Comm

WP5 - 2:30

Reprinted from PROCEEDINGS OF THE 22ND IEEE CONFERENCE ON DECISION & CONTROL, December 1983

CHAOTIC ORBITS BY MELNIKOV'S METHOD: A SURVEY OF APPLICATIONS

Jerrold E. Marsden*

University of California, Berkeley

§1. Introduction

In Melnikov [1963] a method was introduced for proving the existence of chaotic orbits in dynamical systems. This was used by Arnold [1964], the paper in which "Arnold diffusion" was discovered. The method lay relatively dormant until it was revived by Chirikov [1979], Holmes [1980], and Chow, et al. [1980].

In this lecture I will survey a number of recent applications of Melnikov's method to a variety of interesting physical situations.

There are many modifications of the basic technique possible depending on the dimension of the system, whether or not dissipation or forcing are included or whether or not the system is autonomous. However to get the basic idea, it is useful to begin with a conservative but externally forced one degree of freedom Hamiltonian system. After presenting the basic theorem for this case in §2, we shall discuss the generalizations of the theory in §3 and the various applications in §4. For additional background and applications, see Lichtenberg and Lieberman [1983] and Guckenheimer and Holmes [1983].

§2. Forced One Degree of Freedom Hamiltonian Systems

We consider an evolution equation in the plane $\mathbb{R}^2$ of the form

$$\dot{x} = f_0(x) + \varepsilon f_1(x,t) + O(\varepsilon^2) \tag{2.1}$$

where f_0 is a smooth Hamiltonian vector field with energy H_0 (that is, if $x = (q,p)$, then $f_0 = (\partial H_0/\partial p, -\partial H_0/\partial q))$, f_1 is a smooth, time dependent Hamiltonian vector field with energy H_1 and f_1 and H_1 are T-periodic. We assume that when $\varepsilon = 0$, the unperturbed system $\dot{x} = f_0(x)$ possesses a homoclinic orbit $\bar{x}(t)$ to a hyperbolic saddle point x_0; i.e., if $\bar{x}(0) \neq x_0$ is a convenient reference point on the orbit, then

$$\lim_{t\to\infty} \bar{x}(t-t_0) = x_0 = \lim_{t\to-\infty} \bar{x}(t-t_0).$$

There are two convenient ways of visualizing the dynamics of (2.1). One can introduce the Poincare map $P^s_\varepsilon:\mathbb{R}^2 \to \mathbb{R}^2$, which is the time T map for (2.1) starting at time s. For $\varepsilon = 0$, the point x_0 and the homoclinic orbit are invariant under P^s_0, which is independent of s. The hyperbolic saddle x_0 persists as a nearby family of saddles x_ε for $\varepsilon > 0$, small, and we are interested in whether or not the stable and unstable manifolds of the point x_ε for the map P^s_ε intersect transversally (if this holds for one s, it holds for all s). If so, we say (2.1) admits horseshoes for $\varepsilon > 0$ (see Smale [1967], Moser [1973], Holmes and Marsden [1981], Abdel-Salam, Marsden and Varayia [1983a] and Guckenheimer and Holmes [1983] for discussions of why transverse homoclinic orbits lead to Smale horseshoes).

The second way to study (2.1) is to look directly at the suspended system on $\mathbb{R}^2 \times S^1$, where S^1 stands for the circle, elements of which are regarded as the T-periodic variable θ. Then (2.1) becomes the autonomous suspended system

$$\left.\begin{aligned}\dot{x} &= f_0(x) + \varepsilon f_1(x,\theta)\\ \dot{\theta} &= 1\end{aligned}\right\} \tag{2.2}$$

From this point of view, the curve

$$\gamma_0(t) = (x_0,t)$$

is a periodic orbit for (2.2), whose stable and unstable manifolds $W^s_0(\gamma_0)$ and $W^u_0(\gamma_0)$ are coincident. For $\varepsilon > 0$ the hyperbolic closed orbit γ_0 perturbs to a nearby hyperbolic closed orbit γ_ε which has stable and unstable manifolds $W^s_\varepsilon(\gamma_\varepsilon)$ and $W^u_\varepsilon(\gamma_\varepsilon)$. If $W^s_\varepsilon(\gamma_\varepsilon)$ and $W^u_\varepsilon(\gamma_\varepsilon)$ intersect transversally, we again say that (2.1) admits horseshoes. These two definitions of admitting horseshoes are readily seen to be equivalent.

Melnikov's [1963] criterion for the splitting of separatrices is as follows.

Melnikov Criterion. Define the Melnikov function by

$$M(t_0) = \int_{-\infty}^{\infty} \{H_0,H_1\}\, dt \tag{2.3}$$

where the Poisson bracket is evaluated at position $\bar{x}(t-t_0)$ and time t, so the integral is taken around the unperturbed homoclinic orbit. Assume $M(t_0)$ has simple zeros as a T-periodic function of t_0. Then (2.1) has horseshoes.

*Research partially supported by DOE Contract AT03-82ER 12097.

0191-2216/83/0000-0356 $1.00 © 1983 IEEE

Proof. (Melnikov [1963], Arnold [1964]). In the suspended picture, we use the energy function H_0 to measure the first order movement of $W^s_\varepsilon(\gamma_\varepsilon)$ at $\bar{x}(0)$ at time t_0 as ε is varied. Note that points of $\bar{x}(t)$ are regular points for H_0 since H_0 is constant on $\bar{x}(t)$ and $\bar{x}(0)$ is not a fixed point. Thus, the values of H_0 give an accurate measure of the distance from the homoclinic orbit. If $(x^s_\varepsilon(t,t_0),t)$ is the curve on $W^s_\varepsilon(\gamma_\varepsilon)$ that is an integral curve of the suspended system (2.2) and has an initial condition $x^s(t_0,t_0)$ which is the perturbation of $W^s_0(\gamma_0) \cap \{\text{the plane } t = t_0\}$ in the normal direction to the homoclinic orbit, then $H_0(x^s_\varepsilon(t_0,t_0))$ measures this normal distance. But

$$H_0(x^s_\varepsilon(T,t_0))-H_0(x^s_\varepsilon(t_0,t_0)) = \int_{t_0}^{T} \frac{d}{dt} H_0(x^s_\varepsilon(t,t_0))\, dt \tag{2.4}$$

From (2.4), we get

$$H_0(x^s_\varepsilon(T,t_0)) - H_0(x^s_\varepsilon(t_0,t_0)) = \int_{t_0}^{T} \{H_0,H_0 + \varepsilon H_1\}(x^s(t,t_0,t)\, dt \tag{2.5}$$

Since $x^s_\varepsilon(T,t_0)$ is ε-close to $\bar{x}(t-t_0)$ (uniformly as $T \to +\infty$), and $d(H_0 + \varepsilon H^1)(x^\varepsilon(t,t_0),t) \to 0$ exponentially as $t \to +\infty$, and $\{H_0,H_0\} = 0$, (2.5) becomes

$$H_0(x^s_\varepsilon(T,t_0)) - H_0(x^s_\varepsilon(t_0,t_0)) = \varepsilon \int_{t_0}^{T} \{H_0,H_1\}(\bar{x}(t-t_0,t)dt + O(\varepsilon^2) \tag{2.6}$$

Similarly,

$$H_0(x^u_\varepsilon(t_0,t_0)) - H_0(x^u_\varepsilon(-S,t_0)) = \varepsilon \int_{-S}^{t_0} \{H_0,H_1\}(\bar{x}(t-t_0),t)\, dt + O(\varepsilon^2) \tag{2.7}$$

Now $x^s_\varepsilon(T,t_0) \to \gamma_\varepsilon$, a periodic orbit for the perturbed system as $T \to +\infty$. Thus, we can choose T and S such that $H_0(x^s_\varepsilon(T,t_0)) - H_0(x^u_\varepsilon(-S,t_0)) \to 0$ as $T,S \to \infty$. Thus, adding (2.6) and (2.7), and letting $T,S \to \infty$, we get

$$H_0(x^u_\varepsilon(t_0,t_0)) - H_0(x^s_\varepsilon(t_0,t_0)) = \varepsilon\int_{-\infty}^{\infty} \{H_0,H_1\}(\bar{x}(t-t_0),t)\, dt + O(\varepsilon^2) \tag{2.8}$$

It follows that if $M(t_0)$ has a simple zero in time t_0, then $x^u_\varepsilon(t_0,t_0)$ has $x^s_\varepsilon(t_0,t_0)$ must intersect transversally near the point $\bar{x}(0)$ at time t_0. ■

Remark. Since $dH_0 \to 0$ exponentially at the saddle points, the integrals involved in this criterion are automatically convergent.

Example. (Forced Pendulum). Consider the equation

$$\ddot{\phi} + \sin\phi = \varepsilon \cos t \tag{2.9}$$

we claim that it has horseshoes for ε small, $\varepsilon \neq 0$. For $\varepsilon = 0$, the system is Hamiltonian with $H(\phi,v) = \frac{v^2}{2} - \cos\phi$. The homoclinic orbits are given by

$$\phi(t) = \pm 2 \tan^{-1}(\sinh t)$$
$$v(t) = \pm 2 \operatorname{sech} t$$

The perturbation is described by

$$H_1(\phi,v) = (\sin t)\phi \tag{2.10}$$

and so from (2.3) we compute

$$M(t_0) = \pm \int_{-\infty}^{\infty} 2 \operatorname{sech}(t-t_0) \cos(t)\, dt$$

changing variables and using the fact that sech is even and sin is odd,

$$M(t_0) = \pm 2\left(\int_{-\infty}^{\infty} \operatorname{sech} t \cos t\, dt\right) \cos t_0$$

Evluation by residues gives

$$M(t_0) = \pm\pi \operatorname{sech}\left(\frac{\pi}{2}\right) \cos t_0$$

which has simple zeros. Hence the system has horseshoes by Melnikov's criterion.

§3. Extensions of the Melnikov Theory

a) The Melnikov technique has been applied to autonomous Hamiltonian systems with two degrees of freedom by Holmes and Marsden [1982a] with a Hamiltonian of the form

$$H = F(q,p) + G(I) + \varepsilon H_1(q,p,\theta,I) + O(\varepsilon^2) \tag{3.1}$$

where (I,θ) are action angle coordinates for the oscillator G. One assumes that the F system has a homoclinic orbit, $\bar{x}(t) = (\bar{q}(t), \bar{p}(t))$ as in §2 and that

$$M(t_0) = \int_{-\infty}^{\infty} \{F,H_1\}\, dt \tag{3.2}$$

where $\{F,H\}$ is evaluated at $(\bar{x}(t-t_0), \Omega t, I)$, has simple zeros. Then the system has horseshoes on the energy surface corresponding to the unperturbed energy of the homoclinic orbit, oscillator frequency Ω and action I. This applies, for example to a coupled pendulum and harmonic oscillator.

b) If the variables don't split as in (3.1), one needs a more geometric setting: that of S^1 reduction of Hamiltonian systems. This was developed by Holmes and Marsden [1983] and was applied to the heavy top with two nearly equal moments of inertia (some related results were also obtained by Ziglin [1980a]).

c) When there are more than two degrees of freedom present, the new phenomena Arnold diffusion enters, as introduced by Arnold [1964]. This was developed for autonomous Hamiltonian systems by Holmes and Marsden [1982b].

Here a torus replaces the simple oscillator and the results apply, for example, to a pendulum coupled to n (≥ 2) oscillators, with action variables $I_1, \ldots, I_n$.

The basic criterion now involves the Melnikov vector $\vec{M}$ defined by

$$M_0(t_0, \ldots, t_{n-1}) = \int_{-\infty}^{\infty} \{F, H^1\} dt$$

$$M_j(t_0, \ldots, t_{n-1}) = \int_{-\infty}^{\infty} \{I_j, H^1\} dt \qquad (3.2)$$

If $\vec{M}$ has transverse zeros, then the perturbed stable and unstable manifolds of the torus intersect transversally, which ultimately leads to chaotic energy drifts between the oscillators.

d) The method also applies to certain infinite dimensional systems although the techniques require some damping to be present to avoid resonance difficulties and to retain a "pure" horseshoe. This technique was developed by Holmes and Marsden [1981] and was applied to the equations for a forced oscillating beam in which chaos had been seen experimentally by F. Moon.

e) If the forcing function has high frequency, such as in the equation

$$\ddot{\phi} + \sin \phi = \varepsilon \cos\left(\frac{t}{\varepsilon}\right) \qquad (3.3)$$

then the Melnikov theory as it stands is not applicable. Indeed, the Melnikov function is

$$M(t_0) = \pm \pi \operatorname{sech}\left(\frac{\pi}{2\varepsilon}\right) \cos\left(\frac{t_0}{\varepsilon}\right) \qquad (3.4)$$

which is exponentially small. Under the right circumstances, however, the Melnikov criterion is believed to be valid (cf. Chirikov [1979], Sanders [1982]). This situation is currently under investigation (Holmes, Marsden, Scheurle).

It is very important to establish the criterion in these circumstances since this difficulty comes up in a variety of key circumstances, such as KAM theory.

§4. Applications

We list just a few of the other applications of the theory:

a) (Power Systems). The swing equations of a power system have been shown to have horseshoes and Arnold diffusion (appropriate to the number of degrees of freedom) by Melnikov techniques by Kopell and Washburn [1982] and by Abdel-Salam, Marsden and Varaiya [1983b].

b) (Chaotic Motion of Vortices). Chaos in the motion of four vortices was reported by Ziglin [1980b] although hisproof seems to suffer from the exponential smallness disease noted in (3.3) above. Using some configurations suggested by work of Synge, Aref and Pomphrey (see Aref [1983] for a review), Koiller and de Carvalho [1983] have shown the existence of horseshoes by a "normal" Melnikov method with an S^1 symmetry ((b) of §3).

c) (Rigid body with attachments). In Holmes and Marsden [1983] a simplified problem modelling a rigid body with rotary attachments, important for the attitude control of spacecraft, was shown to have horseshoes (and Arnold diffusion if there are at least 2 attachments). A more realistic model was shown to have the same features by Koiller [1983]. See also Krishnaprasad [1983].

d) A one dimensional van der Waals fluid with periodic thermal fluctuations (either spatial or temporal) was shown to have horseshoes in a Hamiltonian truncation by Slemrod and Marsden [1983]. This chaos is of interest because it occurs near the Maxwell line and is believed to be related to phase transitions.

References

F.H. Abdel-Salam, J.E. Marsden and P. P. Varaiya [1983a]. Chaos and Arnold diffusion in dynamical systems. Trans. IEEE (to appear).

F.H. Abdel-Salam, J.E. Marsden and P. P. Varaiya [1983b]. Arnold diffusion in the swing equations of a power system. Trans. IEEE (to appear).

H. Aref [1983]. Integrable, Chaotic and Turbulent Vortex Motion in Two-Dimensional Flows. Ann. Rev. Fluid Mech. 15, 345-389.

V. Arnold [1964]. Instability of dynamical systems with several degrees of freedom, Dokl. Akad. Nauk. SSSR. 156, 9-12.

B.V. Chirikov [1979]. A universal instability of many-dimensional oscillator systems, Physics Reports, 52, 265-379.

S.N. Chow, J.K. Hale, and J. Mallet-Paret [1980]. An example of bifurcation to homoclinic orbits, J. Diff. Eqns. 37, 351-373.

J. Guckenheimer and P. Holmes [1983]. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer, Applied Math. Sciences, Vol. 42.

P. Holmes [1980]. Averaging and chaotic motions in forced oscillations, SIAM J. on Appl. Math. 38, 68-80 and 40, 167-168.

P. J. Holmes and J.E. Marsden [1981]. A partial differential equation with infinitely many periodic orbits: chaotic oscillations of a forced beam, Arch. Rat. Mech. Anal. 76, 135-166.

P.J. Holmes and J.E. Marsden [1982a]. Horseshoes in perturbations of Hamiltonian systems with two degrees of freedom, Comm. Math. Phys. 82, 523-544.

P.J. Holmes and J.E. Marsden [1982b]. Melnikov's method and Arnold diffusion for perturbations of integrable Hamiltonian systems, J. Math. Phys. 23, 669-675.

P. Holmes and J. Marsden [1983]. Horseshoes and Arnold diffusion for Hamiltonian systems on Lie groups, Ind. Univ. Math. J. 32, 273-310.

J. Koiller [1983]. A dynamical system with a wild horseshoe (preprint).

J. Koiller and S. Pinto de Carvalho [1983]. Nonintegrability of a restricted problem of four point vortices (preprint).

N. Kopelland R.B. Washburn [1982]. Chaotic motions in the two-degree-of-freedom swing equations, IEEE Trans. Circuits and Systems 29, 738-746.

P.S. Krishnaprasad [1983]. Lie-Poisson structures and dual-spin spacecraft (preprint).

A. J. Lichtenberg and M.A. Liebermann [1983]. Regular and Stochastic Motion, Springer, Applied Math. Sciences, Vol. 38,

V.K. Melnikov [1963]. On the stability of the center for time periodic perturbations, Trans. Moscow Math. Soc. 12, 1-57.

J. Moser [1973]. Stable and random motions in dynamical systems, Annals of Math. Studies, Princeton University Press.

J. Sanders [1982]. Melnikov's method and averaging, Celest. Mech. 28, 171-181.

M. Slemrod and J. Marsden [1983]. Temporal and spatial chaos in a van der Waals fluid due to periodic thermal fluctuations Arch. Rat. Mech. An. (to appear).

S. Smale [1967]. Differentiable dynamical systems, Bull. Am. Math. Soc. 73, 747-817.

S.L. Ziglin [1980a]. Decomposition of separatrices, branching of solutions and nonexistence of an integral in the dynamics of a rigid body, Trans. Moscow Math. Soc. 41, 287.

S.L. Ziglin [1980b]. Nonintegrability of a problem on the motion of four point vortices, Sov. Math. Dokl. 21, 296-299.

S.L. Ziglin [1981a]. Branching of solutions and nonexistence of integrals in Hamiltonian systems. Doklady Akad Nauk. SSSR 257, 26-29.

S.L. Ziglin [1981b]. Self-intersection of the complex separatrices and the nonexistence of the integrals in the Hamiltonian systems with 1-1/2 degrees of freedom, Prikl. Math. Mek. 45, 564-566; trans. as J. Appl. Math. Mech. 45, 411-413.