Linearization stability and Signorini Series for the traction problem in elastostatics

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Synopsis

This paper uses previous results of Chillingworth, Marsden and Wan on symmetry and bifurcation for the traction problem in three dimensional elastostatics to establish new results on the Signorini expansion. We show that the Signorini compatibility conditions are necessary and sufficient for linearization stability and analogies with results known for other field theories are pointed out. Under an explicit non-degeneracy condition, a new series expansion is given in which successive terms are inductively determined in pairs rather than singly. Our results include as special cases, classical results of Signorini, Tolotti and Scippetti.

I. Introduction

This paper studies linearization stability of the equations of non-linear elastostatics and the closely related notion of the expansion of the solution in a series whose terms are found by solving a hierarchy of linear problems. The latter topic was developed extensively by Signorini starting in [12] and is thoroughly described in Grioli [10] and Truesdell and Noll [16]. Linearization stability originated in perturbation theory for the Einstein equations of general relativity by Fischer and Marsden [7, 8] but is a notion that is useful in the study of non-linear partial differential equations rather generally.

Chillingworth, Marsden and Wan [4, 5] studied the bifurcation of solutions of the traction problem for small loads, as the loads are varied. Some of the results obtained there will be used in an essential way here. In particular, those papers used the Liapunov-Schmidt procedure to study the bifurcations near a given load \( I \); this process reduced the problem to studying the critical points of a reduced potential function on a manifold \( S_A \), where \( A \) is the associated static load. The manifold \( S_A \) is either four points, two points and a circle, one point and \( RP^1 \), two

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disjoint circles or $\mathbb{R}P^3$, depending on the type (either 0, 1, 2, 3 or 4) of the load. Crucial in this bifurcation study is the Betti form, a function on $S_0$ closely related to the symmetric bilinear form on linearized solutions that occurs in the Betti reciprocity theorem (see for instance, Marsden and Hughes [11]).

This paper contains three principal theorems. First of all, in Theorem 1, we relate the Signorini compatibility conditions to the critical points of the Betti form on $S_0$. This has, as a consequence, an extension of a theorem of Tobetti [18]. Secondly, in Theorem 2, we show that the Signorini compatibility conditions are necessary and sufficient for linearization stability. This is analogous to the theorem in general relativity which states that the Taub conditions are necessary and sufficient for linearization stability, a result of Fischer, Marsden and Moncrief [9] and Arms, Marsden and Moncrief [1, 2]. However, the technical details in the two theorems seem to have little if anything in common. Thirdly, in Theorem 3, we establish a generalization and modification of the classical Signorini-Stoppelli schemes for a power series solution which is valid even when the loads have axes of equilibrium. Our scheme is different from and is not restricted by the special series used by Stoppelli [14] in his analysis of the bifurcation of solutions for type 1 loads.

As in [4, 5], we assume the material is hyperelastic, materially frame indifferent, the reference configuration is stress free and the linearized elasticity tensor $e$ at the identity is stable (and hence strongly elliptic).

2. Linearization stability and critical points of the Betti form

We begin with the definition of linearization stability.

**Definition.** Suppose a pair $(u_0, l_0)$ (displacement, load) satisfies the equations of elastostatics linearized about the (stress free) reference state $I$ (= Identity); i.e.

$$Lu_0 = l_0,$$

Here, $Lu_0 = D\Phi(l_0)u_0$ and $\Phi(\phi) = (-\operatorname{DIV} P, P, P, H)$, where $\phi: \mathbb{R} \to \mathbb{R}^3$ is a configuration of the body $\mathbb{R}$, $P$ is the first Piola-Kirchhoff stress at $\phi$ and $H$ is the unit outward normal on $\mathbb{R}$. Let us call the pair $(u_0, l_0)$ linearization stable (or integrable) if there exists a $C^\infty$ curve $(\phi(l), l(l)) \in \mathcal{E} \times \mathcal{L}$ (configuration, loads) such that

(i) $\phi(0) = I$, $l(0) = 0$,
(ii) $\phi(0) = u_0 \in \ker D\Phi(I), l'(0) = I_0$,
and
(iii) $\Phi(\phi(l)) = l(l)$.

Here, $(\Phi(l), l(l))$ should be defined in some interval; say $[0, \varepsilon)$, $\varepsilon > 0$.

For $l_0 \in \mathcal{L}$, we say that $l_0$ is linearization stable when there is a curve $(\phi(l_0), l(l_0)) \in \mathcal{E} \times \mathcal{L}$ satisfying (i) and (iii) above with $l'(0) = I_0$.

Then, $D\Phi(l_0)\phi'(0) = I_0$ is automatic and we can take $u_0 = \phi'(0)$.

A classical result of Signorini and Stoppelli [3] is the following:

**Proposition 1.** Suppose $l_0 \in \mathcal{L}$ has no axis of equilibrium and $D\Phi(l_0)u_0 = I_0$. Then $(u_0, l_0)$ is linearization stable.

**Linearization stability and Signorini Series**

**Proof.** Let $l(l) = l_0$. Then there is a unique smooth curve $\phi(l)$ through $I_0$ such that $\Phi(\phi(l)) = l(l)$ by [4, Theorem 5.1]. By differentiating at $l_0 = 0$, we obtain $D\Phi(l_0)\phi'(0) = I_0$, so $\phi'(0) = u_0 \in \ker D\Phi(l_0)$.

In this proposition and elsewhere in the paper, the spaces of loads and configurations are appropriate Sobolev spaces; see [4, 5] for details. We note, however, that elements of the space $\mathcal{E}$ of configuration are at least $C^1$.

The following Signorini compatibility conditions produce a potential obstruction to linearization stability. Let us write

$$\int_a^b u \times l \quad \text{for} \quad \int_a^b u(X) \times B(X) \cdot dV(X) + \int_a^b u(X) \times \nu(X) \cdot dA(X).$$

where $l = (B, \nu) = (\text{body force, surface traction})$.

**Lemma 2.** Suppose $l_0 \in \mathcal{L}$ is linearization stable with $l(l) \in \mathcal{L}$. On letting $u_0$ be as above, we have

$$\int_a^b l_0 \times u_0 = 0. \quad \text{(C)}$$

**Proof.** We have identity $\int_a^b \phi(l) \times \phi(l) = 0$ from balance of moment. By differentiating twice in $l$ and setting $l = 0$, we obtain

$$\int_a^b l'(0) \times l_0 + 2 \int_a^b l'(0) \times \phi'(0) = 0.$$

Since $l(0) \in \mathcal{L}$, we have $l'(0) \in \mathcal{L}$, so the first integral is zero. Thus $\int_a^b l_0 \times u_0 = 0$.

If we write

$$l(l) = \lambda I_0 + \lambda^2 l_1 + \ldots$$

and

$$\phi(l) = I_0 + \lambda u_0 + \lambda^2 u_2 + \ldots$$

and assume $l(l) \in \mathcal{L}$, then $\int l(l) \times \phi(l) = 0$ gives a hierarchy of conditions:

order $l_0$: $\int l_0 \times l_0 = 0$ (i.e. $l_0 \in \mathcal{L}$),
order $l_1$: $\int l_1 \times u_0 = 0$ (using $l_1 \in \mathcal{L}$),
order $l_2$: $\int l_2 \times u_{0,1} + \int l_2 \times u_{1,2} + \ldots + \int l_{n-1} \times u_n = 0$.

Stoppelli [13] proved that the curve $l(l)$ needed for Proposition 1 can be obtained by the preceding Signorini scheme with $l(l) = \lambda I_0$.

We shall now reformulate the compatibility condition (C) in geometric terms. Let $A = k(I)\operatorname{sym}$ (the symmetric $3 \times 3$ matrices) be the elastic load defined by

$$k(I) \int_a^b B(X) \cdot \delta(X) \cdot dV(X) + \int_a^b \nu(X) \cdot \delta(X) \cdot dA(X),$$

and let $u_0(l) = u_{m}(l)$, the $L^2$ orthogonal complement to $Skow$ in $\mathbb{R} T \mathcal{L}$, the space of linearized displacement, be defined by the linear problem $l u_0(l) = Ql$.
when $Q \in S_3$, i.e. when $Q \in L_2$. (By the linear theory, $L: \mathcal{S}_3 \to L_2$ is an isomorphism; see Fichera [6] or Marsden and Hughes [11, Chapter 6].) Let $B(Q) = (c(\nabla u_0(I)), \nabla u_0(I))$, the Benzi form, defined on $S_3$.

**Lemma 1.** $B$ restricted to $S_3$ has a critical point at $Q \in S_3$, if and only if $(c(\nabla u_0(I)), \nabla u_0(I)) = 0$ for all $W \in \text{skew}$ (the $3 \times 3$ skew matrices) such that $WQ(I) \in \text{skew}$. 

**Proof.** This follows from the definition of $B$. □

The following is readily verified.

**Lemma 2.** Let $A \in \text{sym}$ be fixed and let $p: \text{skew} \times \text{skew} \to \mathbb{R}$ be defined by $p(K, W) = (KA, W)$. Then $p$ is a symmetric bilinear form with kernel $\{K \in \text{skew} | KA = 0\}$.

Note that $(KA, W)$ is the Hessian of $-A(I, Q^T I) = -(Q(I, I))$ at $Q = I$, where

$$
(I, A) = \int_0^1 x B(X) \cdot \phi(X) \, dx(X) + \int_0^1 \phi(X) \cdot a(X) \, dx(X).
$$

Here is our first main result.

**Theorem 1.** Let $l_1 \in L_2$. Then there exists a $u_1 \in \mathcal{U}$ such that $L(u_1) = l_1$ and $l_1 \times u_1 = 0$, if and only if the Benzi form (for $l_1$) restricted to $S_3$, has a critical point at the identity $I$.

**Proof.** First assume $u_1$ exists. Thus, $l_1 \times u_1 = 0$, so $(W_{l_1}, u_1) = 0$ for all $W \in \text{skew}$. We can write $u_1(X) = u_1(X) + k k$ for some $k \in \text{sym}$. Then

$$
(\xi(\nabla u), \nabla u) = (W_{l_1}, u_1) = (W_{l_1}, u_1 + k k) = (W_{l_1}, k k) = (k, k(W_{l_1})),
$$

when $W \in \text{sym}$. Thus, $B$ has a critical point at $l_1$ by Lemma 1. For the converse, we need to find $K \in \text{skew}$ such that $u_1 = u_1 - K$, $W_{l_1} = (W_{l_1}, u_1) = 0$ for all $W \in \text{skew}$. Now $(W_{l_1}, u_1) = (c(\nabla u_0), u_1)$ is a linear function of $W_{l_1} \in \text{skew}$, vanishing for $W_{l_1} \in \text{sym}$ (by hypotheses). Thus, by Lemma 2, there is a $K \in \text{skew}$ such that $(W_{l_1}, u_1) = (W_{l_1}, u_1) = 0$ for all $W \in \text{skew}$. Therefore, $(W_{l_1}, u_1) = (W_{l_1}, u_1) - (W_{l_1}, k k) = (W_{l_1}, k k) = 0$ for all $W \in \text{skew}$. □

The next corollary is an extension of results of Tolotti [15].

**Corollary.** There exist at least 4 notions $Q_0 \in SO(3)$ such that the Signorini $0$th and $1$st order compatibility conditions hold for $Q = I^*$, i.e. $I^* \in L_2$, and $l_1 \times u_1 = 0$ for some $u_1$ satisfying $L(u_1) = l_1$.

**Proof.** Let $Q$ be a critical point of $(c(\nabla u_0(I)), \nabla u_0(I))$ on $S_3$. By Lemma 1, $(c(\nabla u_0(I)), \nabla u_0(I)) = 0$ when $WQ(I)e_{3\times3}$ is skew. Since $u_0(I) = u_0(I) = u_0(I) = c(\nabla u_0(I)) = c(\nabla u_0(I)) = 0$ for all $WQ(I)e_{3\times3}$ is skew. By Lemma 1 and Theorem 1 with $Q = I$ and $l_1 = I$, we see that this critical point has the desired property. Since, by critical point theory, at least 4 such critical points can always be found, the corollary follows. □

### 3. Linearization stability and the compatibility conditions

Besides the case of no-axis of equilibrium, Stoppelli also showed that the Signorini scheme can be made to work for parallel loads. In fact this result, which we now recall, follows directly from [5, Theorem 4.7].

**Proposition 3 (Stoppelli).** Let $l(I) = \lambda I$, where $l$ is a non-trivial load parallel to the $z$-axis. Then there is a solution curve $\phi(I)$, which can be obtained by Signorini's scheme supplemented by the condition $u_2(0) - u_2(0) = 2(\lambda)$, where $\lambda$ is non-zero but is otherwise an arbitrary given function.

We now recall some developments that combine the classical cases of Propositions 1 and 3, following Capriz and Podio-Guidugli [3].

**Definition.** Let $l_1 \in L_2$ and set $\phi(I) = k(l_1, I + \lambda I)$ for $u \in \mathcal{U}$. The load $l_1$ is said to be infinitesimally stable when, for any $u \in \mathcal{U}$, there exists a smooth curve $\phi(I) \in SO(3)$, with $\phi(I) = I$ such that $l_1 \phi(I) \in \mathcal{U}$.

This is motivated by the following. One seeks a solution in the form $\phi(I) = \phi(l_1, I + \lambda I)$, where $\phi = Q(I)$ and $u = u(I)$. Thus, $k(l_1, I + \lambda I) \in \mathcal{U}$, $k(l_1, I + \lambda I) \in \mathcal{U}$. Requiring a solvability condition depending on the load only, leads to the notion of an infinitesimally stable load. The next result follows readily; see the aforementioned reference for details.

**Proposition 4.** A load $l_1 \in L_2$ is infinitesimally stable if and only if

1. $l_1$ has no axis of equilibrium; or
2. $l_1$ is a non-trivial parallel load.

Thus, the load $l_1$ is linearization stable if it is infinitesimally stable.

The next theorem generalizes the classical results by showing that the necessary condition (C) is also sufficient. No special non-degeneracy hypotheses are required.

**Theorem 2.** Let $l_1 \in L_2$. If there is a $u_1 \in \mathcal{U}$ such that $l_1 \times u_1 = 0$, then $l_1$ is linearization stable.

**Proof.** By Theorem 1, $(c(\nabla u_0), \nabla u_0)$ has a critical point at $Q = I$. Choose $l_1$ so that $-2(l_1, Q^T I) - (c(\nabla u_0), \nabla u_0)$ restricted to $S_3$ is non-degenerate at $l_1$. For example, we can choose $l_1 = I$ for a large, $\lambda > 1$. Let $l(I) = \lambda I + M$. Then the reduced bifurcation potential on $S_3$ is, by [5, formula (30)],

$$
\lambda = \lambda l(I, Q^T I) - \frac{\lambda}{2} (c(\nabla u_0), \nabla u_0) + O(l^2).
$$

Thus $l_1$ has non-degenerate critical points which vary smoothly in $\lambda$ and thus, from them, $\phi(I)$ can be reconstructed by the Linearized-Schmidt procedure (see [5, §2]). □

Notice that the second order term $\lambda I^2$ is necessary to allow construction of $\phi(I)$.

An example due to Signorini of a type 4 load $l_1$ for which no $u_1$ exists satisfying (C) is described in Capriz and Podio-Guidugli [3, §9].
Two specific cases in which the construction of a curve \( \phi(\lambda) \) corresponding to \( I(\lambda) = \lambda I_1 \) is possible and which employ non-degeneracy hypotheses on the Betti form are as follows.

**Corollary.** (a) Let \( I_1 \in \mathcal{L}_c \), and suppose (\( e(\nabla \mathbf{u}_0), \nabla \mathbf{u}_0 \)) is non-degenerate along \( S_{\mathbf{u}_0} \) at \( Q = I \). There is a unique solution curve \( \phi(\lambda) \) with \( \phi(0) = I \) such that \( \Phi(\phi(\lambda)) = I_1 \) (here one can choose \( I_2 = 0 \) by examination of the preceding proof).

(b) Let \( I_1 \in \mathcal{L}_c \) be a "trivial" load (i.e., \( A_1 = 0 \)) parallel to the z-axis. Suppose that (\( e(\nabla \mathbf{u}_0), \nabla \mathbf{u}_0 \)) on \( SO(3) \) has a critical point at \( I \) which is non-degenerate transversal to \( S^1 = \{Q | Q_1 = I_1 \} \). Then there is a solution curve \( \phi(\lambda) \) with \( \phi(0) = I \) and \( \Phi(\phi(\lambda)) = \lambda I_1 \).

The classical results Propositions 1 and 3 are also corollaries of Theorem 2.

4. Signorini expansions

Now we turn to the problem of finding a generalization of the Signorini scheme which will work in the generality of Theorem 2. We begin by setting up the perturbation series using slightly different notation.

We consider the problem of solving \( \Phi(\phi(\lambda)) = I(\lambda) \), where \( I(\lambda) \in \mathcal{L}_c \) is a given curve and \( (\mathbf{u}_0) = 0 \). (Note that \( I(\lambda) \) is not assumed to lie in \( \mathcal{L}_c \).) Write the Taylor expansion of \( \Phi \) in \( u \) at \( I \) as \( \Phi = \Phi_0 + \Phi_2 + \Phi_3 + \cdots \). (Thus \( \Phi_0 = L \), and \( \Phi_2 \) is quadratic in \( u, \) etc,) and expand \( I \) as a series in \( \lambda \) by setting \( I(\lambda) = \lambda I_1 + \lambda^2 I_2 + \lambda^3 I_3 + \cdots \). Write the unknown \( \phi(\lambda) \) as \( \phi = I + \lambda u_1 + \lambda^2 u_2 + \cdots \).

Hence,

\[
\Phi(I + \lambda u_1 + \lambda^2 u_2 + \cdots) = \Phi(I + \lambda u_1 + \lambda^2 u_2 + \cdots) + \Phi_2(\lambda u_1, \lambda^2 u_2, \ldots, \lambda u_1 + \lambda^2 u_2 + \cdots) = \lambda I_1 + \lambda^2 I_2 + \cdots.
\]

By comparing orders in \( \lambda \), we get

- order \( \lambda \): \( L(u_1) = I_1 \) (so \( L \in \mathcal{L}_c \)),
- order \( \lambda^2 \): \( L(u_2) + \Phi_2(u_1, u_1) = L_2 \),
- order \( \lambda^3 \): \( L(u_3) + \Phi_3(u_1, u_1, u_1) + \) a polynomial in \( u_1, u_2, u_3 \) \( \in L_3 \) (\( n \geq 2 \)),

Therefore, one hopes to determine \( u_1, \ldots, u_n \) inductively, with the help of the compatibility conditions:

\[
\int I_1 \times u_1 + \cdots + \int L_n \times u_n + \int L_n \times I = 0. \quad (C_n)
\]

Theorem 1 shows that if \( I_1 = I(0) \in \mathcal{L}_c \) and has no axis of equilibrium, then there is a unique solution \( \phi(\lambda) \), and can be obtained by Signorini's scheme. Indeed, suppose \( u_1, \ldots, u_n \) are determined, then \( (C_n) \) and \( (C_n) \) define \( \phi \). The uniqueness and existence of the formal Signorini's scheme follows from a special case of the next two lemmas.

**Lemma 3.** Let \( K \in \text{skew} \), then \( I_1 \times K K = 0 \) if and only if \( K K(I_1) \in \text{sym} \) (i.e., \( K \in T_{S_{\mathbf{u}_0}} \)).

**Lemma 4.** Suppose \( u_1, \ldots, u_n \), satisfy \( (L_1), (C_n), \ldots, (L_n) \). Then \( (C_n) \) is the solvability condition for \( u_n \) in \( (L_n) \).

These simple facts are discussed and proved in \([6]\).

Let us state the results obtained in Theorems 1 and 2 in a slightly different and more general form.

**Theorem 1'.** Let \( I_1 \in \mathcal{L}_c \). Then there exists \( u_n \in \mathcal{U} \), such that \( L(u_n) = I_1 \), and \( I_1 \times u_n + \int L_n \times I = 0 \) if and only if \( 2(L_2, Q^1) + e(\nabla \mathbf{u}_0), \nabla \mathbf{u}_0) \) restricted to \( S_{\mathbf{u}_0} \) has a non-degenerate critical point at \( I \).

**Theorem 2'.** Consider the problem \( \Phi(\phi(\lambda)) = I(\lambda) \), with \( I_1 \in \mathcal{L}_c \) and \( I(\lambda) \in \mathcal{L}_c \) given. Suppose that \( 2(L_2, Q^1) + e(\nabla \mathbf{u}_0), \nabla \mathbf{u}_0) \) restricted to \( S_{\mathbf{u}_0} \) has a non-degenerate critical point at \( I \). Then there is a unique \( \phi(\lambda) \) such that \( \Phi(\phi(\lambda)) = I(\lambda) \), where \( \phi(0) = I \).

These theorems are proved in the same way as Theorems 1 and 2.

We claim that \( \phi(\lambda) \) determined by Theorem 2' can be obtained by a modification of Signorini's scheme. The new scheme determines the solutions in pairs.

**Theorem 3.** Suppose \( u_1, \ldots, u_n \), \( n \geq 2 \), and \( u_{n+1} \mod K \), \( K \in T_{S_{\mathbf{u}_0}} \), are determined; then, equations \( (L_n) \) and \( (C_n) \) define \( u_{n+1} \) and \( u_n \mod K \), where \( K \in T_{S_{\mathbf{u}_0}} \).

From Theorem 1', one can see readily that \( u_n \mod K \), with \( K \in T_{S_{\mathbf{u}_0}} \), is determined by equations \( (L_n) \) and \( (C_n) \), provided that the non-degeneracy hypothesis in Theorem 2' is fulfilled. Thus, starting from \( u_1 \mod K \) with \( K \in T_{S_{\mathbf{u}_0}} \), one can find \( u_1, \ldots, u_n \) inductively by Theorem 3.

Our proof of Theorem 3 consists of a brute force computation. Lemmas 5, 6, and 7 provide relevant facts we have already established, in \([4, 5]\).

**Lemma 5.** (a) The function \( 2(L_2, Q^1) + e(\nabla \mathbf{u}_0), \nabla \mathbf{u}_0) \) has a critical point on \( S_{\mathbf{u}_0} \) at \( I \) if and only if \( 0 = \beta(\nabla \mathbf{u}_0), \nabla \mathbf{u}_0) \); for all \( K \in T_{S_{\mathbf{u}_0}} \).

(b) The Hessian of this function is \( 2(L_2, Q^1) + e(\nabla \mathbf{u}_0), \nabla \mathbf{u}_0) \); for all \( K \in T_{S_{\mathbf{u}_0}} \).

Write the first Piola-Kirchhoff tensor in a perturbation series \( \Phi = \Phi_1 + \Phi_2 + \cdots \), with \( \Phi_1 = I \). Define and write, as above,

\[
\Phi = \Phi_1 + \Phi_2 + \cdots \quad \text{with} \quad \Phi_1 = I.
\]

**Lemma 6.**

(a) \( (L(u), w) = (\nabla \mathbf{w}, \nabla w) \).

(b) \( (\Phi_1(u), w) = (P_1(\nabla u), \nabla w) \), for all \( u, w \in \mathcal{U} \).

Now, write the stored energy function as \( W = W(D) \), where \( D = (F'F - I) \), and \( F = D\Phi \) is the deformation gradient. Thus

\[
\frac{\partial W}{\partial F} = \frac{\partial W}{\partial F} \quad \text{(the second Piola-Kirchhoff stress)},
\]

and

\[
P(F) = FS(D).
\]
Computations show that:

**Lemma 7.**

(a) \( \frac{\partial P}{\partial F} (H) = \frac{\partial S}{\partial D} (H) \) (i.e. \( n = e \)).

(b) \( \frac{\partial^2 P}{\partial D^2} (H, K) = H \left[ \frac{\partial S}{\partial D} (K) + K \frac{\partial S}{\partial D} (H) \right] + \frac{\partial S}{\partial D} (K'H) + \frac{\partial S}{\partial D} (H, K) \).

**Lemma 8.**

(a) \( \phi_2(KX^2) = L(K'X) \).

(b) \( 2\phi_2(u, KX) = Kl + L(K'x) \), for \( K \in T_{\mathcal{S}} \).

Proof. \( L(K'X) \).

(b) \( \frac{\partial S}{\partial D} (K'K), \nabla w \) by Lemmas 6(b) and 7(b).

\( = (L(K'X), w) \) by Lemmas 6(a) and 7(a).

**Lemma 9.** \( (Wl, K'x) - (Wl, KX) \) is symmetric in \( W \) and \( K \) when \( W, K \in T_{\mathcal{S}} \).

Proof. \( \{ I, x, u, + I, x, I = 0 \) means that \( k(l, u) + k(I, l) = 0 \) for all \( E \) is skew. Let \( k = WK - KW \). Then one obtains \( (Wl, u) - (Wl, X) = (Wl, u) - (Wl, X) \) for \( k(1, x) \).

Now, we are ready to prove Theorem 3.

(A) Let \( n > 2 \). We need to show that there is a unique \( K \in T_{\mathcal{S}} \), such that \( u_{n-1} = u_{n-1} + K \) (given by hypotheses) and a corresponding \( u_{n-1} \) obtained by Lemma 4, which solve (1) and (2). For each \( K \in T_{\mathcal{S}} \), from (1) for \( u_1 \) and \( u_2 \) given by Lemma 4,

\( l (u_1 - u_2) + 2\phi_2(u_1, KX) = 0. \)

By Lemma 8, \( u_1 - u_2 + K'x + KX = 0 \) for some \( K \) is skew (to be determined). On substituting in (2), one has

\[ \begin{align*} &l (u_1 - u_2) + 2\phi_2(u_1, KX) \quad \text{or} \quad -k(l, u_1 + K'x + KX) + 2\phi_2(u_1, KX) + l (u_1 - u_2) + \ldots + 2\phi_2(u_1, KX) + M \in \text{sym}, \end{align*} \]

where

\[ M = k(l, u_1 + K'x + KX) + 2\phi_2(u_1, KX) + l (u_1 - u_2) + \ldots + 2\phi_2(u_1, KX). \]

Since \( (W, k(l, KX)) = (W, k(l, KX)) = (W, KX) \), for \( W \in T_{\mathcal{S}} \).

\( (W, -k(l, u_1 + K'x)) + (W, k(l, KX)) = (W, u_1) + (W, K'x) - (W, KX) \),

is a non-degenerate form, by Lemma 9 and (b). Thus, there is a unique \( K \in T_{\mathcal{S}} \), such that equation (2) holds. Now, choose this \( K \) and consider the equation (1) for \( K \), i.e.

\[ k(l, KX) = -k(l, u_1 + K'x) + k(l, KX) + M \text{ mod sym} \]

\[ = N \text{ mod sym}. \]

From \( (K, k(Wl)) = (k(l, KX), W) \), the solvability condition for \( K \) becomes \( (W, N) = 0 \), for \( W \in T_{\mathcal{S}} \). Therefore, for the unique solution determined by equation (2), one can obtain a \( K \) such that equation (1) holds. In other words, for each \( K \) (unique) (K modulo \( T_{\mathcal{S}} \),

\[ u_n = u_{n-1} + KX \]

\[ u_1 - u_2 - K'x - KX \]

have the desired properties.

(B) The proof for \( n = 2 \) is basically the same as in (A), where one needs Lemma 8(a).

\[ \square \]

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References


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