ARCH. RAT. MECH & AN
80 (1982) 295-331

SYMM & BIF
3D ELAST.

Symmetry and Bifurcation in Three-Dimensional Elasticity, Part I

D. R. J. CHILLINGWORTH, J. E. MARSDEN & Y. H. WAN

Communicated by S. ANTMAN & M. GURTIN

Dedicated to CLIFFORD TRUESDELL

Contents

§ 1. Introduction 296
§ 2. Statement of the problem 297
§ 3. The astatic load and axes of equilibrium 300
§ 4. Equivalent reformulations of the problem 304
§ 5. Loads of type 0, having no axis of equilibrium 307
§ 6. Classification of orbits in M_3 311
§ 7. The bifurcation equation and its gradient character 317
§ 8. Bifurcation analysis of problems of type 1 322

Glossary of Notation

$\mathscr{B} \subset \mathbb{R}^3$	reference configuration
$T_X\mathscr{B}$	vectors in $\mathbb{R}^3$ based at the point $X \in \mathscr{B}$
$\phi: \mathscr{B} \to \mathbb{R}^3$, $x = \phi(X)$	deformation
$u: \mathscr{B} \to \mathbb{R}^3$	displacement for the linearized theory
$e = \frac{1}{2}[\nabla u + (\nabla u)^T]$	strain
$\mathscr{C}$	all deformations ϕ
$F = D\phi$	deformation gradient = derivative of ϕ
F^T	transpose of F
$C = F^T F$	Cauchy-Green tensor
W	stored energy function
$P = \dfrac{\partial W}{\partial F}$	first Piola-Kirchhoff stress
$S = 2\dfrac{\partial W}{\partial C}$	second Piola-Kirchhoff stress
$\mathbf{A} = \dfrac{\partial P}{\partial F}$	elasticity tensor

$\mathsf{C} = \dfrac{\partial S}{\partial C}$	(second) elasticity tensor
$\mathsf{c} = 2\mathsf{C}\vert_{\phi = I_{\mathscr{B}}}$	classical elasticity tensor
I or $I_{\mathscr{B}}$	identity map on $\mathbb{R}^3$ or $\mathscr{B}$
$l = (B, \tau)$	a (dead) load
$\mathscr{L}$	all loads with total force zero
$L(T_X\mathscr{B}, \mathbb{R}^3)$	all linear maps of $T_X\mathscr{B}$ to $\mathbb{R}^3$
$L(T_X\mathscr{B}, \mathbb{R})^*$	linear maps of $L(T_X\mathscr{B}, \mathbb{R})$ to $\mathbb{R}$
$\mathrm{sym}\,(T_X\mathscr{B}, T_X\mathscr{B})$	symmetric linear maps of $T\mathscr{B}_X$ to $T\mathscr{B}_X$
$SO(3)$	$\{Q \in L(\mathbb{R}^3, \mathbb{R}^3) \mid Q^T Q = I, \det Q = 1\}$
$\mathbb{RP}^2$	real projective 2-space; lines through $(0, 0, 0)$ in $\mathbb{R}^3$
M_3	$L(\mathbb{R}^3, \mathbb{R}^3)$
sym	symmetric elements of M_3
skew $= so(3)$	skew symmetric elements of M_3
$\hat{v}$	infinitesimal rotation about the axis v
$\mathscr{L}_e$	equilibrated loads
$k: \mathscr{L} \to M_3$	astatic load map
$A = k(l)$	astatic load for a load l
$j = (k \mid (\ker k)^{\perp})^{-1}$	non-singular part of k
Skew $= j\,$(skew)	skew viewed in load space
Sym $= j\,$(sym)	sym viewed in load space
$\Phi: \mathscr{C} \to \mathscr{L}$	$\Phi(\phi) = (-\mathrm{DIV}\, P, P \cdot N)$
$\mathscr{C}_{\mathrm{sym}}$	$\{u: \mathscr{B} \to \mathbb{R}^3 \mid u(0) = 0, \nabla u(0) \in \mathrm{sym}\}$
$\mathscr{N}$	image of $\mathscr{C}_{\mathrm{sym}}$ near $I_{\mathscr{B}}$ under Φ
$F: \mathscr{L}_e \to$ Skew	$\mathscr{N}$ is the graph of F
$\overline{F}: \mathbb{R} \times \mathscr{L}_e \to$ Skew	$\overline{F}(\lambda, l) = F(\lambda l)/\lambda^2$
S_A	Q's in $SO(3)$ that equilibrate A

§ 1. Introduction

This paper is the first of a series of three devoted to the study of the traction problem in three-dimensional nonlinear elasticity by means of geometric techniques and singularity theory. The first two papers in the series treat the traction problem with dead loads for configurations that are nearly stress-free. As was shown by SIGNORINI [1930] and STOPPELLI [1958], this problem has nontrivial solutions. However, their analysis is incomplete for three reasons. First, their load is varied only by a scalar factor; in a *full* neighborhood in load space of a load that has an axis of equilibrium there are additional solutions missed by their analysis. Second, their analysis is only local in the rotation group, so additional nearly stress-free solutions are missed by restricting the rotations to those near the identity. Third, some classes of loads with a degenerate axis of equilibrium are not considered. This series of papers completes their analysis by treating these questions as well as the stability of solutions. The complexity of the answer is indicated by the fact that near certain types of loads, we find up to 40 distinct solutions that are nearly stress-free. Our constitutive hypotheses on the stress tensor are "generic"; for a degenerate stress tensor there can be even more solutions.

The literature on this problem is very extensive, going back to SIGNORINI in the 1930's. Our primary sources have been STOPPELLI [1958], GRIOLI [1962], TRUESDELL & NOLL [1965], VAN BUREN [1968], WANG & TRUESDELL [1973], and CAPRIZ & PODIO GUIDUGLI [1974]. However, none of the references beyond STOPPELLI [1958] gives complete proofs of any of the theorems dealing with nontrivial cases: *i.e.*, loads with axes of equilibrium. However, GRIOLI [1962] is a convenient reference for the statements.

The outline of this paper is as follows. We establish our notation for nonlinear elasticity and we formulate our problem near a natural state in Section 2. In Section 3 the basic properties of the astatic load are reviewed and developed. The problem is reformulated with special reference to the global aspects of the rotation group in Section 4. Here we introduce the bifurcation equation, which plays a crucial role throughout the paper. Our treatment of $SO(3)$ is invariant; if we used Euler angles to parametrize it, unnecessary analytical singularities would result. Section 5 treats loads with no axis of equilibrium; there are three new features in this section. First, the proof of STOPPELLI's results is considerably simplified. Second, the results are global relative to the rotation group. Finally, the stability of the solutions is determined. The number of solutions is classified by load types; this classification scheme is explained in Section 6. (Some work related to the "type classification" was given by OGDEN [1977].) In Section 7 a second-order bifurcation equation is shown to be a gradient. This consequence of Betti reciprocity is basic to our analysis. Section 8 gives a complete bifurcation analysis of loads of type 1 (the case considered by STOPPELLI), including a stability analysis. New local and global solutions are found. The final section makes explicit the comparison with STOPPELLI's theorem.

In Part II of this series we shall analyze the remaining types of loads, using a reformulation of our gradient results. We shall also discuss linearization stability and parallel loads. In Part III we shall investigate general loads and stressed initial configurations.

Acknowledgments. The Signorini-Stoppelli problem was introduced to us by ROBIN KNOPS in 1977. Since then a number of other people have made useful comments, including STUART ANTMAN, JOHN BALL, ROGER BROCKETT, MARTIN GOLUBITSKY, DAVID SCHAEFFER, MORTON GURTIN, and CLIFFORD TRUESDELL.

§ 2. Statement of the Problem

Let $\mathscr{B} \subset \mathbb{R}^3$ be an open bounded set with smooth boundary* and assume for convenience that $0 \in \mathscr{B}$. Let $1 < p < \infty$, $s > (3/p) + 1$ and let $\mathscr{C}$ be the space of maps $\phi: \overline{\mathscr{B}} \to \mathbb{R}^3$ that are of class $W^{s,p}$ (so they are C^1) such that $\phi(0) = 0$,

* We believe that our results also hold when $\mathscr{B}$ has piecewise smooth boundary. This program depends on elliptic regularity for such regions. Except in special cases, this theory is non-existent and seems to depend on a modification of the usual Sobolev spaces near corners. However, for simple shapes like cubes, the necessary regularity can be checked by hand in situations where the linearized elastostatic equations can be solved explicitly.

ϕ has a $W^{s,p}$-inverse on its image, and $J(\phi) > 0$, where $J(\phi)$ is the Jacobian of ϕ.[†]

For example, if $\psi\colon \bar{\mathscr{B}} \to \mathbb{R}^3$ is close to the identity in $W^{s,p}$ and if $\psi(0) = 0$, then $\psi \in \mathscr{C}$. If Q is a linear isomorphism of $\mathbb{R}^3$ to $\mathbb{R}^3$ with $\det Q > 0$, then $Q \circ \psi \in \mathscr{C}$ as well.

Let points in $\mathscr{B}$ be denoted X and points in $\mathbb{R}^3$ be denoted x. The vector from the origin to X is denoted $\mathbf{X}$. Sometimes we write $x = \phi(X)$. Let $T_X\mathscr{B}$ be the tangent space to $\mathscr{B}$ at X, regarded as vectors in $\mathbb{R}^3$ based at X. We do not identify $T_X\mathscr{B}$ and $\mathbb{R}^3$ for conceptual clarity. For $\phi \in \mathscr{C}$, let $F(X) \in L(T_X\mathscr{B}, \mathbb{R}^3)$ be the derivative of ϕ at X; by standard abuse of notation we write $F(X) = D\phi(X)$ or $\nabla\phi(X)$ interchangeably. $L(T_X\mathscr{B}, \mathbb{R}^3)$ denotes the set of all linear maps of $T_X\mathscr{B}$ to $\mathbb{R}^3$. We let $F(X)^T \in L(\mathbb{R}^3, T_X\mathscr{B})$ denote the adjoint of $F(X)$ relative to the Euclidean inner product. Observe that $F(X) \in L^+(T_X\mathscr{B}, \mathbb{R}^3)$, the linear transformations with positive determinant, since $\det F(X) = J(\phi)(X) > 0$. We let $C = F^T F$ (that is, $C(X) = F(X)^T F(X) \in L(T_X\mathscr{B}, T_X\mathscr{B})$) denote the Cauchy-Green tensor. Observe that $C(X) \in \mathrm{sym}_{\mathrm{pos}}(T_X\mathscr{B}, T_X\mathscr{B})$, the positive definite symmetric linear transformations on $T_X\mathscr{B}$.

Assume we are given a *smooth stored energy function* W defined on pairs (X, C) where $C \in \mathrm{sym}_{\mathrm{pos}}(T_X\mathscr{B}, T_X\mathscr{B})$. For $\phi \in \mathscr{C}$, the *stored energy of* ϕ is $\int_{\mathscr{B}} W(X, C(X))\, dV(X)$, where C is the Cauchy-Green tensor of ϕ and dV is the volume element in $\mathscr{B}$. That W depends on C rather than on all of F is a consequence of the Principle of Material Frame-Indifference. (See TRUESDELL & NOLL [1965].) Since C is a function of F, we shall abuse notation by writing $W(\phi)$ and $W(X, F)$ for $W(X, F^T F)$.

The first *Piola-Kirchhoff stress tensor* $P(X, F)$ is defined by $P(X, F) = \frac{\partial}{\partial F} W(X, F)$, the partial derivative of W with respect to F. Thus, $P(X, F) \in L(T_X\mathscr{B}, \mathbb{R}^3)^*$. The *second Piola-Kirchhoff stress tensor* $S(X, C)$ is defined by $S(X, C) = \frac{\partial}{\partial C} W(X, C)$, so that $S(X, C) \in \mathrm{sym}\,(T_X\mathscr{B}, T_X\mathscr{B})^*$. The chain rule implies that

$$P(X, F) \cdot G = \frac{1}{2} S(X, C) \cdot [F^T G + G^T F]$$

for all $G \in L(T_X\mathscr{B}, \mathbb{R}^3)$.[††]

For finite dimensional inner product spaces V, W, the inner product $\langle A, B\rangle = \mathrm{trace}\,(A^T B)$ on $L(V, W)$ and $L(V, V)$ defines isomorphisms $L(V, W)^* \cong L(V, W)$ and $L(V, V)^* \cong L(V, V)$. The latter isomorphism also identifies $\mathrm{sym}\,(V, V)^*$ with $\mathrm{sym}\,(V, V)$. Using these identifications we get $P(X, F) \in L(T_X\mathscr{B}, \mathbb{R}^3)$, $S(X, C) \in \mathrm{sym}\,(T_X\mathscr{B}, T_X\mathscr{B})$ and $P(X, F) = F \cdot S(X, C)$, or $P = FS$ for short.

[†] There is a $W^{s,p}$ inverse function theorem: if φ is in $W^{s,p}$, $s > (n/p) + 1$, and has a C^1 inverse, then the inverse is in $W^{s,p}$. For our analysis one can also use the Hölder spaces $C^{k+\alpha}$.

[††] The mass density does not appear in our formulas as we are building it into the definitions and use, for example, the body force per unit volume rather than per unit mass.

Let $\mathsf{A}(X, F) = \frac{\partial P}{\partial F}(X, F) \in L(L(T_X\mathscr{B}, \mathbb{R}^3), L(T_X\mathscr{B}, \mathbb{R}^3))$ denote the *elasticity tensor*. We may regard A as a bilinear form on $L(T_X\mathscr{B}, \mathbb{R}^3)$ via $\mathsf{A}(X, F)(G, H) = \langle \mathsf{A}(X, F)\, G, H\rangle$. In the hyperelastic case, which is our concern, $\mathsf{A} = \partial^2 W/\partial F\, \partial F$, so this bilinear form is symmetric in G and H.

The *second elasticity tensor* $\mathsf{C}(X, C)$ is similarly defined to be $\frac{\partial S}{\partial C} = 2\frac{\partial^2 W}{\partial C\, \partial C}$ evaluated at (X, C), and so may be regarded as a symmetric bilinear map on $\mathrm{sym}\,(T_X\mathscr{B}, T_X\mathscr{B})$. The chain rule gives

$$2\mathsf{A}(X, F) \cdot (G, H)$$
$$= \mathsf{C}(X, C) \cdot (F^T H + H^T F, F^T G + G^T F) + S(X, C) \cdot (H^T G + G^T H).$$

The following two assumptions will be made in the first two papers of this series:

(H1) *The undeformed state is stress-free*; i.e., $P(X, I) = 0$, *or equivalently*, $S(X, I) = 0$, *where* I *is the identity.*

(H2) *Strong ellipticity holds: there is an* $\varepsilon > 0$ *such that*

$$\mathsf{A}(X, I) \cdot (v \otimes \xi, v \otimes \xi) \geqq \varepsilon\, \|\xi\|^2\, \|v\|^2$$

for all $\xi \in T_X\mathscr{B}^*$ *and* $v \in \mathbb{R}^3$, where $v \otimes \xi \in L(T_X\mathscr{B}, \mathbb{R}^3)$ is defined by $(v \otimes \xi)(V) = \xi(V)\, v$.

The *classical elasticity tensor* is defined by $\mathfrak{c}(X) = 2\mathsf{C}(X, I)$, so $\mathfrak{c}(X)$ is a symmetric bilinear mapping on $\mathrm{sym}\,(T_X\mathscr{B}, T_X\mathscr{B})$ to $\mathbb{R}$; at $\phi = I_{\mathscr{B}}$ we identify $T_X\mathscr{B}$ and $\mathbb{R}^3$ since x and X coincide. By (H1),

$$\mathsf{A}(X, I) \cdot (G, H) = \frac{1}{4} \mathfrak{c}(X) \cdot (G + G^T, H + H^T).$$

If we regard $\mathsf{A}(X, I)$ as belonging to $L(L(T_X\mathscr{B}, T_X\mathscr{B}), L(T_X\mathscr{B}, T_X\mathscr{B}))$ and $\mathfrak{c}(X)$ as belonging to $L(\mathrm{sym}(T_X\mathscr{B}, T_X\mathscr{B}), \mathrm{sym}\,(T_X\mathscr{B}, T_X\mathscr{B}))$, this last equation reduces to

$$2\mathsf{A}(X, I) \cdot G = \mathfrak{c}(X) \cdot (G + G^T),$$

or, if G is symmetric, to

$$\mathsf{A}(X, I) \cdot G = \mathfrak{c}(X)\,(G).$$

By (H2), solvability of the linearized equations of elastostatics can be determined by the Fredholm alternative (see, *e.g.*, MARSDEN & HUGHES [1978]).

We shall let $B\colon \mathscr{B} \to \mathbb{R}^3$ denote a given body force (per unit volume) and $\tau\colon \partial\mathscr{B} \to \mathbb{R}^3$ a given surface traction (per unit area). These are dead loads; in other words, the equilibrium equations for ϕ that we are studying are:

$$\text{(E)} \qquad \begin{cases} \mathrm{DIV}\, P(X, F(X)) + B(X) = 0 & \text{for } X \in \mathscr{B} \\ P(X, F(X)) \cdot N(X) = \tau(X) & \text{for } X \in \partial\mathscr{B} \end{cases}$$

where $N(X)$ is the outward unit normal to $\partial\mathscr{B}$ at $X\in\partial\mathscr{B}$ and DIV P is the divergence† of $P(X, F(X))$ with respect to X.

Let $\mathscr{L}$ denote the space of all pairs $(B, \tau) = l$ of loads (with B of class $W^{s-2,r}$ on $\mathscr{B}$ and with τ of class $W^{s-1-1/p,p}$ on $\partial\mathscr{B}$) with zero resultant:

$$\int_{\mathscr{B}} B(X)\,dV(X) + \int_{\partial\mathscr{B}} \tau(X)\,dA(X) = 0.$$

Here dV and dA are the volume and area elements on $\mathscr{B}$ and $\partial\mathscr{B}$. Using the divergence theorem, observe that if the pair (B, τ) is such that (E) holds for some $\phi\in\mathscr{C}$, then $(B, \tau)\in\mathscr{L}$.

Throughout the paper, the group $SO(3) = \{Q\in L(\mathbb{R}^3, \mathbb{R}^3) \mid Q^TQ = I$ and $\det Q = +1\}$ of proper orthogonal transformations will play a key role.

By (H1), $\phi = I_{\mathscr{B}}$ (the identity map on $\mathscr{B}$) is a solution of (E) with $B = \tau = 0$. By the principle of material frame-indifference, $\phi = Q \mid \mathscr{B}$ is also a solution for any $Q\in SO(3)$. The map $Q\mapsto Q\mid\mathscr{B}$ embeds $SO(3)$ into $\mathscr{C}$; we shall identify its image with $SO(3)$. Thus, the "trivial" solutions of (E) are elements of $SO(3)$.

Our basic problem is:

(P1) *Describe the set of all solutions of* (E) *near the trivial solutions* $SO(3)$ *for various loads* $l\in\mathscr{L}$ *near zero.*

Such a description is to include the counting of solutions, the determination of their stability, and the demonstration that the results are insensitive to small perturbations of the stored energy function and of the load.

§ 3. The Astatic Load and Axes of Equilibrium

This section is devoted to the geometry of the load space $\mathscr{L}$. Many of the results of this section are available in the literature, but we gather them here for convenience.

Before beginning, we shall recall a few notations and facts about the rotation group $SO(3)$. Let

$M_3 = L(\mathbb{R}^3, \mathbb{R}^3)$ = the space of linear transformations of $\mathbb{R}^3$ to $\mathbb{R}^3$,

$$\text{sym} = \{A\in M_3 \mid A^T = A\},$$

$$\text{skew} = \{A\in M_3 \mid A^T = -A\}.$$

We identify skew with $so(3)$, the Lie algebra of $SO(3)$; skew and $\mathbb{R}^3$ are isomorphic by the mapping $v\in\mathbb{R}^3\mapsto\hat{v}\in$ skew, where $\hat{v}(w) = w\times v$. If $v = (p, q, r)$ relative to the standard basis, then

$$\hat{v} = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}.$$

† Recall that $P(X, F(X))\in L(T_X\mathscr{B}, \mathbb{R}^3)$. For any $v\in\mathbb{R}^3$, $P(X, F(X))^T v$ defines a vector field P^Tv on $\mathscr{B}$. Its divergence defines DIV P by $(\text{DIV } P)\cdot v = \text{DIV }(P^Tv)$.

The Lie bracket is $[\hat{v}, \hat{w}] = v\otimes w - w\otimes v = (v\times w)^{\wedge}$ where $v\otimes w\in M_3$ is given by $(v\otimes w)(u) = v\langle w, u\rangle$. The inner product is $\langle v, w\rangle = \frac{1}{2}\,\text{trace}\,(\hat{v}^T\hat{w})$, the Killing form on $so(3)$. Finally, $\exp(\hat{v})$ is the rotation about the vector v in the positive sense through the angle $\|v\|$.

Now we turn to a study of $\mathscr{L}$. For $\phi\in\mathscr{C}$ and $l\in\mathscr{L}$, we say that l is *equilibrated relative to* ϕ if the total torque in the configuration ϕ vanishes:

$$\int_{\mathscr{B}} \phi(X)\times B(X)\,dV(X) + \int_{\partial\mathscr{B}} \phi(X)\times\tau(X)\,dA(X) = 0$$

where $l = (B, \tau)$ and $\phi(X)$ is the *vector* from the origin to the *point* $\phi(X)$. From the symmetry of the stress tensor S, one sees that if $l = (B, \tau)\in\mathscr{L}$ satisfies (E) for some $\phi\in\mathscr{C}$, then l is equilibrated relative to ϕ. (An easy proof uses the Piola transform; *cf.* MARSDEN & HUGHES [1978].)

Let $\mathscr{L}_e$ denote the loads that are equilibrated relative to the identity configuration $I_{\mathscr{B}}$.

Define the *astatic load map* $k: \mathscr{L}\times\mathscr{C}\to M_3$ by

$$k(l, \phi) = \int_{\mathscr{B}} B(X)\otimes\phi(X)\,dV(X) + \int_{\partial\mathscr{B}} \tau(X)\otimes\phi(X)\,dA(X)$$

and write $k(l) = k(l, I_{\mathscr{B}})$.

We have actions of $SO(3)$ on $\mathscr{L}$ and $\mathscr{C}$ given by:

Action of $SO(3)$ *on* $\mathscr{L}$: $Ql(X) = (QB(X), Q\tau(X))$,
Action of $SO(3)$ *on* $\mathscr{C}$: $Q\phi = Q\circ\phi$.

Note that Ql means "the load arrows are rotated, keeping the body fixed." We shall write $\mathcal{O}_l$ and $\mathcal{O}_\phi$ for the $SO(3)$ orbits of l and ϕ. Thus, $\mathcal{O}_{I_{\mathscr{B}}}$ denotes the trivial solutions corresponding to $l = 0$.

The following is a list of basic observations about the astatic load, each of which is readily verified:

(A1) *l is equilibrated relative to ϕ if and only if* $k(l, \phi)\in$ sym. *In particular, $l\in\mathscr{L}_e$ if and only if* $k(l)\in$ sym.

(A2) (*equivariance*). *For* $l\in\mathscr{L}$, $\phi\in\mathscr{C}$, *and* $Q_1, Q_2\in SO(3)$,

$$k(Q_1 l, Q_2\phi) = Q_1 k(l, \phi)\,Q_2^{-1}.$$

In particular, $k(Ql) = Qk(l)$

(A3) (*infinitesimal equivariance*). *For* $l\in\mathscr{L}$, $\phi\in\mathscr{C}$, *and* $W_1, W_2\in$ skew,

$$k(W_1 l, \phi) = W_1 k(l, \phi,) \quad\text{and}\quad k(l, W_2\phi) = -k(l, \phi)\,W_2.$$

In particular, $k(Wl) = Wk(l)$.

Later on we shall be concerned with how the orbit of $l\in\mathscr{L}$ meets $\mathscr{L}_e$. The most basic result in this direction is the following.

3.1. Da Silva's Theorem. *Let* $l\in\mathscr{L}$. *Then* $\mathcal{O}_l\cap\mathscr{L}_e \neq 0$.

Proof. By the polar decomposition theorem, we can write $k(l) = Q^T A$ for some $Q \in SO(3)$ and $A \in \mathrm{sym}$. By (A2), $k(Ql) = Qk(l) = A \in \mathrm{sym}$, so by (A1), $Ql \in \mathscr{L}_e$. ■

Similarly, any load can be equilibrated relative to any chosen configuration by a suitable rotation.

The concept of an axis of equilibrium deals with the case in which $\mathcal{O}_l$ meets $\mathscr{L}_e$ in a degenerate way.

3.2. Definition. Let $l \in \mathscr{L}_e$ and $v \in \mathbb{R}^3$, $\|v\| = 1$. Such a v is an *axis of equilibrium for l* when $\exp(\theta\hat{v})\, l \in \mathscr{L}_e$ for all real θ, *i.e.*, when rotations of l about the axis v do not destroy equilibration relative to the identity.

A number of useful ways of reformulating the condition that v be an axis of equilibrium are as follows.

3.3. Proposition. *Let $l \in \mathscr{L}_e$ and $A = k(l) \in$ sym. The following conditions are equivalent:*

1. *l has an axis of equilibrium v;*
2. *there is a $v \in \mathbb{R}^3$, $\|v\| = 1$ such that $\hat{v}l \in \mathscr{L}_e$;*
3. *$W \mapsto AW + WA$ fails to be an isomorphism of* skew *to itself;*
4. trace *A is an eigenvalue of A.*

Proof. $1 \Rightarrow 2$. This follows by differentiating $\exp(\theta\hat{v})\, l$ in θ at $\theta = 0$.

$2 \Rightarrow 1$. By (A2),

$$k(\exp(\theta\hat{v})\, l) = \left[I + (\theta\hat{v}) + \frac{1}{2}(\theta\hat{v})^2 + \dots\right] k(l).$$

Since $k(\hat{v}l) = \hat{v}k(l)$ is symmetric, each term on the right-hand side of this last equation is symmetric.

$2 \Rightarrow 3$. Since $k(\hat{v}l) = \hat{v}A$ is symmetric, $\hat{v}A + A\hat{v} = 0$, so $W \mapsto AW + WA$ is not an isomorphism.

$3 \Rightarrow 2$. There exists a $v \in \mathbb{R}^3$, $\|v\| = 1$, such that $\hat{v}A + A\hat{v} = 0$, so $k(\hat{v}l) = \hat{v}A$ is symmetric.

$3 \Leftrightarrow 4$. Define $L \in M_3$ by $L = (\mathrm{trace}\, A)\, I - A$. Then

$$(Lv)^\wedge = A\hat{v} + \hat{v}A.$$

(In fact, if $[u, v, w]$ denotes the scalar triple product, the relationship $[Bu, Bv, Bw] = (\det B)[u, v, w]$ gives $[Au, v, w] + [u, Av, w] + [u, v, Aw] = (\mathrm{trace}\, A)[u, v, w]$. This yields $(Lv)^\wedge = \hat{v}A + A^T\hat{v}$, which gives the claimed results for symmetric A). Therefore, $A\hat{v} + \hat{v}A = 0$ if and only if $Lv = 0$, *i.e.*, if and only if v is an eigenvector of A with eigenvalue trace A. ■

3.4. Corollary. *Let $l \in \mathscr{L}_e$ and $A = k(l) \in$ sym. Let the eigenvalues of A be denoted a, b, c. Then l has no axis of equilibrium if and only if $(a + b)(a + c)(b + c) \neq 0$.*

Proof. This condition is equivalent to saying that trace A is not an eigenvalue of A. ■

3.5. Definition. $l \in \mathscr{L}_e$ is said to be a load of *type* 0 if l has no axis of equilibrium and if the eigenvalues of $A = k(l)$ are distinct.

The following proposition shows how the orbits of type 0 loads meet $\mathscr{L}_e$.

3.6. Proposition. *Let $l \in \mathscr{L}_e$ be a type 0 load. Then $\mathcal{O}_l \cap \mathscr{L}_e$ consists of exactly four loads of type 0.*

Proof. We first prove that the $SO(3)$-orbit of A in M_3 under the action $Q \mapsto QA$ meets sym at four points. The matrix of A relative to its basis of eigenvectors is diag (a, b, c). Then $\mathcal{O}_A \cap \mathrm{sym}$ contains the four points

$$\begin{array}{ll} \mathrm{diag}\,(a, b, c) & (Q = I) \\ \mathrm{diag}\,(-a, -b, c) & (Q = \mathrm{diag}\,(-1, -1, 1)) \\ \mathrm{diag}\,(-a, b, -c) & (Q = \mathrm{diag}\,(-1, 1, -1)) \\ \mathrm{diag}\,(a, -b, -c) & (Q = \mathrm{diag}\,(1, -1, -1)) \end{array}$$

These are distinct points since $(a + b)(a + c)(b + c) \neq 0$. Now suppose a, b and c are distinct. Suppose $QA = S \in \mathrm{sym}$. Then $S^2 = A^2$. Let μ_i be an eigenvalue of S with eigenvector u_i. Then $S^2u_i = \mu_i^2u_i = A^2u_i$, so μ_i^2 is an eigenvalue of A^2. Thus, as the eigenspace of A^2 with a given eigenvalue has dimension 1, u_i is an eigenvector of A and $\pm\mu_i$ is the corresponding eigenvalue. Since $\det Q = +1$, it follows that $\det S = \det A$, and we must have one of the four cases above.

By equivariance, $k(\mathcal{O}_l) \cap \mathrm{sym} = \mathcal{O}_{k(l)} \cap \mathrm{sym}$ is a set consisting of four points. Now $\mathcal{O}_l \cap \mathscr{L}_e = k^{-1}(\mathcal{O}_{k(l)} \cap \mathrm{sym})$, so it suffices to show that k is one-to-one on $\mathcal{O}_l$. This is a consequence of the following lemma and (A2).

3.7. Lemma. *Suppose that $A \in$ sym and that* $\dim \ker A \leqq 1$. *Then A has no isotropy*; i.e., *$QA = A$ implies $Q = I$.*

Proof. Every $Q \neq I$ acts on $\mathbb{R}^3$ by rotation through an angle θ about a unique axis, that is, about a line through the origin in $\mathbb{R}^3$. Now $QA = A$ means that Q is the identity on the range of A. Therefore if $Q \neq I$ and $QA = A$, the range of A must be zero-dimensional or one dimensional, so $\dim \ker A \geqq 2$. ■

Finally in this section we study the range and kernel of $k: \mathscr{L} \to M_3$.

3.8. Proposition. 1. ker k *consists in those loads in $\mathscr{L}_e$ for which every axis is an axis of equilibrium.*

2. *$k: \mathscr{L} \to M_3$ is surjective.*

Proof. 1. Let $l \in \ker k$. For $W \in$ skew, $k(Wl) = Wk(l) = 0$ so $Wl \in \mathscr{L}_e$; by 3.3 every axis is an axis of equilibrium. Conversely, if $Wl \in \mathscr{L}_e$ for all $W \in$ skew, then $k(WI) - Wk(I)$ is symmetric for all W; i.e., $k(I)W + Wk(I) = 0$ for all W. From $(Lv)\hat{} = A\hat{v} + \hat{v}A$, where $A = k(I)$, and $L = (\text{trace } A)\, I - A$, we see that $L = 0$. This implies that trace $A = 0$ and hence $A = 0$.

To prove 2 introduce an invariant inner product on $\mathscr{L}$ under $SO(3)$:

$$\langle l, \bar{l}\rangle = \int_{\mathscr{B}} \langle B(X), \bar{B}(X)\rangle \, dV(X) + \int_{\partial\mathscr{B}} \langle \tau(X), \operatorname{tr} \bar{\tau}(X)\, dA(X).$$

Relative to this and the inner product trace $(A^{\mathrm{T}}B)$ on M_3, the adjoint k^{T} $M_3 \to \mathscr{L}$ of k is given by

$$k^{\mathrm{T}}(D) = (B, \tau), \quad \text{where} \quad B(X) = DX - G, \quad \tau(X) = DX - G,$$

and

$$G = \Big[\int_{\mathscr{B}} DX\, dV(X) + \int_{\partial\mathscr{B}} DX\, dA(X)\Big] \Big/ \Big[\int_{\mathscr{B}} dV + \int_{\partial\mathscr{B}} dA\Big].$$

If $k^{\mathrm{T}}(D) = (0, 0)$, then it is clear that $D = 0$. It follows from the Alternative Theorem that k is surjective. ■

3.9. Corollary. 1. *$\ker k$ is the largest subspace of $\mathscr{L}_e$ that is invariant under $SO(3)$.*
2. *$k \,|\, (\ker k)^{\perp} : (\ker k)^{\perp} \to M_3$ is an isomorphism.*

Let $j = (k \,|\, (\ker k)^{\perp})^{-1}$ and write

$$\text{Skew} = j\,(\text{skew}), \qquad \text{Sym} = j\,(\text{sym}).$$

These are linear subspaces of $\mathscr{L}$ of dimension 3 and 6 respectively. Thus we have the decomposition:

$$\mathscr{L} = \overbrace{\text{Skew} \oplus \text{Sym}}^{SO(3)\text{-invariant pieces}} \oplus \overbrace{\ker k}^{} \quad\text{with}\quad \underbrace{\text{Sym} \oplus \ker k}_{\mathscr{L}_e}$$

corresponding to the decomposition $M_3 = \text{skew} \oplus \text{sym}$; $U = \frac{1}{2}(U - U^{\mathrm{T}}) + \frac{1}{2}(U + U^{\mathrm{T}})$.

Note: Skew and $\mathscr{L}_e$ need not be orthogonal.

§ 4. Equivalent Reformulations of the Problem

Define: $\Phi: \mathscr{C} \to \mathscr{L}$ by $\Phi(\phi) = (-\text{DIV}\, P, P \cdot N)$, *i.e.*,

$$\Phi(\phi)(X) = (-\text{DIV}\, P(X, F(X)), \quad P(X, F(X)) \cdot N(X))$$

so the equilibrium equations (E) become $\Phi(\phi) = l$. The principle of material frame-indifference implies the equivariance of Φ: $\Phi(Q\phi) = Q\Phi(\phi)$. Standard Sobolev estimates show that Φ is a smooth mapping (see, for example, PALAIS [1968]). The derivative of Φ is given by

$$D\Phi(\phi) \cdot u = (-\text{DIV}\, (\mathbf{A} \cdot \nabla u), \quad (\mathbf{A} \cdot \nabla u) \cdot N)$$

and at $\phi = I_{\mathscr{B}}$ this becomes

$$D\Phi(I_{\mathscr{B}}) \cdot u = (-\text{DIV}\, (\mathbf{c} \cdot e), \quad (\mathbf{c} \cdot e) \cdot N)$$

where $e = \frac{1}{2}[\nabla u + (\nabla u)^{\mathrm{T}}]$.

If $D\Phi(I_{\mathscr{B}}): T_{I_{\mathscr{B}}}\mathscr{C} \to \mathscr{L}$ were an isomorphism, we could solve $\Phi(\phi) = l$ uniquely for ϕ near $I_{\mathscr{B}}$ and l small. The essence of our problem is that $D\Phi(I_{\mathscr{B}})$ is *not* an isomorphism: since $\Phi(SO(3)) = 0$, $\ker D\Phi(I_{\mathscr{B}})$ contains skew.

Define $\mathscr{C}_{\text{sym}} = \{u \in T_{I_{\mathscr{B}}}\mathscr{C} \mid u(0) = 0 \text{ and } \nabla u(0) \in \text{sym}\}$. From (H2) and from the linear theory of elasticity we have:

4.1. Lemma. *$D\Phi(I_{\mathscr{B}}) \,|\, \mathscr{C}_{\text{sym}} : \mathscr{C}_{\text{sym}} \to \mathscr{L}_e$ is an isomorphism.*

The connection between the astatic load map $k: \mathscr{L} \to M_3$ and Φ is seen from the following computation of $k \circ \Phi$.

4.2. Lemma. *Let $\phi \in \mathscr{C}$ and let $\mathbf{P}$ be the first Piola-Kirchhoff stress tensor at ϕ. Then*

$$k(\Phi(\phi)) = \int_{\mathscr{B}} \mathbf{P}\, dV.$$

This result follows by application of Gauss' theorem to

$$k(\Phi(\phi)) = \int_{\mathscr{B}} (-\text{DIV}\, P) \otimes X\, dV(X) + \int_{\partial\mathscr{B}} (P \cdot N) \otimes X\, dA(X).$$

This expression for $k(\Phi(\phi))$ should be compared with the astatic load relative to the configuration ϕ rather than $I_{\mathscr{B}}$: if σ denotes the Cauchy stress, then

$$k(\Phi(\phi), \phi) = \int_{\mathscr{B}} \sigma\, dV,$$

which is symmetric, while $k(\Phi(\phi)) = k(\Phi(\phi), I_{\mathscr{B}})$ need not be.

To study solutions of $\Phi(\phi) = l$ for ϕ near the trivial solutions and l near a given load l_0, it suffices to take $l_0 \in \mathscr{L}_e$. This follows from DA SILVA's theorem and the equivariance of Φ.

Let $\mathscr{C}_{\text{sym}}$ be regarded as an affine subspace of $\mathscr{C}$ centered at $I_{\mathscr{B}}$. Let $\tilde{\Phi}$ be the restriction of Φ to $\mathscr{C}_{\text{sym}}$. From the implicit function theorem and Lemma 4.1 we get:

4.3. Lemma. *There is a ball centered at $I_{\mathscr{B}}$ in $\mathscr{C}_{\text{sym}}$ whose image $\mathscr{N}$ under $\tilde{\Phi}$ is a smooth submanifold of $\mathscr{L}$ tangent to $\mathscr{L}_e$ at 0 (see Figure 1). The manifold $\mathscr{N}$ is the graph of a unique smooth mapping*

$$F: \mathscr{L}_e \to \text{Skew}$$

such that $F(0) = 0$ and $DF(0) = 0$.

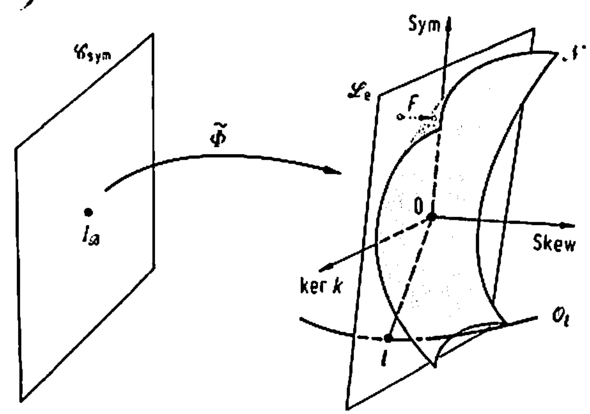


Fig. 1

Later we shall show how to compute $D^2F(0)$ in terms of $D\Phi(I_{\mathscr{B}})^{-1}$ and c. Now we are ready to reformulate problem (P1).

(P2) *For a given* $l_0 \in \mathscr{L}_e$ *near zero, study how* $\mathcal{O}_l$ *meets the graph of F for various* l *near* l_0.

Problems (P1) and (P2) are related as follows. Let ϕ satisfy (E) with $l \in \mathscr{L}$ and let Q be such that $\bar{\phi} = Q\phi \in \mathscr{C}_{\text{sym}}$. Then $\Phi(\bar{\phi}) = Ql$, so the orbit of l meets the graph of F at $\Phi(\bar{\phi})$. Conversely, if the orbit of l meets $\mathscr{N}$ at $\Phi(\bar{\phi}) = Ql$, then $\phi = Q^{-1}\bar{\phi}$ satisfies (E). We claim that near the trivial solutions, the numbers of solutions to each problem also correspond. This follows from the next lemma.

4.4. Lemma. *There is a neighborhood U of* $I_{\mathscr{B}}$ *in* $\mathscr{C}_{\text{sym}}$ *such that if* $\phi \in U$ *and* $Q\phi \in U$, *then* $Q = I$.

Proof. Note that $\mathscr{C}_{\text{sym}}$ is transverse to $\mathcal{O}_{I_{\mathscr{B}}}$ at $I_{\mathscr{B}}$ and $I_{\mathscr{B}}$ has trivial isotropy. Since $SO(3)$ is compact, $\mathcal{O}_{I_{\mathscr{B}}}$ is closed. Thus there is a neighborhood U_0 of $I_{\mathscr{B}}$ in $\mathscr{C}_{\text{sym}}$ such that if $Q \mid \mathscr{B} \in U_0$, then $Q = I$. The same thing is true of orbits passing through a small neighborhood of $I_{\mathscr{B}}$ by the openness of transversality and the compactness of $SO(3)$. ■

If $\mathcal{O}_l$ meets $\mathscr{N}$ in k points $Q_i l = \Phi(\bar{\phi}_i)$, $i = 1, \ldots, k$, then $\bar{\phi}_i$ are distinct as Φ is 1-1 on a neighborhood of $I_{\mathscr{B}}$ in $\mathscr{C}_{\text{sym}}$ (by the implicit function theorem). If this neighborhood is also contained in U of 4.4, then the points $Q_i^{-1}\bar{\phi}_i = \phi_i$ are also distinct by 4.4. *Hence the problems* (P1) *and* (P2) *are equivalent.*

We require some more notation to describe the action $(Q, A) \mapsto QA$ of $SO(3)$ on M_3. Let

$$\text{Skew}\,(A) = \frac{1}{2}(A - A^{\text{T}}) \in \text{skew} \tag{3.2a}$$

and

$$\text{Sym}\,(A) = \frac{1}{2}(A + A^{\text{T}}) \in \text{sym} \tag{3.2b}$$

be the skew symmetric and symmetric parts of A, respectively.

We shall abuse notation by suppressing j and identifying Sym with sym and Skew with skew. Thus we write a load $l \in \mathscr{L}$ as $l = (A, n)$ where $A = k(l) \in M_3$ and $n \in \ker k$; hence $l \in \mathscr{L}_e$ precisely when $A \in$ sym. The action of $SO(3)$ on $\mathscr{L}$ is given by

$$Ql = (QA, Qn).$$

Using this notation, we can reformulate problem (P2) as follows:

(P3) *For a given* $l_0 = (A_0, n_0) \in \mathscr{L}_e$ *near zero, and* $l = (A, n)$ *near* l_0, *find* $Q \in SO(3)$ *such that*

$$\text{Skew}\,(QA) - F(\text{Sym}\,(QA), Qn) = 0.$$

Define the rescaled map $\bar{F}: \mathbb{R} \times \mathscr{L}_e \to \text{Skew}$ by

$$\bar{F}(\lambda, l) = \frac{1}{\lambda^2} F(\lambda l).$$

Since $F(0) = 0$ and $DF(0) = 0$, F is smooth. Moreover, if $F(l) = \frac{1}{2} G(l) + \frac{1}{6} C(l) + \ldots$ is the Taylor expansion of F about zero, then $\bar{F}(\lambda, l) = \frac{1}{2} G(l) + \frac{\lambda}{6} C(l) + \ldots$; here $G(l) = D^2F(0)\,(l, l)$ and $C(l) = D^3F(0)\,(l, l, l)$.

In problem (E) let us measure the size of l by the parameter λ. Thus, replace $\Phi(\phi) = l$ for l near zero by $\Phi(\phi) = \lambda l$ for λ near zero. This scaling enables us to distinguish conveniently the size of l from its 'orientation'. In the literature l has always been fixed and λ taken small. Here we allow l to vary as well. Thus we arrive at the final formulation of the problem.

(P4) *For a given* $l_0 = (A_0, n_0) \in \mathscr{L}_e$, *for* l *near* l_0 *and for* λ *small, find* $Q \in SO(3)$ *such that*

$$\text{Skew}\,(QA) - \lambda \bar{F}(\lambda, \text{Sym}\,(QA), Qn) = 0.$$

The left-hand side of this equation will be denoted $H(\lambda, A, n; Q)$ or $H(\lambda, Q)$ if A and n are fixed.

§ 5. Loads of Type 0, having no Axis of Equilibrium

We shall begin the analysis by giving an (almost trivial) proof of one of the basic theorems of STOPPELLI [1958]*:

* The only other *complete* proof in English we know of is given in VAN BUREN [1968], although sketches are available in GRIOLI [1962], TRUESDELL & NOLL [1965] and WANG & TRUESDELL [1973]. Our proof is rather different; the use of the map $\bar{F}$ avoids a series of complicated estimates used by STOPPELLI and VAN BUREN.

5.1. Theorem. *Suppose $l \in \mathscr{L}_e$ has no axis of equilibrium. Then for λ sufficiently small, there are a unique $\bar\phi \in \mathscr{C}_{\text{sym}}$ and a unique Q in a neighborhood of the identity in $SO(3)$ such that $\phi = Q^{-1}\bar\phi$ satisfies the traction problem*

$$\Phi(\phi) = \lambda l.$$

Proof. Define $H: \mathbb{R} \times SO(3) \to \text{Skew}$ as above by

$$H(\lambda, Q) = \text{Skew}(QA) - \lambda \bar F(\lambda, \text{Sym}(QA), Qn)$$

where $l = (A, n) \in \mathscr{L}_e = \text{Sym} \oplus \ker k$ is fixed. Note that the partial derivative is $D_Q H(0, I)\cdot W = \text{Skew}(WA) = \frac{1}{2}(WA + AW)$. By Proposition 3.3, $D_Q H(0, I)$ is an isomorphism. Hence, by the implicit function theorem, $H(\lambda, Q) = 0$ can be uniquely solved for Q near $I \in SO(3)$ as a function of λ near $0 \in \mathbb{R}$. ■

The geometric reason "why" this proof works and the clue to treating other cases is the following.

5.2. Lemma. *A load $l \in \mathscr{L}_e$ has no axis of equilibrium precisely when $\mathscr{L} = \mathscr{L}_e \oplus T_l\mathscr{O}_l$. In particular, if l has no axis of equilibrium, then $\mathscr{O}_l$ intersects $\mathscr{L}_e$ transversely at l.*

Proof. The tangent space to $\mathscr{O}_l$ at $l \in \mathscr{L}_e$ is $T_l\mathscr{O}_l = \{Wl \mid W \in \text{skew}\}$, and the projection of this into the complement Skew to $\mathscr{L}_e$ is $Wl \mapsto \frac{1}{2}(WA + AW)$ where $A = k(l)$. The result then follows from part 3 of 3.3. ■

We have shown that there is only one solution to $\Phi(\phi) = \lambda l$ near the identity if λ is small and l has no axis of equilibrium. How many solutions are there near the trivial solutions $SO(3)$? As we shall see, this problem has a non-trivial answer which depends on the *type* of l. We analyze the simplest case here. Recall from Definition 3.5 that a load $l \in \mathscr{L}_e$ is said to be of type 0 if l has no axis of equilibrium and if $A = k(l)$ has distinct eigenvalues.

Loads with no axis of equilibrium occur amongst other types of loads classified in the next section, and STOPPELLI's Theorem 5.1 applies to them. However, the global structure of the solutions ("global" being relative to $SO(3)$) is different for the different types. For loads of type 0 the situation is as follows.

5.3. Theorem. *Let $l_0 \in \mathscr{L}_e$ be of type 0. Then for λ sufficiently small, $\Phi(\phi) = \lambda l$ has exactly four solutions ϕ_1, ϕ_2, ϕ_3 and ϕ_4 in a neighborhood of the trivial solutions $SO(3) \subset \mathscr{C}$* (see Figure 2).

Proof. By 3.6, $\mathscr{C}_{\lambda l_0}$ meets $\mathscr{L}_e$ in four points. By 5.1, in a neighborhood of 0 in $\mathscr{L}$, $\mathscr{O}_{\lambda l_0}$ meets $\mathscr{N}$ in exactly four points, the images of $\bar\phi_1$, $\bar\phi_2$, $\bar\phi_3$ and $\bar\phi_4$, say. The problem (P2) has four solutions. By the equivalence of (P1) and (P2), so does (P1). ■

Let $A = k(l_0)$ and $S_A = \{Q \mid QA \in \text{sym}\}$. From the proof of 3.6 we see that S_A is a four-element subgroup of $SO(3)$ isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. By our

earlier discussions, the elements ϕ_i are obtained from $\bar\phi_i$ by applying rotations close to elements of S_A. In particular, as $\lambda \to 0$, the solutions $\{\phi_i\}$ converge to the four-element set S_A (regarded as a subset of $\mathscr{C}$).

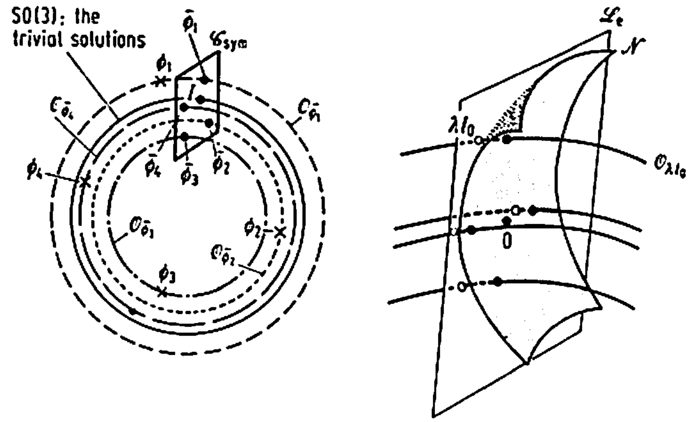


Fig. 2

For l sufficiently close to l_0, the problem $\Phi(\phi) = \lambda l$ will also have four solutions. Indeed, by the openness of transversality, $\mathscr{O}_{\lambda l}$ will also meet $\mathscr{N}$ in four points. In other words, the picture for loads of type 0 in Figure 2 is stable under small perturbations of l_0.

Next we study the stability of the four solutions found in Theorem 5.3. This will be done under the hypothesis that the classical elasticity tensor c is *stable*; i.e., that it satisfies

(13) *There is an $\eta > 0$ such that*

$$\varepsilon(e) := \frac{1}{2}\,c(X)(e, e) \geqq \eta \,\|e\|^2$$

for all $e \in \text{sym}(T_X\mathscr{B}, T_X\mathscr{B})$. (Here $\|\cdot\|$ is the pointwise norm and $\varepsilon(e)$ is the stored energy function for linearized elasticity.)

Because of difficulties with potential wells and dynamical stability in elasticity (see KNOPS & WILKES [1973] and BALL, KNOPS & MARSDEN [1978]) we shall adopt the following "energy criterion" definition of stability.

5.4. Definition. A solution ϕ of $\Phi(\phi) = l$ will be called *stable* if ϕ is a local minimum in $\mathscr{C}$ of the potential function

$$V_l(\phi) = \int_{\mathscr{B}} W(\phi)\, dV - \langle l, \phi\rangle$$

where

$$\langle l, \phi\rangle = \int_{\mathscr{B}} B(X)\cdot\phi(X)\, dV(X) + \int_{\partial\mathscr{B}} \tau(X)\cdot\phi(X)\, dA(X) = \text{trace}\, k(l, \phi).$$

If ϕ is not stable, its *index* is the dimension of the largest subspace of vectors u tangent to $\mathscr{C}$ at ϕ with the property that ϕ decreases along some curve tangent to u. (Thus, index 0 corresponds to stability.)

5.5. Theorem. *Let* (H1)–(H3) *hold and let* l_0 *be as in* 5.3. *For* λ *sufficiently small, amongst the four solutions* $\phi_1, \phi_2, \phi_3, \phi_4$ *given by* 5.3, *exactly one is stable; the others have indices* 1, 2, *and* 3. *Suppose* ϕ *is a solution approaching* $Q \in S_A$ *as* $\lambda \to 0$. *Then* ϕ *is stable if and only if* $QA - \operatorname{trace}(QA)\, I \in \operatorname{sym}$ *is positive definite. In general, the index of* ϕ *is the number of negative eigenvalues of* $QA - \operatorname{trace}(QA)\, I$.

Proof. Let $\phi_0 \in \mathscr{C}$ satisfy $\Phi(\phi_0) = \lambda l_0 = l$. Then ϕ_0 is a critical point of V_l. Consider the orbit $\mathcal{O}_{\phi_0} = \{Q\phi_0 \mid Q \in SO(3)\}$ of ϕ_0. Its tangent space decomposes $T_{\phi_0}\mathscr{C}$ as follows:

$$T_{\phi_0}\mathscr{C} = T_{\phi_0}\mathcal{O}_{\phi_0} \oplus T_{\phi_0}(\mathcal{O}_{\phi_0})^{\perp}.$$

First consider V_l restricted to $(T_{\phi_0}\mathcal{O}_{\phi_0})^{\perp}$. Its second derivative at ϕ_0 in the direction of $u \in (T_{\phi_0}\mathcal{O}_{\phi_0})^{\perp}$ is $\int_{\mathscr{B}} \frac{\partial^2 W}{\partial F\, \partial F}(\phi) \cdot (\nabla u, \nabla u)\, dV$. At $\lambda = 0, \phi_0 \in SO(3)$ this becomes $\int_{\mathscr{B}} c(X) \cdot (e(X), e(X))\, dV(X)$, where $e = \frac{1}{2}(\nabla u + (\nabla u)^T)$. This is larger than a positive constant times the square of the L^2 norm of e, by (H3). As $u \in (T_\phi \mathcal{O}_\phi)^\perp$, $\|e\|^2_{L^2} \geqq (\text{constant}) \|u\|^2_{H^1}$ by Korn's inequality (see FICHERA [1972]). By continuity, we have in general

$$D^2 V_{\lambda l_0}(\phi_0) \cdot (u, u) \geqq \delta \|u\|^2_{H^1}$$

if u is orthogonal to $\mathcal{O}_{\phi_0}$ at ϕ_0 and λ is small. This inequality implies that ϕ_0 is a minimum for $V_{\lambda l_0}$ in directions transverse to $\mathcal{O}_{\phi_0}$. (Actually one can see that ϕ_0 is a local minimum in the topology of $\mathscr{C}$ on $(T_{\phi_0}\mathcal{O}_{\phi_0})^\perp$ by using the version of the Morse lemma given by TROMBA [1976] or by GOLUBITSKY & MARSDEN [1982].)

Next, consider $V_{\lambda l_0}$ restricted to $\mathcal{O}_{\phi_0}$. By material frame-indifference, W is constant on $\mathcal{O}_{\phi_0}$. Since ϕ_0 must be a critical point for $V_{\lambda l_0}$ restricted to $\mathcal{O}_{\phi_0}$, it is also a critical point for $\lambda l_0 = l$ restricted to $\mathcal{O}_{\phi_0}$ (where $l(\phi) = \langle l, \phi \rangle$). It suffices therefore to determine the index of $l \mid \mathcal{O}_{\phi_0}$ at ϕ_0. The result is now a consequence of continuity and the limiting case $\lambda \to 0$ given in the following lemma about loads of type 0.

5.6. Lemma. *Let* l *be of type* 0 *and let* $A = k(l)$. *Then* S_A, *regarded as a subset of* $\mathscr{C}$, *equals the set of critical points of* $l \mid \mathcal{O}_{I_{\mathscr{B}}}$. *These 4 critical points are nondegenerate with indices* 0, 1, 2, *and* 3; *the index of* Q *is the number of negative eigenvalues of* $QA - \operatorname{trace}(QA)\, I$.

Proof. First note that $\mathscr{L}_e = (T_I SO(3))^\perp$ since $D\Phi(I_{\mathscr{B}})$ has kernel $T_I SO(3) = \text{skew}$, has range $\mathscr{L}_e$ and is self-adjoint. Thus $QI \in \mathscr{L}_e$ if and only if $l \perp T_{Q^T} SO(3)$. It follows that $QI \in \mathscr{L}_e$ if and only if Q^T is a critical point of $l \mid \mathcal{O}_{I_{\mathscr{B}}}$. (Recall that elements of $S_A = \{Q \in SO(3) \mid QI \in \mathscr{L}_e\}$ are symmetric.)

To compute the index of $l \mid \mathcal{O}_{I_{\mathscr{B}}}$ at $Q \in S_A$, we compute the second derivative

$$\frac{d^2}{dt^2} l(\exp(tW)\, Q)\big|_{t=0} = l(W^2 Q).$$

Now

$$l(W^2 Q) = \operatorname{trace} k(l, W^2 Q) = \operatorname{trace}\,[k(l, Q)\, W^2]$$

$$= \operatorname{trace}\,[AQ^{-1}W^2] = \operatorname{trace}\,[W^2 QA]$$

because $Q^{-1} = Q$. This quadratic form on skew is represented by the element $QA - \operatorname{trace}(QA)\, I$ of sym as is seen from $\hat{v}A + A\hat{v} = (Lv)^\wedge$ with A replaced by QA and $\operatorname{trace}(\hat{v}^T \hat{w}) = 2v \cdot w$. Using the representations for $\{QA\}$ given in Proposition 3.6, namely

$$\operatorname{diag}(a, b, c), \quad \operatorname{diag}(-a, -b, c), \quad \operatorname{diag}(-a, b, -c) \quad \text{and} \quad \operatorname{diag}(a, -b, -c)$$

one checks that all four indices occur. ∎

Remark. This lemma is a special case of the general problem of studying the critical points of linear functionals on orbits of a representation of a Lie group. This situation will arise again in our analysis of the other load types; *cf.* FRANKEL [1965] and RAMANUJAM [1969].

§ 6. Classification of Orbits in M_3†

The purpose of this section is to classify orbits in M_3 under the action $(Q, A) \mapsto QA$ of $SO(3)$ on M_3 by the way the orbits meet sym. The polar decomposition theorem implies that it is enough to consider orbits $\mathcal{O}_A$ of elements of sym. We begin by recalling Proposition 3.6 (another proof of which will be given below).

6.1. Proposition (Type 0). *Suppose* $A \in \operatorname{sym}$ *has no axis of equilibrium and has distinct eigenvalues. Then* $\mathcal{O}_A \cap \operatorname{sym}$ *consists of four points, at each of which the intersection is transversal.*

We shall let the eigenvalues of $A \in \operatorname{sym}$ be denoted a, b, c. Using the terminology from § 3, we say that A *has no axis of equilibrium* when $(a + b)(b + c)(a + c) \neq 0$; *i.e.*, when $a + b + c \neq a, b$ or c. In this case $\mathcal{O}_A$ intersects sym transversely at A.

6.2. Definition. *A is said to be of type 1 if A has no axis of equilibrium and if exactly two of a, b, c are equal and non-zero (say $a = b \neq c$, $a \neq 0$).*

† The reader may gain some insight by replacing the "abstract" proofs in this section with explicit matrix computations. This is, of course, how we originally obtained the results.

6.3. Propositio[n]. *[If] A is of type 1, then $\mathcal{O}_A \cap$ sym consists of two points (each with no axis of equilibrium) and $\mathbb{RP}^1 \approx S^1$ (each point of which has one axis of equilibrium).*

Before proving this, we give a number of lemmas of general utility. If $l \in \mathbb{RP}^2$ is a line through the origin in $\mathbb{R}^3$, let Q_l be the rotation through angle π about l.

6.4. Lemma. *$l \mapsto Q_l$ is an embedding of $\mathbb{RP}^2$ onto $SO(3) \cap \text{sym} \setminus I$.*

Proof. It is clear that $l \mapsto Q_l$ is a one-to-one map of $\mathbb{RP}^2$ into $SO(3)$. Since $Q_l^2 = I$, it follows that $Q_l = Q_l^{-1} = Q_l^T$. Hence Q_l lies in $SO(3) \cap \text{sym}$.
Every $Q \in SO(3) \setminus I$ is a rotation through some angle θ about some axis l. If such Q also is symmetric then it has three independent real eigenvectors. Hence $\theta = \pi$. ■

6.5. Corollary. *The orbit $\mathcal{O}_I$ of the identity meets* sym *at one point* (I) and at $\mathbb{RP}^2 \cong (SO(3) \cap \text{sym}) \setminus I$.

6.6. Lemma. *Let $A \in$ sym with $\dim \ker A \leqq 1$ and suppose that $Q \in SO(3) \setminus I$ and $QA \in$ sym. Then $Q = Q_l$ for some line l invariant under A, and in particular $Q \in$ sym.*

Proof. We can suppose $Q \neq I$. By Euler's theorem on rotations, there is a unit vector $x \in \mathbb{R}^3$ (unique up to sign) such that $Qx = x$. Since $QA \in$ sym, we have $QA = AQ^T$, so $QAQ = A$. Thus $QAx = Ax$, so $Ax = cx$ for a constant c. Hence Q and A leave $V = x^\perp$, the orthogonal complement of x, invariant, and A is not identically zero on V.
Let $S = QA \in$ sym, so $S^2 = A^2$. Since $Q \mid V$ is a rotation, it follows that $S \mid V = \pm A \mid V$. Thus $Q = I$ or $Q = Q_{l(x)}$ where $l(x)$ is the line through x. Then $Q \in$ sym as in Lemma 6.4. ■

It follows that if $\dim \ker A \leqq 1$ and $QA \in$ sym, then $QA = AQ$. Since Q is both orthogonal and symmetric, A and Q can be simultaneously diagonalized.

[P]roof of 6.1. If A has distinct eigenvalues, its eigenvectors are unique, up to [s]calar factors, so Q is either I or a rotation by π about one eigenvectors. ■

[P]roof of 6.3. Suppose that $0 \neq a = b \neq c$ and let w be an eigenvector corres[p]onding to the eigenvalue c. Let V be the plane orthogonal to w so V is the eigen[s]pace with eigenvalue a. As Q and A can be simultaneously diagonalized and Q [i]s a rotation by π (excluding $Q = I$) we have either $Q = Q_{l(w)}$ or $Q = Q_l$ [f]or l a line in V. In the former case, $Q_{l(w)}A$ is has eigenvalues $(-a, -a, c)$ and so [h]as no axis of equilibrium. In the latter case, Q_lA has eigenvalues $(a, -a, -c)$ [s]o w is an axis of equilibrium (see 3.3). ■

[6].7. Corollary. *The $\mathbb{RP}^1$ in Proposition 6.3 is a right coset of the subgroup S_w^1 [o]f all rotations about w; in fact $\mathbb{RP}^1 = S_w^1\{Q_{l_0} \mid l_0$ is a line in V, the plane ortho[g]onal to $w\}$.*

[Proo]f. $\mathbb{RP}^1 = \{Q_l \mid l \in V\}$ and we have the easily verified identity

$$Q_{l_{\theta/2}} = \exp(\theta\hat{w})\, Q_{l_0}$$

where l_θ makes an angle θ with l_0 (in the positive sense) in V. ■

These lemmas also enable us to handle the next type.

6.8. Definition. A is of *type 2* if A has no axis of equilibrium and all three of a, b, c are equal (and so $\neq 0$).

6.9. Proposition. *If A is of type 2, then $\mathcal{O}_A \cap$ sym consists in one point (A) and an $\mathbb{RP}^2$.*

Proof. This is immediate from 6.5. ■

Notice that each point of $\mathbb{RP}^2$ has a whole circle of axes of equilibrium; namely Q_lA has as axes of equilibrium all vectors orthogonal to l. The eigenvalues of Q_lA are $a, -a, -a$.
Types 0, 1, and 2 exhaust all symmetric matrices with no axis of equilibrium. It is easy to check from the results above that any symmetric A with $\dim \ker A \leqq 1$ lies on the $SO(3)$-orbit of a matrix of type 0, 1, or 2. *From now on we shall say that these orbits, or any representatives of them, are of type* 0, 1, *or* 2.
Finally we turn to the remaining A's with an axis of equilibrium that is not already on an orbit of type 0, 1, or 2.

6.10. Definition. A is of *type 3* if $\dim \ker A = 2$ and A is of *type* 4 if $A = 0$.

6.11. Proposition. *If A is of type 3, then $\mathcal{O}_A \cap$ sym consists in two points, A and $-A$.*

Proof. $S = QA \in$ sym implies that $S^2 = A^2$ and so again $S = \pm A$ as in 6.6, even though possibly $A \mid V = 0$. In this case Q could be any rotation about $l(x)$. ■

All the foregoing information can be summarized as follows:

6.12. Theorem. *The $SO(3)$ orbits in M_3 fall into five distinct types according to the way in which they meet* sym *(see Table 1 below). Furthermore, if $A \in$ sym, $S_A = \{Q \mid QA \in \text{sym}\}$ consists in $I \cup \{Q_l\}$ for all l invariant under A (and hence $S_A \subset$ sym) except if*

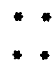
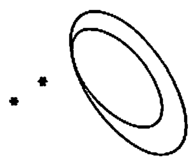
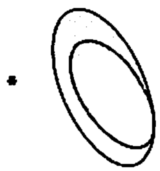
$$(1) \qquad \dim \ker A = 2,$$

in which case S_A also contains the rotations through any angle about the eigenaxis of A corresponding to the non-zero eigenvalue, or if

$$(2) \qquad A = 0$$

in which case $S_A = SO(3)$. (See Table 2 below.)

Table 1. *Orbit types in M_3 under the $SO(3)$-action*

Type	Dim of Orbit	Orbit $\cap$ sym	Picture	Set	Eigenvalues	Isotropy	Dim orbit $\cap$ sym	Axes of Equilibrium
0	3	four points		A, $\{Q_{l_i}A\}$ $l_i \in$ eigenspaces of A	a, b, c distinct $(a+b)(a+c) \times (b+c) \neq 0$	none	0 ⋔	none
1	3	two points and $\mathbb{R}P^1$		A, $Q_{l_i}A : l_i \in 1$ dim'l eigenspace	(a, a, c) $a \neq c$, $a \neq 0$	none	0 ⋔	none
					$(-a, -a, c)$	none	0 ⋔	none
				$\{Q_lA\} : l \in 2$ dim'l eigenspace	$(a, -a, -c)$	none	1	one (l_1)
2	3	one point and $\mathbb{R}P^2$		A	(a, a, a) $a \neq 0$	none	0 ⋔	none
				$\{Q_lA\}$ arbitrary	$(a, -a, -a)$	none	2	circle (any axis orthogonal to l)
3	2	two points		A	$(0, 0, c)$ $c \neq 0$	S^1	0	circle (axis orthogonal to the eigenspace of c)
				$-A$	$(0, 0, -c)$	S^1	0	
4	0	one point		$A = 0$	$(0, 0, 0)$	$SO(3)$	0	$\mathbb{R}P^2$ (any axis)

⋔ denotes transversal intersection of the orbit of A with sym.

Table 2

Type of A	Description of S_A
0	four points
1	two points and $\mathbb{R}P^1 \approx S^1$
2	one point and $\mathbb{R}P^2$
3	two disjoint circles
4	$SO(3)$

Remarks.

1. Table 1 highlights the fact that *having an axis of equilibrium or not* is not an invariant of the $SO(3)$ action on $\mathscr{L}$. This means that there are equilibrated loads having an axis of equilibrium, but which, when rotated globally by a certain amount to another equilibrated load, no longer have one.
2. Thus, by Theorem 5.1, we get *existence* of solutions to the traction problem for *all types* of astatic loads except 3, 4.*
3. The notion of type can be pulled back from M_3 to $\mathscr{L}$ with a little care, as we see below.

6.13. Definition. By analogy with our definition

$$S_A = \{Q \in SO(3) \mid QA \in \text{sym}\}, \quad A \in M_3$$

which we applied when A is of type 0, let us now write

$$S_l = \{Q \in SO(3) \mid Ql \in \mathscr{L}_e\}, \quad l \in \mathscr{L}.$$

From the equivariance of k we clearly have

6.14. Lemma. $S_l = S_{k(l)}$.

Note that the map of S_A to $\mathcal{O}_A \cap \text{sym}$ given by $Q \mapsto QA$ is an embedding for types 0, 1, 2 but *not* for all types 3 and 4, because of the isotropy. Pulling back to $\mathscr{L}$, we see that $Q \mapsto Ql$ is an embedding of S_l to $\mathcal{O}_l \cap \mathscr{L}_e$ if $k(l)$ is of type 0, 1 or 2, so we can refer to l as being of type 0, 1, or 2 according as $k(l)$ is. On the other hand, if $k(l)$ is of type 3 then *either*

(a) $\mathcal{O}_l \cap \mathscr{L}_e = \{l, -l\}$

or

(b) $\mathcal{O}_l \cap \mathscr{L}_e =$ two disjoint circles in $\{l, -l\} + \ker k$.

---

* In particular, STOPPELLI's failure to find solutions for certain loads of type 1 is seen to be due to neglect of the full rotation group (see Section 8). Our results are also consistent with those of BALL [1977].

Finally, if $k(l)$ is of type 4, then $\mathcal{O}_l \subset \ker k \subset \mathcal{L}_e$ and any $SO(3)$ orbit $\mathcal{O}_l$ is allowable.

Figure 3 illustrates some simple examples of loads of different types. These loads are all pure traction, with $B = 0$.

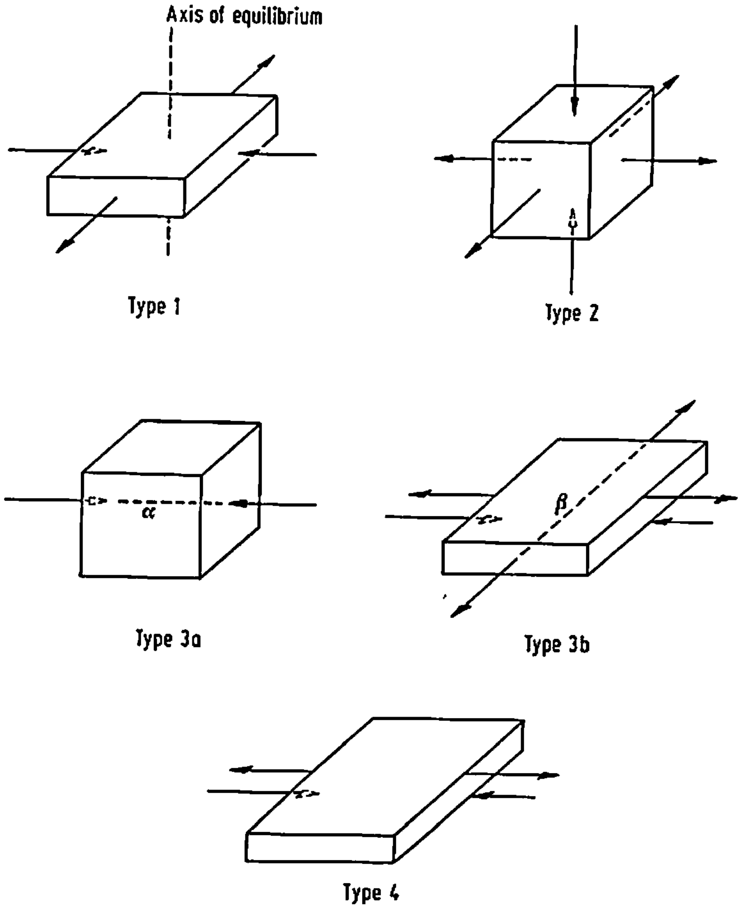


Fig. 3. Load types

Type 1. Rotation by 180° about one of the horizontal axes produces an equilibrated load with no axis of equilibrium.

Type 2. Any horizontal axis is an axis of equilibrium; vertical axis is not an axis of equilibrium. Rotation by 180° about the vertical axis gives an equilibrated load with no axis of equilibrium.

Type 3 (a). The load itself admits a circle group of symmetries about the axis α—which is thus an axis of equilibrium.

Type 3 (b). The load is not symmetric, but the astatic load remains constant under rotation about the axis β—which is thus an axis of equilibrium.

Type 4. The astatic load is zero: all axes are axes of equilibrium.

§ 7. The Bifurcation Equation and its Gradient Character

According to the formulation (P4) of our problem, we wish to solve the equation $H(\lambda, A, n; Q) = 0$ for Q, where

$$H(\lambda, A, n; Q) \equiv \text{Skew}(QA) - \lambda \bar{F}(\lambda, \text{Sym}(QA), Qn),$$

(A, n) is near $(A_0, n_0) \in \mathcal{L}_e$ and λ is small. In this section we perform the Liapunov-Schmidt procedure on this equation and show that the resulting bifurcation equation is essentially a gradient.

Define the right-invariant vector field X_{A_0} on $SO(3)$ by

$$X_{A_0}(Q) = \text{skew}(QA_0) \cdot Q,$$

which is a right translation of $\text{skew}(QA_0) \in so(3) = T_I SO(3)$ to $T_Q SO(3)$. Likewise, we shall regard H as a right-invariant vector field on $SO(3)$ depending on the parameters λ, A, n by setting

$$X(\lambda, A, n; Q) = k(H(\lambda, A, n; Q)) \cdot Q.$$

Thus,

$$X(0, A_0, n_0; Q) = X_{A_0}(Q).$$

Finally, note that S_{A_0} is the zero set of X_{A_0}; *i.e.*,

$$S_{A_0} = \{Q \in SO(3) \mid \text{skew}(QA_0) = 0\}.$$

What S_{A_0} is for various types of loads was given in Table 2 above.

7.1. Lemma. *Suppose* $A_0 \in \text{sym}$ *is of any type. Then for* $Q \in S_{A_0}$,

$$T_Q S_{A_0} = \{WQ \mid W \in \text{skew} \quad \text{and} \quad WQA_0 + QA_0W = 0\} = \ker DX_{A_0}(Q).$$

Proof. The second equality is clear for any A_0, because $DX_{A_0}(Q)\colon WQ \mapsto \text{skew}(WQA_0) \cdot Q$. For the first one, the inclusion $\subset$ immediately follows by differentiation of $X_{A_0}(Q) = 0$ in Q. Equality then follows by a dimension count; recall from 3.3 that $v \mapsto \hat{v}$ gives an isomorphism from the space of axes of equilibrium for A (not necessarily of unit length) to the $W \in \text{skew}$ such that $WA + AW = 0$. ■

Recall that $W \mapsto WQA_0 + (QA_0)^T W$ corresponds to the linear transformation $\text{trace}(QA_0)\, I - QA_0$ under the isomorphism of skew $= so(3)$ with $\mathbb{R}^3$. When $Q \in S_{A_0}$, QA_0 is symmetric, so this transformation is symmetric relative to the Killing form on $so(3)$. This remark and 7.1 yield the next lemma.

7.2. Lemma. *Suppose* A_0 *is of any type. Then at each point* Q *of* S_{A_0}, *the range of* $DX_{A_0}(Q)\colon T_Q SO(3) \to T_Q SO(3)$ *is the orthogonal complement of* $T_Q S_{A_0}$.

Next we recall a general context for the bifurcation of vector fields that will be applied to our situation (*cf.* REEKEN [1973]). Let M and Λ be manifolds and $X\colon M \times \Lambda \to TM$ a smooth vector field on M depending on the parameters $\lambda \in \Lambda$. We seek the zeros of X. For $\lambda = \lambda_0$, suppose the zero set S of X is a known smooth compact submanifold of M. Assume that M carries a Riemannian metric and that for $x \in S$, the range of $D_x X(x, \lambda_0)$ is the orthogonal complement of $T_x S$. The normal bundle E of S trivializes a neighborhood U of S. For each $x \in U$, let $P_x: T_x M \to T_x S_{\pi(x)}$ be the orthogonal projection to the fiber $S_{\pi(x)}$ over $\pi(x)$, where $\pi: E \to S$ is the projection. By the inverse function theorem there is a unique section $\phi_\lambda: S \to E$ such that $P_x X(\phi_\lambda(x), \lambda) = 0$ for $x \in S$ and λ in a neighborhood of λ_0 (by use of the fact that S is compact). Let $\tilde{X}(x, \lambda)$ be the orthogonal projection of $X(x, \lambda)$ onto the tangent space to the graph of ϕ_λ at a point x on the graph. Thus, $\tilde{X}(x, \lambda)$ is a vector field on the graph of ϕ_λ and finding its zeros is clearly equivalent (for small λ) to finding zeros of X. We call the equation $\tilde{X}(x, \lambda) = 0$ on the graph of ϕ_λ the *bifurcation equation.* Since S and the graph of ϕ_λ are diffeomorphic under ϕ_λ, we can equally well regard $\tilde{X}$ as a vector field on S. This reduction of the problem is often known as the Liapunov-Schmidt method.

The above procedure may be applied to our vector field $X(\lambda, A, n; Q)$ with parameters (λ, A, n) and variable $x = Q \in SO(3) = M$. Assume λ is near zero and (A, n) is near a load (A_0, n_0) where A_0 is of arbitrary type. Thus, there is a unique section $\phi_{\lambda,A,n}$ of the normal bundle to S_{A_0} determined by the Liapunov-Schmidt procedure as described above. Let $\Gamma(\lambda, A, n)$ denote the graph of $\phi_{\lambda,A,n}$ and let $\tilde{X}(\lambda, A, n; Q)$ be the orthogonal projection of X to the tangent space of Γ at Q. Thus, $\tilde{X}$ is a vector field on Γ. As above, we may also regard $\tilde{X}$ as a vector field on S_{A_0}.

The rest of this section is devoted to proving that the essential part of $\tilde{X}$ is a gradient. In the general context above, if X is a gradient, then so is $\tilde{X}$ since the orthogonal projection of a gradient vector field to a submanifold is the gradient of the restriction. This simple version does not directly apply to our situation as X need not be a gradient vector field on $SO(3)$. However, the "second order" Taylor approximation $\tilde{X}_2$ of $\tilde{X}$ will be.†

To state our gradient results, recall that in § 4 we defined the quadratic function $G: \mathcal{L}_e \to \text{skew}$ to be the second order term in the Taylor expansion of F about 0. Thus $\tilde{F}(\lambda, l) = \frac{1}{2} G(l) + \frac{\lambda}{6} C(l) + \dots$ where $G(l) = D^2F(0)(l, l)$ is a quadratic function of l. The appropriate second order approximation to the vector field X will thus be defined by

$$X_2(\lambda, A, n; Q) = \left[\text{skew}(QA) - \frac{1}{2}\lambda k\, G(Ql_0)\right] \cdot Q.$$

Let $\tilde{X}_2$ be the second order approximation of the vector field $\tilde{X}$ on S_{A_0} obtained by the Liapunov-Schmidt procedure. Thus, $\tilde{X}_2(Q)$ is the orthogonal projection of X_2 onto the tangent space $T_Q S_{A_0}$ for $Q \in S_{A_0}$.

† A somewhat more invariant procedure for this construction is given in the next paper in this series.

7.3. Theorem. *Suppose that A_0 is of arbitrary type. Then $\tilde{X}_2$ is a gradient vector field on S_{A_0}. In fact, $\tilde{X}_2 = -\text{grad} f$, where*

$$f(Q) = \langle l_0, Q^T I_{\mathcal{B}}\rangle + \left\langle l_0, \frac{1}{2}\lambda\, Q^T u_Q\right\rangle = \langle l_0, Q^T I_{\mathcal{B}}\rangle + \frac{1}{2}\lambda \int_{\mathcal{B}} \langle \nabla u_Q, c(\nabla e_Q)\rangle\, dV$$

and $u_Q = D\tilde{\Phi}(I_{\mathcal{B}})^{-1}(Ql_0)$; i.e., u_Q is the unique solution in $\mathcal{C}_{\text{sym}}$ of the linearized equations with load $Ql_0 \in \mathcal{L}_e$.

Recall that the pairing between loads $l = (B, \tau)$ and configurations (or displacements) is given by

$$\langle l, \phi\rangle = \int_{\mathcal{B}} B(X) \cdot \phi(X)\, dV + \int_{\partial\mathcal{B}} \tau(X) \cdot \phi(X)\, dA = \text{trace}\, k(l, \phi)$$

and physically represents a potential for the working of the loads. Observe that if $l \in \mathcal{L}_e$, then $\langle l, Q^T I_{\mathcal{B}}\rangle = \text{trace}(AQ) = \text{trace}(AQ^T) = \langle l, QI_{\mathcal{B}}\rangle$ for all $Q \in SO(3)$.

Remark. In the second term of X_2 and f we can replace l_0 by l. Indeed, the difference is of higher order, so the use of l_0 is sufficient for subsequent applications.

To prove 7.3, we shall show that X_2 is a gradient field on $SO(3)$ which, by the remarks following 7.2., is sufficient.

We proceed in two parts. Let us first show that $X_A(Q)$ is the gradient of $\langle l, Q^T I_{\mathcal{B}}\rangle$ on all of $SO(3)$.

7.4. Lemma. *Let $l \in \mathcal{L}$ and $A = k(l)$. Let the vector field X_A on $SO(3)$ be defined by $X_A(Q) = \text{skew}(QA) \cdot Q$ as above and let the map $\tilde{l}$ of $SO(3)$ to $\mathbb{R}$ be defined by $\tilde{l}(Q) = \langle l, Q^T I_{\mathcal{B}}\rangle$. Then $X_A = -\text{grad}\, \tilde{l}$.*

Proof. Two simple, but useful observations are that if $E, W \in M_3$, with $W \in$ skew, then

$$\langle E, W\rangle = \langle \text{skew}\, E, W\rangle, \tag{1}$$

and if $E \in M_3$, $l \in \mathcal{L}$ and $\phi \in \mathcal{C}$, then

$$\langle l, E\phi\rangle = \langle E, k(l, \phi)\rangle. \tag{2}$$

To prove 7.4, we compute as follows:

$$\begin{aligned} d\tilde{l}(Q) \cdot (WQ) &= \langle l, (WQ)^T I_{\mathcal{B}}\rangle \\ &= \langle (WQ)^T, k(l, I_{\mathcal{B}})\rangle \quad \text{by (2)} \\ &= \langle (WQ)^T, A\rangle = \langle W^T, QA\rangle \\ &= -\langle W, \text{skew}(QA)\rangle \quad \text{by (1)} \\ &= -\langle WQ, \text{skew}(QA) \cdot Q\rangle = -\langle WQ, X_A(Q)\rangle. \end{aligned}$$ ▼

This result takes care of the first term of $\tilde{X}_2$. To deal with the second term, we need a special case of Betti's reciprocity theorem:

7.5. Lemma. $\langle QI_0, u_{WQ}\rangle = \langle (WQ)\, I_0, u_Q\rangle$ *for* QI_0 *and* $(WQ)\, I_0 \in \mathrm{Sym}$.

This is a direct consequence of the symmetry of $D\Phi(I_{\mathscr{B}})$, *i.e.*, of the elasticity tensor. It is also proved in standard references; for example, see TRUESDELL & NOLL [1965]; p. 325.

To prove 7.3., we shall also need to calculate the second derivative of the skew component of Φ; *i.e.*, of $\mathscr{F}(\phi) = \mathrm{Skew}\,[k(\Phi(\phi))]$. Surprisingly, this second derivative depends *only* on the classical elasticity tensor c. Recall from § 2 that we regard c as a linear map of sym to itself and that we write $e = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$.

7.6. Lemma. Let $\mathscr{F}: \mathscr{C} \to \mathrm{skew}$ *be defined by* $\mathscr{F}(\phi) = \mathrm{Skew}\,[k(\Phi(\phi))]$. *Then* $\mathscr{F}(I_{\mathscr{B}}) = 0$, $D\mathscr{F}(I_{\mathscr{B}}) = 0$ *and*

$$D^2\mathscr{F}(I_{\mathscr{B}})\,(u, u) = 2\,\mathrm{Skew}\left(\int_{\mathscr{B}} \nabla u \cdot \mathsf{c}(e)\, dV\right) = -2\,\mathrm{Skew}\, k(l_u, u)$$

where $l_u = (b_u, \tau_u)$, $b_u = -\mathrm{DIV}\,(\mathsf{c}(e))$ *and* $\tau_u = \mathsf{c}(e) \cdot N$. *If we identify* skew *with* $\mathbb{R}^3$, *this becomes*

$$-D^2\mathscr{F}(I_{\mathscr{B}})\,(u, u) = \int_{\mathscr{B}} b_u \times u\, dV + \int_{\partial\mathscr{B}} \tau_u \times u\, dA.$$

Proof. By Lemma 4.2., $\mathscr{F}(\phi) = \mathrm{Skew}\left[\int_{\mathscr{B}} P\, dV\right]$ where P is the first Piola-Kirchhoff stress tensor. We have $P(I_{\mathscr{B}}) = 0$, so $\mathscr{F}(I_{\mathscr{B}}) = 0$. Also, $D\mathscr{F}(I_{\mathscr{B}}) \cdot u = \mathrm{skew} \int_{\mathscr{B}} \frac{\partial P}{\partial F} \cdot \nabla u\, dV = \mathrm{Skew} \int_{\mathscr{B}} \mathsf{c} \cdot e\, dV = 0$, since $\mathsf{c} \cdot e$ is symmetric and since $\frac{\partial P}{\partial F}(I_{\mathscr{B}}) = \mathsf{c}$. To compute $D^2\mathscr{F}$, we shall need to use the fact that S is symmetric. Write $P = FS$ and use the product rule to obtain $D_F P(F) \cdot \nabla u = \nabla u \cdot S(F) + F D_F S(F) \cdot \nabla u$. Thus, as $S(I_{\mathscr{B}}) = 0$,

$$D_F^2 P(I_{\mathscr{B}}) \cdot (\nabla u, \nabla v)$$
$$= \nabla u \cdot D_F S(I_{\mathscr{B}}) \cdot \nabla v + \nabla v \cdot D_F S(I_{\mathscr{B}}) \cdot \nabla u + D_F^2 S(I_{\mathscr{B}}) \cdot (\nabla u, \nabla v).$$

Now $D_F S(I_{\mathscr{B}}) \cdot \nabla u = D_C S(I_{\mathscr{B}}) \cdot (\nabla u + \nabla u^{\mathrm{T}}) = \mathsf{c} \cdot e$ and $D_F^2 S(I_{\mathscr{B}})$ is symmetric, so

$$D^2\mathscr{F}(I_{\mathscr{B}})\,(u, u) = \mathrm{Skew} \int_{\mathscr{B}} D_F^2 P(I_{\mathscr{B}}) \cdot (\nabla u, \nabla u)\, dV$$
$$= 2\,\mathrm{Skew}\left(\int_{\mathscr{B}} \nabla u \cdot \mathsf{c}(e)\, dV\right).$$

Finally, this equals

$$-2\,\mathrm{Skew}\left\{\int_{\mathscr{B}} b_u \otimes u\, dV + \int_{\partial\mathscr{B}} \tau_u \otimes u\, dA\right\}$$

by the divergence theorem, so the last statement follows. ▼

7.7. Example. For a homogeneous isotropic material,

$$\mathsf{c}(e) = \lambda\,(\mathrm{trace}\; e)\, I + 2\mu e$$

where $e = \frac{1}{2}[\nabla u + (\nabla u)^{\mathrm{T}}]$ and λ, μ are the Lamé moduli. Thus

$$D^2\mathscr{F}(I_{\mathscr{B}})\,(u, u) = 2\,\mathrm{Skew}\left(\int_{\mathscr{B}} \{\lambda \nabla u \cdot [\mathrm{trace}\,(\nabla u)]\, I + 2\mu \nabla u \cdot e\}\, dV\right)$$
$$= 2\,\mathrm{Skew} \int_{\mathscr{B}} \{\lambda\, [\mathrm{trace}\,(\nabla u)]\, \nabla u + \mu \nabla u \cdot \nabla u\}\, dV. \quad ■$$

Let us next see what 7.6 says about the quadratic term G in the Taylor expansion of F. For $\phi \in \mathscr{C}_{\mathrm{sym}}$ we have the identity

$$\mathscr{F}(\phi) = F P_e \Phi(\phi)$$

where $P_e\colon \mathscr{L} \to \mathscr{L}_e$ is the projection and F is the mapping given by 4.3. Thus, because $D\mathscr{F}$ and DF are zero at $I_{\mathscr{B}}$ and 0 respectively, and $P_e\, D\Phi(I_{\mathscr{B}}) = D\Phi(I_{\mathscr{B}})$, we get

$$D^2\mathscr{F}(I_{\mathscr{B}})\,(u, v) = D^2 F(0)\, D(\Phi(I_{\mathscr{B}}) \cdot u, D\Phi(I_{\mathscr{B}}) \cdot v).$$

Let $u_l = D\Phi(I_{\mathscr{B}})^{-1}\, l$. Then for $l \in \mathscr{L}_e$ we have the identity

$$G(l) = D^2\mathscr{F}(I_{\mathscr{B}})\,(u_l, u_l),$$

i.e.,

$$-G(l) = 2\,\mathrm{Skew}\left[\int_{\mathscr{B}} b \otimes u_l\, dV + \int_{\partial\mathscr{B}} \tau \otimes u_l\, dA\right]$$
$$= 2\,\mathrm{Skew}\, k((b, \tau), u_l)$$

where $b = -\mathrm{DIV}\,(\mathsf{c} \cdot (e_l))$, $\tau = \mathsf{c}(e_l) \cdot N$ and $e_l = \frac{1}{2}[\nabla u_l + (\nabla u_l)^{\mathrm{T}}]$. However, these last equations say exactly that $(b, \tau) = l$, and so we get

$$-\frac{1}{2}\, G(l) = \mathrm{Skew}\, k(l, u_l). \tag{3}$$

Completion of the proof of 7.3. The derivative of $Q \mapsto \langle l_0, \frac{1}{2}\lambda\, Q^{\mathrm{T}} u_Q\rangle$ in the direction WQ is given by λ times

$$\left\langle l_0, \frac{1}{2}(WQ)^{\mathrm{T}} u_Q\right\rangle + \left\langle l_0, \frac{1}{2} Q^{\mathrm{T}} u_{WQ}\right\rangle$$
$$= \langle l_0, (WQ)^{\mathrm{T}} u_Q\rangle \quad \text{(by Betti reciprocity, 7.5)}$$
$$= -\langle Q l_0, W u_Q\rangle$$
$$= -\langle W, k(Q l_0, u_Q)\rangle \quad \text{by (2)}$$

$$= -\langle W, \text{skew}\, k(QI_0, u_Q)\rangle \quad \text{by (1)}$$

$$= -\langle WQ, \text{skew}\, k(QI_0, u_Q)\, Q\rangle$$

$$= \left\langle WQ, \frac{1}{2} G(QI_0)\cdot Q\right\rangle \quad \text{by (3).} \quad \blacksquare$$

§ 8. Bifurcation Analysis of Problems of Type I

We now discuss the solutions of the basic equation

$$H(\lambda, A, n; Q) \equiv \text{Skew}\,(QA) - \lambda \bar{F}(\lambda, \text{Sym}\,(QA), Qn) = 0 \qquad (1)$$

for the load $l = (A, n)$ near a load $l_0 = (A_0, n_0)$ of type 1 (having an axis of equilibrium) and for λ near 0. We shall also obtain the stability of the solutions and finally we shall compare our results with those of STOPPELLI [1958]. For loads of type 1 we need to do a bifurcation analysis on the circle S_{A_0} corresponding to the degenerate zero set of H when $\lambda = 0$ and $l = l_0$. The analysis has some features in common with the papers of HALE [1977] and of HALE & TABOADA [1981].

Without loss of generality we can assume that $A_0 = \text{diag}\,(a, -a, -c)$ where $0 \neq a^2 \neq c^2$. Thus, from § 6 the set S_{A_0} of zeros of skew (QA) for $Q \in SO(3)$ is given explicitly by the following two points and circle:

$$S_{A_0} = \{\text{diag}\,(1, -1, -1), \text{diag}\,(-1, 1, -1)\} \cup C_{A_0} \qquad (2)$$

where

$$C_{A_0} = \left\{ Q = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| x = \cos\theta, y = \sin\theta \right\}.$$

The loads corresponding to the two points are $A_0^* = \text{diag}\,(a, a, c)$ and $A_0^{**} = \text{diag}\,(-a, -a, c)$.

From 7.3, we are led to study the critical points of $f(Q) = \langle l, Q^T I_{\mathscr{B}}\rangle + \frac{1}{2}\lambda\langle l_0, Q^T u_Q\rangle$ on C_{A_0}. Note that the divergence theorem implies that

$$\langle l_0, Q^T u_Q\rangle = \int_{\mathscr{B}} \langle \nabla u_Q, c(e_Q)\rangle\, dV \qquad (3)$$

where $u_Q = D\tilde{\Phi}(I_{\mathscr{B}})^{-1}(QI_0)$ and $e_Q = \frac{1}{2}[\nabla u_Q + (\nabla u_Q)^T]$. Thus the function f is computable from linearized elasticity alone, which leads to the curious observation that our "second order" nonlinear elasticity here involves no more data than linear elasticity, but merely processes the information in a different way.

If we write $Q = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then f becomes a polynomial of degree 2 in (x, y). Write the two terms of f as

$$f(Q) = f(x, y) = (b_0 + b_1 x + b_2 y) + \frac{1}{2}\lambda(a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6), \qquad (4)$$

which defines the numerical constants b_0, b_1, b_2 and $a_1, \dots, a_6$. Next, define new parameters $\alpha_1, \dots, \alpha_6$ by writing

$$f^*(x, y) = \frac{2}{\lambda} f(x, y)$$

and letting

$$f^*(x, y) = \alpha_1 x^2 + \alpha_2 xy + \alpha_3 y^2 + \alpha_4 x + \alpha_5 y + \alpha_6. \qquad (5)$$

Note that $\alpha_1, \dots, \alpha_6$ depend on our parameters λ, l as well as on the elastic moduli of the material. Thus,

$$\alpha_1 = a_1, \quad \alpha_2 = a_2, \quad \alpha_3 = a_3,$$
$$\alpha_4 = \frac{2}{\lambda} b_1 + a_4, \quad \alpha_5 = \frac{2}{\lambda} b_2 + a_5, \quad \alpha_6 = \frac{2}{\lambda} b_0 + a_6. \qquad (6)$$

Replacing l_0 by Ql_0, where Q is as in (2), effects a rotation of the x-y plane. Thus, by rotating l_0 if necessary, we can assume $\alpha_2 = 0$.

Let us fix α_1, α_3 and consider the bifurcations of zeros of $\frac{df^*}{d\theta} = 2(\alpha_3 - \alpha_1)\, xy + \alpha_5 x - \alpha_4 y$ on S^1 (*i.e.*, of critical points of f^* on S^1) with α_4 and α_5 as parameters.

Set $M = \left\{(\alpha_4, \alpha_5, \theta) \in \mathbb{R}^2 \times S^1 \middle| \frac{df^*}{d\theta}(\alpha_4, \alpha_5, \theta) = 0\right\}$, the manifold of critical points of f^*. Indeed, M is a manifold and can be parametrized by $\varrho: \mathbb{R} \times S^1 \to M$, $\varrho(\mu, \theta) = (-2(\alpha_1 + \mu)\cos\theta, -2(\alpha_3 + \mu)\sin\theta, \theta)$. Denote by $\pi: \mathbb{R}^2 \times S^1 \to \mathbb{R}^2$ the projection onto the first factor.

8.1. Lemma. *Set*

$$\Delta = [2(\alpha_1 - \alpha_3)^2 - \alpha_4^2 - \alpha_5^2]^3 - 108\alpha_4^2\alpha_5^2(\alpha_1 - \alpha_3)^2. \qquad (7)$$

If $\alpha_1 - \alpha_3 \neq 0$, *then* $\pi: M \to \mathbb{R}^2$ *is a proper stable map in* (α_4, α_5)*-space, and its set of critical values is the astroid defined by* $\Delta = 0$ *(see Figure 4 below).*

Since the number of points in $\pi^{-1}(\alpha)$ $\left(i.e., \text{the zeros of } \frac{df^*}{d\theta} \text{ at } \alpha = (\alpha_4, \alpha_5)\right)$ is a constant over $\Delta < 0$ or $\Delta > 0$, we obtain

8.2. Corollary. $\frac{df^*}{d\theta}$ *has 4 zeros if* $\Delta > 0$, *and has 2 zeros if* $\Delta < 0$.

Proof of Lemma 8.1. The critical set Σ of $\pi \circ \varrho: \mathbb{R} \times S^1 \to \mathbb{R}^2$ is $\{(\mu, \theta) \in \mathbb{R} \times S^1 \mid \alpha_1 \sin^2\theta + \alpha_3\cos^2\theta + \mu = 0\}$. Thus, the set of critical values of π can be parametrized

$$\alpha_4 = -2(\alpha_1 - \alpha_3)\cos^3\theta,$$
$$\alpha_5 = 2(\alpha_1 - \alpha_3)\sin^3\theta.$$

Since Σ consists of 4 cusp points and 4 fold lines and since $\pi \circ \varrho \,|\, \Sigma$ is 1-1, a result of WHITNEY (see MATHER [1969] or GOLUBITSKY & GUILLEMIN [1973]) implies that $\pi \circ \varrho$ is a stable map.

Eliminating θ produces the bifurcation set

$$[2(\alpha_1 - \alpha_3)]^{\frac{2}{3}} = \alpha_4^{\frac{2}{3}} + \alpha_5^{\frac{2}{3}}.$$

For $\alpha_1 - \alpha_3 \neq 0$, (8) describes the astroid shown in Figure 4.

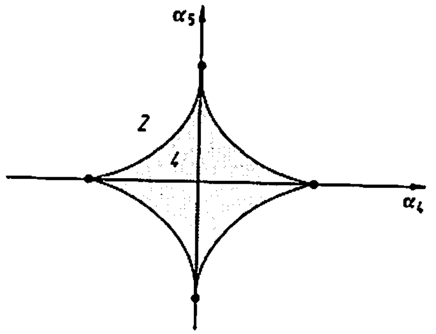


Fig. 4

Next, observe that for real numbers A, B and C,

$$A + B + C = 0 \text{ if and only if } A^3 + B^3 + C^3 = 3ABC \qquad (10)$$

by virtue of the identity $A^3 + B^3 + C^3 - 3ABC = (A + B + C)(A^2 + B^2 + C^2 - AB - BC - CA)$. Applying (10) to (9) shows that (9) is equivalent to $\alpha_4^2 + \alpha_5^2 - 2(\alpha_1 - \alpha_3)^2 = -3\alpha_4^{\frac{2}{3}}\alpha_5^{\frac{2}{3}}(2(\alpha_1 - \alpha_3))^{\frac{2}{3}}$. Cubing both sides gives the stated conclusion. ■

The family $\frac{df^*}{d\theta}$ of functions on S^1 with parameters α_4, α_5 enjoys a universal property. Consider a perturbed family $\frac{df^*}{d\theta} + g(\lambda, p, \theta)$, with $g(0, 0, \theta) = 0$ for $(\lambda, p) \in \mathbb{R} \times \mathbb{R}^m$. To each (λ, p), denote by $M_{\lambda,p} = \left\{(\alpha_4, \alpha_5, \theta) \,\middle|\, \left(\frac{df^*}{d\theta} + g\right)(\lambda, p, \alpha_4, \alpha_5, \theta) = 0\right\}$ the "manifold" of zeros.

8.3. Lemma. *For (λ, p) sufficiently small, the sets $M_{\lambda,p}$ are manifolds and there exist two smooth families of diffeomorphisms $\psi_{\lambda,p}: \mathbb{R}^2 \to \mathbb{R}^2$ and $\Psi_{\lambda,p}: M_{\lambda,p} \to M$ defined for λ, p sufficiently small, such that $\pi \circ \Psi_{\lambda,p} = \psi_{\lambda,p} \circ \pi$, and $\Psi_{0,0} =$* identity, $\psi_{0,0} =$ identity.

Proof. For λ, p sufficiently small, the map $\varrho_{\lambda,p}: \mathbb{R} \times S^1 \to M_{\lambda,p}$ with $\varrho_{\lambda,p}(\mu, \theta) = \left(-2(\alpha_1 + \mu)\cos\theta + \sin\theta \frac{\partial g}{\partial\theta},\ -2(\alpha_3 + \mu)\sin\theta - \cos\theta \frac{\partial g}{\partial\theta}, \theta\right)$ defines a parametrization of $M_{\lambda,p}$. By Lemma 8.1, $\pi \circ \varrho_{\lambda,p}: \mathbb{R} \times S^1 \to \mathbb{R}^2$ is an unfolding of the proper stable map $\pi \circ \varrho$. Thus, one can find diffeomorphisms $\Psi^*_{\lambda,p}$, $\psi_{\lambda,p}$ of $\mathbb{R} \times S^1$ and $\mathbb{R}^2$ respectively such that $\psi_{\lambda,p} \circ (\pi \circ \varrho_{\lambda,p}) = (\pi \circ \varrho) \circ \Psi^*_{\lambda,p}$. This lemma follows by letting $\Psi_{\lambda,p} = \varrho \circ \Psi^*_{\lambda,p} \circ \varrho^{-1}_{\lambda,p}$. ■

Now we are ready to state our main result on the number of solutions near loads of type 1. Let $l = l(p)$ depend smoothly on a parameter p in $\mathbb{R}^m$, with $l(0) = l_0$. Recall that Δ is defined by (7), a_1, a_2 and a_3 by (4).

8.4. Theorem. *Let l_0 be a load of type 1 with $k(l_0) = (a, -a, -c)$, $0 \neq a^2 \neq c^2$, $a_2 = 0$, and $a_1 \neq a_3$. Then there exists a (smooth) function $\tilde{\Delta}(\lambda, p)$, $\tilde{\Delta}(\lambda, 0) = \Delta(a_4, a_5) + O(\lambda)$ defined for (λ, p) sufficiently small and $\lambda > 0$ such that the traction problem has:*

(i) four solutions for the load $\lambda l(p)$ if $\tilde{\Delta}(\lambda, p) < 0$ (two of them near C_{A_0}).

(ii) six solutions for the load $\lambda l(p)$ if $\tilde{\Delta}(\lambda, p) > 0$ (four of them near C_{A_0}).

Proof. Finding zeros of $\tilde{X}$ (*cf.* § 7) on $C_{A_0}\,(= S^1)$ is the same as finding zeros of $\left\langle -\tilde{X}, \frac{\partial}{\partial\theta}\right\rangle = \left\langle -\tilde{X}_2, \frac{\partial}{\partial\theta}\right\rangle + \frac{1}{2}\lambda g(\lambda, p, \theta) = \frac{df}{d\theta} + \frac{1}{2}\lambda g$ or $\frac{df^*}{d\theta} + g$, where $g(0, 0, \theta) = 0$. Let $\psi_{\lambda,p}$ be the family of diffeomorphisms found in Lemma 8.3. Take $\tilde{\Delta}(\lambda, p) = \Delta \circ \psi_{\lambda,p} \circ k_\lambda$, where $k_\lambda(p) = \left(\frac{zb_1 l(p)}{\lambda} + a_4, \frac{zb_2 l(p)}{\lambda} + a_5\right)$, which has the desired property. ■

Next, we wish to determine the "generic" structure of the bifurcation set $\mathscr{K} = \{\tilde{\Delta} = 0\}$ in (λ, p) space, $\lambda > 0$.

If $m = 0$ and $\Delta(\alpha_4, \alpha_5) = \Delta(a_4, a_5) \neq 0$, then it is clear that $\mathscr{K} = \emptyset$. Indeed, our traction problem has two solutions near C_{A_0} if $\Delta(a_4, a_5) < 0$, and four solutions near C_{A_0} if $\Delta(a_4, a_5) > 0$.

For $m = 1$, consider $k_1: \mathbb{R} \to \mathbb{R}^2$. This represents a line assumed to intersect the astroid transversely if they meet. Notice that $\mathscr{K}_\lambda = \{p \mid (\lambda, p) \in \mathscr{K}\}$ is the inverse image of the astroid (defined by equation (9)), under the map $h_\lambda: p \mapsto \psi_{\lambda,p} \circ k_\lambda(p)$. Recall that $l(\lambda p) = \lambda l(p) + O(\lambda^2)$ and consider the map

$$\tilde{h}_\lambda: p \mapsto h_\lambda(\lambda p) = \psi_{\lambda,\lambda p} \circ (k_1(p) + O(\lambda)).$$

Since the astroid is bounded and $\psi_{\lambda,p}$ is close to the identity, there exists an interval $(-M, M)$ such that $\tilde{\mathscr{K}}_\lambda = \{p \mid \lambda p \in \mathscr{K}_\lambda\} \subset (-M, M)$ for $\lambda > 0$ and for λp sufficiently small. Applying the isotopy theorem for transversal maps (see *e.g.*, HIRSCH [1976]) to the family $\tilde{h}_\lambda$ through $\tilde{h}_0 = k_1$, we conclude that the bifurcation set $\mathscr{K}$ consists in 0, 2, or 4 curves with slopes given by the inverse image under k_1 of the astroid) (see Figures 5, 6). Thus, for example, by choosing $p \neq 0$ sufficiently small, and letting $\lambda \to 0$ (consider the load $\lambda l(p)$), one can pass from ı parameter region where there are two solutions near the circle (four in all) to ›ne where there are four near the circle (six in all) and back again to the two- ›olution region (see Figure 5). Such a situation is not dealt with in the analysis ›f STOPPELLI [1958].

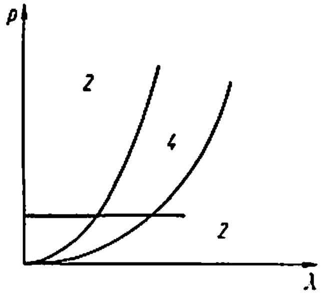


Fig. 5

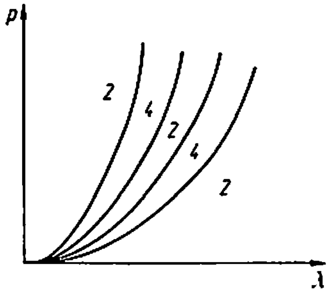


Fig. 6

For $m \geqq 2$, let us suppose that the affine map: $k_1 : \mathbb{R}^m \to \mathbb{R}^2$ is surjective Without loss of generality, we may also assume that $b_1(p) = p_1$ and $b_2(p) = p_2$, where $p = (p_1, p_2, z)$. Notice that $\mathscr{K}_{\lambda,z} = \{(p_1, p_2) \mid (\lambda, p_1, p_2, z) \in \mathscr{K}\}$ is the inverse image of the astroid under the map $h_{\lambda,z} : (p_1, p_2) \mapsto \psi_{\lambda,p_1,p_2,z} \circ k_\lambda(p_1, p_2)$ and consider the map

$$\tilde{h}_{\lambda,z} : (p_1, p_2) \mapsto h_{\lambda,z}(\lambda p_1, \lambda p_2) = \psi_{\lambda,\lambda p_1,\lambda p_2,z}(2p_1 + a_4, 2p_2 + a_5).$$

As before, $\tilde{\mathscr{K}}_{\lambda,z} = \{(p_1, p_2) \mid (\lambda p_1, \lambda p_2) \in \mathscr{K}_{\lambda,z}\}$ is bounded uniformly for $\lambda > 0$ and for λp sufficiently small. Applying the isotopy theorem for transversal maps to the family $\tilde{h}_{\lambda,z}$ through $\tilde{h}_{0,0} = k_1$ we conclude that the bifurcation set is a cylinder-like set along the z-axis with base a cone over the astroid in p_1, p_2 space. The first order approximation of this cone is given by the cone over the astroid in the plane $\lambda = 1$, centered at $(-\frac{1}{2}a_4, -\frac{1}{2}a_5)$ with "size" $4\,|a_1 - a_3|$ (see Figure 7).

Next we discuss the stability of the solutions corresponding to loads near a load of type 1. This can be determined by combining our stability results for loads

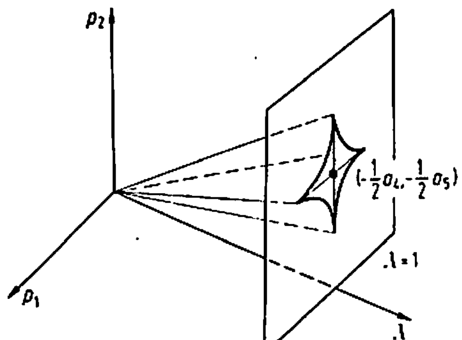
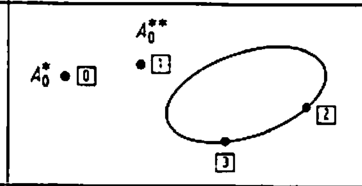
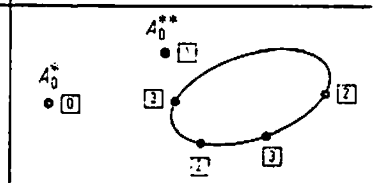
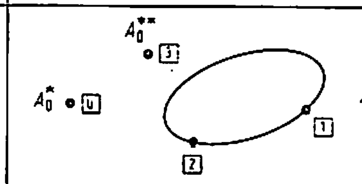
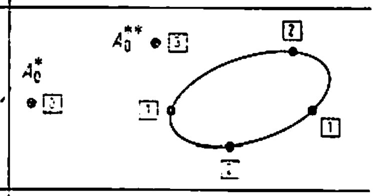
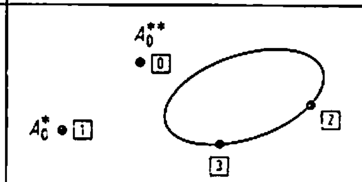
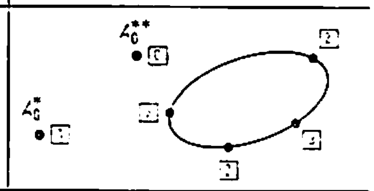
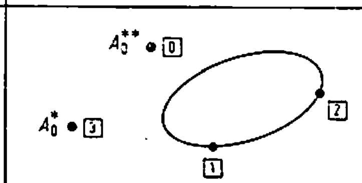
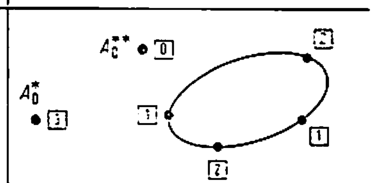
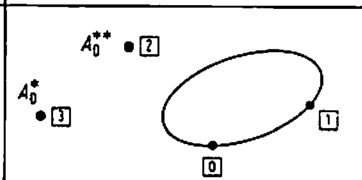
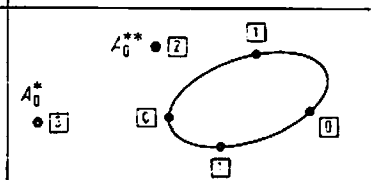
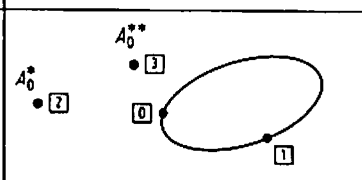
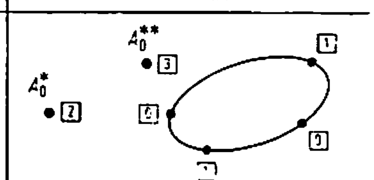


Fig. 7

of type 0 (Theorem 5.5) together with well-known stability results for the cusp. We make the same assumptions as those in Theorem 8.4.

8.5. Theorem. *Let* $A_0 = \operatorname{diag}(a, -a, -c)$, $A_0^* = \operatorname{diag}(a, a, c)$ *and* $A_0^{**} = \operatorname{diag}(-a, -a, c)$ *as above. The indices of the bifurcating solutions are given by the boxed numbers in Table 3. (Recall that stable solutions have index* $= 0$.) *In each case the circle represents* C_{A_0} *defined by equation (2).*

Table 3

Values of c, a	Values of $\tilde{\Delta}$ (see Theorem 8.4)	
	$\tilde{\Delta} < 0$	$\tilde{\Delta} > 0$
$c < \lvert a \rvert$, $a < 0$ $c < a$		
$c < \lvert a \rvert$, $a < 0$ $c > a$		
$c < \lvert a \rvert$, $a > 0$ $c < -a$		
$c < \lvert a \rvert$, $a > 0$ $c > -a$		
$c > \lvert a \rvert$, $a > 0$		
$c > \lvert a \rvert$, $a < 0$		

Note that the stable solutions bifurcate off the circle when $c > |a|$. In all other cases the solutions near the circle are unstable.

8.4. Example. Let $\mathscr{B} \subset \mathbb{R}^3$ be a region with unit volume and let the load be given by $l_0 = (0, \tau_0)$ where

$$\tau_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix} N, \quad 0 \neq a^2 \neq c^2,$$

where N is the unit outward normal on $\partial\mathscr{B}$. Consider a homogeneous isotropic hyperelastic material whose linearized elasticity tensor $\mathfrak{c}$ has Lamé moduli λ, μ (see Example 7.7) and is stable and strongly elliptic; *i.e.*, $\mu > 0$, $2\mu + 3\lambda > 0$. Thus, $k(l_0) = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}$ by the divergence theorem, and so l_0 is a load of type 1. It is easy to check that

$$u_Q(X) = \mathfrak{c}^{-1}\left[Q \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix} \right] X = \mathfrak{c}^{-1} \begin{bmatrix} ax & ay & 0 \\ ay & -ax & 0 \\ 0 & 0 & -c \end{bmatrix} X,$$

for

$$Q = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$x^2 + y^2 = 1$, where $\mathfrak{c}^{-1}(F) = \dfrac{F}{2\mu} - (\text{trace } F)\dfrac{\lambda I}{2\mu(2\mu + 3\lambda)}$. Hence,

$$\langle l_0, Q^T u_Q \rangle = \int_{\mathscr{B}} \langle \nabla u_Q, \mathfrak{c}(\nabla u_Q) \rangle \, dV \quad (\text{by } (3))$$

$$= \left\langle \mathfrak{c}^{-1} Q \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}, \; Q \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix} \right\rangle$$

$$= \left\langle \frac{Q \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}}{2\mu} + \frac{c\lambda I}{2\mu(2\mu + 3\lambda)}, \; Q \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix} \right\rangle$$

$$= \frac{2a^2 + c^2}{2\mu} - \frac{\lambda c^2}{2\mu(2\mu + 3\lambda)},$$

which is a constant (independent of x, y). In this situation, $\alpha_1 = \alpha_3 = 0$ and so our theorems do not apply. This degenerate problem is discussed in Part II.

8.5. Example. Consider the same traction problem as above, but with a homogeneous anisotropic hyperelastic material whose linearized elasticity tensor is given by $\mathfrak{c}(e) = e - \frac{1}{2} \operatorname{diag} e$. In this case,

$$u_Q(X) = \mathfrak{c}^{-1}\left[Q \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix} \right] X, \quad Q = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\mathfrak{c}^{-1}(F) = F + \operatorname{diag} F$. Then

$$\langle l_0, Q^T u_Q \rangle = \int_{\mathscr{B}} \langle \nabla u_Q, \mathfrak{c}(\nabla u_Q) \rangle \, dV$$

$$= \left\langle \begin{pmatrix} 2ax & ay & 0 \\ ay & -2ax & 0 \\ 0 & 0 & -2c \end{pmatrix}, \begin{pmatrix} ax & ay & 0 \\ ay & -ax & 0 \\ 0 & 0 & -c \end{pmatrix} \right\rangle$$

$$= 4a^2x^2 + 2a^2y^2 + 2c^2.$$

Hence, $\Delta = 8a^{12} > 0$, and so our traction problem for $\lambda_0 l$ has six solutions (four near C_{A_0}), with stability determined by Table 3. ∎

Next we shall discuss how to obtain the results of STOPPELLI [1958] as a special case of our analysis. We refer the reader to the statements of STOPPELLI's results by GRIOLI [1962, p. 58]. In this approach one focuses attention on bifurcations that occur on the circle by examining what happens near a particular location on the circle and $l = l_0$. We can assume that this point is $(1, 0)$, *i.e.*, that $\theta = 0$, with no loss of generality.

First of all, if $\alpha_2 + \alpha_5 \neq 0$, then $(1, 0)$ is not a critical point of f^*, so there are no solutions near $(1, 0)$. We may assume then that $\alpha_2 + \alpha_5 = 0$, in which case the Taylor expansion of f^* about $\theta = 0$ becomes

$$f^*(\theta) = (\alpha_1 + \alpha_6) + \left(-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2}\right)\theta^2 - \frac{\alpha_2}{2}\theta^3$$

$$+ \frac{1}{3}\left(\alpha_1 - \alpha_3 + \frac{\alpha_4}{8}\right)\theta^4 + (\text{higher order terms}).$$

For critical points, we are seeking zeros of

$$\frac{df^*}{d\theta} = 2\left(-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2}\right)\theta - \frac{3}{2}\alpha_2\theta^2 + \frac{4}{3}\left(\alpha_1 - \alpha_3 + \frac{\alpha_4}{8}\right)\theta^3 + O(\theta^4).$$

Case 1. If $-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2} \neq 0$, then $\frac{df^*}{d\theta} = 2\left(-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2}\right)\theta + O(\theta^2)$ and so there it just one solution. This is Theorem F on p. 58 of GRIOLI [1962].

Case 2. If $-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2} = 0$ and $\alpha_2 \neq 0$, then $\frac{df^*}{d\theta} = -\frac{3}{2}\alpha_2\theta^2 + O(\theta^3)$ and so there are 0, 1 or 2 solutions (fold point). This is Theorem G on p. 58 of GRIOLI [1962].

Case 3. If $-\alpha_1 + \alpha_3 - \frac{\alpha_4}{2} = 0$, $\alpha_2 = 0$ but $\alpha_1 - \alpha_3 + \frac{\alpha_4}{8} \neq 0$ then $\frac{df^*}{d\theta} = \frac{4}{3}\left(\alpha_1 - \alpha_3 + \frac{\alpha_4}{8}\right)\theta^3 + O(\theta^4)$, so there are 1, 2 or 3 solutions (cusp point). This is Theorem H on p. 58 of GRIOLI [1962].

Furthermore, if we express our constants α_j $(= a_i)$ in terms of the elasticity nsor c and solutions of the linearized problem using (3) above, we find the same onditions for these three cases as is given on p. 57 of GRIOLI [1962].

Thus we recover the results of STOPPELLI on loads of type I. As was explained the Introduction, however, his analysis is only local on the circle and does not ve the full story of the bifurcation picture, even in this case. The complete furcation analysis, including stability, is summarized by our Figure 7 and Table 3.

The research reported here was partially supported by the U.S. National Science oundation under Grant MCS-78-06718, by the U.S. Army Research Office, contract AAG-29-79-C-0086, and the Miller Institute.

References

M. BALL [1977]. Convexity Conditions and Existence Theorems in Nonlinear Elasticity, *Arch. Rational Mech. Anal.* **63**, 337–403.

M. BALL, R. J. KNOPS & J. E. MARSDEN [1978]. Two examples in nonlinear elasticity, Springer Lecture Notes in Math. # 466, 41–49.

CAPRIZ & P. PODIO GUIDUGLI [1974]. On Signorini's Perturbation Method in Nonlinear Elasticity, *Arch. Rational Mech. Anal.* **57**, 1–30.

FICHERA [1972]. Existence theorems in elasticity, *Handbuch der Physik*, Bd. VI a/2, 347–389, Springer-Verlag.

FRANKEL [1965]. Critical submanifolds of the classical groups and Stiefel manifolds, in *Differential and Combinatorial Topology*, S. S. CAIRNS (ed.), Princeton University Press.

GOLUBITSKY & V. GUILLEMIN [1973]. *Stable Mappings and Their Singularities*, Graduate Texts in Math. # 14, Springer-Verlag.

GOLUBITSKY & J. MARSDEN [1982]. *The Morse lemma in infinite dimensions via singularity theory* (preprint).

GOLUBITSKY & D. SCHAEFFER [1979a]. A theory for imperfect bifurcation via singularity theory, *Comm. Pure Appl. Math.*, **32**, 21–98.

M. GOLUBITSKY & D. SCHAEFFER [1979b]. Imperfect bifurcation in the presence of symmetry, *Commun. Math. Phys.* **67**, 205–232.

G. GRIOLI [1962]. *Mathematical Theory of Elastic Equilibrium*, Ergebnisse der Ang. Mat. # 67, Springer.

J. K. HALE [1977]. Bifurcation near families of solutions, *Proc. Int. Conf. on Differential Equations*, Uppsala, 91–100.

J. K. HALE & P. Z. TABOAS [1980]. Bifurcation near degenerate families, *Journal of Applicable Anal.* **11**, 21–37.

M. HIRSCH [1976]. *Differential Topology*, Graduate Texts in Math. # 33, Springer-Verlag.

R. KNOPS & E. WILKES [1973]. Theory of elastic stability, *Handbuch der Physik* VI a/3, C. TRUESDELL (ed.), Springer-Verlag.

J. MARSDEN & T. HUGHES [1978]. Topics in the Mathematical Foundations of Elasticity, in *Nonlinear Analysis and Mechanics*, Volume II, R. J. KNOPS (ed.), Pitman.

J. MATHER [1969]. Stability of C^∞ mappings: II. *Ann. of Math.* (2) **89**, 254–291.

R. OGDEN [1977]. Inequalities associated with the inversion of elastic stress-deformation relations and their implications. *Math. Proc. Camb. Phil. Soc.* **81**, 313–324.

R. PALAIS [1968]. *Foundations of Global Non-linear Analysis*, Benjamin.

S. RAMANUJAM [1969]. Morse Theory of Certain Symmetric Spaces, *J. Diff. Geometry* **3**, 213–229.

M. REEKEN [1973]. Stability of critical points under small perturbations, *Manuscripta Math.* **8**, 69–92.

A. SIGNORINI [1930]. Sulle deformazioni termoelastiche finite, *Proc. 3rd Int. Cong. Appl. Mech.* **2**, 80–89.

F. STOPPELLI [1958]. Sull'esistenza di soluzioni delle equazioni dell'elastostatica isoterma nel caso di sollecitazioni dotate di assi di equilibrio, *Ricerche Mat.* **6**, (1957) 241–287, **7** (1958) 71–101, 138–152.

A. J. TROMBA [1976]. Almost Riemannian structures on Banach manifolds, the Morse lemma and the Darboux theorem. *Can. J. Math.* **28**, 640–652.

C. TRUESDELL & W. NOLL [1965]. *The Nonlinear Field Theories of Mechanics*, *Handbuch der Physik* III/3, S. FLÜGGE (ed.), Springer-Verlag.

W. VAN BUREN [1968]. On the Existence and Uniqueness of Solutions to Boundary Value Problems in Finite Elasticity, Thesis, Carnegie-Mellon University; Westinghouse Research Laboratories Report 68-107-MEKMARI.

C. C. WANG & C. TRUESDELL [1973]. *Introduction to Rational Elasticity*, Nordhoff.

R. WASSERMAN [1974]. *Stability of Unfoldings*. Springer Lecture Notes in Mathematics, # 393.

University of Southampton
U.K.

and

University of California at Berkeley
U.S.A.

and

State University of New York at Buffalo
U.S.A.

(Received September 2, 1981)