On the Rotated Stress Tensor and the Material Version of the Doyle-Ericksen Formula

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Dedicated to J. L. Ericksen

Contents

1. Introduction

DOYLE & ERICKSEN [1956, p. 77] observed that the Cauchy stress tensor $\boldsymbol{\sigma}$ can be derived by varying the internal free energy ψ with respect to the Riemannian metric $\mathbf{g}$ on the ambient space: $\boldsymbol{\sigma} = 2\varrho\, \partial\psi/\partial\mathbf{g}$. Their formula has gone virtually unnoticed in the literature of continuum mechanics. In this paper we shall establish the material version of this formula: $\boldsymbol{\Sigma} = 2\varrho\partial\overline{\Psi}/\partial\mathbf{G}$ for the *rotated* stress tensor $\boldsymbol{\Sigma}$, and shall address some of the reasons why these formulae are of fundamental significance. Making use of these formulae one can derive elasticity tensors and establish rate forms of the *hyperelastic* constitutive equations for the Cauchy stress tensor and the rotated stress tensor, as discussed in Sections 4 and 5. The role of the rotated stress tensor in the formulation of continuum theories has been noted by GREEN & NAGHDI [1965]. Generalizations of hypoelasticity based on the use of the rotated stress tensor have been considered by GREEN & MCINNIS [1967].

Some additional reasons for the importance of these formulae follow. First of all, the original (spatial) formula of DOYLE & ERICKSEN allows for a rational derivation of the spatial form of the Duhamel-Neumann hypothesis on a decomposition of the rate of deformation tensor (see SOKOLNIKOFF [1956], p. 359), which is useful in the identification problem for constitutive equations. This derivation, due to HUGHES, MARSDEN & PISTER, is described in MARSDEN & HUGHES ([1983], pp. 204–207). Our material version of the Doyle-Ericksen formula together with some additional results discussed in this paper, allows one to carry out the same argument materially. This is described in Section 5.1. Second, these formulae play a crucial role in extending the balance of energy principle (using invariance under superposed rigid body motions) to a covariant theory which allow arbitrary spatial diffeomorphisms. † The spatial formulation of the covariant version of this principle is described in Section 2.4 of MARSDEN & HUGHES [1983]. The material formulae here enable one to derive a material version of a covariant balance of energy principle. This is described in Section 5 of this paper. The covariant energy principle not only makes classical hyperelasticity a fully covariant theory, but also allows one to obtain *directly* the principle of virtual work. We note that from the principle of invariance of energy under superposed rigid body motions it is not possible to obtain the principle of virtual work *directly* [see *e.g.* SEWELL 1966]. An alternative derivation of this principle based on the integral form of the balance laws is found in ANTMAN & OSBORNE [1979].

Finally, in classical relativistic field theory, it has been standard since the pioneering work of BELINFANTE [1939] and ROSENFELD [1940] to regard the stress-energy-momentum tensor as the derivative of the Lagrangian density with respect to the space-time (Lorentz) metric; see for example, HAWKING & ELLIS [1973, Sect. 3.3] and MISNER, THORNE & WHEELER [1973, Sect. 21.3]. This modern point of view has largely replaced the construction of *canonical stress-energy-momentum tensors*. Thus, for the Lagrangian formulation of elasticity (relativistic or not), the Doyle-Ericksen formula plays the same role as the Belinfante-Rosenfeld formula, and brings it into line with developments in other areas of classical field theory.

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† In the context of classical continuum mechanics, a derivation of this principle was given by GREEN & RIVLIN [1964a], and applied to multipolar media in GREEN & RIVLIN [1964b, c]. An alternative although equivalent formulation of this principle can be given based on closely related ideas of invariance due to NOLL [1963] (see MARSDEN & HUGHES [1983] pp. 145–152, for this and a comparison of both approaches). As noted by NAGHDI ([1972], footnote 17 in p. 490) the idea of obtaining balance laws from balance of energy and invariance under spatial isometries was known to ERICKSEN [1961]. A version of this principle in the context of Cosserat Continua was given by TOUPIN [1964], who gives much credit to the COSSERATS. A comprehensive account of the applications to Cosserat continua is found in NAGHDI [1972] and references therein. In the context of classical Lagrangian field theory the application of this principle amounts to imposing invariance of the Lagrangian density under one parameter groups of translations and rotations and using NOETHER's theorem (see, *e.g.*, MARSDEN & HUGHES [1983], Sect. 5.5).

2. Notation. Basic Relations

Let B and S be orientable smooth n-manifolds equipped with Riemannian metrics $\mathbf{G}_o$ and $\mathbf{g}$ respectively, with associated volume elements denoted by dV and dv. We refer to $(B, \mathbf{G}_o)$ as the *fixed reference configuration* of the body of interest, and to $(S, \mathbf{g})$ as the *ambient space* in which its deformation takes place. A configuration of the body is an embedding $\phi: B \to S$, and the set of all configurations is denoted by $\mathscr{C}$.

A motion of the body is a curve $t \in \mathbb{R} \to \phi_t \in \mathscr{C}$, and for $X \in B$ we write $x = \phi_t(X) \equiv \phi(X, t)$. The material velocity is the vector field $\mathbf{V}_t: B \to T_x S$ defined by $\mathbf{V}_t(X) = \partial\phi_t(X)/\partial t$, for $X \in B$, where $T_x S$ is the tangent space to S at $x = \phi_t(X)$. The material acceleration $\mathbf{A}_t: B \to T_x S$ of a point $X \in B$ is defined by $\mathbf{A}_t(X) = \partial\mathbf{V}_t(X)/\partial t$. The spatial velocity $\mathbf{v}_t(x)$ and the spatial acceleration $\mathbf{a}_t$ are vector fields on $\phi_t(B)$ defined by the relations $\mathbf{v}_t = \mathbf{V}_t \circ \phi_t^{-1}$ and $\mathbf{a}_t = \mathbf{A}_t \circ \phi_t^{-1}$, respectively.

The deformation gradient, $\mathbf{F} = T\phi_t$, is the tangent of the map $\phi_t: B \to S$, $t \in \mathbb{R}$, with components F^a_A relative to coordinate charts $\{X^A\}$ and $\{x^a\}$ in B and S, respectively. The components of the right *Cauchy-Green* deformation tensor are then defined as

$$C_{AB} = F^a_A F^b_B g_{ab} \circ \phi_t^{-1}. \tag{2.1a}$$

To emphasize the geometric meaning, it is convenient to employ the standard *pull-back/push-forward* notation of analysis on manifolds (Lang [1972], Marsden & Hughes [1983]). Equation (2.1a), then, is simply the coordinate expression for the *pull-back* of the spatial metric $\mathbf{g}$; *i.e.*,

$$\mathbf{C} = \phi_t^*(\mathbf{g}). \tag{2.1b}$$

Next, we consider the classical *polar decomposition* theorem in a slightly more general setting. Consider the manifold B endowed with two *different* Riemannian metrics $\mathbf{G}_o$ and $\mathbf{G}$, and let $(T_X B, \mathbf{G}_o)$ and $(T_X B, \mathbf{G})$ be the associated tangent spaces at $X \in B$ with the inner products $\mathbf{G}_o(X)$ and $\mathbf{G}(X)$, respectively. The polar decomposition theorem states that

$$T\phi_t \equiv \mathbf{F} = \mathbf{RU} \quad \text{i.e.,} \quad F^a_A = R^a_B U^B_A \tag{2.2}$$

where $\mathbf{R}(X): (T_X B, \mathbf{G}) \to (T_{\phi_t(X)} S, \mathbf{g})$ is a $(\mathbf{G}, \mathbf{g})$-orthogonal transformation called the *rotation tensor*, and $\mathbf{U}(X): (T_X B, \mathbf{G}_o) \to (T_X B, \mathbf{G})$ is the *material stretch* tensor. Although we may choose $\mathbf{G}_o = \mathbf{G}$, the discussion between the metrics $\mathbf{G}_o$ and $\mathbf{G}$ becomes crucial for later developments. The metric $\mathbf{G}_o$ is a *fixed* Riemannian metric assigned to the reference configuration B, and is *invariant* with respect to superposed *spatial* diffeomorphisms. (The metric $\mathbf{G}$ can be arbitrarily chosen and *may change* under *spatial* diffeomorphisms as we shall see later. The following diagram summarizes the domains and ranges of $\mathbf{F}$, $\mathbf{U}$ and $\mathbf{R}$. The fact that $\mathbf{R}$ is a two-point $(\mathbf{G}, \mathbf{g})$-*orthogonal* tensor is expressed by the relation

$$G_{AB} = R^a_A R^b_B g_{ab} \circ \phi_t \tag{2.3a}$$

which simply states that the pull-back of the (spatial) metric tensor $\mathbf{g}$ by the rotation part $\mathbf{R}$ of the tangent $\mathbf{F} = T\phi_t$ yields the metric tensor $\mathbf{G}$. To emphasize this interpretation we introduce the notion of *pull-back under* $\mathbf{R}$, by rewriting (2.3a) as

$$\mathbf{G} = \mathbf{R}^*(\mathbf{g}). \tag{2.3b}$$

$$\begin{tikzcd}
(T_X B, \mathbf{G}_o) \arrow[r, "\mathbf{U}(X)"] \arrow[dr, "\mathbf{F}(X)"'] & (T_X B, \mathbf{G}) \arrow[d, "\mathbf{R}(X)"] \\
& (T_{\phi_t(X)} S, \mathbf{g})
\end{tikzcd}$$

From the stretching part $\mathbf{U}$ of the polar decomposition (2.2), we recover the right Cauchy-Green tensor $\mathbf{C}$ by the formula

$$C_{AB} = U^I_A G_{IJ} U^J_B. \tag{2.4a}$$

Thus, we may regard the tensor $\mathbf{C}$ as the pull-back of the metric tensor $\mathbf{G}$ by the stretching part $\mathbf{U}$ of $\mathbf{F}$. Again this interpretation is emphasized by rewriting (2.4a) in pull-back notation as

$$\mathbf{C} = \mathbf{U}^*(\mathbf{G}). \tag{2.4b}$$

In view of (2.1) and (2.4) we may think of $\mathbf{C}$ either (a) as the pull-back of the metric $\mathbf{g}$ by $\mathbf{F}$, or (b) as the pull-back of the metric $\mathbf{G}$ by $\mathbf{U}$. In the next section a simple argument employing the chain rule will show that point of view (a) yields the (*spatial*) formula of DOYLE & ERICKSEN [1956], whereas point of view (b) leads to the material version of this formula. A more fundamental approach based on the notion of *covariance* will be pursued in Section 6. This approach will reveal the fundamental role played by the formula of DOYLE & ERICKSEN in a covariant formulation of classical hyperelasticity.

3. The Rotated Stress Tensor and the Doyle-Ericksen Formula

Recall that standard constitutive assumptions and arguments (COLEMAN-NOLL [1959]) imply that the free energy functional Ψ for a thermoelastic material depends on the motion $\phi_t(X)$ through the *point values* of the deformation tensor $\mathbf{C}(X)$ and the temperature variable $\Theta(X)$. We write $\hat{\Psi}(X, \mathbf{C}(X), \Theta(X), \mathbf{G}_o)$ to express this functional dependence†. Let $\varrho_{\mathrm{Ref}}(X)$ be the density in the reference configuration B. If $\mathbf{S}$ denotes the (symmetric) second Piola-Kirchhoff stress

† The metric tensor $\mathbf{G}_o$ must be included since it is needed to form a scalar from the tensor $\mathbf{C}$, such as $\operatorname{tr} \mathbf{C} = C_{AB} G_o^{AB}$.

tensor, one has the constitutive equation

$$\mathbf{S} = 2\varrho_{\mathrm{Ref}} \frac{\partial \hat{\Psi}(X, \mathbf{C}, \Theta, \mathbf{G}_o)}{\partial \mathbf{C}} . \tag{3.1}$$

By equation (2.1), $\mathbf{C}$ is a function of the deformation gradient $\mathbf{F}$ (the spatial metric $\mathbf{g}$ assumed *fixed*). Hence, on defining $\overline{\overline{\Psi}}(X, \mathbf{F}, \Theta, \mathbf{G}_o) = \hat{\Psi}(X, \mathbf{F}^T\mathbf{F}, \Theta, \mathbf{G}_o)$, the constitutive equation (3.1) takes the equivalent alternative form:

$$\mathbf{P} = \varrho_{\mathrm{Ref}} \frac{\partial \overline{\overline{\Psi}}}{\partial \mathbf{F}} \mathbf{g}^{-1} \tag{3.2}$$

where $\mathbf{P}$ is the first Piola-Kirchhoff (two-point) stress tensor, with components P^I relative to coordinate charts $\{X^I\}$ and $\{x^i\}$.

In this section, we shall develop the *material* version of the formula of DOYLE & ERICKSEN by a chain rule type of argument analogous to that employed by DOYLE & ERICKSEN [1956]. (See also MARSDEN & HUGHES [1983], p. 196.) We recall the basic idea.

(*a*) *Spatial Form.* Making use of (2.1), one defines the (spatial) free energy functional $\tilde{\psi}$ as a function of the point values of the spatial metric $\mathbf{g}$ and the deformation gradient $\mathbf{F}$, by the relation

$$\psi(x, \mathbf{g}(x), \mathbf{F}(X), \Theta(X), \mathbf{G}_o(X)) = \hat{\Psi}(X, \mathbf{C}(\mathbf{g}(x), \mathbf{F}(X)), \Theta(X), \mathbf{G}_o(X)) \tag{3.3}$$

where $\mathbf{C}(\mathbf{g}(x), \mathbf{F}(X)) \equiv \phi_t^*(\mathbf{g}(x))$. The chain rule together with the Piola-transformation and the constitutive equation (3.1) then leads to

$$\boldsymbol{\sigma} = 2\varrho \frac{\partial \tilde{\psi}}{\partial \mathbf{g}} \tag{3.4}$$

where ϱ is the density in the current configuration $\phi_t(B)$ given by (*conservation of mass*) $\varrho = \varrho_{\mathrm{Ref}}/J$, where J is the Jacobian of $\phi_t : B \to S$ taken relative to $\mathbf{G}_o$ and $\mathbf{g}$.

The *material* version of formula (3.4) may be developed by a similar argument as follows.

(*b*) *Material Form.* We make use of of (2.4) to express the free energy as a function of the point values of the metric $\mathbf{G}$ and the stretching tensor $\mathbf{U}$. The chain rule and use of (3.1) then yields the desired result. Explicitly, set

$$\overline{\Psi}(X, \mathbf{U}(X), \mathbf{G}(X), \Theta(X), \mathbf{G}_o(X)) = \hat{\Psi}(X, \mathbf{C}(\mathbf{U}(X), \mathbf{G}(X)), \Theta(X), \mathbf{G}_o(X)) \tag{3.5}$$

where, by virtue of (2.4),

$$\mathbf{C}(\mathbf{U}, \mathbf{G}) = \mathbf{U}^*(\mathbf{G}) \equiv \mathbf{U} \cdot \mathbf{G} \cdot \mathbf{U}. \tag{3.6}$$

The chain rule and (3.1) then gives, in coordinates,

$$\varrho_{\mathrm{Ref}} \frac{\partial \overline{\Psi}}{\partial G_{IJ}} = \varrho_{\mathrm{Ref}} \frac{\partial \hat{\Psi}}{\partial C_{AB}} \frac{\partial C_{AB}}{\partial G_{IJ}} = \varrho_{\mathrm{Ref}} \frac{\partial \hat{\Psi}}{\partial C_{AB}} U_A^I U_B^J \equiv \frac{1}{2} (\mathbf{U}^*(\mathbf{S}))^{IJ}. \tag{3.7}$$

Next, recall that the *rotated* stress tensor (GREEN & NAGHDI [1965], MARSDEN & HUGHES [1983], pp. 137), denoted by $\boldsymbol{\Sigma}$, is defined as the pull-back by the rotation tensor $\mathbf{R}$ of the Cauchy stress tensor $\boldsymbol{\sigma}$. Accordingly,

$$\boldsymbol{\sigma} = \mathbf{R}_*(\boldsymbol{\Sigma}) \quad \text{i.e.,} \quad \sigma^{ab} = R^a_A R^b_B \Sigma^{AB} \circ \phi_t^{-1}. \tag{3.8}$$

By the Piola transformation and the polar decomposition, we have

$$\boldsymbol{\sigma} = \frac{1}{J}\,\phi_{t_*}(\mathbf{S}) \equiv \frac{1}{J}\,\mathbf{R}_* \cdot (\mathbf{U}_*(\mathbf{S})) \tag{3.9}$$

and from (3.8) and (3.9) we obtain the following expression for $\boldsymbol{\Sigma}$,

$$\boldsymbol{\Sigma} = \frac{1}{J}\,\mathbf{U}_*(\mathbf{S}) \quad \text{i.e.,} \quad S^{IJ} = J U^I_A U^J_B \Sigma^{AB}. \tag{3.10}$$

Substitution of (3.10) into (3.7) and use of conservation of mass (*i.e.*, $\varrho = \varrho_{\mathrm{Ref}}/J$) leads to the formula

$$\boldsymbol{\Sigma} = 2\varrho\,\frac{\partial \overline{\Psi}}{\partial \mathbf{G}} \tag{3.11}$$

which is the material version of the Doyle-Ericksen formula.

Remark. Formula (3.11) requires careful interpretation. Although we may take $\mathbf{G} = \mathbf{G}_o$, the variation is done only with respect to the dependence on $\mathbf{G}$ through $\mathbf{C}$. The dependence of $\overline{\Psi}$ on $\mathbf{G}_o$ plays *no role* in formula (3.11). Our covariant argument of Section 6 will be consistent with and will clarify the reason for this.

For the purpose of comparison, the equivalent forms in which constitutive equations for hyperelasticity may be expressed have been summarized in Table 1 of Section 5. In the next section, we examine the appropriate expression for the elasticity tensor associated with the rotated stress tensor $\boldsymbol{\Sigma}$.

4. The Rotated Elasticity Tensor

The elasticity tensors appear naturally in the formulation and linearization of boundary value problems in nonlinear elasticity. (See, *e.g.*, TRUESDELL & NOLL [1965] or MARSDEN & HUGHES [1983].) These tensors also appear in a natural manner when the hyperelastic constitutive equations are formulated in rate form. The second point is illustrated in the next section.

The following two tensors are often referred to as *material* and *spatial* versions of the *second elasticity tensor*:

$$\mathsf{C} = 2\varrho_{\mathrm{Ref}}\,\frac{\partial^2 \hat{\Psi}}{\partial \mathbf{C}\,\partial \mathbf{C}}, \quad \mathsf{c} = 4\varrho\,\frac{\partial^2 \tilde{\psi}}{\partial \mathbf{g}\,\partial \mathbf{g}}. \tag{4.1}$$

These tensors are related by the Piola-transformation,

$$\mathbf{c} = \frac{2}{J}\,\phi_{t*}(\mathbf{C}) \quad \text{i.e.,} \quad c^{abcd} = \frac{1}{J}\,F^a_A F^b_B F^c_C F^d_D C^{ABCD} \circ \phi_t^{-1}. \tag{4.2}$$

$\mathbf{c}$ appears naturally in connection with the spatial formula (3.4) in terms of $\boldsymbol{\sigma}$ and $\mathbf{g}$, whereas $\mathbf{C}$ is naturally associated with the stress tensor S and the constitutive equation (3.1), In addition, one defines the *first elasticity tensor* by the expression

$$\mathbf{A} = \varrho_{\mathrm{Ref}}\frac{\partial^2\overline{\overline{\Psi}}}{\partial\mathbf{F}\,\partial\mathbf{F}} = \frac{\partial\mathbf{P}}{\partial\mathbf{F}}. \tag{4.3}$$

The tensor $\mathbf{A}$, associated with the first Piola-Kirchhoff two-point tensor, is connected to $\mathbf{C}$ and $\mathbf{c}$ by the relations

$$A^{aBcD} = F^a_A F^c_C C^{ABCD} + g^{ac}S^{BD} - c^{abcd}(F^{-1})^B_b\,(F^{-1})^D_d + \sigma^{ac}G^{BD}. \tag{4.4}$$

In connection with constitutive equation (3.11), the material version of the Doyle-Ericksen formula, we define the *rotated* elasticity tensor $\boldsymbol{\Xi}$ by the expression

$$\boldsymbol{\Xi} = 4\varrho\,\frac{\partial^2\overline{\Psi}}{\partial\mathbf{G}\,\partial\mathbf{G}}. \tag{4.5}$$

In next section, we shall that this tensor appears naturally in the rate form of constitutive equation (3.11). By making use of the chain rule and relation (4.2) we obtain the following expressions relating $\boldsymbol{\Xi}$, $\mathbf{C}$ and $\mathbf{c}$:

$$\boldsymbol{\Xi} = \frac{2}{J}\,\mathbf{U}_*(\mathbf{C}), \qquad \boldsymbol{\Xi} = \mathbf{R}^*(\mathbf{c}). \tag{4.6a}$$

In coordinates, these transformation formulae read

$$\Xi^{IJKL} = \frac{2}{J}\,U^I_A U^J_B U^K_C U^L_D C^{ABCD}, \qquad c^{abcd} = R^a_I R^b_J R^c_K R^d_L \Xi^{IJKL} \circ \phi_t^{-1}. \tag{4.6b}$$

5. Rate Form of Hyperelastic Constitutive Equations

In this section we examine rate forms of constitutive equations for *hyperelasticity*, based on the spatial and material versions of the Doyle-Ericksen formula, in the context of *thermoelasticity*. First, we summarize the relevant results in the spatial picture.

(*a*) *Spatial Form.* Recall that the *flow* of the spatial velocity field $\mathbf{v}_t(x)$ is the map $\gamma_{t,s} = \phi_s \circ \phi_t^{-1} : \phi_t(B) \to \phi_s(B)$. The *Lie derivative* of a (spatial) tensor field in the direction of the flow of the $\mathbf{v}_t(x)$ may then be defined as

$$\mathbf{L}_{\mathbf{v}}(\mathbf{t}_t) - \left.\frac{d}{ds}\right|_{s=t}(\gamma^*_{t,s}\mathbf{t}_s) = \phi_{t*}\left(\frac{\partial}{\partial t}\,\phi^*_t(\mathbf{t}_t)\right) \tag{5.1}$$

where $\mathbf{t}_t$ denotes an arbitrary, possibly time dependent, tensor field (i.e., for fixed t, $\mathbf{t}_t$ is a section of a tensor bundle constructed on S). By holding t fixed, one obtains the *autonomous Lie derivative* $\mathscr{L}_{\mathbf{v}}(\mathbf{t}_t)$; that is: $\mathbf{L}_{\mathbf{v}}(\mathbf{t}_t) = \frac{\partial \mathbf{t}_t}{\partial t} + \mathscr{L}_{\mathbf{v}}(\mathbf{t}_t)$. For the metric tensor $\mathbf{g}$, a direct application of definition (5.1) leads to the formula

$$\mathbf{L}_{\mathbf{v}}(\mathbf{g}) = \phi_{t*}(\dot{\mathbf{C}}) \equiv 2\mathbf{d} \tag{5.2}$$

where $\mathbf{d} = \frac{1}{2}\,\phi_{t*}(\dot{\mathbf{C}})$ is the *spatial rate of deformation tensor*. In addition, note that

$$\mathbf{L}_{\mathbf{v}}(J) \equiv \dot{J} = J \operatorname{div}(\mathbf{v}_t). \tag{5.3}$$

Taking Lie derivatives on both sides of the (spatial) Doyle-Ericksen formula (3.4) and substitutions (5.2), (5.3), yields the rate constitutive equation:

$$\begin{aligned}\mathbf{L}_{\mathbf{v}}(\boldsymbol{\sigma}) + \operatorname{div}(\mathbf{v}_t)\,\boldsymbol{\sigma} &= \frac{1}{J}\,\mathbf{L}_{\mathbf{v}}(J\boldsymbol{\sigma}) \\ &= 4\varrho\,\frac{\partial^2\tilde{\psi}}{\partial\mathbf{g}\,\partial\mathbf{g}} : \mathbf{d} + 2\varrho\,\frac{\partial^2\tilde{\psi}}{\partial\mathbf{g}\,\partial\Theta}\cdot\dot{\Theta} \equiv \mathbf{c} : \mathbf{d} + \mathbf{m}\cdot\dot{\Theta}\end{aligned} \tag{5.4}$$

where $\mathbf{m} = 2\varrho\,\partial^2\tilde{\psi}/\partial\mathbf{g}\,\partial\Theta$ are the *thermal stress coefficients*. One calls $\boldsymbol{\tau} \equiv J\boldsymbol{\sigma}$ the *Kirchhoff* stress tensor. The left hand side of (5.4) is the Truesdell rate of Cauchy stresses, defined as $\mathbf{L}_{\mathbf{v}_t}(\boldsymbol{\tau})/J$ (TRUESDELL [1955]), and the spatial second elasticity tensor $\mathbf{c}$ given by $(4.1)_2$ appear naturally on the right hand side of (5.4).

(*b*) *Material Form.* By analogy with definition (5.1) we may define the *material stretch* Lie derivative of an arbitrary *material* tensor $\mathbf{T}_t$ by

$$\mathbf{L}_{\mathbf{U}}(\mathbf{T}_t) = \mathbf{U}_* \cdot \left(\frac{\partial}{\partial t}\,\mathbf{U}^*(\mathbf{T}_t)\right). \tag{5.5}$$

That is, the pull-back/push-forward operations with the tensor $\mathbf{F}$ in (5.1) are replaced by its stretching part $\mathbf{U}$. An interpretation of definition (5.5) in terms of the spatial velocity field is given by part (i) of the proposition below. The introduction of (5.5) is motivated by part (ii), which is the material version of (5.2).

Proposition.

(i) The material stretch Lie derivative defined by (5.5) is the Lie derivative with respect to the rotated spatial velocity field $\mathbf{R}^*(\mathbf{v}_t)$; *i.e.*,

$$\mathbf{L}_{\mathbf{U}}(\mathbf{T}_t) = \mathbf{L}_{\mathbf{R}^*(\mathbf{v}_t)}(\mathbf{T}_t). \tag{5.6}$$

(ii) If Λ is the *rotated rate of deformation* tensor defined by

$$\Lambda = \mathbf{R}^*(\mathbf{d}) \quad \text{i.e.,} \quad \Lambda_{AB} = R^a_A R^b_B\, d_{ab} \circ \phi_t, \tag{5.7}$$

then one has the formula

$$\mathbf{L}_{\mathbf{U}}(\mathbf{G}) = 2\Lambda. \tag{5.8}$$

Proof.

(i) Making use of standard properties of pull-backs and the Lie derivative, by definitions (5.1) and (5.5), we have

$$\begin{aligned}\mathbf{L_U}(\mathbf{T}_t) &= \mathbf{R}^*\left(\phi_{t*}\frac{\partial}{\partial t}\phi_t^*(\mathbf{R}_*\mathbf{T}_t)\right)\\ &= \mathbf{R}^*\mathbf{L_v}(\mathbf{R}_*\mathbf{T}_t)\\ &\equiv \mathbf{L}_{(\mathbf{R}^*\mathbf{v}_t)}(\mathbf{T}_t).\end{aligned}$$

(ii) By definition (5.5), together with (2.4) and (5.2) it follows that

$$\begin{aligned}\mathbf{L_U}(\mathbf{G}) &= \mathbf{U}_*(\dot{\mathbf{C}})\\ &= 2\mathbf{U}_*(\phi_t^*\,\mathbf{d})\\ &= 2\mathbf{R}^*(\mathbf{d}) \equiv 2\mathbf{\Lambda}. \quad \square\end{aligned}$$

Taking the *material stretch* Lie derivative (according to (5.5)) on both sides of the material version (3.11) of the Doyle-Ericksen formula, and using (5.3) together with (5.8) yields the following rate constitutive equation for thermoelasticity:

$$\begin{aligned}\mathbf{L_U}(\mathbf{\Sigma}) + \mathbf{\Sigma}\,\mathrm{tr}\,(\mathbf{\Lambda}) &\equiv \frac{1}{J}\mathbf{L_U}(J\mathbf{\Sigma})\\ &= 4\varrho\frac{\partial^2\overline{\Psi}}{\partial\mathbf{G}\,\partial\mathbf{G}} : \mathbf{\Lambda} + 2\varrho\frac{\partial^2\overline{\Psi}}{\partial\mathbf{G}\,\partial\Theta}\cdot\dot{\Theta} \equiv \mathbf{\Xi} : \mathbf{\Lambda} + \mathbf{M}\cdot\dot{\Theta} \qquad (5.9)\end{aligned}$$

which is the material version of (5.4). The tensor $\mathbf{T} \equiv J\mathbf{\Sigma}$ is simply the *rotated* Kirchhoff stress tensor; *i.e.*, $\mathbf{T} = \mathbf{R}^*(\boldsymbol{\tau})$.

Remarks. (i) Clearly, the material stretch Lie derivative $\mathbf{L_U}(\mathbf{\Sigma})$ may also be defined as the $\mathbf{R}$-rotated Lie derivative $\mathbf{L_v}(\boldsymbol{\sigma})$, since

$$\mathbf{L_U}(\mathbf{\Sigma}) = \mathbf{L}_{(\mathbf{R}^*\mathbf{v}_t)}(\mathbf{R}^*\boldsymbol{\sigma}) \equiv \mathbf{R}^*(\mathbf{L_v}(\boldsymbol{\sigma})). \qquad (5.10\text{a})$$

In coordinates, we have the formula

$$(\mathbf{L_v}(\boldsymbol{\sigma}))^{ab} = R^a_A R^b_B (\mathbf{L_U}(\mathbf{\Sigma}))^{AB} \circ \phi_t^{-1}. \qquad (5.10\text{b})$$

(ii) The rate constitutive equation (5.9) may be derived by a direct computation as follows. Taking the material time derivative on both sides of expression (3.10) yields

$$\dot{\mathbf{\Sigma}} - (\dot{\mathbf{U}}\mathbf{U}^{-1})\,\mathbf{\Sigma} - \mathbf{\Sigma}(\dot{\mathbf{U}}\mathbf{U}^{-1})^T + \mathbf{\Sigma}\,tr\,(\mathbf{\Lambda}) = \frac{1}{J}\mathbf{U}\cdot\dot{\mathbf{S}}\cdot\mathbf{U}^T. \qquad (5.11)$$

We note that the left hand side of (5.11) is $\mathbf{L_U}(J\mathbf{\Sigma})$. Since

$$\mathbf{U}\cdot\dot{\mathbf{S}}\cdot\mathbf{U}^T = \mathbf{U}\cdot\left(\frac{\partial\mathbf{S}}{\partial\mathbf{C}} : \dot{\mathbf{C}} + \frac{\partial\mathbf{S}}{\partial\Theta}\cdot\dot{\Theta}\right)\cdot\mathbf{U} \equiv J(\mathbf{\Xi} : \mathbf{\Lambda} + \mathbf{M}\cdot\dot{\Theta}); \qquad (5.12)$$

the equivalence of (5.11) and (5.9) follows.

(iii) By taking the **R**-push-forward of the *rate of rotated Cauchy stress tensor* $\dot{\boldsymbol{\Sigma}}$, one obtains the following *objective* rate of Cauchy stresses (GREEN & NAGHDI [1965], DIENES [1979])

$$\overset{\square}{\boldsymbol{\sigma}} \equiv \mathbf{R}_* \frac{\partial}{\partial t} \mathbf{R}^*(\boldsymbol{\sigma}) = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\Omega}^T \tag{5.13}$$

where the spatial tensor $\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$ is *skew-symmetric*. Thus, $\overset{\square}{\boldsymbol{\sigma}}$ is simply the Lie derivative of $\boldsymbol{\sigma}$, with the pull-back/push-forward operations by $\mathbf{F}$ replaced by its rotation part $\mathbf{R}$. Introducing the *left* polar decomposition $\mathbf{F} = \mathbf{V}\mathbf{R}$, we have the following expression for $\overset{\square}{\boldsymbol{\sigma}}$ in terms of the spatial velocity field:

$$\overset{\square}{\boldsymbol{\sigma}} = \mathbf{V}^* \left(\phi_{t_*} \frac{\partial}{\partial t} \phi_t^*(\mathbf{V}_* \boldsymbol{\sigma}) \right) = \mathbf{V}^*(\mathbf{L}_\mathbf{v}(\mathbf{V}_* \boldsymbol{\sigma})) \equiv \mathbf{L}_{(\mathbf{V}^* \mathbf{v}_t)}(\boldsymbol{\sigma}). \tag{5.14}$$

Notice that the *rotated* elasticity tensor $\boldsymbol{\Xi}$ is associated with the material stretch Lie derivative $\mathbf{L}_\mathbf{U}(\boldsymbol{\Sigma})$ of the rotated stress tensor through (5.9), and not with $\dot{\boldsymbol{\Sigma}} = \mathbf{R}^*(\overset{\square}{\boldsymbol{\sigma}})$.

(iv) The Lie derivative provides the natural way of measuring the rate of change of a tensor field with respect to a vector field in differential geometry. In the context of continuum mechanics, several definitions to measure the rate of change of the Cauchy stress tensor have been proposed and called objective rates. (See, *e.g.*, TRUESDELL & TOUPIN [1960] Sects. 147–152.) As noted in MARSDEN & HUGHES [1983] (pp. 99–102) all spatial objective rates are in fact different manifestations of the Lie derivative. The spatial formula (5.4) shows that the Lie derivative of the Kirchhoff stress tensor (essentially the TRUESDELL rate of Cauchy stresses) is naturally associated with the second elasticity tensor $\mathbf{c}$. Obviously, any objective rate could be used in (5.4) provided the right hand side is properly adjusted.

The alternative representations of constitutive equations for (isothermal) hyperelasticity have been summarized for convenience in Table 1 below.

Table 1

Stress Tensor	**Elasticity Tensor**	**Rate Equation**
$\mathbf{S} = 2\varrho_{\mathrm{Ref}} \dfrac{\partial \hat{\Psi}}{\partial \mathbf{C}}$	$\mathbf{C} = 2\varrho_{\mathrm{Ref}} \dfrac{\partial^2 \hat{\Psi}}{\partial \mathbf{C}\, \partial \mathbf{C}}$	$\dot{\mathbf{S}} = \mathbf{C} : \dot{\mathbf{C}}$
$\mathbf{P} = \varrho_{\mathrm{Ref}} \dfrac{\partial \overline{\Psi}}{\partial \mathbf{F}}$	$\mathbf{A} = \varrho_{\mathrm{Ref}} \dfrac{\partial^2 \overline{\Psi}}{\partial \mathbf{F}\, \partial \mathbf{F}}$	$\dot{\mathbf{P}} = \mathbf{A} : \dot{\mathbf{F}}$
$\boldsymbol{\sigma} = 2\varrho \dfrac{\partial \tilde{\psi}}{\partial \mathbf{g}}$	$\mathbf{c} = 4\varrho \dfrac{\partial^2 \tilde{\psi}}{\partial \mathbf{g}\, \partial \mathbf{g}}$	$\mathbf{L}_\mathbf{v}(J\boldsymbol{\sigma}) = J\mathbf{c} : \mathbf{d}$
$\boldsymbol{\Sigma} = 2\varrho \dfrac{\partial \overline{\Psi}}{\partial \mathbf{G}}$	$\boldsymbol{\Xi} = 4\varrho \dfrac{\partial^2 \overline{\Psi}}{\partial \mathbf{G}\, \partial \mathbf{G}}$	$\mathbf{L}_\mathbf{U}(J\boldsymbol{\Sigma}) = J\boldsymbol{\Xi} : \boldsymbol{\Lambda}$

5.1. An Application: The Material Form of the Duhamel-Neumann Hypothesis

The results of this section allow for a derivation of the material version of the *Duhamel-Neumann hypothesis* (SOKOLNIKOFF [1956], p. 359). The material version of the Doyle-Ericksen formula (3.11) and our definition (5.5) of material stretch Lie derivative are the two key ingredients in the derivation.

As used in Table 1, the free energy function, depending on the point values of $\mathbf{C}$, admits the various representations

$$\hat{\Psi}(\mathbf{C}, \Theta, \mathbf{G}_o) = \overline{\Psi}(\mathbf{G}, \Theta, \mathbf{G}_o) = \tilde{\psi}(\mathbf{g}, \Theta, \mathbf{G}_o) \tag{5.15}$$

where, for notational simplicity, the dependence of $\overline{\Psi}$ on $\mathbf{U}$ and the dependence of $\tilde{\psi}$ on $\mathbf{F}$ has been suppressed. By performing a *Legendre* transformation on $\overline{\Psi}(\mathbf{G}, \Theta, \mathbf{G}_o)$ (see ARNOLD [1978], p. 61 for a geometric interpretation), we define the *material complementary free energy* $\bar{\chi}$ by

$$2\varrho\bar{\chi}(\mathbf{T}, \Theta, \mathbf{G}_o) = \boldsymbol{\Sigma} : \mathbf{G} - 2\varrho\overline{\Psi}(\mathbf{G}, \Theta, \mathbf{G}_o) \tag{5.16}$$

where $\mathbf{T} = \mathbf{R}^*(\boldsymbol{\tau})$ is the rotated Kirchhoff tensor. Since $\varrho = \varrho_{\mathrm{Ref}}/J$, the chain rule gives

$$2\varrho_{\mathrm{Ref}}\frac{\partial\bar{\chi}}{\partial\mathbf{T}} = \mathbf{G} + J\boldsymbol{\Sigma} : \frac{\partial\mathbf{G}}{\partial\mathbf{T}} - 2\varrho_{\mathrm{Ref}}\frac{\partial\overline{\Psi}}{\partial\mathbf{G}} : \frac{\partial\mathbf{G}}{\partial\mathbf{T}}. \tag{5.17}$$

Since by (3.11) the last two terms cancel, we are led to the relation

$$\mathbf{G} = 2\varrho\frac{\partial\bar{\chi}}{\partial\mathbf{T}} \tag{5.18}$$

which is *dual* to the formula $\mathbf{T} = 2\varrho_{\mathrm{Ref}}\,\partial\Psi/\partial\mathbf{G}$. Applying the material stretch Lie derivative on both sides of (5.17) and making use of (5.8) we obtain

$$\boldsymbol{\Lambda} = \varrho_{\mathrm{Ref}}\frac{\partial^2\bar{\chi}}{\partial\mathbf{T}\,\partial\mathbf{T}} : \mathbf{L}_{\mathbf{U}}(\mathbf{T}) + \varrho_{\mathrm{Ref}}\frac{\partial^2\bar{\chi}}{\partial\mathbf{T}\,\partial\Theta}\cdot\dot{\Theta} = \boldsymbol{\Gamma} : \mathbf{L}_{\mathbf{U}}(\mathbf{T}) + \mathbf{N}\cdot\dot{\Theta} \tag{5.19}$$

where $\boldsymbol{\Gamma}$ and $\mathbf{N}$ are the *rotated material compliance tensors*. Therefore, the total *rotated* rate of deformation may be decomposed into the *mechanical rate* and the *thermal rate*, according to (5.19). It is clear that $\boldsymbol{\Gamma}$ and $J\boldsymbol{\Xi}$ are inverse tensors.

6. Covariant Energy Balance and the Doyle-Ericksen Formula

The key step in the development of a covariant formulation of classical hyperelasticity, is the extension of the balance of energy principle so that it holds not only for superposed spatial isometries, but for arbitrary diffeomorphisms. This extension, due to MARSDEN & HUGHES, reveals the fundamental role played by the (*spatial*) Doyle-Ericksen formula. To develop the material counterpart by a covariant procedure a simple although non-trivial construction is required. In addition, the *convected* version of this formula will also be considered. First, we summarized the basic ingredients involved in the covarant approach, which are relevant to our development. A complete account can be found in (MARSDEN & HUGHES [1983], sec. 2.4).

Balance of Energy. Let $\phi_t: B \to S$ be a *fixed* motion and $\Omega \subset B$ a compact region with smooth boundary $\partial\Omega$, mapped onto $\phi_t(\Omega) \subset S$ with boundary $\partial\phi_t(\Omega)$. Let $\mathbf{b}(x, t)$ be the external body force per unit of mass, and $\mathbf{t}(x, t, \mathbf{n})$ the Cauchy traction vector. The motion ϕ_t satisfies balance of energy if

$$\frac{d}{dt}\int_{\phi_t(\Omega)} \varrho(e + \tfrac{1}{2}\langle \mathbf{v}_t, \mathbf{v}_t\rangle)\, dv = \int_{\phi_t(\Omega)} \varrho(\langle \mathbf{b}, \mathbf{v}_t\rangle + r)\, dv + \int_{\partial\phi_t(\Omega)} (\langle \mathbf{t}, \mathbf{v}_t\rangle + h)\, ds \tag{6.1}$$

where $e(x, t)$ is the *internal energy* per unit of mass $r(x, t)$ is the *heat supply* per unit mass, and $h(x, t, \mathbf{n})$ the *heat flux* across the boundary $\partial\phi_t(\Omega)$ with unit normal $\mathbf{n}$.

Covariant Assumption. For the *fixed* motion $\phi_t: B \to S$, which satisfies balance of energy, consider an *arbitrary diffeomorphism* $\xi_t: S \to S$. Postulate that the new motion $\bar{\phi}_t = \xi_t \circ \phi_t$ also satisfies balance of energy provided the metric $\mathbf{g}$ is replaced by $\xi_t^*\mathbf{g}$ and velocities, forces and accelerations are transformed according to the standard dictates of the (Cartan) theory of the classical spacetime. (See MARSDEN & HUGHES [1983], sec. 2.4.) That is, the key assumption is

$$\bar{e}(\bar{x}, t, \mathbf{g}) = e(x, t, \xi_t^*\mathbf{g}), \qquad \bar{x} = \xi_t(x). \tag{6.2}$$

Remark. Assumption (6.2) is rather natural and may be motivated as follows. If $\xi_t: (S, \mathbf{g}) \to (S, \bar{\mathbf{g}})$ is a *spatial* diffeomorphism, one may ask what the relation between the metrics $\mathbf{g}$ and $\bar{\mathbf{g}}$ must be so that the tensor $\mathbf{C}$, a *material* tensor, remain unchanged. Since $\bar{\mathbf{C}} = \bar{\phi}_t^*(\bar{\mathbf{g}})$, the condition $\bar{\mathbf{C}} = \mathbf{C}$ implies that

$$\bar{\mathbf{g}} = \xi_{t_*} \circ \phi_{t_*}(\bar{\mathbf{C}}) \equiv \xi_{t_*} \circ \phi_{t_*}(\mathbf{C}) = \xi_{t_*}(\mathbf{g}). \tag{6.3}$$

That is, $\mathbf{C}$ remains unchanged if $\bar{\mathbf{g}} = \phi_{t_*}(\mathbf{g})$; note that $\bar{\mathbf{g}}$ is in the set

$$O_{\mathbf{g}} \equiv \{\xi_{t_*}(\mathbf{g}) \mid \xi_t: S \to S \text{ is a diffeomorphism}\}$$

called the *orbit* of $\mathbf{g}$. Thus, the assumption that the internal energy transforms tensorially according to $e(x, t, \bar{\mathbf{g}}) = \bar{e}(\bar{x}, r, \xi_{t_*}(\bar{\mathbf{g}}))$ is consistent with the classical result of constitutive theory since $\mathbf{C}$ does not change.

Prior to considering the covariant argument which leads to our material version of the formula of DOYLE & ERICKSEN, we recall the fundamental role played by its spatial counterpart in the covariant formulation of the balance of energy principle.

6.1. Spatial Form

The basic idea is to evaluate the balance of energy equation (6.1) for the motion $\bar{\phi}_t = \xi_t \circ \phi_t$ at time $t = t_o$ for which

$$\xi_t|_{t=t_o} = \text{Identity}, \qquad \left.\frac{\partial \xi_t}{\partial t}\right|_{t=t_o} = \mathbf{w}. \tag{6.4}$$

Use of the transport theorem, the divergence theorem and the Cauchy tetrahedron construction gives, as in the Green-Rivlin-Naghdi argument, conservation if mass, balance of momentum, balance of moment of momentum; together with the additional identity

$$\int_{\phi_t(\Omega)} [\varrho(\dot{\bar{e}} - \dot{e}) - \boldsymbol{\sigma} : \mathbf{k}]\, dv = 0 \tag{6.5}$$

where

$$\mathbf{k} = \frac{1}{2}\mathbf{L}_{\mathbf{w}}(\mathbf{g}); \qquad \text{i.e.,} \qquad k_{ab} = \frac{1}{2}(w_{a|b} + w_{b|a}). \tag{6.6}$$

In the Green-Rivlin-Naghdi argument one has $\dot{\bar{e}} = \dot{e}$ since spatial *isometries* leave the metric $\mathbf{g}$ unchanged. Under arbitrary spatial diffeomorphisms, however, (6.2) and the definition of Lie derivative gives

$$\dot{\bar{e}} = \dot{e} + \frac{\partial e}{\partial \mathbf{g}} : \frac{d}{dt}\bigg|_{t=t_o} \xi_t^*(\mathbf{g}) = \dot{e} + \frac{\partial e}{\partial \mathbf{g}} : \mathbf{L}_{\mathbf{w}}(\mathbf{g}). \tag{6.7}$$

Substitution of (6.7) into (6.5) and noting that $\mathbf{L}_{\mathbf{w}}(\mathbf{g})$ can be arbitrarily specified, yields

$$\boldsymbol{\sigma} = 2\varrho \frac{\partial e}{\partial \mathbf{g}}. \tag{6.8}$$

Thus, the Doyle-Ericksen formula appears as the essential condition which serves the purpose of relaxing the "rigidity" part of the assumption that balance of energy must hold under arbitrary *spatial* diffeomorphisms.

In terms of the polar decomposition, the above argument leads to the situation described in the following diagram

$$\begin{tikzcd}
(T_X B, \mathbf{G}_o) \arrow[r, "\mathbf{U}"] & (T_X B, \mathbf{G}) \arrow[r, "\mathbf{R}"] \arrow[dr, "\bar{\mathbf{R}}"'] & (T_{\phi_t(X)} S, \mathbf{g}) \arrow[d, "T\xi_t"] \\
 & & (T_{\bar{\phi}_t(X)} S, \bar{\mathbf{g}})
\end{tikzcd}$$

Where

$$\bar{\mathbf{g}} = \xi_{t_*}(\mathbf{g}), \qquad \bar{\mathbf{R}} = T\xi_t \circ \mathbf{R} \circ \xi_t^{-1}, \qquad \bar{\mathbf{U}} = \mathbf{U}. \tag{6.9}$$

Notice that the metric $\mathbf{G}$ and $\mathbf{C} = \mathbf{U}^*(\mathbf{G})$, *remain unchanged* through the argument. Clearly, to obtain the material version of the Doyle-Ericksen formula, one must introduce a framework in which the metric $\mathbf{G}$ varies under spatial diffeomorphisms.

6.2. Material Form

Suppose we hold the rotation tensor $\mathbf{R} = \overline{\mathbf{R}}$ fixed under spatial diffeomorphisms ξ_t. Since $\mathbf{R}$ is an *orthogonal* two-point tensor, and $\bar{\mathbf{g}} = \xi_{t_*}(\mathbf{g}) \in O_{\mathbf{g}}$, then the metric $\mathbf{G}$ must change to say $\overline{\mathbf{G}}$ so that the orthogonal character of $\mathbf{R}$, expressed by (2.3a), is preserved. The precise situation is summarized in the following diagram:

$$\begin{tikzcd}
(T_X B, \mathbf{G}_o) \arrow[r, "\mathbf{U}"] \arrow[dr, "\overline{\mathbf{U}}"] & (T_X B, \mathbf{G}) \arrow[r, "\mathbf{R}"] \arrow[d, "\mathbf{H}"] & (T_{\phi_t(X)} S, \mathbf{g}) \arrow[d, "T\xi_t"] \\
& (T_X B, \overline{\mathbf{G}}) \arrow[r, "\mathbf{R}"] & (T_{\bar{\phi}_t(X)} S, \bar{\mathbf{g}})
\end{tikzcd}$$

where $\mathbf{H} : (T_X B, \mathbf{G}) \to (T_X B, \overline{\mathbf{G}})$ is defined as

$$\mathbf{H} = \mathbf{R}^{-1} \circ T\xi_t \circ \mathbf{R} \circ \phi_t^{-1} \equiv \mathbf{R}^* \circ \xi_{t_*}. \tag{6.10}$$

Notice that the metric $\mathbf{G}_o$ in the reference configuration, remains *unchanged.* The metric $\mathbf{G}$, however, transforms tensorially according to

$$\overline{\mathbf{G}} = \mathbf{H}^{-1} \circ \mathbf{G} \circ \mathbf{H}^{-1}; \quad \text{i.e.,} \quad \overline{G}_{IJ} = (H^{-1})^A_I \, (H^{-1})^B_J \, G_{AB} \tag{6.11}$$

which preserves the orthogonal character of $\mathbf{R}$ with respect to $\overline{\mathbf{G}}$ and $\bar{\mathbf{g}} = \xi_{t_*}(\mathbf{g})$. The linear transformation $\overline{\mathbf{U}} : (T_X B, \mathbf{G}_o) \to (T_X B, \overline{\mathbf{G}})$ is obtained by composition as

$$\overline{\mathbf{U}} = \mathbf{H} \circ \mathbf{U}; \quad \text{i.e.,} \quad \overline{U}^I_A = H^I_B U^B_A. \tag{6.12}$$

Thus, since $\mathbf{U}^*(\mathbf{G}) = \overline{\mathbf{U}}^*(\overline{\mathbf{G}})$, the material tensor $\mathbf{C}$ remains unchanged.

The material form of the Doyle-Ericksen formula can now be derived by a covariant argument analogous to that used in the spatial case summarized above. We define the material form $E(X, t, \mathbf{G})$ of the internal energy $e(x, t, \mathbf{g})$ in the natural manner by setting

$$E(X, t, \mathbf{G}) = e(\phi_t(X), t, \mathbf{R}_*(\mathbf{G})). \tag{6.12}$$

By use of this definition and (6.10), the material version of the covariant assumption (6.2) now takes the form

$$\begin{aligned}
\overline{E}(X, t, \mathbf{G}) &= \bar{e}(\bar{\phi}_t(X), t, \mathbf{R}_*(\mathbf{G})) \\
&= e(\phi_t(X), t, \xi_t^* \circ \mathbf{R}_*(\mathbf{G})) \\
&= E(X, t, \mathbf{H}^*(\mathbf{G}))
\end{aligned} \tag{6.13}$$

and the definitions of material time derivative and Lie derivative lead to

$$\dot{\bar{E}} = \dot{E} + \frac{\partial E}{\partial \mathbf{G}} : \frac{d}{dt}\bigg|_{t=t_o} H^*(\mathbf{G})$$

$$= \dot{E} + \frac{\partial E}{\partial \mathbf{G}} : \mathbf{L}_{\mathbf{W}}(\mathbf{G}) \tag{6.14}$$

where $\mathbf{W}$ is "rotated" velocity of ξ_t at time $t = t_o$; *i.e.*,

$$\mathbf{W} = \mathbf{R}^*(\mathbf{w}). \tag{6.15}$$

To complete the argument we note that, by making use of standard properties of pull-backs and the Lie derivative, the term $\boldsymbol{\sigma} : \mathbf{k}$ can be expressed as

$$\begin{aligned} \boldsymbol{\sigma} : \mathbf{k} &= \mathbf{R}_*(\boldsymbol{\Sigma}) : \mathbf{k} \\ &= \boldsymbol{\Sigma} : \mathbf{R}^*(\mathbf{k}) \\ &= \boldsymbol{\Sigma} : \frac{1}{2}\mathbf{R}^*(\mathbf{L}_{\mathbf{w}}(\mathbf{g})) \\ &= \boldsymbol{\Sigma} : \frac{1}{2}\mathbf{R}^*(\mathbf{L}_{(\mathbf{R}^*\mathbf{w})}(\mathbf{R}^*\mathbf{g})) \\ &= \frac{1}{2}\boldsymbol{\Sigma} : \mathbf{L}_{\mathbf{W}}(\mathbf{G}). \end{aligned} \tag{6.16}$$

By substitution of (6.14) and (6.16), the identity (6.5) now reads

$$\int_{\Omega} \left(\varrho \frac{\partial E}{\partial \mathbf{G}} - \frac{1}{2}\boldsymbol{\Sigma}\right) : \mathbf{L}_{\mathbf{W}}(\mathbf{G})\, J\, dV \equiv 0. \tag{6.17}$$

Since $\mathbf{L}_{\mathbf{W}}(\mathbf{G})$ can be arbitrarily specified, our covariant argument yields the material version of the Doyle-Ericksen formula:

$$\boldsymbol{\Sigma} = 2\varrho \frac{\partial E}{\partial \mathbf{G}}. \tag{6.18}$$

Finally, we show that the argument leading in the covariant formulation to the Doyle-Ericksen formula (6.18) leads in the *convected* picture to an expression closely related to constitutive equation (3.1) for the second Piola-Kirchhoff stress tensor.

6.3. Convected Form

In the *convected picture*, the basic objects are obtained from the spatial objects by pulling back to the reference configuration. Thus, the *convected velocity* $\boldsymbol{\nu}_t$ and the *convected acceleration* $\boldsymbol{\alpha}_t$ of a given motion $t \to \phi_t \in \mathscr{C}$ are vector fields on B defined as

$$\boldsymbol{\nu}_t = \phi_t^*(\mathbf{v}_t), \qquad \boldsymbol{\alpha}_t = \phi_t^*(\mathbf{a}_t). \tag{6.19}$$

Similarly, the *convected* stress tensor $\mathfrak{S}$ is defined as the pull-back of the Cauchy stress tensor; *i.e.*,

$$\mathfrak{S} = \phi_t^*(\sigma) \equiv J\mathbf{S}. \tag{6.20}$$

The *convected* form of the Doyle-Ericksen formula may be derived by the argument leading to the material formula (6.24). The key point to note is that in performing the polar decomposition of the deformation gradient $\mathbf{F}$, the metric $\mathbf{G}$ *can be arbitrarily chosen*. Regardless of the choice of $\mathbf{G}$ the covariant argument of section 6.2 must always hold. In particular, by choosing the *convected* metric $\mathbf{G} = \phi_t^*\mathbf{g}$, since the relations $\mathbf{C} = \mathbf{U}^*(\mathbf{G})$, $\mathbf{G} = \mathbf{R}^*(\mathbf{g})$ and $\mathbf{C} = \phi_t^*(\mathbf{g})$ must hold, we are led to the situation described in the following diagram:

$$\begin{tikzcd}
(T_X B, \mathbf{G}_o) \arrow[r, "\mathbf{I}"] \arrow[dr, "\overline{\mathbf{U}}"'] & (T_X B, \mathbf{C}) \arrow[r, "\mathbf{F}"] \arrow[d, "\mathbf{H}"] & (T_{\phi_t(X)} S, \mathbf{g}) \arrow[d, "T\xi_t"] \\
 & (T_X B, \overline{\mathbf{C}}) \arrow[r, "\mathbf{F}"] & (T_{\overline{\phi}_t(X)} S, \overline{\mathbf{g}})
\end{tikzcd}$$

That is, the choice of metric $\mathbf{G} = \mathbf{C}$ implies that

$$\mathbf{U}(X) = \mathbf{I}, \quad \mathbf{U}(X) = \mathbf{H}(X), \quad \mathbf{R}(X) = \overline{\mathbf{R}}(X) \equiv \mathbf{F}(X) \tag{6.21}$$

where $\mathbf{H} = \mathbf{F}^{-1} \circ T\xi_t \circ \mathbf{F}$. We then have the relations

$$\boldsymbol{\Sigma} \equiv \mathbf{R}^*(\sigma) = \phi_t^*(\sigma) \equiv J\mathbf{S} \tag{6.22}$$

and

$$2\boldsymbol{\Lambda} = \mathbf{L}_{\mathbf{U}}(\mathbf{G}) = \mathbf{L}_{\mathbf{I}}(\mathbf{C}) \equiv \dot{\mathbf{C}}. \tag{6.23}$$

Therefore, when we make use of (6.20)–(6.23), the argument that led to the material formula (6.18) now yields

$$\mathbf{S} = \frac{2}{J}\frac{\partial E}{\partial \mathbf{C}} \tag{6.24}$$

which is the *convected* form of the Doyle-Ericksen formula. Notice that the covariant argument in the convected picture yields, in addition to formula (6.24), the *convected* form of the equations of motion; *i.e.*,

$$\mathrm{DIV}_C\,\mathbf{S} + \varrho_{\mathrm{Ref}}\phi_t^*(\mathbf{B}) = \alpha_t, \qquad \mathbf{S} = \mathbf{S}^T \tag{6.25}$$

where $\mathrm{DIV}_C\,\mathbf{S}$ is the divergence of $\mathbf{S}$ with respect to the metric $\mathbf{C} \equiv \phi_t^*(\mathbf{g})$, and $\mathbf{B} = \mathbf{b} \cdot \phi_t$.

Remarks. (i) Our covariant argument of section 6.2 (or section 6.3) is essentially a *material formulation* of the notion of invariance under superposed *spatial* diffeomorphisms. This argument does not involve, nor does it imply, the assumption of *material covariance* which embodies the notion of invariance under superposed *material* diffeomorphisms. Explicitly, the function $\hat{\Psi}(\mathbf{C}, \Theta, \mathbf{G}_o)$ is said to be *materially covariant* if for all material diffeomorphisms $\Pi_t: B \to B$ one has

$$\hat{\Psi}(\mathbf{C}, \Theta, \mathbf{G}_o) \circ \Pi_t = \hat{\Psi}(\Pi_{t_*}(\mathbf{C}), \Theta, \Pi_{t_*}(\mathbf{G}_o)).$$

In the context of a Euclidean structure, material covariance and *material isotropy* are equivalent.

(ii) The covariant formulation of balance of energy *directly* yields the principle of virtual work (*i.e.*, the weak form of the balance equations) for hyperelasticity expressed either in the spatial, material or convected pictures. Notice, however, that the material form involves the rotated stress tensor, not the first Piola-Kirchhoff stress tensor.

7. Concluding Remarks

In Section 6, we have focussed our attention on a covariant formulation of hyperelasticity based on the covariant balance of energy principle, expressed either spatially or materially. This approach reveals the fundamental role played by the spatial or material versions of the Doyle-Ericksen formula. However, this is not the only possible approach leading to a fully covariant theory.

(i) The Hamiltonian formalism can be used as an alternative to the covariant form of the balance of energy principle. Again we may proceed materially (MARSDEN & HUGHES [1983], Sect. 5.3) or spatially (MARSDEN, RATIU & WEINSTEIN [1983]).

(ii) A deep understanding of the Hamiltonian formalism for incompressible fluids enabled ARNOLD [1966a, b] to prove the nonlinear stability of plane flows studied by Rayleigh in a situation where one would otherwise expect the usual difficulties with potential wells (KNOPS & WILKES [1973] and MARSDEN & HUGHES [1983, Sect. 6.6]). A similar stability result for compressible plane flow was found by HOLM, MARSDEN, RATIU & WEINSTEIN [1983]. It is conceivable that a similar understanding in elasticity will shed light on the energy criterion; see, however, BALL and MARSDEN [1984].

(iii) Finally, we note that in the context of general relativity, the Doyle-Ericksen formula is the spatial part of the stress-energy-momentum tensor that naturally arises when one couples elasticity to the gravitational field in EINSTEIN's theory (MARSDEN & HUGHES [1983], Sect. 5.7).

References

ANTMAN, S. S., & J. E. OSBORNE, 1979. "The Principle of Virtual Work and Integral Laws of Motion," *Arch. Rational Mech. Anal.* **69**, pp. 231–262.

ARNOLD, V. 1966a. "Sur la Géometrie Différentielle des Groupes de Liede Dimension Infinie et ses Applications à l'hydrodynamique des Fluids Parfaits," *Ann. Inst. Fourier, Grenoble*, **16**, pp. 319–361.

ARNOLD, V. 1966b. "An a priori Estimate in the Theory of Hydrodynamic Stability," *Trans. Amer. Math. Soc.*, **79** (1969), pp. 267–269.

ARNOLD, V. 1978 *Mathematical Methods of Classical Mechanics*, New York: Springer-Verlag.

BALL, J. M., & J. E. MARSDEN, 1984. Quasiconvexity at the boundary, second variations, and stability in nonlinear elasticity. *Arch. Rational Mech. Anal.* (to appear).

BELINFANTE, F. 1939. "On the Current and the Density of the Electric Charge, the Energy, the Linear Momentum, and the Angular Momentum of Arbitrary Fields," *Physica*, **6**, p. 887 and **7**, pp. 449–474.

COLEMAN, B. D., & W. NOLL, 1959. "On the Thermodynamics of Continuous Media," *Arch. Rational Mech. Anal.*, **4**, pp. 97–128.

DIENES, J. K., 1979. "On the Analysis of Rotation and Stress Rate in Deforming Bodies," *Acta Mechanica*, **32**, pp. 217–232.

DOYLE, T. C., & J. L. ERICKSEN, 1956. *Nonlinear Elasticity*, in *Advances in Applied Mechanics IV*. New York: Academic Press, Inc.

ERICKSEN, J. L., 1961. "Conservation Laws for Liquid Crystals," *Tran. Soc. Rheol.*, **5**, pp. 23–34.

GREEN, A. E., & B. C. MCINNIS, 1967. "Generalized Hypo-Elasticity," *Proc. Roy. Soc. Edinburgh*, A **57**, p. 220.

GREEN, A. E., & P. M. NAGHDI, 1965. "A General Theory of an Elastic-Plastic Continuum," *Arch. Rational Mech. Anal.*, **18**, p. 251.

GREEN, A. E., & R. S. RIVLIN, 1964a. "On Cauchy's Equations of Motion," *J. Appl. Math. and Physics (ZAMP)*, **15**, pp. 290–292.

GREEN, A. E., & R. S. RIVLIN, 1964b. "Multipolar Continuum Mechanics," *Arch. Rational Mech. Anal.*, **17**, pp. 113–147.

GREEN, A. E., & R. S. RIVLIN, 1964c. "Simple Force and Stress Multipoles," *Arch. Rational Mech. Anal.*, **16**, pp. 327–353.

HAWKING, S., & G. ELLIS, 1973. *The Large Scale Structure of Spacetime*, Cambridge, England: Cambridge Univ. Press.

HOLM, D., & B. KUPERCHMIDT, 1983. "Poisson Brackets and Clebch Representations for Magnetohydrodynamics, Multifluid Plasmas and Elasticity," *Physica D* **6**, 347–356.

HOLM, D. D., J. E. MARSDEN, T. RATIU & A. WEINSTEIN, 1983. "Nonlinear Stability Conditions and *a priori* Estimates for Barotropic Hydrodynamics" *Physics Lett.* **98** A 15–21.

KNOPS, R., & E. WILKES, 1973. *Theory of Elastic Stability*, in *Handbuch der Physik*, VIa/3, S. FLÜGGE, ed., Belin: Springer-Verlag.

LANG, S., 1972. *Differential Manifolds*, Reading, MA: Addison-Wesley Publishing Co. Inc.

MARSDEN, J. E., & T. J. R. HUGHES, 1983. *Mathematical Foundations of Elasticity*, Englewood-Cliffs, N. Y. Prentice-Hall, Inc.

MARSDEN, J. E., T. RATIU & A. WEINSTEIN, 1982. "Semidirect Products and Reduction in Mechanics," *Trans. Am. Math. Soc.* (to appear).

MISNER, C., K. THORNE & J. WHEELER, 1982. *Gravitation*, S. Francisco: W. H. Freman & Company, Publishers, Inc.

NAGHDI, P. M., 1972. *The Theory of Shells and Plates*, in *Handbuch der Physik*, C. TRUESDELL ed., Vol. VIa/2, Berlin: Springer-Verlag.

NOLL, W., 1963. La Méthode Axiomatique dans les Mécaniques Classiques et Nouvelles, Paris, pp. 47–56.

ROSENFELD, L., 1940. "Sur le Tenseur d'Impulsion-Energy," *Mem. Acad. R. Belg. Sci.*, **18**, no. 6, pp. 1–30.

SEWELL, M. J., 1966. "On Configuration-Dependent Loading," *Arch. Rational Mech. Anal.*, **23**, pp. 327–351.

SOKOLNIKOFF, I. S., 1956. *The Mathematical Theory of Elasticity* (2d ed.), New York: McGraw-Hill Book Company.

TOUPIN, R. A., 1964. "Theories of Elasticity with Couple Stress," *Arch. Rational Mech. Anal.*, **11**, pp. 385–414.

TRUESDELL, C., 1955. "Hypo-elasticity," *J. Rational Mech. Anal.*, **4**, pp. 83–133.

TRUESDELL, C., & R. A. TOUPIN, 1960. *The Classical Field Theories*, in *Handbuch der Physik*, Vol. III/1, S. FLÜGGE, ed., Berlin: Springer-Verlag.

TRUESDELL, C., & W. NOLL, 1965. *The Non-Linear Field Theories of Mechanics*, in *Handbuch der Physik*, Vol. III/3, S. FLÜGGE, ed., Berlin: Springer-Verlag.

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