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Offprint from "Archive for Rational Mechanics and Analysis",
Volume 83, Number 4, 1983, pp. 363–395

Symmetry and Bifurcation in Three-Dimensional Elasticity. Part II

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Communicated by S. ANTMAN

Contents

Glossary of Notation

$\mathcal{B} \subset \mathbf{R}^3$	reference configuration
$T_X\mathcal{B}$	vectors in $\mathbf{R}^3$ based at the point $X \in \mathcal{B}$
$\phi: \mathcal{B} \to \mathbf{R}^3,\ x = \phi(X)$	deformation
$u: \mathcal{B} \to \mathbf{R}^3$	displacement for the linearized theory
$e = \frac{1}{2}[\nabla u + (\nabla u)^T]$	strain
$\mathcal{C}$	all deformations ϕ
$F = D\phi$	deformation gradient = derivative of ϕ
F^T	transpose of F
$C = F^TF$	Cauchy-Green tensor
W	Stored energy function
$P = \dfrac{\partial W}{\partial F}$	first Piola-Kirchhoff stress
$S = 2\dfrac{\partial W}{\partial C}$	second Piola-Kirchhoff stress
$\mathsf{A} = \dfrac{\partial P}{\partial F}$	elasticity tensor
$\mathsf{C} = \dfrac{\partial S}{\partial C}$	(second) elasticity tensor

$\mathsf{c} = 2\mathsf{C}\vert_{\phi = I_{\mathscr{B}}}$	classical elasticity tensor
I or $I_{\mathscr{B}}$ or 1	identity map on $\mathbb{R}^3$ or $\mathscr{B}$
$l = (B, \tau)$	a (dead) load
$\mathscr{L}$	all loads with total force zero
$L(T_X\mathscr{B}, \mathbb{R}^3)$	all linear maps of $T_X\mathscr{B}$ to $\mathbb{R}^3$
$L(T_X\mathscr{B}, \mathbb{R})^*$	linear maps of $L(T_X\mathscr{B}, \mathbb{R})$ to $\mathbb{R}$
$\mathrm{sym}\,(T_X\mathscr{B}, T_X\mathscr{B})$	symmetric linear maps of $T_X\mathscr{B}$ to $T_X\mathscr{B}$
$SO(3)$	$\{Q \in L(\mathbb{R}^3, \mathbb{R}^3) \mid Q^TQ = I, \det Q = 1\}$
$\mathbb{R}P^2$	real projective 2-space; lines through $(0, 0, 0)$ in $\mathbb{R}^3$
M_3	$L(\mathbb{R}^3, \mathbb{R}^3)$
sym	symmetric elements of M_3
$\mathrm{skew} = so(3)$	skew symmetric elements of M_3
$\hat{v}$	infinitesimal rotation about the axis v
$\mathscr{L}_e$	equilibrated loads
$k: \mathscr{L} \to M_3$	astatic load map
$A = k(l)$	astatic load for a load l
$j = (k \mid (\ker k)^{\perp})^{-1}$	non-singular part of k
$\mathrm{Skew} = j\,(\mathrm{skew})$	skew viewed in load space
$\mathrm{Sym} = j\,(\mathrm{sym})$	sym viewed in load space
$\Phi: \mathscr{C} \to \mathscr{L}$	$\Phi(\phi) = (-\mathrm{DIV}\,P, P \cdot N)$
$\mathscr{U} = T_I\mathscr{C}$	the space of linearized displacements
$\mathscr{U}_{\mathrm{sym}}$	orthogonal complement to Skew in $\mathscr{U}$
$L: \mathscr{U}_{\mathrm{sym}} \to \mathscr{L}_e$	linearized operator: $L = D\Phi(I)$
l_e	the equilibrated part of l according to the decomposition $\mathscr{L} = \mathscr{L}_e \oplus \mathrm{Skew}$
$u_l \; (u_Q^0 = u_{Ql_0})$	linearized solution: $Lu_l = l_e$
$\langle , \rangle$	L^2 pairing
$B(l_1, l_2) = \langle l_1, u_{l_2} \rangle = \langle \mathsf{c}(\nabla u_{l_1}), \nabla u_{l_2} \rangle$	Betti form
S_A	Q's in $SO(3)$ that equilibrate A
ϱ	tubular neighborhood for $SO(3)$ in $\mathscr{C}$
$V(\phi) = \int W(F)\,dV - \lambda\langle l, \phi \rangle$	potential function for the static problem
$V_\varrho = V \circ \varrho$	potential function in new coordinates
$f(Q) = V_\varrho(Q, \phi_Q)$	reduced potential function on $SO(3)$
$\tilde{f}(Q) = -\langle Q^T, l \rangle - \frac{\lambda}{2} \langle \mathsf{c}(\nabla u_Q^0), \nabla u_Q^0 \rangle + O(\lambda^2) + O(\lambda \vert l - l_0 \vert)$	second reduced potential on S_{A_0}

§ 1. Introduction

In Part I of this paper (CHILLINGWORTH, MARSDEN & WAN [1982]—hereafter referred to as [I]), we reformulated the traction problem in elastostatics in various forms, gave a classification of loads and gave a complete analysis of solutions of the traction problem that are nearly stress-free for loads near loads of type 0 and type 1. This part develops the basic theory as well as giving an analysis of

solutions for loads of types 2, 3 and 4. It includes a count of the numbers of solutions and an analysis of their stability and the structural stability of the bifurcation diagrams.

We begin in Section 2 with a derivation of a potential formulation of the problem on $SO(3)$. The "second order potential" used in [I] can be recovered as a special case. It follows from this that the traction problem always has at least four solutions, at least one of which is neutrally stable. For loads of type 0, we showed in [I] that there are exactly four solutions near $SO(3)$; for the other types there can be many more ... up to 40. Sections 3, 4 and 5 examine types 2, 3 and 4 respectively, in a manner analogous to our treatment of types 0 and 1 in [I]. Loads of type 3 and 4 have some special features already studied in the literature in connection with parallel loads. These special features will be discussed and other connections with the existing literature will be made at appropriate points throughout the paper.

In a related paper MARSDEN & WAN [1983] study the linearization stability of the traction problem, which is related to the power series methods in the literature (see for example TRUESDELL & NOLL [1965]). One of the main results we prove is that even without the assumptions of non-degeneracy, the Signorini compatibility conditions at first order are sufficient for linearization stability; this means that one can obtain a Signorini-type expansion for the solution just under the assumption of compatibility at first order. The classical expansions occur as special cases.

We begin by recalling some of the principal notations used in [I].

Let $\mathscr{B} \subset \mathbb{R}^3$ denote the reference configuration and let $\mathscr{C} = \{\phi: \mathscr{B} \to \mathbb{R}^3 \mid \phi(0) = 0\}$ denote the set of all deformations (with the $W^{s,p}$ topology, $s > (3/p) + 1$). The space of all loads $l = (B, \tau)$ with total force zero is denoted $\mathscr{L}$. The astatic load map is denoted $k: \mathscr{L} \to M_3$, where M_3 denotes the set of 3×3 matrices. Thus

$$k(l) = \int_{\mathscr{B}} B(X) \otimes X\,dV(X) + \int_{\partial\mathscr{B}} \tau(X) \otimes X\,dA(X). \tag{1}$$

We have $k(l) = k(l, I)$ where I is the identity and where

$$k(l, \phi) = \int_{\mathscr{B}} B(X) \otimes \phi(X)\,dV(X) + \int_{\partial\mathscr{B}} \tau(X) \otimes \phi(X)\,dA(X). \tag{2}$$

We let $\mathrm{sym} \subset M_3$ denote the symmetric matrices and $\mathrm{skew} \subset M_3$ denote the skew symmetric ones. The equilibrated loads are denoted $\mathscr{L}_e = k^{-1}(\mathrm{sym})$.

Let F denote the deformation gradient $D\phi$ and let $W(F)$ denote a materially frame indifferent stored energy function. We assume $W(I) = 0$ without loss of generality. Let $P = \partial W/\partial F$ denote the first Piola-Kirchhoff stress and $\mathsf{A} = \partial P/\partial F$ the elasticity tensor. As in [I] we assume that the material is frame indifferent and that

(H1) *the undeformed state is stress free*;

(H2) *the strong ellipticity condition holds, and, moreover, the linearized theory satisfies the stability condition.*

Let $l_0 \in \mathscr{L}_e$ be a given load and λ a small parameter. We seek solutions of

$$\Phi(\phi) = \lambda l, \tag{3}$$

where l is near l_0 and $\Phi(\phi) = (-\mathrm{DIV}\, P, P \cdot N)$. Solving (3) is equivalent (under sufficient regularity) to finding critical points of

$$V = V_{\lambda l} : \mathscr{C} \to \mathbf{R}; \quad V(\phi) = \int_{\mathscr{B}} W(F)\, dV - \langle \lambda l, \phi \rangle, \tag{4}$$

where $\langle\ ,\rangle$ denotes the L^2 pairing, given by

$$\langle l, \phi \rangle = \int_{\mathscr{B}} B(X) \cdot \phi(X)\, dV(X) + \int_{\partial\mathscr{B}} \tau(X) \cdot \phi(X)\, dA(X)$$
$$= \mathrm{tr}\, [k(l, \phi)].$$

To see this equivalence, observe that for $u \in T_\phi \mathscr{C} = \mathscr{C}$

$$DV(\phi) \cdot u = \int_{\mathscr{B}} \frac{\partial W}{\partial F} \cdot \nabla u\, dV - \langle \lambda l, u \rangle$$

$$= -\int_{\mathscr{B}} (\mathrm{DIV}\, P) \cdot u\, dV + \int_{\partial\mathscr{B}} (P \cdot N) \cdot u\, dA - \langle \lambda l, u \rangle$$

$$= \langle \Phi(\phi) - \lambda l, u \rangle. \tag{5}$$

The group $SO(3)$ of proper orthogonal linear transformations of $\mathbf{R}^3$ plays a key role in our work. Its Lie algebra is skew, the collection of 3×3 skew symmetric matrices. We identify skew with $\mathbf{R}^3$ by the map $\hat{}: \mathbf{R}^3 \to$ skew, given by

$$\hat{v}(w) = w \times v. \tag{6}$$

The inner product we use on M_3 is $\langle A, B \rangle = \mathrm{tr}\,(AB^{\mathrm{T}})$; we note that $\langle v, w \rangle = \frac{1}{2}\langle \hat{v}, \hat{w} \rangle$. The map $\hat{}$ has an additional useful property: if $A \in$ sym and we let $L_A = (\mathrm{tr}\, A)\, I - A \in M_3$, then

$$(L_A v)\hat{} = A\hat{v} + \hat{v}A \quad \text{for} \quad v \in \mathbf{R}^3. \tag{7}$$

The group $SO(3)$ acts on $\mathscr{C}$ and $\mathscr{L}$ by

$$Q\phi = Q \circ \phi \quad \text{and} \quad Ql(X) = (QB(X), Q\tau(X)).$$

The algebra skew acts by the same formulas.

The astatic load map satisfies

$$\left.\begin{array}{l} k(l, Q\phi) = k(l, \phi)\, Q^{\mathrm{T}},\ k(Ql, \phi) = Qk(l, \phi),\ Q \in SO(3) \\ \text{and} \\ k(l, W\phi) = k(l, \phi)\, W^{\mathrm{T}},\ k(Wl, \phi) = Wk(l, \phi),\ W \in \text{skew}. \end{array}\right\} \tag{8}$$

From (8) and (5) we have, for example,

$$\langle l, W\phi \rangle = \mathrm{tr}\,(k(l, \phi)\, W^{\mathrm{T}}) = \langle k(l, \phi), W \rangle. \tag{9}$$

The divergence theorem enables one to establish readily the following identities from [I]:

$$k(\Phi(\phi), \phi) = \int_{\phi(\mathscr{B})} \sigma\, dv \tag{10}$$

and

$$k(\Phi(\phi)) = \int_{\mathscr{B}} P\, dV, \tag{11}$$

where σ is the Cauchy stress; $P = J\sigma F^{-\mathrm{T}}$, $J = \det F$. From (10) it follows that $k(\Phi(\phi), \phi) \in$ sym; *i.e.*, the torque in the configuration ϕ is zero.

The linearization of Φ is given by

$$D\Phi(\phi) \cdot u = (-\mathrm{DIV}\,(\mathrm{A} \cdot \nabla u), (\mathrm{A} \cdot \nabla u) \cdot N), \tag{12}$$

where A is regarded as a linear operator from $L(T_X\mathscr{B}, \mathbf{R}^3)$ to itself, as in [I], and $u \in \mathscr{U} = T_I\mathscr{C}$ is a displacement for the linearized theory. At $\phi = I$, (12) becomes

$$D\Phi(I) \cdot u = (-\mathrm{DIV}\,(\mathbf{c} \cdot e), (\mathbf{c} \cdot e) \cdot N), \tag{13}$$

where $\mathbf{c}$ is the classical elasticity tensor, regarded as a linear map of sym to itself (see [I]) and where $e = \frac{1}{2}[\nabla u + (\nabla u)^{\mathrm{T}}]$ is the linearized strain tensor. We sometimes write $\mathbf{c} \cdot \nabla u$ for $\mathbf{c} \cdot e$.

Let $L = D\Phi(I)$ denote the linear operator of classical elasticity, given by (13). This has a kernel equal to skew (there are no translations since we have demanded $\phi(0) = 0$ and $u(0) = 0$) and range equal to $\mathscr{L}_e$, the equilibrated loads. This follows from the stability condition, as was explained in [I]. A convenient complement to skew in $\mathscr{U} = T_I\mathscr{C}$ is obtained as follows.

Let $j: M_3 \to \mathscr{L}$ be a right inverse for $k: \mathscr{L} \to M_3$ (for example, $j = (k|(\ker k)^{\perp})^{-1}$ as in [I]) and let

$$\text{Skew} = j(\text{skew}).$$

Thus, we have the algebraic decomposition

$$\mathscr{L} = \mathscr{L}_e \oplus \text{Skew},$$

where $\mathscr{L}_e$ denotes the equilibrated loads, related to k by $\mathscr{L}_e = k^{-1}(\text{sym})$ (see Figure 1).

Now let $\mathscr{U}_{\mathrm{sym}}$ denote the orthogonal complement to Skew in the pairing (5). That is,

$$\mathscr{U}_{\mathrm{sym}} = \{u \in \mathscr{U} \mid \langle l, u \rangle = 0 \quad \text{for all} \quad l \in \text{Skew}\}.$$

Since the pairing (4) is (weakly) non-degenerate between $\mathscr{L}$ and $\mathscr{U}$, $\mathscr{U}_{\mathrm{sym}}$ is a complement to skew in $\mathscr{U}$. Note that $\mathscr{U}_{\mathrm{sym}}$ and skew need not be L^2 orthogonal in $\mathscr{U}$, however. What is more convenient for later use is to have $\mathscr{U}_{\mathrm{sym}}$ the orthogonal complement of Skew (see Lemma 2.2 below).

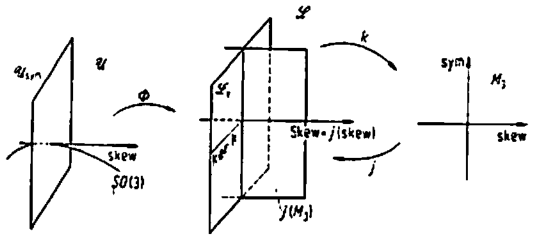

Fig. 1

It follows from the theory of elliptic equations that

$$L: \mathscr{U}_{sym} \to \mathscr{L}_e$$

is an isomorphism. Given $l \in \mathscr{L}_e$, let $u_l \in \mathscr{U}_{sym}$ satisfy $L(u_l) = l$. Define the *Betti form* $B: \mathscr{L}_e \times \mathscr{L}_e \to \mathbb{R}$ by

$$B(l_1, l_2) = \langle l_1, u_{l_2} \rangle \tag{14}$$

(the inner product is defined as in equation (4)). The divergence theorem shows that

$$B(l_1, l_2) = \langle \mathbf{c} \cdot \nabla u_{l_1}, \nabla u_{l_2} \rangle. \tag{15}$$

Here the inner product means

$$\langle \mathbf{c} \cdot \nabla u_{l_1}, \nabla u_{l_2} \rangle = \int_{\mathscr{B}} \operatorname{tr} [(\mathbf{c} \cdot \nabla u_{l_1}) (\nabla u_{l_2})^T] \, dV.$$

Since $\mathbf{c}$ is symmetric, $B(l_1, l_2)$ is symmetric in l_1 and l_2. This is the *Betti reciprocity theorem*, which will be useful in the next section. Notice that (15) is unchanged if u_l is replaced by $u_l + K$ for $K \in$ skew. Thus the same formula (15) holds independent of the choice of complement to skew in $\mathscr{U}$. This freedom is convenient for computations that will be given later in the paper.

Next we recall that loads l are classified into five types according to the way in which the orbit of $A = k(l)$, under the left action of $SO(3)$ on M_3, meets sym. See [I], § 6. An important set is

$$S_A = \{Q \in SO(3) \mid QA \in \text{sym}\}. \tag{16}$$

In [I] we established the following descriptions of S_A:

Load Type	S_A
0	four points
1	two points $\cup\, S^1 \approx \mathbb{RP}^1$
2	one point $\cup\, \mathbb{RP}^2$
3	$\mathbb{RP}^1 \cup \mathbb{RP}^1$ (disjoint)
4	$SO(3) \approx \mathbb{RP}^3$

Notice that

$$T_I S_A = \{W \in \text{skew} \mid WA + AW = 0\}.$$

Since $\hat{}$ is an isomorphism, (7) implies that

$$T_I S_A \approx \{v \in \mathbb{R}^3 \mid Av = (\operatorname{tr} A)\, v\}; \tag{17}$$

i.e., the eigenspace of A with eigenvalue $\operatorname{tr} A$. Thus, from Proposition 3.3 of [I], $T_I S_A$ consists of the axes of equilibria for l.

Under hypotheses of non-degeneracy on the equations of linear elasticity, we shall prove in Sections 3, 4 and 5, the existence of the following numbers of solutions for the nonlinear traction problem:

Load Type	Number of Solutions
0	4
1	$4 \leqq n \leqq 6$
2	$4 \leqq n \leqq 14$
3	$4 \leqq n \leqq 8$
4	$4 \leqq n \leqq 40$

A formula for the index will be given in Section 2. In particular, this will enable us to determine the stable solutions which have index = 0. The key to determining the number of solutions is the quadratic function $Q \mapsto B(Ql, Ql)$ restricted to S_A.

The number of solutions is related to the vanishing of derivatives of real non-degenerate quadratic forms on $\mathbb{RP}^s$, $s = 1, 2, 3$. In fact, using Bezout's theorem applied to associated *cubic* polynomials on the *double* covering, we find that the number of solutions branching out from $\mathbb{RP}^s$ in the above table is at most $\dfrac{3^{s+1} - 1}{2}$. For instance the maximum in type 2 is

$$1 \text{ (for the single point)} + \frac{3^3 - 1}{2} = 14.$$

We also show that the bifurcation diagrams obtained are structurally stable; that is, in a sense made precise in [I] and in § 3, 4, 5, insensitive to small perturbations. Finally we note that cusps occur for loads of type 1 (see [I]) and double cusps occur for loads of type 2.

The role of symmetry in the present problem is somewhat different from that discussed by others. Our group $SO(3)$ acts freely on $\mathscr{C}$ and also acts on $\mathscr{L}$, whose elements play the role of parameters. The orbit of the identity of $SO(3)$ in $\mathscr{C}$ comprises the trivial solutions. In all the papers we have seen (Golubitsky & Schaeffer [1979], Dancer [1980], Arms, Marsden & Moncrief [1981] and Hale & Taboas [1980] are examples) the trivial solutions have some isotropy and there is still some symmetry left when one passes to a slice for the group action. In these problems the bifurcation equation is on the slice. In our problem however, the bifurcation equation is on the orbit itself. However, when one is considering bifurcations in the traction problem near a stressed state or when the loads have special symmetries, a combination of the two methods is necessary. The treatment of this topic is given in Part III of this series of papers.

Finally we note that some information in related problems can be obtained by the methods here. Specifically, in RIVLIN's problem of homogeneous incompressible deformations of a cube, BALL & SCHAEFFER [1982] have continued RIVLIN's original analysis by examining perturbations from a neo-Hookean to a Mooney-Rivlin material by using the Golubitsky-Schaeffer bifurcation theory for problems with S_3 symmetry. Methods of the present paper enable one to show that for any isotropic material, the solutions near $SO(3)$ for small tractions are all homogeneous and are in one-to-one correspondence with the union of a point with $\mathbb{RP}^2$. (The tractions can be positive or negative and the material can be compressible or incompressible.) Details are given in Section 2, in Part III and in WAN [1983].

§ 2. A Potential Function on $SO(3)$ and S_{A_0}

Recall from § 1 that $\mathscr{U}_{sym}$ is the L^2 orthogonal complement to Skew in $\mathscr{U} = T_I\mathscr{C}$. We first note that a neighborhood of 0 in $\mathscr{U}_{sym}$ yields a slice for the action of $SO(3)$ in the sense that when translated around the orbit of I (which we identify with $SO(3)$ itself), it becomes a tubular neighborhood of $SO(3)$.

2.1 Lemma. *There is a neighborhood U of $0 \in \mathscr{U}_{sym}$ such that the map*

$$\varrho : SO(3) \times (I + U) \to \mathscr{C},$$

defined by

$$\varrho(Q, I + u) = Q^{-1} + Q^{-1}u, \tag{18}$$

is a diffeomorphism onto a neighborhood of $SO(3)$ in $\mathscr{C}$.

This follows by a standard argument using compactness of $SO(3)$ and the implicit function theorem; *cf.* Lemma 4.1 of [I]. We use Q^{-1} and not Q in (18) only for consistency with [I].

Recall that we are seeking critical points of the function $V_{\lambda l} = V$ given by (4). Let

$$V_\varrho = V \circ \varrho : SO(3) \times (I + U) \to \mathbb{R}.$$

Thus, if $\phi = I + u$, then

$$V_\varrho(Q, \phi) = \int_{\mathscr{B}} W(Q^{-1}F)\, dV(X) - \lambda\langle l, Q^{-1}\phi\rangle = \int_{\mathscr{B}} W(F)\, dV(X) - \lambda\langle Ql, \phi\rangle \tag{19}$$

by material frame-indifference.

Clearly (Q, ϕ) is a critical point of V_ϱ if and only if $Q^{-1}\phi$ is a critical point of V.

Next we break up the problem of finding a critical point of V_ϱ into a transverse and tangential part relative to $SO(3) \subset \mathscr{C}$. Note that for $\lambda l = 0$, each point of $SO(3)$ is a critical point; the set of these points are the "trivial solutions".

Now we may regard Φ as the gradient of V (relative to the L^2 pairing between $\mathscr{L}$ and $\mathscr{C}$). This gradient takes values in $\mathscr{L}$ which can be decomposed into the two components along $\mathscr{L}_e$ and Skew. In terms of V_ϱ, we are led to the following.

2.2 Lemma. *Let $(Q, \phi) \in SO(3) \times (I + U)$. Then (Q, ϕ) is a critical point of V_ϱ if and only if*

(i) $\qquad \Phi(\phi) - \lambda Ql \in \text{Skew}$

and (20)

(ii) $\qquad \langle \lambda WQl, \phi\rangle = 0$ *for all* $W \in$ skew.

Proof. We have

$$D_\phi V_\varrho \cdot u = \int_{\mathscr{B}} \frac{\partial W}{\partial F} \cdot \nabla u \, dV - \langle \lambda Ql, u\rangle$$

$$= \int_{\mathscr{B}} (-\operatorname{DIV} P) \cdot u \, dV + \int_{\partial\mathscr{B}} (P \cdot N) \cdot u \, dA - \langle \lambda Ql, u\rangle$$

$$= \langle \Phi(\phi) - \lambda Ql, u\rangle.$$

This is zero for all $u \in \mathscr{U}_{sym}$ if and only if $\Phi(\phi) - \lambda Ql \in \text{Skew}$ since $\mathscr{U}_{sym}$ and Skew are L^2 orthogonal.

Next, $D_Q V_\varrho \cdot (WQ) = -\langle \lambda WQl, \phi\rangle$, which vanishes if and only if (ii) holds. ■

2.3 *Remark.* We can rephrase lemma 2.2 as follows: Conditions (i) and (ii) together are equivalent to $\Phi(\phi) = \lambda Ql$; *i.e.* $\Phi(Q^{-1}\phi) = \lambda l$, for $\phi \in I + U$. It is instructive to see that the equivalence remains valid for *Cauchy* materials (*i.e.*, materials for which a stored energy function need not exist). Since $\varrho : (Q, \phi) \mapsto Q^{-1}\phi$ is a diffeomorphism, $T_{Q^{-1}\phi}\mathscr{C} \approx \mathscr{U} = \{-Q^{-1}W\phi \mid W \in \text{skew}\} \ominus \{Q^{-1}u \mid u \in \mathscr{U}_{sym}\}$, and thus $\mathscr{U} = Q\mathscr{U} = \{-W\phi \mid W \in \text{skew}\} \ominus \mathscr{U}_{sym}$. Hence, $\Phi(\phi) - \lambda Ql = 0$ if and only if

(i)′ $\qquad \langle \Phi(\phi) - \lambda Ql, u\rangle = 0$ for all $u \in \mathscr{U}_{sym}$

and

(ii)′ $\qquad \langle \Phi(\phi) - \lambda Ql, -W\phi\rangle = 0$ for all $W \in$ skew.

From the fact that $k(\Phi(\phi), \phi) = \int \sigma \, dv \in \text{sym}$ (see equation (10)), $\langle \Phi(\phi), -W\phi\rangle = \langle -W, k(\Phi(\phi), \phi)\rangle = 0$. Thus equation (ii)′ becomes $-\langle \lambda WQl, \phi\rangle = 0$ for all $W \in$ skew. Therefore $\Phi(\phi) = \lambda Ql$ if and only if conditions (i) and (ii) hold.

Now we are ready to perform the Liapunov-Schmidt procedure on our equation $\Phi(\phi) = \lambda Ql$. We wish to do this in a way that retains the potential form. A convenient way to do this is to use the ideas in the splitting lemma of GROMOLL & MEYER [1969] and the related bifurcation theory of REEKEN [1973] and WEINSTEIN [1978]. Our construction proceeds directly as follows:

2.4 Lemma. *There is a unique function from $SO(3)$ to $I + U$ (shrinking U if necessary) denoted $Q \mapsto \phi_Q$ (and depending on λl) such that equation (20i) is satisfied*; i.e.,

$$\Phi(\phi_Q) - \lambda Ql \in \text{Skew}. \tag{21}$$

Proof. This follows from the fact that $D\Phi(I): \mathscr{U}_{\text{sym}} \to \mathscr{L}_e$ is an isomorphism and from the implicit function theorem. ■

Now define $f: SO(3) \to \mathbb{R}$ by

$$f(Q) = V_e(Q, \phi_Q). \tag{22}$$

Then we have

2.5 Theorem. *The set of solutions of $\Phi(\phi) = \lambda l$ in a neighborhood of $SO(3)$ in $\mathscr{C}$ is put in one-to-one correspondence with critical points of f by the correspondence $Q^{-1}\phi_Q \leftrightarrow Q$.*

Proof. We have

$$Df(Q) = D_Q V_e(Q, \phi_Q) + D_\phi V_e(Q, \phi_Q) \cdot D_Q \phi_Q.$$

However, ϕ_Q was chosen to make $D_\phi V_e(Q, \phi_Q)$ vanish. Thus $Df(Q) = 0$ precisely when $DV_e = (D_Q V_e, D_\phi V_e)$ vanishes at (Q, ϕ_Q); *i.e.*, when $DV(Q^{-1}\phi_Q) = 0$, which is equivalent to $\Phi(Q^{-1}\phi_Q) = \lambda l$ by (5). ■

Recall that the *index* of a critical point is the dimension of the largest subspace on which the second derivative is negative-definite. Now the second derivative of V in a direction orthogonal to $SO(3)$ is always positive-definite, by the stability of the elasticity tensor c and Korn's inequality (see FICHERA [1972] and [I, Theorem 5.5]). Thus we have

2.6 Proposition. *Let Q be a critical point for f so that $\phi = Q^{-1}\phi_Q$ is a critical point for V. Then*

$$\text{index}(V, \phi) = \text{index}(f, Q).$$

In particular, if Q is a strict local minimum for f, then ϕ is a strict local minimum for V.

A point will be called *stable* if it is a strict local minimum for V. If it is a minimum, but not necessarily strict, it will be called *neutrally stable*.

2.7 Corollary. *For l given and for λ sufficiently small, the traction problem $\Phi(\phi) = \lambda l$ has at least four solutions. One of them is neutrally stable.*

Proof. The (Liusternik-Shnirel'man) category of $SO(3) \approx \mathbb{RP}^3$ is 4, so any smooth real valued function on it has at least four critical points, one of which is the minimum. Now use 2.5 and 2.6. ■

Notice that the existence of at least four solutions has nothing to do with the load type. However, for loads of type 0 we proved in [I] that there are exactly four solutions and exactly one is stable. For loads near a load of type 1 we similarly proved that the number of solutions is between 4 and 6, and at least one is stable.

The load classification will enter through the following development.

From (19) and (22) we have

$$f(Q) = \int_{\mathscr{B}} W(F_Q)\, dV(X) - \lambda\langle Ql, \phi_Q\rangle, \tag{23}$$

where $F_Q = D\phi_Q$. By the construction of ϕ_Q,

$$\phi_Q = I + \lambda u_{Ql} + O(\lambda^2), \tag{24}$$

where $L(u_{Ql}) = (Ql)_e$ and $(Ql)_e$ denotes the equilibrated part of Ql according to the decomposition $\mathscr{L} = \mathscr{L}_e \oplus \text{Skew}$. Since $W(I) = 0$ and $P(I) = \frac{\partial W}{\partial F}(I) = 0$, it follows that

$$\begin{aligned}\int_{\mathscr{B}} W(F_Q)\, dV(X) &= \int_{\mathscr{B}} \Big[W(I) + \lambda \frac{\partial W}{\partial F}(I) \cdot \nabla u_{Ql} \\ &\quad + \frac{\lambda^2}{2} \frac{\partial^2 W}{\partial F\, \partial F}(I) \cdot (\nabla u_{Ql}, \nabla u_{Ql}) + O(\lambda^3)\Big]\, dV(X) \\ &= \frac{\lambda^2}{2} \int_{\mathscr{B}} \langle c(\nabla u_{Ql}), \nabla u_{Ql}\rangle\, dV(X) + O(\lambda^3).\end{aligned} \tag{25}$$

Also, using (15), we obtain

$$\begin{aligned}\langle Ql, \phi_Q\rangle &= \langle Ql, I\rangle + \lambda\langle Ql, u_{Ql}\rangle + O(\lambda^2) \\ &= \langle l, Q^T I\rangle + \lambda\langle c(\nabla u_{Ql}), \nabla u_{Ql}\rangle + O(\lambda^2).\end{aligned} \tag{26}$$

Substituting (25) and (26) into (23) gives

$$f(Q) = -\lambda\left[\langle l, Q^T I\rangle + \frac{\lambda}{2}\langle c(\nabla u_{Ql}), \nabla u_{Ql}\rangle + O(\lambda^2)\right]. \tag{27}$$

Let us write $u_Q^0 = u_{Ql_0}$ and consider the case in which $|l - l_0|$ and λ are small. Then (27) yields the following

2.8 Proposition. *We have*

$$f(Q) = -\lambda\left[\langle l, Q^T I\rangle + \frac{\lambda}{2}\langle c(\nabla u_Q^0), \nabla u_Q^0\rangle + O(\lambda^2) + O(\lambda\,|l - l_0|)\right]. \tag{28}$$

It is instructive to see the derivation of (28) in an abstract form. Let E be a Banach space with $0 \in E$ a nondegenerate critical point of a C^2 function $g: E \to \mathbb{R}$ i.e. $Dg(0) = 0 \in E^* = L(E, \mathbb{R})$ and $D^2g(0) = T \in L(E, E^*)$ is invertible. (In examples, including ours, one must replace E^* by a suitable Banach space in duality with E.) Let $h: E \to \mathbb{R}$ be another C^2 function; then the implicit function theorem shows that for small $\lambda \in \mathbb{R}$, the perturbed function $g + \lambda h$ has a unique critical point near 0 of the form $u(\lambda) = \lambda u^h + O(\lambda^2)$:

$$Dg(\lambda u^h + O(\lambda^2)) + \lambda\, Dh(\lambda u^h + O(\lambda^2)) = 0 \in E^*.$$

Comparing terms of order λ we find that $Tu^h = -Dh(0)$. Evaluating $g + \lambda h$ at this critical point gives

$$(g + \lambda h)(u(\lambda)) = (g + \lambda h)(0) + \lambda^2\, Dh(0)(u^h) + \frac{\lambda^2}{2}\langle Tu^h, u^h\rangle + O(\lambda^3)$$

$$= g(0) + \lambda h(0) - \frac{\lambda^2}{2}\langle Tu^h, u^h\rangle + O(\lambda^3). \tag{27$'$}$$

Let us apply this formula to the case in which

$E = \mathscr{U}_{sym}$ and identify $E^* \approx \mathscr{L}_e$ via $\langle \cdot, \cdot \rangle$,

$g(u) = \int W(I + \nabla u)\, dV$ so that $g(0) = 0$ and $T = L \mid (\mathscr{U}_{sym})$,

$h(u) = -\langle Ql, I + u\rangle$ (so that $Dh(0) = -(Ql)$, and $u^h = u_{Ql}$).

Since $\langle Tu^h, u^h\rangle = \langle c(\nabla u_{Ql}), \nabla u_{Ql}\rangle$ by the divergence theorem, the formula (27)$'$ gives the formula (27).

Now we are ready to link this result up with S_{A_0} (see equation (16)) and hence with the type classification. Recall that $A_0 = k(l_0) \in \text{sym}$ is the astatic load of l_0.

2.9 Proposition. *The set $S_{A_0} \subset SO(3)$ is a non-degenerate critical manifold for $Q \mapsto -\langle l_0, Q^T I\rangle$. The index in the direction $(T_Q S_{A_0})^\perp$ is the index of $QA_0 - \operatorname{tr}(QA_0)\, I$.*

Proof. See [I], Lemma 5.6. ■

2.10 Corollary. *For λ small and l near l_0, all critical points of $f(Q)$ lie in an neighborhood of S_{A_0}.*

Proof. Since $Df(Q) \cdot WQ = -\lambda\langle WQl_0, I\rangle + O(\lambda^2) + O(\lambda\, |l - l_0|)$ it follows that Q can be a critical point for f only if $-\langle WQl_0, I\rangle = \langle Ql_0, W\rangle$ vanishes up to $O(\lambda^2, \lambda\, |l - l_0|)$ for all $W \in$ skew, *i.e.* $Ql_0 \in$ Sym (equivalently $Q \in S_{A_0}$) up to $O(\lambda^2, \lambda\, |l - l_0|)$. ■

Because of Proposition 2.9, we are led to carry out a second Liapunov-Schmidt reduction. This proceeds as follows. Let $N(S_{A_0})$ be a normal bundle neighborhood of S_{A_0} in $SO(3)$ with fiber at Q orthogonal to $T_Q S_{A_0}$. Since the normal bundle is a non-degenerate direction for the second derivative of $\frac{1}{\lambda} f(Q)$ for λ small and $|l - l_0|$ small, we can solve uniquely for critical points of $\frac{1}{\lambda} f(Q)$ restricted to fibers of the normal bundle to produce a smooth mapping on S_{A_0}, $Q \mapsto n(Q) \in (T_Q S_{A_0})^\perp$ such that $n(Q)$ is the critical point of $\frac{1}{\lambda} f$ restricted to the fiber of $N(S_{A_0})$ through $Q \in S_{A_0}$. Note that $n = O(\lambda)$ but $\langle l_0, n\rangle = O(\lambda^2)$.

2.11 Proposition. *Critical points of f are in one-to-one correspondence with critical points of*

$$\tilde{f}: S_{A_0} \to \mathbb{R},$$

defined by

$$\tilde{f}(Q) = \frac{1}{\lambda} f(Q, n(Q)), \tag{29}$$

and we have

$$\tilde{f}(Q) = \langle -l, Q^T I\rangle - \frac{\lambda}{2}\langle c(\nabla u^0_Q), \nabla u^0_Q\rangle + O(\lambda^2) + O(\lambda\, |l - l_0|). \tag{30}$$

This proposition agrees with Theorem 7.3 of [I]; the present derivation, however, seems more satisfactory. The proof of 2.11 follows from the usual Liapunov-Schmidt process.

We summarize what we have obtained as follows.

2.12 Theorem. *For $\lambda > 0$ small and l near l_0, the solutions of the problem $\Phi(\phi) = \lambda l$ are in one-to-one correspondence with critical points of $\tilde{f}$ on S_{A_0} where $\tilde{f}$ is given by (29) and (30). The index of the solution corresponding to a critical point at Q is given by* $\operatorname{index}(QA_0 - \operatorname{tr}(QA_0)\, I) + \operatorname{index}(\tilde{f}, Q)$.

We remark that the critical points of the Betti form on S_{A_0} are intimately and simply related to the compatability conditions and series expansion methods of Signorini. See MARSDEN & WAN [1983] for details.

In the following sections the leading terms in (30) will play the crucial role in our bifurcation analysis. As in [I], suitable hypotheses of non-degeneracy on the Betti form $B(Q, Q) := B(Ql_0, Ql_0)$ (the second term in (30)) will guarantee that the bifurcation diagrams obtained are structurally stable.

There are, however, cases of interest in which the Betti form is degenerate and no bifurcation occurs. We conclude this section by studying such a case for an isotropic homogeneous material with a homogeneous load.

Let us call a load "homogeneous" if $l_0 = \begin{pmatrix} 0 \\ \tau_0 \end{pmatrix}$, where $\tau_0 = KN$, $K \in \text{sym}$ is a constant matrix, and N is the outward unit normal on $\partial\mathscr{B}$. The astatic load is $A_0 = K(\text{vol}\, \mathscr{B})$.

Consider an isotropic, homogeneous material with $c(e) = \lambda \operatorname{trace} e + 2\mu e$

and with a homogeneous load l_0. One verifies that $u_Q^0 = c^{-1}K$, a homogeneous solution, for $Q \in S_{A_0}$. Thus the Betti form $B(Q, Q)$ is a constant on S_{A_0}, so we have a degenerate case.

2.13 Theorem. *Let* $l_0 = \begin{pmatrix} 0 \\ KN \end{pmatrix}$ *be a homogeneous load, where* $K \in \text{sym}$. *Then for small* λ *the solutions are homogeneous deformations* $\phi_Q(\lambda, X)$, *parametrized in a unique fashion by elements* Q *of* S_{A_0}.

In other words, for small λ, the solution set near $SO(3)$ has the form $\{\phi_Q(\lambda, X) \mid Q \in S_{A_0}\}$ and is homeomorphic to S_{A_0}. Thus, in this case one expects that "no" bifurcation occurs in the solution set and so non-homogeneous solutions do not exist. To prove 2.13, we prepare a lemma.

2.14 Lemma. *The first Piola-Kirchhoff stress* P *maps* sym *to* sym; i.e., $F \in \text{sym}$ *implies* $P(F) \in \text{sym}$.

Proof. This follows directly from the standard representation of P for isotropic materials (see TRUESDELL & NOLL [1965], p. 140). ■

Proof of 2.13. As we have remarked, $A_0 = k(l_0) = K \operatorname{vol}(\mathscr{B})$. Now $Q \in S_{A_0}$ if and only if $k(Ql_0) = QK \operatorname{vol}(\mathscr{B}) \in \text{sym}$. Let $Q \in S_{A_0}$. By the stability assumption (H2), $DP(I)$ is an isomorphism of sym to sym and so by 2.14 and the inverse function theorem, there is a unique element $E_{\lambda,Q} \in \text{sym}$ such that $P(I + \lambda E_{\lambda,Q}) = \lambda QK$ for small λ. Let $\phi_Q(\lambda, X) = Q^{-1}(X + \lambda E_{\lambda,Q}X)$. Clearly, $\phi_Q(\lambda, X)$ is *homogeneous*. By the Principle of Material Frame Indifference, $P(QF) = QP(F)$ and so $P(\phi_Q(\lambda X)) = Q^{-1}P(I + \lambda E_{\lambda Q}) = Q^{-1}(\lambda QK) = K$. Hence

$$-\mathrm{DIV}\, P = 0$$

and

$$P \cdot N = \lambda KN = \lambda \tau_0.$$

Consequently, ϕ_Q satisfies the traction problem. Observe that the ϕ_Q's are distinct for small λ. ■

For example, if $K = \operatorname{diag}(T, T, T)$ then S_{A_0} is a point together with RP^2, so the solutions in this case are in one-to-one correspondence with this set. The solution near the identity is easily checked to be a multiple of the identity.

A similar theorem holds in the incompressible case, provided that $\mathscr{C}$ is replaced by $\mathscr{C}_{J=1}$, the volume-preserving deformations. Even with the constraint $J = 1$, the solutions are still homogeneous. For $\tau = TN$, we again may conclude that the solutions near the identity are in one-to-one correspondence with the set $\{I\} \cup \mathrm{RP}^2$. (See Part III for additional details.) As is noted by BALL & SCHAEFFER [1982], the only homogeneous solution for small tractions near the identity is the trivial one. Therefore the traction problem for Rivlin's cube with small tractions admits a further set of homogeneous solutions in one-to-one correspondence with RP^2. (This set is invariant under conjugation by elements of $SO(3)$, a fact consistent with the results of ADELEKE [1980].)

§ 3. Analysis of Loads of Type 2

For loads of type 2, we can assume that $k(l_0) = \operatorname{diag}(a, a, a)$, where $a \neq 0$. See [I], § 6. In this case, $S_{A_0} = SO(3) \cap \text{sym} = \{I\} \cup \mathrm{RP}^2$. As is well known, RP^2 has the double covering $\varrho \mid S^2 : S^2 \to \mathrm{RP}^2$, defined by $X \mapsto 2XX^T - 1$, where $S^2 = \{X \in \mathrm{R}^3 \mid \|X\| = 1\}$. For $Y \in S_{A_0}$, $k(Yl_0) = aY$ (see equation (8)), and one denotes by u_Y^0 the solution in $\mathscr{U}_{\text{sym}}$ to the linearized problem $L(u_Y^0) = Yl_0 \in \mathscr{L}_e$. Recall from (30) that we seek to study the critical points of $\bar{f}(Y) = -\langle l, Y^T 1\rangle - \frac{\lambda}{2}\langle c(\nabla u_Y^0), \nabla u_Y^0\rangle + O(|\lambda|\, \|l - l_0\|) + O(\lambda^2)$ for $Y \in S_{A_0} = SO(3) \cap \text{sym}$. For small $\lambda > 0$, it is natural to study the function $\bar{h}(Y) = \bar{l}(Y) + B_{l_0}(Y)$, where $\bar{l}(Y) = \frac{2}{\lambda}\langle l, Y^T 1\rangle$, and $B_{l_0}(Y) = \langle c(\nabla u_Y^0), \nabla u_Y^0\rangle$, $Y \in SO(3) \cap \text{sym}$. As before, we call B_{l_0} the *Betti form* (see (14) and (15)). We can regard it as a quadratic form on sym.

Fix a region $\mathscr{B}$ with unit volume. Let us first study the case in which l_0 has the form $l_0 = \begin{pmatrix} B_0 \\ \tau_0 \end{pmatrix} = \begin{pmatrix} 0 \\ N \end{pmatrix}$. Clearly, $k(l_0)$ is the identity and l_0 is of type 2.

3.1 Proposition. *Given any positive-definite quadratic form* B *on* sym, *there exists a homogeneous hyperelastic material with a stable* (i.e., *positive-definite*) *elasticity tensor* c, *such that* $B = B_{l_0}$ *the Betti form on* sym.

Proof. Define a symmetric elasticity tensor $c : \text{sym} \to \text{sym}$ by $\langle Y, c^{-1}Y\rangle = B(Y)$. Set $W(F) = \frac{1}{2}\langle D, c(D)\rangle$, where $D = \frac{1}{2}(F^TF - 1)$. Clearly, $W(F)$ is a stored energy function with c as its elasticity tensor.

It is easy to verify that $u_Y^0(X) = (c^{-1}Y)X$; that is

$$-\mathrm{DIV}\, c(\nabla u_Y) = 0,$$

$$c(\nabla u_Y^0)N = YN.$$

Thus $B_{l_0}(Y) = \langle c(\nabla u_Y^0), \nabla u_Y^0\rangle = \langle c(c^{-1}Y), c^{-1}Y\rangle = \langle Y, c^{-1}Y\rangle = B(Y)$. ■

3.2 Corollary. *Given any quadratic form* B *on* sym, *there exists a hyperelastic material with a stable elasticity tensor such that* $B + c = B_{l_0}$ *on* RP^2, *for some constant* c.

Proof. Choose c large, so that $B + \frac{c}{3}\operatorname{trace}(Y^TY)$ is positive-definite on sym. By the previous proposition, $B + \frac{c}{3}\operatorname{trace}(Y^TY) = B_{l_0}$ on sym for some Betti form B_{l_0}. On $\mathrm{RP}^2 = \text{sym} \cap SO(3) \setminus \{I\}$, this becomes $B + c = B_{l_0}$. ■

The above corollary implies that in $\bar{h}$, B_{l_0} can in principle be any quadratic form B. Let us first carry out a local study of the critical points of $\bar{h}$. Given any $Y_0 \in \mathrm{RP}^2$, we can write $Y_0 = Q \operatorname{diag}(1, -1, -1)\, Q^T$ for some $Q \in SO(3)$. Thus the linear map $Y \mapsto QYQ^T$ leaves RP^2 invariant, leaves the form of $\bar{h}$ invariant, and sends $\operatorname{diag}(1, -1, -1)$ to Y_0. Therefore, without loss of generality, we take $\operatorname{diag}(1, -1, -1)$ as a typical point near which to study $\bar{h} = \bar{l} + B_{l_0}$. Let us use a local chart

$$\phi: \mathrm{RP}^2 \setminus \mathrm{RP}^1 \to \mathrm{R}^2,$$

where

$$\phi^{-1}(x, y) = \varrho\left(\frac{x}{\sqrt{1 + x^2 + y^2}}, \frac{y}{\sqrt{1 + x^2 + y^2}}, \frac{1}{\sqrt{1 + x^2 + y^2}}\right),$$

where $\varrho: S^2 \to \mathrm{RP}^2$ is the double covering defined earlier and where we identify $\mathrm{RP}^1 = \varrho(S^1)$, and $S^1 = \{(x, y, 0) \in S^2 \mid x^2 + y^2 = 1\}$.

Set

$$\bar{h}(x, y) = \bar{h}(\phi^{-1}(x, y)) = \frac{r^2 \bar{l}(2XX^T - r^2 1) + B(2XX^T - r^2 1)}{r^4} = \frac{\xi(x, y)}{r^4}, \quad \text{where}$$

$r = \sqrt{1 + x^2 + y^2}$ and $X^T = (x, y, 1)$. Thus, ξ is a polynomial of degree ≤ 4.

3.3 Lemma. *Given any polynomial $\xi(x, y)$ of degree ≤ 4, there exists a quadratic form B on sym such that $\xi(x, y) = B(2XX^T - r^2 1)$.*

Proof. Consider the linear map of the set of quadratic forms B on sym to the set of polynomials ξ in x, y of degree ≤ 4 given by $\xi(x, y) = B(2XX^T - r^2 1)$. Let

$$Y = 2XX^T - r^2 1 = \begin{pmatrix} x^2 - y^2 - 1 & 2xy & 2x \\ 2xy & y^2 - x^2 - 1 & 2y \\ 2x & 2y & 1 - x^2 - y^2 \end{pmatrix}.$$

By symmetry considerations, it suffices to observe that $1, x, x^2, xy, x^3, x^2y, x^4, x^3y, x^2y^2$ are the images of $B_1, B_x, \ldots, B_{x^2y^2}$ defined respectively as follows:

$$B_1(Y) = \left(\frac{y_{11} + y_{22}}{-2}\right)^2, \qquad B_{x^2y}(Y) = \frac{y_{12}y_{13}}{4},$$

$$B_x(Y) = \left(\frac{y_{11} + y_{22}}{-2}\right)\left(\frac{y_{13}}{2}\right), \qquad B_{x^4}(Y) = \left(\frac{y_{22} + y_{33}}{-2}\right)^2,$$

$$B_{x^2}(Y) = \left(\frac{y_{13}}{2}\right)^2, \qquad B_{x^3y}(Y) = \left(\frac{y_{22} + y_{33}}{-2}\right)\left(\frac{y_{12}}{2}\right),$$

$$B_{xy}(Y) = \frac{y_{13}y_{23}}{4}, \qquad B_{x^2y^2}(Y) = \left(\frac{y_{12}}{2}\right)^2.$$

$$B_{x^3}(Y) = \left(\frac{y_{22} + y_{33}}{-2}\right)\left(\frac{y_{13}}{2}\right). \quad ■$$

Given any function g defined near a point X, $j^{(4)}g(X)$ denotes the 4^{th} order Taylor polynomial of g at X. Consider a function $\bar{h}$ in the form $\bar{h}(x, y) = \dfrac{\xi(x, y)}{r^4}$ for some polynomial ξ of degree ≤ 4.

3.4 Lemma. (a) *$j^{(4)}\bar{h}(0)$ can be any polynomial $\eta(x, y)$ of degree ≤ 4.*

(b) *If $j^{(4)}\bar{h}(0) = c$, a constant, then $\bar{h} = c$ identically.*

Proof. (a) Define $\xi = j^{(4)}(r^4\eta)(0)$. Thus $j^{(4)}(r^4(\bar{h} - \eta))(0) = j^{(4)}(\xi - r^4\eta)(0) = 0$, which implies $j^{(4)}\bar{h}(0) = \eta$.

(b) $\xi - cr^4 = j^{(4)}[r^4(\bar{h} - c)(0)] = 0$. Thus $\bar{h} = \dfrac{cr^4}{r^4} = c$. ■

Combining Corollary 3.2 and Lemma 3.3, we obtain a description of the possible singularities of $\bar{h}$ on RP^2.

3.1 Proposition. (a) *The 4^{th} order Taylor expansion of $\bar{h}$ at any point Y in RP^2 can be arbitrary.*

(b) *If $j^{(4)}\bar{h}(Y) = c$, a constant, then $\bar{h}(Y) = c$ identically. As usual, $\bar{h}(Y) = \dfrac{2}{\lambda}\langle l, Y^T 1\rangle + \langle c(\nabla u_Y^0), \nabla u_Y^0\rangle$.*

Next, we consider global aspects of the function $\bar{f}$ on RP^2. Denote by $\bar{H} = \{\bar{h} \mid \bar{h} = \bar{l} + B\}$ the space of polynomials of degree ≤ 2 on sym, which vanish at the origin. Define $\Sigma = \{\bar{h} \in \bar{H} \mid \bar{h}(Y)$ on RP^2 has a degenerate critical point$\}$. Thus $\bar{h} \in \bar{H} \setminus \Sigma$ if and only if $\bar{h}$ is a Morse function. Clearly, the bifurcation set Σ is a closed set invariant under the actions $Q \cdot \bar{h}(Y) = \bar{h}(QYQ^{-1})$, $Q \in SO(3)$, and $\lambda \cdot \bar{h}(Y) = \lambda\bar{h}(Y)$, $\lambda \in \mathrm{R}$.

3.5 Proposition. *Σ is a semialgebraic set of codimension ≥ 1 in $\bar{H}$.*

Proof. Consider the polynomial map $\Psi: \bar{H} \times \mathrm{R}^3 \times \mathrm{R} \to \mathrm{R} \times \mathrm{R}^3$ given by $(\bar{h}, X, \mu) \mapsto (X^TX - 1, D_X\mathscr{L})$, where $\mathscr{L}(X, \mu) = \bar{h}(\varrho(X)) + \mu(X^TX - 1)$ stands for the Lagrangian function with multiplier μ. Since $\varrho \mid S^2$ is a local diffeomorphism onto RP^2, by varying $\bar{l}$ in $\bar{h} = \bar{l} + B$ and X, one sees that the map Ψ has $\{0\}$ as a regular value. Thus $\Psi^{-1}(\{0\})$ is an algebraic manifold with the same dimension as $\bar{H}$. The critical point set $\tilde{\Sigma}$ of the projection $\pi: \Psi^{-1}(\{0\}) \to \bar{H}$, $\pi(\bar{h}, X, \mu) = \bar{h}$, is $\{(\bar{h}, X, \mu) \in \Psi^{-1}(\{0\}) \mid \det D_{X,\mu}\Psi = 0\}$, and $\pi(\tilde{\Sigma}) = \Sigma$. Therefore, by the Seidenberg-Tarski theorem and Sard's theorem, our proposition follows. ■

Next, we want to estimate the number of critical points for $\bar{h} = \bar{l} + B$ not in Σ.

3.6 Example. Let $\mathscr{B} \subset \mathbb{R}^3$ be a region with unit volume. Set $l_0 = \begin{pmatrix} 0 \\ N \end{pmatrix}$ a type 2 load where N is the outward unit normal vector along $\partial\mathscr{B}$. Consider a hyperelastic material with elastic tensor

$$c(e) = e - \frac{1}{2}\operatorname{diag}(e_{11}, e_{22}, e_{33}) = (e_{ij}) - \frac{1}{2}\begin{pmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{pmatrix}.$$

We shall show that

(α) $$B_{l_0}(Y) = (y_{11})^2 + (y_{22})^2 + (y_{33})^2 + \langle Y, Y \rangle$$

and

(β) $h = B_{l_0}$ is a Morse function on $\mathbb{RP}^2$ with 13 critical points.

Proof. (α) For $Y \in \mathrm{sym}$, we have

$$u_Y(X) = (c^{-1}Y)X$$

and so

$$\begin{aligned} B_{l_0}(Y) &= \langle Y, c^{-1}Y \rangle = \langle Y, Y + \operatorname{diag} Y \rangle \ (\text{since } c^{-1}Y = Y + \operatorname{diag} Y) \\ &= \langle Y, Y \rangle + (y_{11})^2 + (y_{22})^2 + (y_{33})^2. \end{aligned}$$

(β) We use the method of Lagrange multipliers to find the critical points of $B = B_{l_0}$ on $\mathbb{RP}^2$ (or of $B \circ \varrho$ on S^2). Set

$$\begin{aligned} \mathscr{L} = {} & [(x^2 - y^2 - z^2)^2 + (y^2 - x^2 - z^2)^2 + (z^2 - x^2 - y^2)^2 + 3] \\ & + \mu(x^2 + y^2 + z^2 - 1). \end{aligned}$$

Then the conditions for a critical point are

$$\left.\begin{aligned} \mathscr{L}_x &= 4x[3x^2 - y^2 - z^2] + 2\mu x = 0, \\ \mathscr{L}_y &= 4y[3y^2 - x^2 - z^2] + 2\mu y = 0, \\ \mathscr{L}_z &= 4z[3z^2 - x^2 - y^2] + 2\mu z = 0, \\ & x^2 + y^2 + z^2 - 1 = 0. \end{aligned}\right\}$$

It is easy to see this system has the following solutions:

$$(x, y, z) = \frac{1}{\sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}}(\bar{x}, \bar{y}, \bar{z}),$$

where $\bar{x}, \bar{y}, \bar{z} = 0, 1$, or -1 except $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 0)$; consequently, B_{l_0} on $\mathbb{RP}^2$ has exactly 13 critical points. [Further computations show that B_{l_0} is a Morse function, having 4 critical points $\varrho(x, y, z)$, $x, y, z = \pm\frac{1}{\sqrt{3}}$ of index 2.]

3.7 Proposition. *The number $c(\bar{h})$ of critical points for a Morse function of the form $\bar{h} = \bar{l} + B$ on* $\mathbb{RP}^2$ (i.e., $\bar{h} \notin \Sigma$) *is between* 3 *and* 13.

Proof. (α) for $\mathbb{RP}^2$, the Betti numbers over $\mathbb{Z}_2$ are $b_0 = 1$, $b_1 = 1$, $b_2 = 1$. By the Morse inequality, we have $m_0 + m_1 + m_2 \geqq b_0 + b_1 + b_2 = 3$, where m_i is the number of critical points of index i. Now $c(\bar{h}) = m_0 + m_1 + m_2$, so $c(\bar{h}) \geqq 3$.

(β) Set $\mathscr{L} = \bar{l}(2XX^T - 1) + B(2XX^T - 1) + \mu(X^TX - 1)$ with $X^T = (x, y, z)$.

The equations for critical points are:

$$\left.\begin{aligned} \mathscr{L}_x &= \bar{l}_x + B_x + 2\mu x = 0, \\ \mathscr{L}_y &= \bar{l}_y + B_y + 2\mu y = 0, \\ \mathscr{L}_z &= \bar{l}_z + B_z + 2\mu z = 0, \\ & x^2 + y^2 + z^2 - 1 = 0. \end{aligned}\right\}$$

Consider the homogeneous system (31) in x, y, z, ν over the *complex field* $\mathbb{C}$:

$$\left.\begin{aligned} l_x^* + B_x^* + 2\nu^2 x &= 0, \\ l_y^* + B_y^* + 2\nu^2 y &= 0, \\ l_z^* + B_z^* + 2\nu^2 z &= 0, \end{aligned}\right\} \qquad (31)$$

where $l_x^* + B_x^*$ is obtained by replacing each constant term A in $\bar{l}_x + B_x$ by $A(x^2 + y^2 + z^2)$, *etc.* Clearly, $\pi = \{\bar{h} \mid$ the system (31) has degenerate ray solutions or a solution in the form $(x, y, z, 0)\}$ is an algebraic set. The previous example 3.6 shows that π is proper (*i.e.* $\pi \neq \bar{H}$). Thus by introducing a perturbation, if it is necessary, one may assume that the system (31) has only simple ray solutions and that they are not in the form $(x, y, z, 0)$. By Bezout's theorem, the system (31) has exactly 27 ray solutions. Now each critical point $(\pm x, \pm y, \pm z, \mu)$ gives rise to two ray solutions $(\pm x, \pm y, \pm z, \pm\sqrt{\mu})$ of the system (31). Since $(0, 0, 0, 1)$ is always a solution of the system (31), $2c(\bar{h}) + 1 \leqq 27$ or $c(\bar{h}) \leqq 13$. ■

Our main result on global bifurcation from $\mathbb{RP}^2$ is as follows.

3.8 Theorem. *Let l_0 be a load of type* 2. *Assume that the Betti form $B_{l_0}(Y)$ is a Morse function on* $\mathbb{RP}^2$. *Then for $\lambda > 0$, and $\frac{\|l - l_0\|}{\lambda}$ small, the number of critical points of $\bar{f}$ on* $\mathbb{RP}^2$ *is between* 3 *and* 13. *Therefore, our traction problem has between* 4 *and* 14 *equilibrium solutions.*

Proof. The function $\left(-\frac{2}{\lambda}\right)\bar{f}$ is a small perturbation of the Morse function B_{l_0}, with $3 \leqq c(B_{l_0}) \leqq 13$ by Proposition 3.7. ■

We note that as λ and l are varied, the solutions vary smoothly. In particular, as $\lambda \to 0$ the solutions tend to the critical points of the Betti form on S_{A_0}.

By Proposition 3.1 (a) double cusps can occur as singularities of the Betti form B_{l_0}. *E.g.*, suppose that $B_{l_0} = x^4 + kx^2y^2 + y^4$ (with $k < -2$). Then this double cusp accounts for 9 critical points. Since its gradient has vector field index -3, $\mathbb{RP}^2$ must have 4 other critical points, so that the total number of critical points is the maximum permitted. Thus the existence of (a certain type of) double cusp at one point in $\mathbb{RP}^2$ imposes strong restrictions over what happens elsewhere on $\mathbb{RP}^2$.

§ 4. Analysis of Loads of Type 3; Parallel Loads

As in § 3 we can, without loss of generality, take $k(l_0) = \mathrm{diag}(0, 0, -c)$ where $c \neq 0$. In this case, S_{A_0} is a union of two circles: $S_{A_0} = C \cup C^*$, where

$$C = \left\{ \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x^2 + y^2 = 1 \right\}$$

and

$$C^* = \left\{ \begin{pmatrix} u & v & 0 \\ v & -u & 0 \\ 0 & 0 & -1 \end{pmatrix} \,\middle|\, u^2 + v^2 = 1 \right\}.$$

From Section 2, we have

$$\bar{f}(Q) = -\langle l, Q^{T}1\rangle - \frac{\lambda}{2}\langle c(\nabla u_Q^0), \nabla u_Q^0\rangle + O(|\lambda|\,\|l - l_0\|) + O(\lambda^2).$$

We now regard the Betti form $B(Y) = \langle c(\nabla u_Y^0), \nabla u_Y^0\rangle$ as defined on the linear span $\left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} \right\}$ of the union $C \cup C^*$. Therefore we can write

$$B(Q) = \begin{cases} a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6, & Q \in C, \\ a_1^*u^2 + a_2^*uv + a_3^*v^2 + a_4^*u + a_5^*v + a_6^*, & Q \in C^*. \end{cases}$$

For small $\lambda > 0$, one needs to examine the function

$$\bar{h}(Q) = \frac{2}{\lambda}\langle l, Q^{T}1\rangle + B(Q)$$

$$= \begin{cases} \alpha_1x^2 + \alpha_2xy + \alpha_3y^2 + \alpha_4x + \alpha_5y + \alpha_6 & \text{on } C \\ \alpha_1^*x^2 + \alpha_2^*xy + \alpha_3^*y^2 + \alpha_4^*x + \alpha_5^*y + \alpha_6^* & \text{on } C^*. \end{cases}$$

At this point, it is useful to recognize that the bifurcation problem for type 3 loads from the circle C or the circle C^* is formally the same as that for type 1 loads analyzed in [I, § 8].

For a local study of the critical points of $\bar{h}$ on C (or C^*), we may assume that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\left(\text{or } \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ is a critical point of $\bar{h}$ or equivalently $\alpha_2 + \alpha_5 = 0$ (or $\alpha_2^* + \alpha_5^* = 0$). Thus, in terms of polar angles θ and ψ on the two circles,

$$\bar{h} = \begin{cases} (\alpha_1 + \alpha_4 + \alpha_6) + \left(-\alpha_1 + \alpha_3 - \dfrac{\alpha_4}{2}\right)\theta^2 - \dfrac{\alpha_2}{2}\theta^3 + \dfrac{1}{3}\left(\alpha_1 - \alpha_3 + \dfrac{\alpha_4}{8}\right)\theta^4 \\ \qquad + \text{higher order terms in } \theta, \\ (\alpha_1^* + \alpha_4^* + \alpha_6^*) + \left(-\alpha_1^* + \alpha_3^* - \dfrac{\alpha_4^*}{2}\right)\psi^2 - \dfrac{\alpha_2^*}{2}\psi^3 + \dfrac{1}{3}\left(\alpha_1^* - \alpha_3^* + \dfrac{\alpha_4^*}{8}\right)\psi^4 \\ \qquad + \text{higher order terms in } \psi. \end{cases}$$

In other words, folds and cusps can be the singularities of $\bar{h}$.

For a global study of $\bar{h}$ or $\bar{f}$, we may assume $\alpha_2 = 0$ and $\alpha_2^* = 0$. This can be achieved by rotations in the (x, y) plane and the (u, v) plane separately. Carrying out the same analysis as that for type 1 loads in [I], we obtain the bifurcation set:

$$[2(\alpha_1 - \alpha_3)]^{\frac{2}{3}} = \alpha_4^{\frac{2}{3}} + \alpha_5^{\frac{2}{3}}$$

or

$$[2(\alpha_1^* - \alpha_3^*)]^{\frac{2}{3}} = (\alpha_4^*)^{\frac{2}{3}} + (\alpha_5^*)^{\frac{2}{3}}.$$

Alternatively, $\Delta \cdot \Delta^* = 0$, where

$$\Delta = [2(\alpha_1 - \alpha_3)^2 - \alpha_4^2 - \alpha_5^2]^3 - 108\alpha_4^2\alpha_5^2(\alpha_1 - \alpha_3)^2$$

and

$$\Delta^* = [2(\alpha_1^* - \alpha_3^*)^2 - \alpha_4^{*2} - \alpha_5^{*2}]^3 - 108\alpha_4^{*2}\alpha_5^{*2}(\alpha_1^* - \alpha_3^*)^2.$$

One could phrase our results on loads of type 3 in terms of generic bifurcations with corresponding bifurcation diagrams. However, in keeping up with the other results on bifurcation in this Part II, we shall be content with the following version.

4.1 Theorem. *Let* $k(l_0) = \mathrm{diag}(0, 0, -c)$, $c \neq 0$, *and suppose that* $B(Q)$ *is a Morse function on* $C \cup C^*$ (i.e., $\Delta \cdot \Delta^* \neq 0$). *Then, for small* $\dfrac{\|l - l_0\|}{\lambda}$ *and small* $\lambda > 0$, *the number of equilibrium solutions of our traction problem is between 4 and 8.*

The next example shows that the upper bound 8 is indeed sharp.

4.2 Example. Let the reference configuration $\mathscr{B}$ be the unit ball in $\mathbf{R}^3$ with the load $l_0 = (B_0, \tau_0)$ where $\tau_0 = \operatorname{diag}(x, x, -c_*) N$, $c_* \neq 0$, and where $B_0 = (-1, 0, 0)$. Consider a homogeneous hyperelastic material with the elasticity tensor $\mathsf{c}(e) = e - \frac{1}{2} \operatorname{diag} e$.

Direct computations show that $l_0 \in \mathscr{L}$, and $k(l_0) + \operatorname{diag}\left(0, 0, -\frac{4c_*\pi}{3}\right)$. Thus l_0 is a load of type 3. We claim that

$$(\alpha)\quad B(Y) = \begin{cases} \frac{8\pi}{15}(2s^2 + 5t^2) + \frac{8}{3}c_*^2, & \text{where } Y = \left(\begin{array}{cc|c} s & -t & 0 \\ t & s & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \in C, \\[2ex] \frac{8\pi}{15}(2u^2 + v^2) + \frac{8}{3}c_*^2, & \text{where } Y = \left(\begin{array}{cc|c} u & v & 0 \\ v & -u & 0 \\ \hline 0 & 0 & -1 \end{array}\right) \in C^*, \end{cases}$$

and

(β) B has 4 critical points on C and 4 critical points on C^*.

Proof. To each

$$Y = \left(\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & w \end{array}\right)$$

set

$$c_Y(X) = \left(\begin{array}{cc|c} ax - cy + by & cx & 0 \\ cx & dx & 0 \\ \hline 0 & 0 & -c_* w \end{array}\right), \text{ where } X^T(x, y, z)^T \in \mathscr{B}.$$

There exists exactly one displacement field u_Y (which is linear + quadratic) such that $\mathsf{c}(\nabla u_Y) = c_Y$. It is easy to establish that

$$-\operatorname{DIV}(\mathsf{c}(\nabla u_Y)) = YB_0$$

and

$$\mathsf{c}(\nabla u_Y) \cdot N = Y\tau_0,$$

where

$$\frac{1}{2}(\nabla u_Y + \nabla u_Y^T)(X) = e_Y(X) = \mathsf{c}^{-1}(c_Y(X)) = c_Y(X) + \operatorname{diag} c_Y(X)$$

$$= \left(\begin{array}{cc|c} 2(ax - cy + by) & cx & 0 \\ cx & 2dx & 0 \\ \hline 0 & 0 & -2c_* w \end{array}\right).$$

Consequently,

$$B(Y) = \langle \mathsf{c}(\nabla u_Y), \nabla u_Y \rangle = \langle \mathsf{c}(\nabla u_Y), e_Y \rangle$$

$$= \frac{8\pi}{15}[a^2 + (b - c)^2 + d^2 + c^2] + \frac{8\pi}{3} c_*^2 w^2.$$

This proves (α) and (β) follows from it. ■

4.3 *Remarks.* (1) It is not hard to see that the Betti form is a constant for a homogeneous material (isotropic or not) with a "homogeneous" load of type 3 (*i.e.*, $B_0 = 0$, $\tau_0 = KN$ for some $K \in \operatorname{sym}$). See Theorem 2.13.

(2) A special class of loading of type 3 is given by the non-trivial parallel systems in which the load vectors are a scalar multiple of a fixed vector. For such loads, the Betti form has to be a constant by symmetry. A study of our traction problem in this degenerate case will be given immediately after the next remark.

(3) Combining remarks (1) and (2), one realizes that to get a non-trivial example for homogeneous material with a loading of type 3, one must take a non-homogeneous and non-parallel system of loadings of type 3 (like the one in Example 4.2).

We now examine a special class of loads of type 3, which occur very frequently in the literature.

4.4 Definition. A load l is called a *parallel* system of loads if $l(X) = f(X)\,a$, where $f: \mathscr{B} \to \mathbf{R}$, $0 \neq a \in \mathbf{R}^3$. A parallel system l is said to be *non-trivial* if

$$\bar{f} = \int f(X)\,X\,dV + \int f(X)\,X\,dA \neq 0.$$

4.5 Proposition. *Let l_0 be a equilibrated load, parallel to $a \in \mathbf{R}^3$, $a \neq 0$. Then the load l_0 is non-trivial if and only if it is of type 3.*

Proof. Suppose l_0 is of type 3. Then $k(l_0) = (a_i \bar{f}_j) \neq 0$ and so l_0 must be non-trivial. On the other hand, suppose the equilibrated load l_0 is non-trivial. Then the symmetry of $k(l_0) = (a_i \bar{f}_j)$ implies that $\bar{f}_i = -ca_i$ for some non-zero number c. Therefore, $k(l_0) = (-ca_i a_j) = -ca \otimes a$. The matrix $-c(a \otimes a)$ has eigenvalues $0, 0, -c\|a\|^2$, with eigenvectors p, q, a in which p and q are orthogonal to a. Hence the equilibrated load l_0 is of type 3. ■

For a non-trivial parallel load l_0, there exists a built-in symmetry in our traction problem with $l = l_0$. Without loss of generality, let us take the equilibrated load l_0 parallel to the z-axis. Clearly, the isotropy group of l_0, namely

$$\{Q \in SO(3) \mid Ql_0 = l_0\} = \left\{ \left(\begin{array}{cc|c} x & -y & 0 \\ y & x & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \,\middle|\, x^2 + y^2 = 1 \right\},$$ is the circle group S^1. By the material frame indifference of the stored energy function W, and by the identity $g^{-1} l_0 = l_0$ for $g \in S^1$, the potential function $V(\phi) = \int W(F)\,dV$

$-\langle\lambda l_0, \phi\rangle$ is S^1-invariant (*i.e.* $V(g\phi) = V(\phi)$ for $g \in S^1$). The action $g \cdot (Q, \phi) = (Qg^{-1}, \phi)$, for $g \in SO(3)$, makes the map ϱ equivariant (see § 2). Hence the function $V_\varrho = V \circ \varrho$ is also S^1-invariant under this action.

4.6 Proposition. (a) *The function* $f(Q) = V_\varrho(Q, \phi_Q)$ *on* $SO(3)$ *is* S^1-*invariant under the action* $g \cdot Q = Qg^{-1}$ *for* $g \in S^1$.

(b) S_{A_0} *consists of two* S^1-*orbits* C *and* C^*.

(c) *If an* S^1-*invariant normal bundle is used in the construction of* $\tilde{f}$, *then* $\tilde{f}$ *also becomes* S^1-*invariant on* $C \cup C^*$.

Proof. (a) Since $\phi_{Qg^{-1}} = \phi_Q$ for $g^{-1} \in S^1$, we get

$$f(g \cdot Q) = V_\varrho(Qg^{-1}, \phi_{Qg^{-1}}) = V_\varrho(Qg^{-1}, \phi_Q) = V_\varrho(Q, \phi_Q) = f(Q).$$

(b) Straightforward computations imply that $S_{A_0} = C \cup C^*$, where

$$C = S^1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C^* = S^1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \blacksquare$$

From Section 2, we know that the set of equilibrium solutions for our traction problem is in one-to-one correspondence with the critical points of $\tilde{f}$. By (b), (c) of the proposition above $\tilde{f}$ must be a constant on C and on C^*. Thus every point in $C \cup C^*$ is a critical point of $\tilde{f}$. Therefore we obtain:

4.7 Theorem. *Let* l_0 *be an equilibrated non-trivial parallel load. Then, for* $\lambda > 0$ *small, there exist exactly two circles of equilibrium solutions to our traction problem. One of them is* (neutrally) *stable.*

The theorem above is a global, geometric version of a theorem of STOPPELLI (*cf.* Theorem I, p. 58 in GRIOLI [1962]).

§ 5. Analysis of Loads of Type 4

For loads of type 4, $k(l_0) = 0$, $S_{A_0} = SO(3)$, and $\tilde{f} = -\langle l, Q^T l\rangle - \frac{\lambda}{2}\langle c(\nabla u_Q^0), \nabla u_Q^0\rangle + O(|\lambda|\,\|l - l_0\|) + O(\lambda^2)$. Thus one needs to consider the function $\tilde{h}(Q) = L(Q) + B(Q)$, where $L(Y) = \frac{2}{\lambda}\langle l, Y^T l\rangle$ and $B(Y) = \langle c(\nabla u_Y^0), \nabla u_Y^0\rangle$, the Betti form on M_3. We start our investigation by considering linear and quadratic forms on M_3. It seems plausible that any quadratic form B on M_3 can be the Betti form for some hyperelastic material (*cf.* Corollary 3.2 in § 3). We do not prove this, but we do construct enough Betti forms to obtain sharp bounds on the number of solutions.

The standard double covering $\varrho \mid S^3: S^3 \to SO(3)$ is defined in terms of

a quadratic form ϱ on $\mathbb{R}^4$. This is described as follows: Let

$$\mathbb{H} = \{X = x_0 + ix_1 + jx_2 + kx_3\}, \text{ the quaternions,}$$

and

$$\mathbb{H}_1 = \{X \in \mathbb{H} \mid \|X\|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}, \text{ the unit quaternions.}$$

Identify $\mathbb{H}_1$ with S^3 in $\mathbb{R}^4$ and $\{ix_1 + jx_2 + kx_3 \mid x_1, x_2, x_3 \in \mathbb{R}\}$ with $\mathbb{R}^3$, in an obvious way. To each $X \in \mathbb{H}$, define

$$\varrho(X): \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{by} \quad q \mapsto Xq\bar{X}$$

$$(\bar{X} = x_0 - ix_1 - jx_2 - kx_3 \text{ is the } \textit{conjugate} \text{ of } X).$$

Then $\varrho(X)$ is well defined and $\varrho(X) \in SO(3)$ for $X \in \mathbb{H}_1$. Indeed,

$$x_0 + ix_1 + jx_2 + kx_3 \overset{\varrho}{\mapsto} \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_0x_2 + x_1x_3) \\ (2x_1x_2 + x_0x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_0x_1 + x_2x_3) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}.$$

For a local study of critical points, we use a local chart $\phi: SO(3) \setminus \mathbb{RP}^2 \to \mathbb{R}^3$ (where $\mathbb{RP}^2 = \varrho(S^2) = SO(3) \cap \text{sym} \setminus \{1\}$) so that $\phi^{-1}(x, y, z) = \varrho\left(\frac{1 + ix + jy + kz}{r}\right)$ where $r = \sqrt{1 + x^2 + y^2 + z^2}$. For $Y_0 \in SO(3)$, the linear map $Y \mapsto Y_0 Y$ leaves $SO(3)$, the form of B invariant, and sends the identity to Y_0. Without loss of generality, we can assume that $\phi^{-1}(0) = \text{diag}(1, 1, 1)$ is a typical point of $SO(3)$.

Let

$$\tilde{h}(x, y, z) = \tilde{h}(\phi^{-1}(x, y, z)) = \frac{r^2 L(\varrho(1 + ix + jy + kz)) + B(\varrho(1 + ix + jy + kz))}{r^4} = \frac{\xi(x, y, z)}{r^4}.$$

Hence ξ is a polynomial of degree $\leqq 4$ depending on L and B. Conversely, we have

5.1 Lemma. *Given any polynomial* $\xi(x, y, z)$ *of degree* $\leqq 4$, *there exists a quadratic form* B *on* M_3 *such that*

$$\xi(x, y, z) = B(\varrho(1 + ix + jy + kz)).$$

Proof. We have $\dim\{B \mid B \text{ is a quadratic form on } M_3\} = 45$, and $\dim\{\xi \mid \xi$ is a polynomial in x, y, z of degree $\leqq 4\} = 35$. Now B lies in the kernel of the linear map $B \mapsto \xi$ defined via $\xi(x, y, z) = B(\varrho(1 + ix + jy + kz))$ if and only if $B \mid SO(3) = 0$. Thus it suffices to prove that $\dim\{B \mid B$ is quadratic, and

$B \mid SO(3) = 0\} = 10\,(= 45 - 35)$. In fact, a basis of the kernel can be given explicitly as follows:

$$\sum_i x_{ij} x_{it} (j < t), \quad \sum_i x_{ij} x_{sj} (i < s),$$

$$\sum_i x_{i1}^2 - \sum_i x_{i2}^2, \quad \sum_i x_{i2}^2 - \sum_i x_{i3}^2, \quad \sum_i x_{1i}^2 - \sum_i x_{2i}^2, \text{ and } \sum_i x_{2i}^2 - \sum_i x_{3i}^2,$$

where $B = (x_{ij})$. ■

5.2 Lemma. (a) *The 4th order Taylor expansion $j^{(4)}\tilde{h}(0)$ of $\tilde{h}(x, y, z)$ at 0 can be any polynomial $\eta(x, y, z)$ of degree $\leqq 4$.*

(b) *If $j^{(4)}\tilde{h}(0) = c$, then $\tilde{h} = c$ identically.*

The proof of this lemma is basically the same as that of Lemma 3.4 and so we omit the proof.

Using these two lemmas, we obtain the next proposition, which provides a description of the singularities of $\tilde{h}$ on $SO(3)$.

5.3 Proposition. (a) *If the Betti form can be any quadratic form on M_3 for loads of type 4, then the 4th order Taylor expansion of $\tilde{h}$ at any given point X in $SO(3)$ is arbitrary.*

(b) *If $j^{(4)}\tilde{h}(X) = c$, a constant, then $\tilde{h}(X) = c$ identically on $SO(3)$.*

Recall that here $\tilde{h}(Q) = L(Q) + B(Q) = \frac{2}{\lambda}\langle l, Q^T l\rangle = \langle c(\nabla u_Q^0), \nabla u_Q^0\rangle$.

Now let us consider the global aspects:

Denote by $\tilde{H} = \{\tilde{h} \mid \tilde{h} = L + B\}$ the space of polynomials of degree $\leqq 2$ in M_3, vanishing at the origin. Define $\Sigma = \{\tilde{h} \in \tilde{H} \mid \tilde{h}(Y) = L(Y) + B(Y)$ on $SO(3)$ has a degenerate critical point$\}$. Replacing the double covering $S^2 \to \mathbb{RP}^2$ in the proof of Proposition 3.5 by the double covering $S^3 \to SO(3)$ here, we obtain a proof of the following:

5.4 Proposition. *The set Σ is a semi-algebraic set of codimension $\geqq 1$ in $\tilde{H}$.*

Now we want to estimate the number of critical points for $\tilde{h} = L + B$ not in Σ.

5.5 Example. Consider a hyperelastic material with elasticity tensor $c(e) = e + \mu \operatorname{diag} e$ where $-1 < \mu$, which occupies the unit ball in $\mathbb{R}^3$. Let $l_0 = (B_0, \tau_0)$, with $B_0 = (-1, -1, -1)$, and $\tau_0 = (x^2, y^2, z^2)$. Since $k(l_0) = 0$, this load is of type 4. We claim that

(1) The Betti form $B_{l_0}(Y) = \frac{4\pi}{15}\left\{-2(y_{11}^2 + y_{22}^2 + y_{33}^2) + \left(2 + \frac{1}{1+\mu}\right)\langle Y, Y\rangle\right\}$,

and

(2) $\tilde{h}(Y) = B_{l_0}(Y)$ is a Morse function on $SO(3)$, with 40 critical points.

To prove (1), let $Y = (y_{ij})$, and consider

$$c_Y(X) = \begin{pmatrix} y_{11}x - y_{21}y - y_{31}z & y_{21}x + y_{12}y & y_{31}x + y_{13}z \\ y_{21}x + y_{12}y & -y_{12}x + y_{22}y - y_{32}z & y_{32}y + y_{23}z \\ y_{31}x + y_{13}z & y_{32}y + y_{23}z & -y_{13}x - y_{23}y + y_{33}z \end{pmatrix}.$$

It is easy to see that there exists exactly one displacement field u_Y (which is quadratic) such that $c(\nabla u_Y) = c_Y$.

Since

$$\frac{1}{2}(\nabla u_Y + \nabla u_Y^T) = e_Y = c^{-1}(c(\nabla u_Y))$$

$$= \begin{pmatrix} \frac{y_{11}x - y_{21}y - y_{31}z}{1+\mu} & y_{21}x + y_{12}y & y_{31}x + y_{13}z \\ y_{21}x + y_{12}y & \frac{-y_{12}z + y_{22}y - y_{32}z}{1+\mu} & y_{32}y + y_{23}z \\ y_{31}x + y_{13}z & y_{32}y + y_{23}z & \frac{-y_{13}x - y_{23}y + y_{33}z}{1+\mu} \end{pmatrix},$$

a simple computation shows that

$$\begin{cases} -\mathrm{DIV}\,(c\,\nabla u_Y) = YB_0, \\ c(\nabla u_Y)\cdot N = Y\tau_0. \end{cases}$$

Thus

$$B_{l_0}(Y) = \langle c(\nabla u_Y), \nabla u_Y\rangle = \langle c(\nabla u_Y), e_Y\rangle$$

$$= k\left\{\sum_i \frac{1}{1+\mu} y_{ii}^2 + \sum_{i\neq j}\left(2 + \frac{1}{1+\mu}\right) y_{ij}^2\right\}, \quad \text{where} \quad k = \int x^2\, dV$$

$$= \frac{4\pi}{15}\left\{-2(y_{11}^2 + y_{22}^2 + y_{33}^2) + \left(2 + \frac{1}{1+\mu}\right)\langle Y, Y\rangle\right\}.$$

To prove assertion (2), we use the method of Lagrange to find the critical points of B_{l_0} on $SO(3)$ or equivalently of $B_{l_0} \circ \varrho$ on S^3.

Set

$$L = -\frac{4\pi}{15}\Big\{2[(x_0^2 + x_1^2 - x_2^2 - x_3^2)^2 + (x_0^2 + x_2^2 - x_1^2 - x_3^2)^2$$
$$+ (x_0^2 + x_3^2 - x_1^2 - x_2^2)^2] - \left(2 + \frac{1}{1+\mu}\right)3\Big\} + \lambda(x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1).$$

Thus

$$\begin{cases} L_{x_i} = -\dfrac{32\pi}{15}x_i\left[3x_i^2 - \sum_{j\neq i} x_j^2\right] + 2\lambda x_i = 0, \quad i = 0, 1, 2, 3, \\ \sum_i x_i^2 = 1. \end{cases}$$

It is easy to see that this system has the solutions

$$(x_0, x_1, x_2, x_3) = \frac{1}{\sqrt{\bar{x}_0^2 + \bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2}}(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3),$$

where $\bar{x}_i = 0, 1$ or -1, $i = 0, 1, 2, 3$, except $(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 0, 0)$. Consequently, B_{l_0} on $SO(3)$ has exactly 40 critical points. Further straightforward computations show B_{l_0} is a Morse function, having 8 points $(\varrho(x_0 + ix_1 + jx_2 + kx_3)$, $x_i = \pm\frac{1}{2})$ of index 3. Indeed, replacing λ by μ^2, we see from Bezout's theorem that our system $L_{x_i} = 0, i = 0, \ldots, 3$, has exactly 81 solutions in x_i, μ including multiplicity. Since our system has 81 solutions, the multiplicities have to be 1, so each ray solution is simple.

5.6 Proposition. *The number $c(\bar{h})$ of critical points for a Morse function of the form $\bar{h} = L + B$ on $SO(3)$ (i.e., $\bar{h} \notin \Sigma$) is between 4 and 40.*

The proof of this proposition is basically the same as that of Proposition 3.7. Thus we omit the proof.

Our main result on the global problem is the following:

5.7 Theorem. *Let l_0 be a load of type 4. Suppose the Betti form $B_{l_0}(Y)$ is a Morse function restricted to $SO(3)$. Then, for $\lambda > 0$ and $\dfrac{\|l - l_0\|}{\lambda}$ small, the number of critical points $\bar{f}$ on $SO(3)$ is between 4 and 40. Therefore, our traction problem has between 4 and 40 equilibrium solutions.*

Proof. The function $\left(-\dfrac{2}{\lambda}\right)\bar{f}$ is a small perturbation of the Morse function B_{l_0}, where $4 \leq c(B_{l_0}) \leq 40$ by Proposition 5.6. ∎

Finally, in this section, we analyze our problem for a non-zero parallel system l_0 of type 4 (*i.e.* $k(l_0) = 0$) and $l = l_0$. Without loss of generality, we can assume that l_0 is parallel to the z-axis. Thus the isotropy group of l_0, namely

$$\{Q \in SO(3) \mid Ql_0 = l_0\} \quad \text{is} \quad \left\{\left(\begin{array}{cc|c} x & -y & 0 \\ y & x & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \middle| x^2 + y^2 = 1\right\} = S^1,$$

the circle group.

Clearly, the function $V(\phi)$ is S^1-invariant, *i.e.* $V(\phi) = V(g\phi)$ for all $g \in S^1$. The action $g \cdot (Q, \phi) = (Qg^{-1}, \phi)$, $g \in SO(3)$, makes the map equivariant. Thus the function V_ϱ is S^1-invariant under this action.

5.8 Proposition. (a) *The function $f(Q) = V_\varrho(Q, \phi_Q)$ is S^1-invariant under the action $g \cdot Q = Qg^{-1}$ for $g \in S^1$.*

(b) *There exist at least two invariant circles of critical points of f.*

Proof. (a) Since, $\phi_{Qg^{-1}} = \phi_Q$ for $g^{-1} \in S^1$,

$$f(g \cdot Q) = V_\varrho(Qg^{-1}, \phi_{Qg^{-1}}) = V_\varrho(Qg^{-1}, \phi_Q) = V_\varrho(Q, \phi_Q) = f(Q).$$

(b) From (a), it suffices to say that f has a maximum and a minimum on $SO(3)$. ∎

Remark. The action of S^1 on $SO(3)$ via $g \cdot Q = Qg^{-1}$ is free, and the orbit space $SO(3)/S^1$ is diffeomorphic to S^2 via $[Q] \mapsto Q\begin{pmatrix}0\\0\\1\end{pmatrix}$. The circles of critical points of f on $SO(3)$ correspond to the critical points of an induced function f on $SO(3)/S^1 \approx S^2$. One expects that an example with exactly two circles of solutions for our problem does exist.

From the expression $f(Q) = -\dfrac{\lambda}{2}\langle c(\nabla u_Q), \nabla u_Q\rangle + O(\lambda^2)$ given in §2 (here $u_Q = u_Q^0$, $l = l_0$, and $\langle l_0, Q^T 1\rangle = 0$), one needs to examine the S^1-invariant function $\langle c(\nabla u_Q), \nabla u_Q\rangle$ on $SO(3)$. Notice that here $L(\nabla u_K) = Kl_0$ since $k(Kl_0) = Kk(l_0) = 0$.

5.9 Proposition. *Let l_0 be a non-zero parallel system of type 4, parallel to the z-axis. Then*

(a) $Q \in SO(3)$ *is a critical point of* $\langle c(\nabla u_Q), \nabla u_Q\rangle$ *if and only if*

$$k(Ql_0, u_Q) = \int c(\nabla u_Q)\nabla u_Q^T\, dV \in \text{sym}.$$

(b) *The Hessian of* $\langle c(\nabla u_Q), \nabla u_Q\rangle$ *is given by*

$$\frac{1}{2}\mathscr{H}(WQ) = \langle c(\nabla u_{W^2Q}), \nabla u_Q\rangle + \langle c(\nabla u_{WQ}), \nabla u_{WQ}\rangle$$

or

$$\frac{1}{2}\mathscr{H}(QW) = \langle c(\nabla u_{QW^2}), \nabla u_Q\rangle + \langle c(\nabla u_{QW}), \nabla u_{QW}\rangle.$$

Proof. For $W \in$ skew,

$$\langle c(\nabla u_{e^{Wt}Q}), \nabla u_{e^{Wt}Q}\rangle = \langle c(\nabla u_Q), \nabla u_Q\rangle + 2\langle c(\nabla u_{WQ}), \nabla u_Q\rangle t + \{\langle c(\nabla u_{W^2Q}), \nabla u_Q\rangle + \langle c(\nabla u_{WQ}), \nabla u_{WQ}\rangle\} t^2 + O(t^3).$$

Since $\langle c(\nabla u_{WQ}), \nabla u_Q\rangle = \langle WQl_0, u_Q\rangle = -\langle W, k(Ql_0, u_Q)\rangle$, Q is a critical point if and only if $k(Ql_0, u_Q) \in$ sym. That $k(Ql_0, u_Q) = \int c(\nabla u_Q)\, \nabla u_Q^T\, dV$ follows from the divergence theorem as usual. ■

Let $S^1 \cdot Q$ be a circle of critical points. Then $\mathscr{H}\left(Q\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right) = 0$, and the nullity of $\mathscr{H}$ is $\geqq 1$. The Hessian $\mathscr{H}$ is said to be *non-degenerate* if the nullity of $\mathscr{H}$ is 1.

5.10 Theorem. *Let l_0 be a non-zero parallel system of type 4 (parallel to the z-axis). Suppose that $S^1 \cdot Q$ is a non-degenerate circle of critical points of $\langle c(\nabla u_Q), \nabla u_Q\rangle$ on $SO(3)$. Then for small $\lambda > 0$, the traction problem $\Phi(\phi) = \lambda l_0$ has a circle of solutions $S^1\phi$ near $S^1 \cdot Q$.*

Proof. It suffices to observe that

$$\frac{f(Q)}{\lambda} = -\frac{1}{2}\langle c(\nabla u_Q), \nabla u_Q\rangle + O(\lambda), \quad \text{for} \quad \lambda > 0,$$

and to use elementary results in equivariant differential topology. ■

5.11 Example. Consider a homogeneous hyperelastic material with elastic tensor $c(e) = e - \frac{1}{2}\operatorname{diag} e$, and with reference configuration $\mathscr{B}$ the unit ball in $\mathbb{R}^3$. Let $l_0 = (B_0, \tau_0)$ be the parallel load with $B_0 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ on $\mathscr{B}$, and $\tau_0 = \begin{pmatrix}0\\0\\z^2\end{pmatrix}$ on $\partial\mathscr{B}$. Clearly, $k(l_0) = 0$. We claim that (1) the circles of critical points of $\langle c(\nabla u_Q), \nabla u_Q\rangle$ on $SO(3)$ in the orbit space $SO/S^1 \approx S^2$ correspond to the north pole $S^1 \cdot \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, the south pole $S^1 \cdot \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$, and the equator; (2) the invariant circles $S^1 \cdot \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $S^1 \cdot \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$ are non-degenerate, with

Hessians of index 0. Therefore, for small $\lambda > 0$, the traction problem has solutions $S^1\phi$ with ϕ near $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$, and $\left\{\begin{pmatrix}0 & s & -t\\ 0 & t & s\\ 1 & 0 & 0\end{pmatrix}\middle| s^2 + t^2 = 1\right\}$. (There are at least two of the last form.)

Proof. (1) To each $Y = (y_{ij})$, set

$$c_Y(X) = \begin{pmatrix}0 & 0 & y_{13}z\\ 0 & 0 & y_{23}z\\ y_{13}z & y_{23}z & h\end{pmatrix}, \text{ where } h = -y_{13}x - y_{23}y + y_{33}z.$$

Then there exists exactly one (indeed, quadratic) displacement field u_Y, such that $c(\nabla u_Y) = c_Y$. Clearly

$$\begin{cases}-\operatorname{DIV} c(\nabla u)_Y = YB_0,\\ c(\nabla u_Y)\cdot N = Y\tau_0,\end{cases}$$

and

$$\nabla u_Y = c^{-1}(c(\nabla u_Y)) = c(\nabla u_Y) + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & h\end{pmatrix}.$$

Note that if $Q = (g_{ij})$, then

$$\int c(\nabla u_Q)\, \nabla u_Q^T\, dV = \int c_Q^2\, dV + \int c\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & h\end{pmatrix} dV$$

$$= \int c_Q^2\, dV + \begin{pmatrix}0 & 0 & g_{13}g_{33}\int z^2\, dV\\ 0 & 0 & g_{23}g_{33}\int z^2\, dV\\ 0 & 0 & \int h^2\, dV\end{pmatrix}.$$

Thus $\int c(\nabla u_Q)\, \nabla u_Q^T\, dV \in$ sym if and only if (a) $g_{13} = g_{23} = 0$, $g_{33} = \pm 1$ or (b) $g_{33} = 0$.

(2) Direct computations, using the formula

$$\frac{1}{2}\mathscr{H}(QW) = \langle c(\nabla u_{QW^2}), \nabla u_Q\rangle + \langle c(\nabla u_{QW}), \nabla u_{QW}\rangle,$$

give

$$\frac{1}{2}\mathscr{K}(QW) = \frac{4\pi}{15}(2a^2 + 2b^2) \quad \text{where} \quad W = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & a \\ b & -a & 0 \end{pmatrix},$$

and $Q = I$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. ■

J. E. MARSDEN's research was supported in part by the U.S. National Science Foundation under Grant MCS-81-07086, by the Miller Institute, and by a contract from the Department of Energy, DE-AT03-82ER12097. Y. H. WAN's research was partially supported by the U.S. National Science Foundation under Grant MCS-81-02463 and the Department of Energy, Contract DE-AT03-82ER12097. D. R. J. CHILLINGWORTH's research was partially supported by the U.K. Science Research Council through the University of Warwick, 1980.

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(Received December 15, 1982)