YORK MAP

The York map is a canonical transformation

JAMES ISENBERG (*)
Department of Mathematics University of Oregon
Eugene, Oregon 97403

JERROLD E. MARSDEN (*)
Departement of Mathematics University of California
Barkeley, CA 94720

Abstract. *The York mapping from the space of freely chosen conformal data to the space of constraint-satisfying physical data is shown to be a canonical transformation for both the vacuum Einstein theory and the Einstein-Maxwell theory.*

INTRODUCTION

Building on funcamental work of Lichnerowicz and Choquet-Bruhat, in the early 1970's, York and coworkers developed a program for solving the constraint equations of Einstein's theory. The original work is contained in York [1971]; for a survey plus additional references, see Choquet-Bruhat and York [1980]. In the vacuum case, one may think of this procedure in terms of a map $\mathscr{Y}_\tau$ from the space $T^*_{TT}\mathscr{M}(\Sigma)$ of metrics λ on a 3-surface Σ and transverse traceless conjugate momenta σ, to the space $\mathscr{C}_\tau(\Sigma)$ of gravitational initial data (γ, π) having constant mean curvature τ and satisfying the constraints on Σ. In the non-vacuum case (including, e.g. the Einstein-Yang-Mills theory) the procedure is roughly the same, although the domain and range space must be bigger (they include the nongravitational fields along with the gravitational ones).

Since the York map was largely motivated by Hamiltonian considerations, it has often been speculated (for example in Fischer and Marsden [1979]) that $\mathscr{Y}_\tau$ is a symplectic (i.e., canonical) map. We prove here that it is.

In the elementary examples familiar from classical mechanics, canonical trans-

(*) Research partially supported by NSF grant MCS 81 - 07086.

Journal of Geometry and Physics 1 (1984) 85-105

formations are usually nondegenerate (locally invertible). This is not the case with $\mathscr{Y}_\tau : T^*_{TT}\mathscr{M} \to \mathscr{C}_\tau$, as evidenced by its invariance under conformal transformations. Also somewhat different from the usual simple examples is the fact that both the domain and range of $\mathscr{Y}_\tau$ are presymplectic varieties (manifolds with singularities and with a degenerate symplectic form) rather than symplectic manifolds.

If, however, we quotient out by the action of appropriate groups (discussed below) we obtain reduced spaces $\mathscr{P}_{\text{York}}$ and $\mathscr{P}_\tau$ which are symplectic (though still containing singularities), and we obtain a reduced York map $[\mathscr{Y}_\tau]$ which is nondegenerate (invertible). Since $\mathscr{P}_\tau$ is the space of gravitational degrees of freedom (see Isenberg and Marsden [1982]), this reduced map provides an equivalent, simpler space to represent these degrees of freesom. These results also establish the compatibility between the York [1973] field decomposition and that of Moncrief [1975].

This paper is entirely directed toward proving that the York map is symplectic: We define the important spaces of fields and their reductions, describe the York map, estabilsh its properties, show that it reduces properly, and then state and prove the result. We do this first for the vacuum Einstein theory and then for the Einstein-Maxwell equations. Some applications of the results herein to other connections between the conformal and «ADM» pictures are planned for a future publication.

Acknowledgements

We thank Robert Jantzen for showing us his calculation on the canonical nature of the York map for homogeneous cosmological models.

1. THE VACUUM EINSTEIN CASE

A) Spaces of Field and Group Actions

For simplicity, we shall work in space of C^∞ fields; this can be generalized to Sobolev $W^{s,p}$ or Holder $C^{k+\alpha}$ spaces (weighted for nonspatially compact spaces) in a routine fashion following Fischer and Marsden [1979], Choquet-Bruhat, Fischer and Marsden [1979], Choquet-Bruhat and York [1980] and Isenberg and Marsden [1982].

Fix an oriented connected smooth 3-manifold Σ and define the following spaces:

$\mathscr{M}(\Sigma)$ — the space of all Riemannian matrics γ on Σ. In local coordinates on Σ we write $\gamma_{ij}(x)$ for γ.

$T^*\mathscr{M}(\Sigma)$ — the natural L^2 contangent bundle of $\mathscr{M}(\Sigma)$ consisting of pairs (γ, π) of Riemannian metrics γ and symmetric contravariant tensor densities π; in coordinates on Σ we write (γ_{ij}, π^{ij}) for (γ, π). [we shall also find the notation $\mu := \sqrt{\det \gamma}$ useful].

$\mathscr{C}_\tau(\Sigma)$ the subset of $T^*\mathscr{M}(\Sigma)$ which satisfies the vacuum Einstein constraints:

(1) $$0 = \mathscr{J}(\gamma, \pi) = \delta_\gamma \pi; \quad \text{or} \quad \nabla_a \pi^{ab} = 0$$

(2) $$0 = \mathscr{H}(\gamma, \pi) = -\mu R + \left(\pi^{k\ell}\pi_{k\ell} - \frac{1}{2}(tr\,\pi)^2\right)\frac{1}{\mu}$$

ans also satisfies the constant mean curvature condition:

(3) $$\frac{1}{2\mu}\, tr\,\pi = \tau$$

(for some constant τ).

$T^*_{TT}\mathscr{M}(\Sigma)$ the subspace of $T^*\mathscr{M}(\Sigma)$ consisting of pairs $(\lambda, \sigma) \in T^*\mathscr{M}(\Sigma)$ for which

(4) $$tr\,\sigma = 0$$

and

(5) $$\delta_\gamma \sigma = 0.$$

The space $T^*\mathscr{M}(\Sigma)$ is a (weak) symplectic manifold. We identify tangent vectors to $\mathscr{M}(\Sigma)$ at γ with symmetric covariant two tensors k and denote the pairing between vectors and covectors as

(6) $$\langle \pi, k \rangle = \int_\Sigma \pi \cdot k$$

where $\pi \cdot k = \pi^{ij} k_{ij}$ is the natural contraction, producing a density. The symplectic form Ω at (γ, π) is then the skew form

(7) $$\Omega((k_1, \pi_1), (k_2, \pi_2)) = \langle \pi_2, k_1 \rangle - \langle \pi_1, k_2 \rangle$$

which is the canonical symplectic form on $T^*\mathscr{M}$, (cf. Abraham and Marsden [1978], p. 178 - 9). This same Ω defines a presymplectic form on $\mathscr{C}_\tau(\Sigma)$ and $T^*_{TT}\mathscr{M}(\Sigma)$ by restriction.

In addition to these spaces of geometric fields, we shall need

$\mathscr{D}(\Sigma)$ – the group of diffeomorphisms of Σ and

$\Theta(\Sigma)$ – the space of positive real-valued functions on Σ

which generate group action on the above defined spaces as follows: The group

$\mathscr{D}(\Sigma)$ acts on $\mathscr{M}(\Sigma)$ by pull-back

(8) $$(\eta, \gamma) \mapsto \eta^*\gamma.$$

This (right) action extends naturally to a symplectic action on $T^*\mathscr{M}$

(9) $$(\eta, (\gamma, \pi)) \mapsto (\eta^*\gamma, \eta^*\pi).$$

Since $\mathscr{C}_\tau(\Sigma)$ and $T^*_{TT}\mathscr{M}(\Sigma)$ are both mapped to themselves by this action, we regard $\mathscr{D}(\Sigma)$ as acting on them as well.

The set $\Theta(\Sigma)$ forms a group under pointwise multiplication. This group acts on $\mathscr{M}(\Sigma)$ by

(10) $$(\theta, \gamma) \mapsto \theta^4\gamma.$$

the induced symplectic action on $T^*\mathscr{M}(\Sigma)$ is

(11) $$(\theta, (\gamma, \pi)) \mapsto (\theta^4\gamma, \theta^{-4}\pi).$$

Since $T^*_{TT}\mathscr{M}(\Sigma)$ is mapped to itself by this action, we can regard $\Theta(\Sigma)$ as acting on it. Note that $\mathscr{C}_\tau(\Sigma)$ is *not* mapped to itself by the action of $\Theta(\Sigma)$.

Now consider

(12) $$\mathscr{G}(\Sigma) := \mathscr{D}(\Sigma) \ltimes \Theta(\Sigma),$$

the semidirect product of $\mathscr{D}(\Sigma)$ and $\Theta(\Sigma)$. The group multiplication is

(13) $$(\eta_1, \theta_1)\cdot(\eta_2, \theta_2) = (\eta_1 \circ \eta_2, (\theta_1 \circ \eta_2)\cdot \theta_2).$$

Clearly G acts on (Σ), via

(14) $$((\eta, \theta), \gamma) \mapsto \theta^4\eta^*\gamma,$$

and also on $T^*\mathscr{M}(\Sigma)$ and $T^*_{TT}\mathscr{M}(\Sigma)$ in the obvious way. In view of the identity

(15) $$\theta_2\eta_2^*(\theta_1\eta_1^*\gamma) = (\theta_1 \circ \eta_2)\,\theta_2(\eta_1 \circ \eta_2)^*\gamma,$$

we see that $\mathscr{G}$ acts on the right on all these spaces (again, excluding $\mathscr{C}_\tau(\Sigma)$).

As noted in the introduction, certain reductions play an important role in our analysis. The first one we consider is the reduction of $T^*_{TT}\ \mathscr{M}$ (the domain of the York map) by the action of $\mathscr{G}$. The key here is the recognition that

(16) $$T^*_{TT}\mathscr{M}(\Sigma) = J_{\mathscr{G}}^{-1}(0),$$

where $J_{\mathscr{G}}$ is the momentum map

(17) $$J_{\mathscr{G}} : T^*M(\Sigma) \to \mathfrak{g}^*$$

corresponding to the action of $\mathscr{G}$ on $T^*\mathscr{M}(\Sigma)$, and $\mathfrak{g}$ is the Lie algebra of G:

$$\mathfrak{g} = (\text{vector fields}) \ltimes (\text{scalar fields})$$

and $\mathfrak{g}^*$ is its natural L^2-dual:

$$\mathfrak{g}^* = (\text{one form densities}) \times (\text{scalar densities}).$$

To verify (16), we compute J from the general formula for the momentum of a cotangent lift (see Abraham and Marsden [1978, p. 283]) and find

$$J_{\mathscr{G}}(\gamma, \pi) = (2\delta_\gamma \pi, tr\, \pi)) = (2\nabla_l \pi^{lj}, \pi^i{}_i); \tag{18}$$

then (16) is obvious. It now follows from the reduction theorem (see Marsden and Weinstein [1974]) that the quotient space (1)

$$\tilde{\mathscr{P}}_{\text{York}} := T^*_{TT}\mathscr{M}/\mathscr{G} \tag{19}$$

is a symplectic manifold (almost everywhere), whose symplectic form Ω_γ is inherited naturally from Ω on $T^*\mathscr{M}$.

The other reduction we need, that of $\mathscr{C}_\tau$ by the action of $\mathscr{D}$, is not so obviously a symplectic manifold. To see roughly that it is, we note that $\mathscr{C}_\tau$ is the zero set of the momentum map corresponding to space *plus time* diffeomorphisms acting on $T^*\mathscr{M}$, with the additional condition that all points $(\gamma, \pi) \in \mathscr{C}_\tau$ satisfy $tr\, \pi/2\mu = = \tau$. This latter condition freezes the time translations. Hence if we factor out the space translations,

$$\mathscr{P}_\tau := \mathscr{C}_\tau/\mathscr{D} \tag{20}$$

becomes (almost everywhere) a symplectic manifold; again, with symplectic Ω_τ induced from Ω on $T^*\mathscr{M}$ (see Isenberg and Marsden [1982] for a more rigorous discussion of this reduction).

Note the caveat «almost everywhere» appearing both with $\tilde{\mathscr{P}}_{\text{York}}$ and $\mathscr{P}_\tau$. This reminds us that both $\tilde{\mathscr{P}}_{\text{York}}$ and $\mathscr{P}_\tau$ have singularities (i.e., they are both stratified manifolds). Their singularities are inherited from $T^*_{TT}\mathscr{M}$ and $\mathscr{C}_\tau$ respectively, they are present even before the quotient operation. See Fischer and Marsden [1977], Arms, Marsden and Moncrief [1982] and Isenberg and Marsden [1982] for discussion of these stratified structures.

B) The York Map and Some of Its Properties

For any chosen constant τ, the York procedure takes data $(\lambda, \sigma) \in T^*_{TT}\mathscr{M}(\Sigma)$ into data $(\gamma, \pi) \in \mathscr{C}_\tau$ by solving the Lichnerowicz equation

$$\nabla^2\phi = \frac{R}{8}\,\phi - \frac{1}{8\mu^2}\,\sigma^{ab}\sigma_{ab}\phi^{-7} + \frac{1}{12}\,\tau^2\phi^5 \tag{21}$$

(1) Since the space we shall be really be working with is a bit smaller than $T^*_{TT}M/G$ (see § 1.B, below), we include the « ~ » here.

for the positive scalar ϕ (here ∇^2, R, μ, and the contractions on σ are all constructed from λ) and then forming

$$\gamma = \phi^4 \lambda \tag{22}$$

and

$$\pi = \phi^{-4} \sigma + \frac{1}{2} \mu \lambda^{-1} \tau \tag{23}$$

one easily verifies that the Einstein constraint equations (1) and (2) are satisfied by any set (γ, τ) constructed in this way.(The identity

$$R(\phi^4 \lambda) = \phi^{-4} R(\lambda) - 8\phi^{-5} \nabla^2 \phi \tag{24}$$

is useful for this verification).

The York procedure is defined at $(\lambda, \sigma) \in T^*_{TT}\mathcal{M}(\Sigma)$ iff Eq. (21) has a unique solution for that choice of data. While there are points (λ, σ) of $T^*_{TT}\mathcal{M}(\Sigma)$ for which this fails, York and O'Murchadha [1974] show that the Lichnerowicz equation has unique solutions on an open dense subset (2) of $T^*_{TT}\mathcal{M}(\Sigma)$, which we shall call $T^*_{TT}\mathcal{M}_0(\Sigma)$. This we have a well-defined map

$$\mathcal{Y}_\tau : T^*_{TT}\mathcal{M}_0(\Sigma) \to \mathcal{C}_\tau(\Sigma) \tag{25}$$

whose action is specified by Eqs. (21)-(23).

We now discuss a number of important properties of the York map y_τ.

They are presented in the form of a series of lemmas.

LEMMA 1. *(Surjectivity).* $\mathcal{Y}_\tau : T^*_{TT}\mathcal{M}_0(\Sigma) \to \mathcal{C}_\tau$ *is a surjective map.*

Proof. Let (γ, π) be a set of data in $\mathcal{C}_\tau$. Define $\lambda = \gamma$ and $\sigma = \pi - 2\gamma^{-1}/3\, \tau\mu$. Since $tr\, \pi = 2\mu\tau$ (by definition of $\mathcal{C}_\tau$) and since $\mathcal{J}(\lambda, \sigma) = \mathcal{J}(\gamma, \pi) = 0$, we see that $(\lambda, \sigma) \in T^*_{TT}\mathcal{M}$. We easily verify that $\sigma = 1$ solves the Lichnerowicz equation for this (λ, σ) and thus find that $\mathcal{Y}_\tau(\lambda, \sigma) = (\gamma, \pi)$. ■

---

(2) While the exact form of $T^*_{TT}\mathcal{M}_0(\Sigma)$ has not yet been determined, we do know the following:

$\tau \neq 0$ case: if σ is nonzero for at least one point $x \in \Sigma$, then $(\lambda, \sigma) \in T^*_{TT}\mathcal{M}_0(\Sigma)$

If $\sigma = 0$ on Σ and if there exists a conformal factor θ such that $R(\theta^4 \lambda) \geq 0$ for all $x \in \Sigma$, then $(\lambda, \sigma) \in T^*_{TT}\mathcal{M}_0(\Sigma)$.

$\tau = 0$ case: If there exists a conformal factor θ such that $R(\theta^4 \lambda) \leq 0$ for all $\lambda \in \Sigma$, the $(\lambda, \sigma) \in$ $\in T^*_{TT}\mathcal{M}_0(\Sigma)$.

LEMMA 2. (Θ Invariance). *For any* $(\lambda, \sigma) \in T^*_{TT}\mathcal{M}_0(\Sigma)$ *and any* $\theta \in \Theta(\Sigma)$, *one has*

(26) $$\mathcal{Y}_\tau(\theta^4 \lambda, \theta^{-4}\sigma) = \mathcal{Y}_\tau(\lambda, \sigma).$$

Proof. Suppose ϕ solves the Lichnerowicz equation for (λ, σ). Then we claim that $\bar{\phi} = (1/\theta)\,\phi$ staisfies the same equation for $(\theta^4\lambda, \theta^{-4}\sigma)$. Clearly if we prove this claim then the lemma is proven.

Let $\bar{\lambda} = \theta^4\lambda, \bar{\sigma} = \theta^{-4}\sigma$, and let $\bar{R}$, $\bar{\nabla}$, $\bar{\mu}$, denote the quantities defined from $\bar{\lambda}$. Then $\bar{\phi}$ must satisfy

(27) $$\bar{\nabla}^2\bar{\phi} = \frac{1}{8}\bar{R}\bar{\phi} - \frac{1}{8}\frac{\bar{\sigma}^2}{\mu}\bar{\phi}^{-7} + \frac{1}{12}\tau^2\bar{\phi}^5.$$

Using Eq. (24) as well as the more obvious transformations, and multiplying through by θ^5, we transform Eq. (27) into

(28) $$\theta^5\bar{\nabla}^2\bar{\phi} + (\nabla^2\theta)\bar{\phi} = \frac{1}{8}R(\theta\bar{\phi}) - \frac{1}{8}\frac{\sigma^2}{\mu^2}(\theta\bar{\phi})^{-7} + \frac{1}{12}\tau(\theta\,\phi)^5$$

Now

$$\bar{\nabla}^2\bar{\phi} = \frac{1}{\sqrt{\det\theta^4\lambda}}\frac{\partial}{\partial x^i}\left(\sqrt{\det\theta^4\lambda}\;\theta^{-4}\lambda^{ij}\frac{\partial}{\partial x^j}\bar{\phi}\right)$$

$$= \frac{1}{\theta^6\mu}\frac{\partial}{\partial x^i}\left(\theta^2\mu\lambda^{ij}\frac{\partial}{\partial x^j}\bar{\phi}\right)$$

(29) $$= \frac{1}{\theta^4}\nabla^2\bar{\phi} + \frac{2}{\theta^5}\nabla\theta\cdot\nabla\bar{\phi}.$$

Thus

$$\theta^5\bar{\nabla}^2\bar{\phi} + (\nabla^2\theta)\,\bar{\phi} = \theta\,\nabla^2\bar{\phi} + 2(\nabla\phi)\cdot(\nabla\bar{\phi}) + (\nabla^2\theta)\,\bar{\phi}$$

(30) $$= \nabla^2(\bar{\theta}\,\bar{\phi}),$$

so we have

(31) $$\nabla^2(\theta\bar{\phi}) = \frac{1}{8}R(\theta\,\bar{\phi}) - \frac{1}{8}\frac{\sigma^2}{\mu^2}(\theta\,\bar{\phi})^{-1} + \frac{1}{12}\tau(\theta\,\phi)^5.$$

Since, by assumpition $(\lambda, \sigma) \in T^*_{TT}\mathcal{M}_0(\Sigma)$, the solution of (31) must be unique. Therefore $\theta\,\bar{\phi} = \phi$, as was to be proven. ∎

LEMMA 3. *(D Equivariance). For any* $(\lambda, \sigma) \in T^*_{TT}\mathcal{M}_0(\Sigma)$ *and any* $\eta \in \mathcal{D}(\Sigma)$, *one has*

$$\mathcal{Y}_\tau(\eta^*\lambda, \eta^*\sigma) = \eta^* y_\tau(\lambda, \sigma). \tag{32}$$

Proof. The scalar curvature and Laplace operators are both covariant. Thus if ϕ satisfies the Lichnerowicz equation for (λ, σ) then $\eta^*\phi = \phi \circ \eta$ satisfies it for $(\eta^*\lambda, \eta^*\sigma)$. The result then follows. ∎

Lemma 2 shows that $\mathcal{Y}_\tau$ cannot be one-one. To compensate, however, it permits us to define a reduced version of $\mathcal{Y}_\tau$ on $T^*_{TT}\mathcal{M}_0(\Sigma)/\Theta$. Further, from Lemma 3, we can reduce by $\mathcal{D}$, and define

$$[\mathcal{Y}_\tau] : T^*_{TT}\mathcal{M}_0(\Sigma)/\mathcal{G} \to \mathcal{C}_\tau/\mathcal{D}. \tag{33}$$

We have earlier identified $\mathcal{C}_\tau/\mathcal{D}$ as the stratified symplectic manifold $\mathcal{P}_\tau$. As for the domain, this resembles $\mathcal{P}_{\text{York}}$, but it involves $T^*_{TT}\mathcal{M}_0(\Sigma)$ rather than $T^*_{TT}\mathcal{M}(\Sigma)$. However since $T^*_{TT}\mathcal{M}_0(\Sigma)$ is invariant under $\mathcal{G}$, the same considerations used earlier show that

$$\mathcal{P}_{\text{York}} := T^*_{TT}\mathcal{M}_0(\Sigma)/\mathcal{G} \tag{34}$$

is also a stratified symplectic manifold. Thus we have

$$[\mathcal{Y}_\tau] : \mathcal{P}_{\text{York}} \to \mathcal{P}_\tau. \tag{35}$$

LEMMA 4. *(Bijectivity of* $[\mathcal{Y}_\tau]$*). The map* $[\mathcal{Y}_\tau]$ *is one-one as well as onto.*

Proof. The reduction procedure preserves surjectivity, so we need only demostrate that $[\mathcal{Y}_\tau]$ is one-one. In this proof (as well as in later discussions, we shall use brackets [ ] to denote equivalence classes (under $\mathcal{G}$ in the domain; under $\mathcal{D}$ in the range).

Now suppose that

$$[\mathcal{Y}_\tau](\lambda, \sigma) = [\mathcal{Y}_\tau](\bar{\lambda}, \bar{\sigma}).$$

By definition of $[\mathcal{Y}_\tau]$, it follows that

$$\left[\left(\phi^4\lambda, \phi^{-4}\sigma + \frac{2}{3}\,\phi^2\lambda^{-1}\mu\tau\right)\right] = \left[\left(\bar{\phi}^4\bar{\lambda}, \bar{\phi}^{-4}\bar{\sigma} + \frac{2}{3}\,\bar{\phi}^2\bar{\lambda}^{-1}\bar{\mu}\tau\right)\right]. \tag{37}$$

This implies (by definition of the equivalence classes) that there exists $\eta \in \mathcal{D}(\Sigma)$ such that

$$\eta^*(\phi^4\lambda) = \bar{\phi}^4\bar{\lambda} \tag{38}$$

and

$$(39) \qquad \eta^*\left(\phi^{-4}\sigma + \frac{2}{3}\phi^2\lambda^{-1}\mu\tau\right) = \bar{\phi}^{-4}\bar{\sigma} + \frac{2}{3}\bar{\phi}^2\bar{\lambda}^{-1}\bar{\mu}\tau.$$

If we *define* $\theta := \eta^*\phi/\bar{\phi}$ then (38) becomes

$$(40) \qquad \bar{\lambda} = \theta^4\eta^*\lambda.$$

Concentrating on the second term on both sides of (39), we find

$$\eta^*\left(\frac{2}{3}\phi^2\lambda^{-1}\mu\tau\right) = \frac{2}{3}(\eta^*\phi)^2(\eta^*\lambda)^{-1}(\eta^*\mu)\tau$$

$$= \frac{2}{3}\theta^2\bar{\phi}^2\theta^{+4}\bar{\lambda}^{-1}\theta^{-6}\bar{\mu}\tau$$

$$(41) \qquad = \frac{2}{3}\bar{\phi}^2\bar{\lambda}^{-1}\bar{\mu}\tau.$$

Hence we can cancel in Eq. (40), and obtain;

$$(42) \qquad \eta^*\phi^{-4}\sigma = \bar{\phi}^{-4}\bar{\sigma}$$

which implies

$$(43) \qquad \bar{\sigma} = \theta^{-4}\eta^*\lambda.$$

Equs (40) and (43) together state

$$(44) \qquad [(\lambda, \sigma)] = [(\bar{\lambda}, \bar{\sigma})],$$

so $[\mathscr{Y}_\tau]$ is one-one. ∎

C) The York Map is Symplectic

Our main theorem (for the vacuum case) is the following.

THEOREM. (*Vacuum Einstein Case*). *The maps* $\mathscr{Y}_\tau : T^*_{TT}\mathscr{M}_0 \to \mathscr{C}_\tau$ *and* $[\mathscr{Y}_\tau] : \mathscr{P}_{\text{York}} \to \mathscr{P}_\tau$ *are both symplectic* (3).

The rest of this section constitutes a proof of this theorem.

---

(3) Where $T^*_{T}T\mathscr{M}_0$ are $\mathscr{P}_{York}$ are singular [i.e., for (λ, σ) with a simulteneous conformal killing vector field] the statement is true *in the stratified sense*.

We first wish to establish that if $\mathcal{Y}_\tau$ is symplectic, then so too is $[\mathcal{Y}_\tau]$. This turns out to be essentially a consequence of the following result:

LEMMA 5. *Symplectivity of Reduced Maps. Let* (P_1, ω_1) *and* (P_2, ω_2) *be symplectic manifolds. Let* G_i *be a group acting by canonical transformations on* P_i *with* Ad*-*equivariant momentum map* $J_i : P_i \to \mathfrak{g}_i^*$, *for* $i = 1, 2$. *Let* $P_i = J_i^{-1}(0)/G_i$ *denote the corresponding reduced space (which, according to Arms, Marsden and Moncrief* [1981], *may have singularities). Let*

(45) $$\Psi : J_1^{-1}(0) \to J_2^{-1}(0)$$

be a given mapping (smooth at nonsingular points).

Let $k : G_1 \to G_2$ *be a surjective group homomorphism and suppose* Ψ *is k--equivariant:*

(46) $$\Psi(g_1 \cdot p) = k(g_1) \cdot \Psi(p)$$

for $g_1 \in G_1$ *and* $p \in J_1^{-1}(0)$. *Then the induced map* $[\Psi] : P_1 \to P_2$ *is symplectic if and only if*

(47) $$\omega_1(v, w) = \omega_2(T\psi \cdot v,\ T\psi \cdot w)$$

for all $v, w \in T_p J_1^{-1}(0) \subset T_p P_1$.

Proof. This lemma follows readily form the definitions of the reduced symplectic forms on P_1 and P_2. [Note that the action of ω_1 on $v, w \notin T_p J_1^{-1}(0)$is irrelevant to what happens on the reduced space]. ■

As a (slightly indirect) consequence of Lemma 5, we have

COROLLARY 6. *If* $\mathcal{Y}_\tau$ *is symplectic, then* $[\mathcal{Y}_\tau]$ *is symplectic.*

Proof. If we choose $P_1 = T^*\mathcal{M}$, $G_1 = \mathcal{G}$, $P_1 = \mathcal{P}_{\text{York}}$ $P_2 = T^*\mathcal{M}$, $G_2 = \mathcal{D}$, $P_2 = \mathcal{P}_\tau$, $k : \mathcal{D} \ltimes \Theta \to \mathcal{D}$ (by projection) and $\psi = \mathcal{Y}_\tau$, then Lemma 5 almost applies. There are just holes to patch: Firstly we have $\mathcal{P}_{\text{York}} = T^*_{TT}\mathcal{M}_0/\mathcal{G} \neq J_1(0)/\mathcal{G}$ (because of the points of $T^*_{TT}\mathcal{M}$at which $\mathcal{Y}_\tau$ is not defined).

Secondly, we have $\mathcal{P}_\tau = \mathcal{C}_\tau/\mathcal{D} \neq J_2(0)/\mathcal{G}$ (because of the super Hamiltonian constraint which is included in the Einstein equations). However, since (as we have seen) $\mathcal{P}_\tau$ and $\mathcal{P}_{\text{York}}$ are both symplectic (stratified) manifolds in spite of these complications, then in fact this corollary does follow from Lemma 5. ■

The most straightforward way to complete the proof of the theorem is to consider a general pair of vectors ξ_1 and ξ_2 tangent to $T^*_{TT}\mathcal{M}_0$ at a general point

(λ, σ) [i.e., $\xi_1, \xi_2 \in T_{(\lambda,\tau)} T^*_{TT} \mathscr{M}_0$] and verify directly by a brute-force and lengthy calculation that

$$\Omega(T\mathscr{Y}^*_\tau(\xi_1), T\mathscr{Y}^*_\tau(\xi_2)) = \Omega(\xi_1, \xi_2). \tag{48}$$

This method, which relies upon the linearization of the York map, is in fact how we first proved the result. However there is another way to proceed which shows more clearly why the result is true. This alternative way, which we shall use, splits $\mathscr{Y}_\tau$ into the composition of two maps: The first is almost an extended point transformation (i.e., cotangent lift) while the other is a fiber translation. Both of the maps will be shown to be canonical, and hence the composition $\mathscr{Y}_\tau$ is also. The two maps we need are

$$\begin{aligned} {}_\tau : T^*_{TT} \quad {}_0 &\to T^*_{TT} \quad {}_0 \\ (\lambda, \dot{\sigma}) &\mapsto (\phi^4 \lambda, \phi^{-4} \sigma) \end{aligned} \tag{49}$$

where ϕ satisfies the Lichnerowicz equation (21) (for some constant τ), and

$$\begin{aligned} {}_\tau : T^* \quad &\to T^* \\ (\lambda, \pi) &\to \left(\gamma, \pi + \frac{2}{3}\, \gamma^{-1} \mu \tau\right). \end{aligned} \tag{50}$$

One easily verifies that both $\mathscr{W}_\tau$ and $\mathscr{X}_\tau$ are $\mathscr{D}$-equivariant, that $\mathscr{W}_\tau$ is Θ-invariant, and an assortment of other properties. Of more immediate concern is the following.

LEMMA 7. [*Two Step York Map*].

a) $\mathscr{Y}_\tau = \mathscr{X}_\tau \circ \mathscr{W}_\tau$

b) *If W_τ is symplectic and $\mathscr{X}_\tau$ is symplectic, then $\mathscr{Y}_\tau$ is symplectic.*

Proof. Part a) is verified by calculating the composition of (49) - (50), and comparing with (22) - (23). To prove part b), we do some strightforward mapchasing recalling that the symplectic form on $T^*_{TT}\mathscr{M}_0$ and on $\mathscr{C}_\tau$ is that induced from $T^*\mathscr{M}$. ■

We now show that the maps both symplectic. We start with $\mathscr{X}_\tau$ (since it is the easier one to check).

LEMMA 8. [*$\mathscr{X}_\tau$ is symplectic*]. *The map $\mathscr{X}_\tau : T^*\mathscr{M} \to T^*\mathscr{M}$ preserves the symplectic form Ω on $T^*\mathscr{M}$.*

Proof. The action of $\mathscr{X}_\tau$ [see Eq. (50)] clearly leaves the base of $T^*\mathscr{M}$ alone; so it is a fibre translation. To show that the amount of translation— $\frac{2}{3}\gamma^{-1}\mu\tau$— is in fact the exact differential of a function $f:\mathcal{M}\to\mathbb{R}$, we consider an arbitrary $k\in T_\gamma\mathcal{M}$ and calculate [using the pairing given in Eq. (6)] (4)

$$(51)\qquad \langle \frac{2}{3}\gamma^{-1}\mu\tau, k\rangle = \int_\Sigma \left(\frac{2}{3}\mu\tau\gamma^{-1}\cdot k\right) = \int_\Sigma \frac{2}{3}\mu\tau\, tr\, k = D_\gamma\left[\frac{4}{3}\tau\int_\Sigma \mu\right]\cdot k$$

Hence $\frac{2}{3}\gamma^{-1}\mu\tau = Df = D\left[\frac{4}{3}\tau\int_\Sigma \mu\right]$. But if $\mathcal{X}_\tau$ is a fibre-translation by an exact differential, then it must preserve the symplectic form (see Abraham and Marsden [1978, ex. 3.2E, p. 186]). ■

LEMMA 9. [$\mathcal{W}_\tau$ *is Symplectic*]. *The map* $\mathcal{W}_\tau : T^*_{TT}\mathcal{M}_0 \to T^*_{TT}\mathcal{M}_0$ *preserves the symplectic form* Ω *on* $T^*_{TT}\mathcal{M}_0$.

Proof. If ϕ depended only on λ, then $\mathcal{W}_\tau$ would be a cotangent lift and therefore automatically be symplectic. But ϕ depends upon σ as well as λ, so we must verify the preservation of Ω by explicit calculation.

Let $\xi_1, \xi_2 \in T_{(\lambda,\sigma)}T^*_{TT}\mathcal{M}_0$, so we can write (5)

$$(52)\qquad \xi_1 = h_1\cdot\frac{\partial}{\partial\lambda} + k_1\cdot\frac{\partial}{\partial\sigma} \quad\text{and}\quad \xi_2 = h_2\cdot\frac{\partial}{\partial\lambda} + k_2\cdot\frac{\partial}{\partial\sigma}.$$

Since ξ_1 is tangent to $T^*_{TT}\mathcal{M}_0$, itscomponents h_1 and k_1 must satisfy

$$(53)\qquad h_1\cdot\sigma + \lambda\cdot k_1 = 0$$

which follows from the linearization of the traceless condition. (There is also a constraint on h_1 and k_1 which follows from the transverse condition, but we won't need it). The components of ξ_2 obey similar constraints.

The action of the tangent of $\mathcal{W}_\tau$ is easily calculated to be

$$(54)\qquad (T\mathcal{W}_\tau)[\xi_i] = (\phi^4 h_i + (D_{\xi_i}\phi^4)\lambda)\cdot\frac{\partial}{\partial\lambda} + (\phi^{-4}k_i + D_{\xi_i}(\phi^{-4})\sigma)\cdot\frac{\partial}{\partial\sigma}$$

(4) In language more familiar to physicists, we calculate

$$\frac{\delta}{\delta\gamma_{ij}}\int \frac{4}{3}\tau\sqrt{\gamma} = \frac{2}{3}\tau\gamma^{ij}\sqrt{\gamma}$$

to show the same thing.

(for $i \in \{1, 2\}$). Plugging this into Ω, which we may write symbolically as

(55) $$\Omega = \int d\sigma \cdot\!\cdot\, d\lambda,$$

we get

$$\Omega(T\mathscr{W}_\tau[\xi_1], T\mathscr{W}_\tau[\xi_2]) =$$

$$= \int_\Sigma \{[\phi^{-4}h_1 + D_{\xi_i}(\phi^4)\lambda]\cdot[\phi^{-4}k_2 + D_{\xi_2}(\phi^{-4})\sigma] - [\overleftrightarrow{1,2}]\}$$

where «$[\overleftrightarrow{1,2}]$» means that the second term is obtained from the first by interchanging the subscripts 1 and 2. Expanding out this first term, we get

$$[\phi^4 h_1 + D_{\xi_1}(\phi^4)\lambda]\cdot[\phi^{-4}k_2 + D_{\xi_2}(\phi^{-4})\sigma] =$$
$$= h_1\cdot k_2 + \phi^4 D_{\xi_2}(\phi^{-4})h_1\cdot\sigma + \phi^{-4}D_{\xi_1}(\phi^4)\lambda\cdot k_2 + D_{\xi_1}(\phi^4)D_{\xi_2}(\phi^{-4})\lambda\cdot\sigma$$

(57) $$= h_1\cdot k_2 - 4\phi^{-1}(D_{\xi_2}\phi)h_1\cdot\sigma + 4\phi^{-1}(D_{\xi_1}\phi)\lambda\cdot k_2$$

(using $tr\,\sigma = 0$). Then plugging back into (56), we find

$$\Omega(T\mathscr{W}_\tau[\xi_1], T\mathscr{W}_\tau[\xi_2]) = \int_\Sigma \{h_1\cdot k_2 - h_2\cdot k_1 - 4\phi^{-1}(D_{\xi_2}\phi)[h_1\cdot\sigma + \lambda\cdot k_1]$$
$$+ 4\phi^{-1}(D_{\xi_1}\phi)[h_2\cdot\sigma + \lambda\cdot k_2]\}$$

$$= \int_\Sigma [h_1\cdot k_2 - h_2\cdot k_2]$$

(58) $$= \Omega(\xi_1, \xi_2)$$

where we have used Eq. (53) to kill the ϕ dependent terms. Hence Ω is preserved by $\mathscr{W}_\tau$. ■

This completes the proof of the theorem. Two aspects of it are worth noting. Firstly, while the proof uses the condition $tr\,\sigma = 0$ as well as its linearization, it

---

(5) In coordinates, this is

$$\xi_1 = \int_\Sigma d^3x\; h_{1\,ab}(x)\left[\frac{\partial}{\partial\lambda_{ab}(x)} + k_1^{ab}(x)\,\frac{\partial}{\partial\sigma^{ab}(x)}\right].$$

does not use the condition $\delta\sigma = 0$. Of course $\delta\sigma = 0$ is needed for $\mathscr{Y}_\tau$ to map into $\mathscr{C}_\tau$, and $\delta\sigma = 0$ is needed for the reduction, but we never use it explicitly in showing that $\mathscr{Y}_\tau$ (or W_τ or $\mathscr{Z}_\tau$) is canonical. Secondly, the proof also uses little of the explicit form of the Lichnerowicz equation. Again, the exact form of this equation (21) is critical if $\mathscr{Y}_\tau$ is to map into $\mathscr{C}_\tau$; but we don't need it in the proof that the symplectic form is preserved.

2. THE EINSTEIN-MAXWELL CASE

The York map is easily extended into a procedure for solving the constrained equations of many Einstein-source field theories: e.g., Einstein-Maxwell, Einstein--Yang-Mills, Einstein-Dirac, Einstein-Higgs, and Einstein-fluid. (See Isenberg and Nester [1977]). One might expect that for many of these theories, the map is still symplectic. At least for Einstein-Maxwell and Einstein-Yang-Mills, this is the case. We show this here, following roughly the same order of discussion as in the last section (leaving some of the trivially duplicated steps). For simplicity, we do only the Einstein-Maxwell case.

Before proceeding, we want to emphasize that the map which proves to be symplectic in these nonvacuum theories is *not* that which involves solving the «LW» equation along with the Lichnerowicz equation. The LW part of the York procedure (which is used to obtain the longitudinal part of the gravitational momentum) does not preserve Ω. We get a symplectic map by starting with data obtained *after* solving the LW equation. We shall see this illustrated in the Einstein-Maxwell theory, and comment further after the proof has been completed.

A) Spaces of Fields and Group Actions

We work on a principal $U(1)$ bundle Ξ over Σ (our oriented connected smooth 3-manidold). In addition to the spaces $\mathscr{M}(\Sigma)$ and $T^*\mathscr{M}(\Sigma)$ introduced in the last section, we shall use the following spaces:

the space of $U(1)$ connections A on Ξ; locally, we write the components of A as $A_i(x)$ and regard it as a one-form on Σ.

$T^*\mathrm{A}$ — the L^2 cotangent bundle of A, consisting of pairs (A, Y), where locally we regard Y as a vector field density, and denote its components $Y^i(x)$. The electric field is $E = -Y/\mu$.

$T_T^*\mathrm{A}$ — the subspaces of $T^*\mathrm{A}$ consisting of pairs (A, Y) for which

(59) $\delta Y = 0$

[Note that condition (59) depends upon a choice of metric (or volume element). It is independent of A (this is not the case for non-Abelian Yang-Mills)].

$\mathscr{C}^{\mathrm{EM}}$ The subset of $T^*\mathscr{M} \times T^*\mathrm{A}$ which satisfies the Einstein Maxwell constraint equation:

$$0 = \mathscr{J}(\gamma, \pi, A, Y) = \delta Y \tag{60}$$

$$0 = \mathscr{T}^{\mathrm{EM}}(\gamma, \pi, A, Y) = -2\delta\pi - Y \times B \tag{61}$$

$$0 = \mathcal{H}^{\mathrm{EM}}(\gamma, \pi, A, Y) = -\mu R + \left(\pi^{k\ell}\pi_{k\ell} - \frac{1}{2}(tr\,\pi)^2\right)\frac{1}{\mu} \tag{62}$$

$$+ \frac{1}{2\mu}\gamma^2 + \frac{\mu}{2}B^2 \tag{62}$$

and also satisfies the constant mean curvature condition (3). [$B = \mathrm{d}A$ is the magnetic field 2-form and $Y \times B = Y^i B_{ij}$ in index form].

$\mathscr{B}^{\mathrm{EM}}$ the subset of $T^*\mathscr{M} \times T^*\mathrm{A}$ which satisfies conditions (59), (60) and has vanishing $tr\,\pi$, but generally fails to satisfy condition (61).

The groups we need, in addition to $\mathscr{D}(\Sigma)$ and $\Theta(\Sigma)$, are

$\mathrm{Aut}(\Xi)$ the group of automorphisms of the $U(1)$-bundle Ξ. Note that each $\psi \in \mathrm{Aut}(\Xi)$ covers an element $\eta \in \mathscr{D}(\Sigma)$.

$\mathrm{Aut}_{id}(\Xi)$ the subgroup of $\mathrm{Aut}(\Xi)$ consisting of all those elements which cover the identity in $\mathscr{D}(\Sigma)$. [These are often called the «pure gauge transformations»].

and $\mathscr{G}^{\mathrm{EM}}(\Xi)$ the semidirect product

$$\mathscr{G}^{\mathrm{EM}}(\Xi) := \mathrm{Aut}(\Xi) \ltimes \Theta(\Sigma). \tag{63}$$

Note that $\mathscr{G}^{\mathrm{EM}}(\Xi)$ is isomorphic to the direct product $\mathscr{G}(\Sigma) \times \times \mathrm{Aut}_{\mathrm{Id}}(\Xi)$.

With all these groups and all these spaces, we have lots of group actions. The actions of most immediate interest here are that of $\mathrm{Aut}(\Xi)$ on $T^*\mathscr{M} \times T^*\mathrm{A}$ and $\mathscr{C}_\tau^{\mathrm{EM}}$, and that of $\mathscr{G}^{\mathrm{EM}}(\Xi)$ on $T^*\mathscr{M} \times T^*\mathrm{A}$ and $\mathscr{B}^{\mathrm{EM}}$. $\mathrm{Aut}(\Xi)$'s action is the standard pullback. Its momentum map $J_{\mathrm{Aut}(\Xi)}$ is found to be

$$J_{\mathrm{Aut}(\Xi)}(\gamma, \pi, A, Y) = (\delta Y, 2\delta\pi + Y \times B). \tag{64}$$

Thus $\mathscr{C}_\tau^{\mathrm{EM}}$ is the subset of $J^{-1}_{\mathrm{Aut}(\Xi)}(0)$ *with* the added conditions that $\mathcal{H}^{\mathrm{EM}}$ vanishes and that $\frac{tr\,\pi}{2\mu} = \tau$. As with $\mathscr{C}_\tau$ for the vacuum Einstein theory, the $tr\,\pi$ condition freezes the action of the time translations which are generated by $\mathcal{H}^{\mathrm{EM}}$, and hence one can show that

(65) $$\mathscr{P}_\tau^{\mathrm{EM}} := \mathscr{C}_\tau^{\mathrm{EM}}/\mathrm{Aut}(\Xi)$$

is a (stratified) symplectic manifold. The symplectic 2-form for $\mathscr{P}_\tau^{\mathrm{EM}}$ is obtained (by pull-back and quotient) from Ω^{EM}, the natural symplectic 2-form on $T^*\mathscr{M} \times T^*\mathrm{A}$.

The action of $\mathscr{G}^{\mathrm{EM}}$ on $T^*\mathscr{M} \times T^*\mathrm{A}$ is obtained by extending $\Theta(\Sigma)$ to $T^*\mathscr{M} \times T^*\mathrm{A}$ via

(66) $$(\theta, (\gamma, \pi, A, Y)) \mapsto (\theta^4\gamma, \theta^{-4}\pi, A, Y),$$

and then combining the action of $\Theta(\Sigma)$ with that of $\mathrm{Aut}(\Xi)$ as per the semi-direct product. The resulting momentum map is

(67) $$J_{\mathscr{G}^{\mathrm{EM}}}(\gamma, \pi, A, Y) = (\delta Y, 2\delta\pi + Y \times B, tr\, \pi),$$

so the space $\mathscr{B}^{\mathrm{EM}}$ *is* the zero set of $J_{\mathscr{G}^{\mathrm{EM}}}$. This permits a straightforward reduction (via the reduction theorem), so

(68) $$\tilde{\mathscr{P}}_{\mathrm{York}}^{\mathrm{EM}} := \mathscr{P}^{\mathrm{EM}}/\mathscr{G}^{\mathrm{EM}}$$

is a (stratified) symplectic manifold. The roles of $\tilde{\mathscr{P}}_{\mathrm{York}}^{\mathrm{EM}}$ and $\mathscr{P}_\tau^{\mathrm{EM}}$ in the York procedure are clearly presaged by the notation.

B) The York Map

The York procedure (as we define it here) for the Einstein-Maxwell-theory takes data $(\lambda, \sigma, A, Y) \in \mathscr{B}^{\mathrm{EM}}$ into data $(\gamma, \pi, A, Y) \in \mathscr{B}_\tau^{\mathrm{EM}}$ by solving the modified Lichnerowicz equation

(69) $$\nabla^2\phi = \frac{R}{8}\,\phi - \frac{1}{8\mu^2}\,\sigma\cdot\sigma\phi^{-7} + \frac{1}{12}\,\tau^2\phi^5 - \left[\frac{1}{2}\,\frac{Y^2}{\mu} + \frac{1}{2}\,B^2\mu\right]\phi^{-3}$$

and the setting $\gamma = \phi^4\lambda$, $\pi = \phi^{-4}\sigma + \frac{1}{2}\,\mu\lambda^{-1}\tau$, with A and Y left unchanged. As in the vacuum Einstein case, the procedure works for an open dense subset of $\mathscr{B}^{\mathrm{EM}}(\Xi)$, which we shall call $\mathscr{B}_0^{\mathrm{EM}}(\Xi)$ (See Isenberg, O'Murchadha and York [1976]). We thus have a well-defined York map

(70) $$^{\mathrm{EM}} : \mathscr{B}_0^{\mathrm{EM}} \to \mathscr{C}^{\mathrm{EM}}.$$

The properties of $\mathscr{Y}_\tau^{\mathrm{EM}}$ essentially mirror those of $\mathscr{Y}_\tau$. We collect them in the following lemma:

LEMMA 10. (*Properties of* $\mathscr{Y}_\tau^{\mathrm{EM}}$). $\mathscr{Y}_\tau^{\mathrm{EM}}$ *is surjective,* $\Theta(\Sigma)$*-invariant, and* $\mathrm{Aut}(\Xi)$*-equivariant.*

Proof. The verification of these three properties very closely follows the proofs given for Lemmas 1, 2, and 3, so we omit the details here. With A and Y invariant under $\mathscr{Y}_\tau^{\mathrm{EM}}$, it is not surprising that no new complications arise. ∎

From Lemma 10 and the discussion of reduced space given above, we are led to define a reduced York map

$$[\mathscr{Y}_\tau^{\mathrm{EM}}] : = \mathscr{P}_{\mathrm{York}}^{\mathrm{EM}} \to \mathscr{P}_\tau^{\mathrm{EM}} \tag{71}$$

(with $\mathscr{P}_{\mathrm{York}}^{\mathrm{EM}} := \mathscr{B}_0^{\mathrm{EM}}/\mathscr{G}^{\mathrm{EM}}$). We then have

LEMMA 11. (*Bijectivity of* $[\mathscr{Y}_\tau^{\mathrm{EM}}]$). *The map* $[\mathscr{Y}_\tau^{\mathrm{EM}}]$ *is one-one as well as onto.*

Proof. Follow the steps outlined in Lemma 4. ∎

C) **The York Map is Symplectic**

Our main result here is

THEOREM. *(Einstein-Maxwell Case). The maps* $\mathscr{Y}_\tau^{\mathrm{EM}} : \mathscr{B}_0^{\mathrm{EM}} \to \mathscr{C}_\tau^{\mathrm{EM}}$ *and* $[\mathscr{Y}_\tau^{\mathrm{EM}}] : \mathscr{P}_{\mathrm{York}}^{\mathrm{EM}} \to \mathscr{P}_\tau^{\mathrm{EM}}$ *are both symplectic.*

Proof. We start by arguing that if $\mathscr{Y}_\tau^{\mathrm{EM}}$ is symplectic, then so too is $[\mathscr{Y}_\tau^{\mathrm{EM}}]$. This is essentially a corollary of Lemma 5, with $P_1 = T^*M \times T^*\Lambda$, $G_1 = \mathscr{G}^{\mathrm{EM}}$, $P_1 = \mathscr{P}_{\mathrm{York}}^{\mathrm{EM}}$, $P_2 = T^*\mathscr{M} \times T^*\Lambda$, $G_2 = \mathrm{Aut}$, $P_2 = \mathscr{P}_\tau^{\mathrm{EM}}$, $k : \mathrm{Aut} \times \Theta \to \mathrm{Aut}$ (by projection) and $\psi = \mathscr{Y}_\tau$. The same complications described in Corollary 6 arise here, and are handled essentially the same way.

Next, we carry out the split of $\mathscr{Y}_\tau$: We define

$$\mathscr{W}_\tau^{\mathrm{EM}} : \mathscr{B}_0^{\mathrm{EM}} \to \mathscr{B}_0^{\mathrm{EM}}$$

$$(\lambda, \sigma, A, E) \mapsto (\phi^4 \lambda, \phi^{-4}\sigma, A, E) \tag{72}$$

with ϕ satisfying Eq. (68), and also

$$\mathscr{X}_\tau^{\mathrm{EM}} : T^*\mathscr{M} \times T^*\Lambda \to T^*\mathscr{M} \times T^*\Lambda$$

$$(\gamma, \pi, A, E) \mapsto (\gamma, \pi + (2/3)\,\gamma^{-1}\mu\tau, A, E). \tag{73}$$

Clearly

$$\mathscr{Y}_\tau^{\mathrm{EM}} = \mathscr{X}_\tau^{\mathrm{EM}} \circ \mathscr{W}_\tau^{\mathrm{EM}}, \tag{74}$$

and clearly if $\mathscr{X}_\tau^{\mathrm{EM}}$ and $\mathscr{W}_\tau^{\mathrm{EM}}$ are both symplectic, then $\mathscr{Y}_\tau^{\mathrm{EM}}$ is as well.

The map $\mathscr{X}_\tau^{\mathrm{EM}}$ may be written as

$$\mathscr{X}_\tau^{\mathrm{EM}} = \mathscr{X}_\tau \times \mathrm{Id}_{T^*\Lambda} \tag{75}$$

Hence, since $\mathscr{Z}_\tau$ is symplectic (see Lemma 9), $\mathscr{Z}_\tau^{EM}$ must be also.

For $\mathscr{W}_\tau^{EM}$, we don't have such a simple decomposition (both because the domain doesn't split, and because the scalar ϕ depends upon A and Y as well as upon λ, and σ). So we must again calculate directly.

Let $\xi_1, \xi_2 \in \mathscr{B}_0(\Xi)$, so we can write

$$\xi_i = h_i \cdot \frac{\partial}{\partial\lambda} + k_i \cdot \frac{\partial}{\partial\sigma} + a_i \cdot \frac{\partial}{\partial A} + e_i \cdot \frac{\partial}{\partial Y} \tag{76}$$

for $i \in \{1, 2\}$. The tangency conditions requires that the components (h_i, k_i, a_i, e_i) satisfy three identities, one of which still $h_i \cdot \sigma + \lambda \cdot k_i = 0$. Now, applying $T\mathscr{W}_\tau^{EM}$ to ξ_i, we get

$$(T\mathscr{W}_\tau^{EM})[\xi_i] = (\phi^4 h_i + (D_{\xi_i}\phi^4)\lambda) \cdot \frac{\partial}{\partial\lambda} + (\phi^4 k_i + D_{\xi_i}(\phi^{-4})\sigma) \cdot \frac{\partial}{\partial\sigma} + a_i \cdot \frac{\partial}{\partial A} + e_i \cdot \frac{\partial}{\partial Y}. \tag{77}$$

Then, if we substitute into

$$\Omega^{EM} = \int [d\sigma \cdot \hat{} \, d\lambda + dA \cdot \hat{} \, dY], \tag{78}$$

we obtain

$$\Omega^{EM}(T\mathscr{W}_\tau^{EM}[\xi_1], T\mathscr{W}_\tau^{EM}[\xi_2]) =$$

$$= \int_\Sigma \{[\phi^4 h_1 + D_{\xi_1}(\phi^4)\lambda] \cdot [\phi^{-4} k_2 + D_{\xi_2}(\phi^{-4})\sigma] + a_1 \cdot e_2 - [\overleftrightarrow{1, 2}]\}. \tag{79}$$

Part of Eq. (79) seems to match the right hand side of Eq. (56) exactly. While this is not true–since ϕ depends upon A and Y and since ξ_1 contains $a_i \dfrac{\partial}{\partial A} + {}$ $+ e_i \dfrac{\partial}{\partial y}$ – the difference are irrelevant to the calculations done in going from (56) to (58). Hence, we find

$$\Omega^{EM}(T\mathscr{W}_\tau^{EM}[\xi_1], T\mathscr{W}_\tau^{EM}[\xi_2]) =$$

$$= \int_\Sigma (h_1 \cdot k_2 + a_1 \cdot e_2 - h_2 \cdot k_1 - a_2 \cdot e_1)$$

$$= \Omega^{\text{EM}}(\xi_1, \xi_2). \tag{80}$$

This shows that Ω^{EM} is preserved by $\mathscr{W}_\tau^{\text{EM}}$, and as it follows that $\mathscr{Y}_\tau^{\text{EM}}$ and $[\mathscr{Y}_\tau^{\text{EM}}]$ are symplectic maps, and so the theorem is proved. ∎

D) Comments on the LW Part of the York Map

As decribed in the standard references (e.g. Choquet-Bruhat and York [1980])' the York procedure for the Einstein-Maxwell theory starts with data (λ, σ, A, Y) in $T^*_{TT}\mathscr{M}_0 \times T^*\mathbf{A}$ (rather than in $\mathscr{B}_0^{\text{EM}}$) and obtains data (γ, π, A, Y) in $\mathscr{C}_\tau^{\text{EM}}$ by solving

$$\nabla_a (LW)^{ab} = -\frac{1}{2\mu} Y^a B_a{}^b \tag{81}$$

for the vector field W^b where $\left[(LW)^{ab} := \nabla^a W^b + \nabla^b W^a - \frac{2}{3} \lambda^{ab} \nabla \cdot W\right]$, then solving

$$\nabla^2 \phi = \frac{R}{8}\phi - \frac{1}{8\mu^2}(\sigma + LW)\cdot(\sigma + LW)\phi^{-7} + \frac{1}{2}\tau^4\phi^5 - \frac{1}{2}\left[\frac{Y^2}{\mu} + B_\mu^2\right] \tag{82}$$

for ϕ, and finally setting $\gamma = \phi^2\lambda$, $\pi = \phi^{-4}(\sigma + LW) + \frac{1}{2}\mu\lambda^{-1}\tau$ with A and Y unchanged. We may denote this by a map

$$\widetilde{\mathscr{Y}}_\tau^{\text{EM}} : T^*_{TT}\mathscr{M}_0 \times T^*\mathbf{A} \to \mathscr{C}_\tau, \tag{83}$$

and we readily show that one can write

$$\widetilde{\mathscr{Y}}_\tau^{\text{EM}} = \mathscr{Y}_\tau^{\text{EM}} \circ \mathscr{X}^{\text{EM}} \tag{84}$$

where

$$\begin{aligned} &\mathscr{X}^{\text{EM}} : T^*_{TT}\mathscr{M}_0 \times T^*\mathbf{A} \to \mathscr{B}_0 \\ &(\lambda, \sigma, A, Y) \mapsto (\lambda, \sigma + LW, A, Y) \end{aligned} \tag{85}$$

for W satisfying (81).

From a practical standpoint, it makes sense to include $\mathscr{X}^{\text{EM}}$ in the York map, since the linear elliptic equation (81) is well-behaved, and it is much easier to choose data in $T^*_{TT}\mathscr{M}_0 \times T^*\mathbf{A}$ than it is to choose data in $\mathscr{B}_0$. Unfortunately the map $\widetilde{\mathscr{Y}}_{\tau}^{EM}$ is not Θ-invariant (6), and the space $T^*_{TT}\mathscr{M}_0 \times T^*A$ does not factor to a symplectic manifold. We therefore have no well-defined reduced York map. Moreover, it is unlikely that $\widetilde{\mathscr{Y}}_{\tau}^{EM}$ itself is canonical. So, while $\widetilde{\mathscr{Y}}_{\tau}^{EM}$ is useful for solving the full set of Einstein-Maxwell constraint equations, only the $\mathscr{Y}_{\tau}^{EM}$ portion of it is a symplectic map.

References

R. ABRAHAM and J. MARSDEN [1978]. *Foundations of Mechanics*, Second Edition. Addison Wesley, Reading, Mass.

J. ARMS, J. MARSDEN and V. MONCRIEF [1982]. The Structure of the Space Solutions of Einstein's Equations; II Many Killing Fields, *Ann. of Phys.* 144, 81 - 106.

J. ARMS, J. MARSDEN and V. MONCRIEF [1981]. Bifurcations of Momentum Mappings. *Comm. Math. Phys.* 78, 455 - 478.

Y. CHOQUET-BRUHAT, A. FISCHER and J. MARSDEN [1979]. Maximal Hypersurfaces and Positivity of Mass, in *Isolated Gravitating Systems and General Relativity*, J. Ehlers, ed., Italian Physical Society, 322 - 395.

Y. CHOQUET-BRUHAT and J.W. YORK [1980]. The Cauchy Problem, in *General Relativity and Gravitation*, Vol. I, A. Held (ed.). Plenum Press, N. Y., 99 - 172.

A. FISCHER and J. MARSDEN [1977]. The Manifold of Confromally Equivalent Metrics, *Can. J. Math.* 29, 193 - 209.

A. FISCHER and J. MARSDEN [1979]. Topics in the Dynamics of General Relativity in «Isolated Gravitating Systems in General Relativity», J. Ehlers, ed., *Italian Physical Society*, 322 - 395.

J. ISENBERG and J. MARSDEN [1982]. A Slice Theorem for the Space of Solutions of Einstein's Equations, *Physics Repors* 89, 179 - 122.

J.A. ISENBERG and J.M. NESTER [1977]. Extension of the York Field Decomposition to General Gravitationally Coupled Fields, Ann. of Phys. 198, 368 - 386.

J.A. ISENBERG, N. O'MURCHADHA and J.W. YORK [1976]. Initial-value Problem of General Relativity III. Coupled Fields and the Scalar-tensor Theory, *Phys. Rev. D.*, 13, 1532 - 1537.

J. MARSDEN and A. WEINSTEIN [1974]. Reduction of Symplectic Manifolds with Symmetry. *Rep. Mat. Phys.*, 5, 121 - 310.

V. MONCRIEF [1975]. Decompositions of Gravitational Perturbations. J. Math. *Phys.*, 16, 1556 - 60.

N. O'MURCHADHA and J.W. YORK [1974]. The Initial-value Problem of General Relativity. *Phys. Rev.*, D10, 428 - 446.

(6) There is another version of $\mathscr{Y}_{\tau}^{EM}$ which *is* Θ invariant (see O'Murchadha and York [1974]). However, in that version, the LW and ϕ are coupled, and no one has been able to prove any existence or uniqueness theorems for the coupled set. Moreover, even if the equations are well-behaved the doman space still doesn't factor, so one has no well-defined $[\mathscr{Y}_{\tau}^{EM}]$.

J.W. YORK [1971]. Gravitational Degrees of Freedom and the Initial-value Problem. *Phys. Rev. Lett.*, 26, 1656 - 8.

J.W. YORK [1973]. Conformally Invariant Orthogonal Decomposition of Symmetric tensors on Riemannian Manifolds and the Initial-value Problem of General Relativity. *J.Math. Phys.*, 14, 456 - 64.

Manuscript received: July 30, 1983.

Invited paper