_e(r_\Sigma))}{2\omega_e'(|\mathbf{x}|)} \geq 0 \tag{2.7}$$

and

$$\frac{3\tau}{r_\Sigma^3} > \left(\frac{\omega_e(r_\Sigma)}{2}\right)^2. \tag{2.8}$$

Condition (2.7) is equivalent to the interior vorticity condition given by Sedenko and Iudovich; condition (2.8) differs from the analogous surface tension condition in Sedenko and Iudovich by a factor of three. (The derivation of this inequality from the variation of the mean curvature is not explained in their paper, so, as before, we were unable to determine the source of the difference.)

If we consider the annulus $r_\Sigma \leq |\mathbf{x}| \leq r_0$ with fixed outer ring and free inner ring, moving with velocity $\mathbf{v}_e$ as before, then the flow is formally stable iff condition (2.7) holds. The analogue of (2.8) is

$$\left(\frac{\omega(r_\Sigma)}{2}\right)^2 + \frac{3\tau}{r_\Sigma^3} > 0.$$

The case of a rigidly rotating annulus, i.e. constant ω_e, is analogous to that of a rigidly rotating circle; in this case we take $\lambda = \Phi'(\omega_e) = 0$ and $\mu = \frac{\omega_e(r_\Sigma)}{2} = \frac{\omega_e}{2}$. The resulting stability condition for an annulus with fixed inner boundary and free outer boundary is (2.8), with $\omega_e(r_\Sigma)$ replaced by the constant ω_e. The rigidly rotating annulus with free inner boundary and fixed outer boundary is always stable.

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