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A SLICE THEOREM FOR THE SPACE OF SOLUTIONS OF EINSTEIN'S EQUATIONS

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Contents:

Abstract:

A slice for the action of a group G on a manifold X at a point $x \in X$ is, roughly speaking, a submanifold S_x which is transverse to the orbits of G near x. Ebin and Palais proved the existence of a slice for the diffeomorphism group of a compact manifold acting on the space of all Riemannian metrics. We prove a slice theorem for the group $\mathcal{D}$ of diffeomorphisms of spacetime acting on the space $\mathscr{E}$ of spatially compact, globally hyperbolic solutions of Einstein's equations. New difficulties beyond those encouncerted by Ebin and Palais arise because of the Lorentz signature of the spacetime metrics in $\mathscr{E}$ and because $\mathscr{E}$ is not a smooth manifold – it is known to have conical singularities at each spacetime metric with symmetries. These difficulties are overcome through the use of the dynamic formulation of general relativity as an infinite dimensional Hamiltonian system (ADM formalism) and through the use of constant mean curvature foliations of the spacetimes in $\mathscr{E}$. (We devote considerable space to a review and extension of some special properties of constant mean curvature surfaces and foliations that we need.) The conical singularity structure of $\mathscr{E}$, the symplectic aspects of the ADM formalism, and the uniqueness of constant mean curvature foliations play key roles in the proof of the slice theorem for the action of $\mathcal{D}$ on $\mathscr{E}$. As a consequence of this slice theorem, we find that the space $\mathcal{G} = \mathscr{E}/\mathcal{D}$ of gravitational degrees of freedom is a stratified manifold with each stratum being a symplectic manifold. The spaces for homogeneous cosmologies of particular Bianchi types give rise to special finite dimensional symplectic strata in this space $\mathcal{G}$. Our results should extend to such coupled field theories as the Einstein–Yang–Mills equations, since the Yang–Mills system in a given background spacetime admits a slice theorem for the action of the gauge transformation group on the space of Yang–Mills solutions, since there is a satisfactory Hamiltonian treatment of the Einstein–Yang–Mills system, and since the singularity structure of the solution set is known.

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Introduction

In gravitational physics, two spacetime solutions of Einstein's equations are considered to be physically equivalent if one is isometric to the other. Therefore, the space $\mathscr{G}$ of isometry classes of spacetime solutions is important in the study of gravitational waves, in spacetime perturbation theory, and in attempts to find a quantum theory of gravity.

We may define $\mathscr{G}$ to be the quotient $\mathscr{E}(V)/\mathscr{D}(V)$, where $\mathscr{E}(V)$ is the space of Einstein solutions on the spacetime manifold V and $\mathscr{D}(V)$ is the group of diffeomorphisms of V. A useful tool for studying such quotient spaces is the concept of a "slice". Roughly speaking, a slice for the action of a group G on a manifold X at a point $x \in X$ is a manifold S_x transversal to the orbit $\mathcal{O}_x$ of x. If the isotropy group G_x of x is trivial, S_x gives a local chart for the space X/G of orbits near x. If x has isotropy, the existence of a slice gives information on the isotropy of nearby points and on the stratification of the space X/G of orbits.

In this paper, we prove a slice theorem for a slightly modified version of $\mathscr{G}$. Define $\bar{\mathscr{E}}(V)$ to be those solutions of Einstein's vacuum field equations which contain a compact, constant mean curvature Cauchy surface. We show that there is a slice for the action of $\mathscr{D}(V)$ on $\bar{\mathscr{E}}(V)$ at every point of $\bar{\mathscr{E}}(V)$. Just how restrictive these assumptions on $\bar{\mathscr{E}}(V)$ are is discussed below.

In the case of Lie groups acting on finite dimensional manifolds, much is known about slice theorems and there is a fairly standard procedure for proving them (see Palais [1960] for a fairly comprehensive review). To go from such results to our theorem, however, involves a number of difficulties: (1) $\bar{\mathscr{E}}$ is infinite dimensional as is the group $\mathscr{D}$ which acts on it. Since the implicit function theorem as well as certain tangent space decompositions are needed in proving slice theorems, care is needed in setting up $\bar{\mathscr{E}}$ and $\mathscr{D}$ properly as Banach manifolds. (2) After arranging (1), one finds that the action of $\mathscr{D}$ on $\bar{\mathscr{E}}$ is only that of a topological group and not that of a Lie group. One must therefore carefully loosen the smoothness requirements on the slice and the group action. (3) As shown by Fischer, Marsden and Moncrief [1980], the space $\bar{\mathscr{E}}$ of vacuum solutions of Einstein's equations is not a manifold; it has conical singularities in the neighborhood of spacetime metrics with nontrivial Killing vector fields. Near such points of $\bar{\mathscr{E}}$, the definition of "slice" must be suitably generalized. (4) The Lorentz signature of the spacetime metrics results in timelike and spacelike diffeomorphisms acting in a significantly different way on $\bar{\mathscr{E}}$. They must therefore be handled separately in the proof of the existence of slices.

Techniques for handling the first two of these problems have been developed in the important works of Ebin [1970] and Palais [1970]. They prove a slice theorem for $\mathscr{D}(\Sigma)$, the group of diffeomorphisms of a compact C^∞ manifold Σ, acting on $\mathcal{M}(\Sigma)$, the manifold of Riemannian metrics on Σ. It is fairly straightforward to use the Ebin–Palais techniques to prove a number of other interesting slice theorems, including (a) the group of conformal transformations acting on $\mathcal{M}(\Sigma)$ (see Fischer and Marsden [1977]) and (b) the group of gauge transformations acting on the space of connections on a principal bundle (i.e., the space of Yang–Mills potentials; see Singer [1978] and Kondracki and Rogulski [1981]). Indeed, their techniques rather easily handle a portion of our theorem. To finish it, however, we need something new.

The results which enable us to complete the slice theorem for $\mathscr{D}(V)$ acting on $\bar{\mathscr{E}}(V)$ concern the uniqueness of CMC (constant mean curvature) hypersurfaces for spacetimes in $\bar{\mathscr{E}}(V)$. Using these results (which are discussed in detail in Choquet-Bruhat, Fischer and Marsden [1979] and in Marsden and Tipler [1980]) together with the formalism and some of the theorems of the 3+1 dynamical formulation of general relativity, we prove our slice theorem essentially "surface-by-surface". The use of 3+1 techniques entails a price since the action of $\mathscr{D}(V)$ is rather awkward (and not manifestly a group action) when $\tilde{\mathscr{E}}(V)$ is broken up into 3+1 data. We emphasize that this is merely an awkwardness in the language of the proof; the theorem itself is stated in spacetime covariant language.

The strongest statement of our slice theorem depends upon the conjecture that every spatially closed, globally hyperbolic, vacuum solution of Einstein's equations admits a complete foliation by constant mean curvature hypersurfaces. Although the evidence supporting this conjecture is strong, no general proof has yet been found (nor have any counterexamples). Even if the conjecture is wrong, however, we still have a slice theorem if we consider a modified version of $\tilde{\mathscr{E}}(V)$ (see section 7).

What are the consequences of a slice theorem for $\mathscr{D}(V)$ acting on $\tilde{\mathscr{E}}(V)$? Firstly, it tells us about the distribution of spacetimes with symmetry in the set of all spacetimes in $\tilde{\mathscr{E}}(V)$. We find, in analogy with $\mathscr{M}(\Sigma)$, that the points corresponding to spacetimes with no symmetry are open and dense in $\tilde{\mathscr{E}}(V)$. The openness follows from the Montgomery–Zippin corollary (see 1.2 below and Palais [1960]) which shows that for any given spacetime g_0 in $\tilde{\mathscr{E}}(V)$, there exists a neighborhood in $\tilde{\mathscr{E}}(V)$ containing no spacetimes with more symmetry than g_0.

Secondly, in the course of proving the slice theorem, we show that the orbits of $\mathscr{D}(V)$ are closed submanifolds in $\tilde{\mathscr{E}}(V)$. We thus find, in accord with Tipler [1980], that closed cosmologies cannot be "almost periodic".

Finally, and most importantly (in light of our original motivation), the slice theorem sheds light on the structure of $\tilde{\mathscr{E}}(V)/\mathscr{D}(V)$, the "degrees of freedom of the gravitational field". Specifically, in combination with the methods of Arms, Marsden and Moncrief [1981], it leads in section 8 to a demonstration that, near generic points (i.e., spacetimes with no killing vector fields), $\tilde{\mathscr{E}}(V)/\mathscr{D}(V)$ is a symplectic manifold; as a whole, it is a stratified symplectic manifold (see Fischer [1970] and Bourguignon [1979] for a discussion of the case $\mathscr{M}(\Sigma)/\mathscr{D}(\Sigma)$). Moreover the slice theorem provides us with a fairly natural set of local coordinates for $\tilde{\mathscr{E}}(V)/\mathscr{D}(V)$ in terms of the conformal "York data" (see, for example, O'Murchadha and York [1974] and section 9). Both the symplectic structure and the local coordinates for $\tilde{\mathscr{E}}(V)/\mathscr{D}(V)$ could prove useful in some of the physical applications mentioned earlier. Similar, but technically less intricate techniques using the structure theory of Arms, Marsden and Moncrief [1981] and the global existence theory of Eardley and Moncrief [1981] can be used to show that the space of gauge equivalent solutions of the Yang–Mills equations form a stratified manifold with symplectic strata.

In considering some of these applications (e.g., perturbations of physical spacetimes), one should recall that thus far our slice theorem deals only with spacetimes which are spatially compact and which have a constant mean curvature hypersurface. Let us discuss each of these restrictions in turn: Most of the machinery necessary for generalizing to asymptotically flat spacetimes is already at hand (see Cantor [1978] and Ashtekar and Hansen [1978]). However it is not clear whether or not one should expect a slice for the action of "asymptotically identity diffeomorphisms" on the space of asymptotically flat spacetimes. This question is currently under study. On the other hand it is believed that a spacetime with a compact Cauchy surface always has one of constant mean curvature (see Marsden and Tipler [1980] for a discussion). This is related to the global existence of solutions of Einstein's equations and the cosmic censorship hypothesis.

The organization of this paper is as follows: In section 1, we discuss slices and slice theorems in general. We state the definition of a slice, discuss the general procedure for proving slice theorems in the finite dimensional case, and describe some of the Ebin–Palais techniques for dealing with infinite dimensional problems. In section 2, we briefly review some of the 3+1 formulation of general relativity and discuss the structure of $\tilde{\mathscr{E}}$. Section 3 summarizes and extends known properties of constant mean curvature hypersurfaces that are needed in our proof. Then section 4 presents the precise statement of our slice theorem. In section 5, we go through the first step in the proof, which is to show that the orbits of $\mathscr{D}(V)$ are closed submanifolds in $\mathscr{E}(V)$. We then complete the proof of the slice theorem in section 6, by providing an explicit construction of a slice in a neighborhood containing a given spacetime. All of the discussion in sections 3–5 assumes that every spacetime containing a constant mean curvature foliation in fact admits an entire foliation by such slices. In section 7, we provide a version of our slice theorem which does not depend upon that assumption. We conclude the paper in sections 8 and 9 with a discussion of some of the implications of the slice theorem, namely the stratification of $\mathscr{E}/\mathscr{D}$ by symplectic manifolds and the parametrization of the slices in terms of York data.

Acknowledgements

An early attempt at a slice theorem which contained some of the ideas of this paper was done in collaboration with Arthur Fischer around 1975. We also thank Judy Arms, Robert Jantzen, Vincent Moncrief, Dick Palais, Iz Singer, Abe Taub and Tony Tromba for their encouragement and suggestions.

1. Generalities on slice theorems

The notion of a slice, and the basic procedures for proving slice theorems, grew out of work of Gleason, Kozul, Montgomery, Yang, Mostow and Palais. As an introduction to our work, we now review a few of these ideas. A thorough survey of the subject is presented by Palais [1960]. For convenience, let us start by presuming that we have a Lie group G which acts smoothly (C^∞) on a manifold X. (Below, we shall weaken these assumptions and thereby obtain a more general definition of slice.) Let gx denote the action of $g \in G$ on $x \in X$, let $\mathcal{O}_x := Gx := \{gx | g \in G\}$ denote the orbit of x, and let $I_x := \{g | gx = x\}$ be the isotropy subgroup of x.

1.1. **Definition.** *A slice at x for the action of G is a submanifold $S_x \subset X$ containing x such that*
(S1) *if $g \in I_x$, then $gS_x = S_x$;*
(S2) *if $g \in G$ and $(gS_x) \cap S_x \neq \emptyset$, then $g \in I_x$; and*
(S3) *there is a local cross-section $\mu : G/I_x \to G$ defined in a neighborhood U of the identity coset such that the map $F : U \times S_x \to X$ defined by $F(u, y) = \mu(u)\, y$ is a diffeomorphism onto a neighborhood V of x.*†

To illustrate some of the properties of slices, it is useful to keep an elementary example in mind (see fig. 1). Take $M = \mathbb{R}^2$ and $G = S^1$ acting by rotations about the origin (0, 0). Here, $\mathcal{O}_{(0,0)} = \{(0, 0)\}$, $I_{(0,0)} = G$, and the slice $S_{(0,0)}$ may be chosen to be any open disc centered at (0, 0). For any other point $(x_1, x_2) \neq (0, 0)$, $\mathcal{O}_{(x_1,x_2)}$ is the circle with center (0, 0) and radius $\sqrt{x_1^2 + x_2^2}$, $I_{(x_1,x_2)}$ is the identity, and we may choose $S_{(x_1,x_2)}$ to be any sufficiently short line segment which is transverse to $\mathcal{O}_{(x_1,x_2)}$.

As this simple example shows, slices are not unique. That is, at each point $x \in X$ there is generally a large family of submanifolds S_x which fulfill the requirements (S1)–(S3) of a slice. In proving a slice theorem, of course, one need only show that at least one slice exists at every point.

A rule for the dimensionality of slices follows directly from condition (S3) in the slice definition: $\dim(S_x) = \dim(X) - \dim(\mathcal{O}_x)$. This rule guarantees that all slices at some particular point x have the same

†R. Palais has emphasized that a slice may be regarded as an equivariant retraction of a neighborhood of $\mathcal{O}_x$ onto $\mathcal{O}_x$. However we shall stress properties (S1)–(S3) for technical convenience.

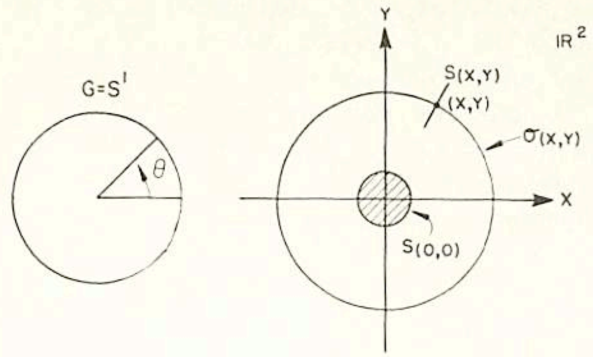

Fig. 1.

dimension. But it also leads in general to slices at different points of X having different dimension (as for example, in fig. 1). The bigger the isotropy group I_x, the larger the dimension of the slice. In the extreme case of a fixed point of the group action, the slice is open in X.

Implicit in the definition of a slice is a mix of localness (in X) and globalness (in G). That is, one can (and usually one must) pick S_x to be quite limited in extent as a submanifold of X, but one must consider all of G in checking that the orbit $\mathcal{O}_x$ intersects S_x only at x. This local-global mix plays an important role in the proof of most slice theorems (including ours).

Let us now review the standard procedures for proving a slice theorem for a given Lie group G acting on a manifold X. One has to prove first that all orbits $\mathcal{O}_x$ are closed embedded submanifolds of X. (Each of them therefore has a well-defined tubular neighborhood.) Next, one finds a G-invariant Riemannian metric $\langle\, ,\rangle_x$ on X. Using this metric, one can define $(T_x\mathcal{O}_x)^\perp$ to be the orthogonal complement to $T_x\mathcal{O}_x$, the tangent space to $\mathcal{O}_x$ at x (see fig. 2). Let $N_\rho(x)$ be the open ρ-ball in $(T_x\mathcal{O}_x)^\perp$; i.e., $N_\rho(x) := \{v \in (T_x\mathcal{O}_x)^\perp | \langle v, v\rangle^{1/2} < \rho\}$. Then by using the smoothness and invertibility of the exponential map in tubular neighborhoods, one proves that for ρ sufficiently small, $S_x := \exp[N_\rho(x)]$ is a slice at *all* $x \in X$.

For a Lie group G acting on a finite dimensional manifold X, most of these steps are automatic. Indeed, if G is compact and X is finite dimensional, then the orbits are automatically closed embedded manifolds and a group-invariant metric can be constructed by averaging. Even for a noncompact Lie group G, the orbits $\mathcal{O}_x$ are always *immersed* submanifolds, but some of them may accumulate on themselves. As long as they don't, and an invariant metric can be found, we have a slice.

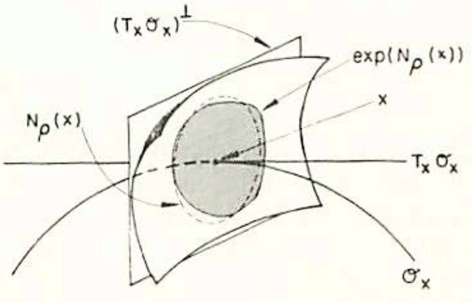

Fig. 2.

In some instances, a modification of the above "exponential method" for proving slice theorems is useful. If (perhaps aided by a choice of local coordinates) we can treat X as an open subspace of a linear space L, then we can forego the exponential map and take S_x to be the ρ-ball $N_\rho(x)$ centered at x in the orthogonal complement to the tangent space to the orbit at x. This "affine slice" is in fact the one we use to prove our theorem.

In a rough sense, the procedure which we have just outlined (exponential or affine version) can be applied to prove slice theorems for groups acting on infinite dimensional as well as finite dimensional manifolds. However, it is necessary to use the proper function spaces to carry this out. To illustrate what these are, we shall now briefly discuss the Ebin–Palais theorem.

Let Σ be a compact manifold (any finite dimension) and let $\mathcal{M}(\Sigma)$ be the set of all Riemannian (positive definite) metrics on Σ. A diffeomorphism $\eta: \Sigma \to \Sigma$ maps one metric $h \in \mathcal{M}(\Sigma)$ to another by pull back $h \mapsto \eta^* h$; symbolically,

$$\eta^* : h \mapsto (h \circ \eta) \mathrm{D}\eta\, \mathrm{D}\eta\,, \tag{1.1}$$

so one finds that the group of diffeomorphisms $\mathcal{D}(\Sigma)$ acts on $\mathcal{M}(\Sigma)$. (Note that $(\eta\zeta)^* h = \zeta^* \eta^* h$, so this is sometimes called a *right* action.) One wants to show that at each $h \in \mathcal{M}(\Sigma)$ there is a slice for this action.

If one allows $\mathcal{M}(\Sigma)$ to include only the smooth (C^∞) Riemannian metrics, then one cannot directly use the implicit function theorem to prove that the orbits are manifolds, nor can one use elementary linear analysis to carry out the tangent space decompositions needed to prove the slice theorem. Such operations can, however, be done in Sobolev spaces and so (following Ebin [1970]) one is led to introduce $\mathcal{M}^s(\Sigma)$, the set of Riemannian metrics of Sobolev class H^s. (See the appendix for a review of the definitions and a discussion of the properties of Sobolev spaces.) $\mathcal{M}^s(\Sigma)$ itself is not a Sobolev space; but as long as $s > \dim(\Sigma)/2$, one finds that $\mathcal{M}^s(\Sigma)$ is an open subset of the Banach space of H^s symmetric 2-tensors on Σ. In the Ebin–Palais theorem one can work with the Sobolev spaces $W^{s,p}$, $s > \dim(\Sigma)/p$. However the case $p = 2$ is most relevant for relativity since it is in these spaces that the initial value problem is well-posed.

One needs a Sobolev-like space for the diffeomorphisms too, and so one chooses $\mathcal{D}^{s'}(\Sigma)$, the diffeomorphisms of class $H^{s'}$. We assume $s' > 1 + \dim(\Sigma)/2$ so that $\mathcal{D}^{s'}(\Sigma)$ is open in the Sobolev manifold $H^{s'}(\Sigma, \Sigma)$ of maps of Σ to itself. One also must have $s' \geq s + 1$ so that $\mathcal{D}^{s'}(\Sigma)$ maps $\mathcal{M}^s(\Sigma)$ to itself (see formula (1.1) and the appendix). Clearly these three conditions on s and s' are compatible.

Unfortunately, though, since multiplication of elements of $\mathcal{D}^{s'}(\Sigma)$ is only a continuous operation, $\mathcal{D}^{s'}(\Sigma)$ is only a topological (C^0) group and not a C^∞ Lie group. (For the gauge group of the Yang–Mills equations one does not have this difficulty.) A slice, as defined in 1.1, can then never be found. One easily broadens definition 1.1, however, by replacing the word "diffeomorphism" by the word "homeomorphism" in condition (S3).

Can one now construct a slice for the action of $\mathcal{D}^{s+1}(\Sigma)$ at any chosen metric $h \in \mathcal{M}^s(\Sigma)$ (assuming $s > \dim(\Sigma)/2$)? In general the answer is no; however the answer is yes for every metric which lives in $\mathcal{M}^{s+1}(\Sigma) \subset \mathcal{M}^s(\Sigma)$. This is because it follows from the Sobolev composition property (see the appendix) that the orbit $\mathcal{O}_h$ is a closed, C^1 embedded, submanifold of $\mathcal{M}^s(\Sigma)$ if $h \in \mathcal{M}^{s+1}(\Sigma)$. Thus Ebin and Palais prove a slice theorem at sufficiently regular metrics. This result is augmented by a "regularization argument", which shows that in the limit $s \to \infty$, the slices in $\mathcal{M}^s(\Sigma)$ continue to exist. Therefore, one obtains a slice theorem for C^∞ diffeomorphisms acting on C^∞ metrics.

We now consider some consequences of the existence of slices. The results hold (in an appropriate form) for both finite and infinite dimensional manifolds. The first is a result obtained by Montgomery and Zippin by other methods.

1.2. **Proposition.** *Let the action of* G *on* X *have a slice at* x. *Then there is a neighborhood* V *of* x *such that* (a) *if* $y \in V \cap S_x$, *then* $I_y \subset I_x$; *and* (b) *if* $y \in V$, *then* I_y *is conjugate to a subgroup of* I_x.

Proof. Conclusion (a) follows directly from property (S2) in definition 1.1. Specifically, consider some $y \in S_x$ and some $g \in I_y$, so that $gy = y$. It follows that $gS_x \cap S_x \neq \emptyset$, and so from (S2) we must have $g \in I_x$. As for conclusion (b), let V be the neighborhood described in property (S3) of the slice definition. Then there must exist some $k \in G$ such that $k^{-1}y \in S_x$. Applying conclusion (a) to $k^{-1}y$, we then find that $k^{-1}I_y k \subset I_x$. ■

What this proposition says, then, is that if a slice exists at x, then x is locally maximal as far as isotropy is concerned. A slice theorem therefore implies that *every* point in X is a local maximum for isotropy.

The next result concerns the structure of the quotient X/G at generic points.

1.3. **Proposition.** *Let the action of* G *on* X *have a slice* S_x *at* x *and let the isotropy group* I_x *be trivial. Then there exists a neighborhood* V *containing* x *such that the collection of* G*-orbits passing through* V *is a manifold which is diffeomorphic to* S_x. (*If* G *is only a topological group, then this collection of orbits is a* C^0*-manifold, locally homeomorphic to* S_x.)

Proof. Since $I_x = \mathrm{id}_G$, property (S3) of the slice definition tells us that there is a neighborhood U of id_G in G and a neighborhood V of x in X such that $U \times S_x$ is diffeomorphic to V. The collection of orbits passing through V is just V/G and thus is diffeomorphic to S_x. ■

If we prove a slice theorem for G acting on X, and if the "generic points" of X (i.e., those for which $I_x = \mathrm{id}_G$) are open in X, then it follows from this corollary that $\hat{X}/G$ has a manifold structure, where $\hat{X} := \{x \in X | I_x = \mathrm{id}_G\}$. What about X itself (the points with isotropy being left in)? Clearly one can't expect X/G to be a manifold, since the space of orbits changes as I_x changes. Still, one can use the slice theorem to show that X/G is a union of manifolds $M_1 \cup M_2 \cup \cdots \cup M_n$ where $M_1 = \hat{X}/G$ as described above and $M_{i+1} \subset \partial M_i$ (the boundary of M_i); thus, X/G is a *stratified manifold*. The stratum containing $\mathcal{O}_{x_0}$, the orbit of $x_0 \in X$, is described as follows. Let $N_{x_0} = \{x \in X | I_{x_0} \text{ is conjugate to } I_x\}$. It is known that N_x is a smooth manifold (see, for instance, Hermann [1968] for a quick proof). Clearly G acts on N_{x_0} and since I_x does not change dimension, it can be shown using the slice theorem that N_{x_0}/G is a manifold; then N_{x_0}/G is the stratum through $\mathcal{O}_{x_0}$. Notice that the strata are ordered by decreasing dimensionality, i.e., increasing symmetry. Unfortunately, much of stratification theory is buried in works on singularity theory by Thom and Mather. However, what we need can be obtained from Palais [1960], Fischer [1970] and Bourguignon [1975].

We close this section with an important further modification of the definition of "slice", which is needed for our slice theorem of $\mathscr{D}(V)$ acting on $\mathscr{E}(V)$. We want to be able to define slices on point sets X which are themselves stratified manifolds. Thus X will contain some singular points. At the nonsingular points of X, the definition 1.1 is still to be used. To see what to do at the singular points, we consider a finite dimensional example: Let X be the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying $x^2 + y^2 - z^2 = 0$ (i.e., a pair of cones joined at the vertex; see fig. 3) and let $G = S^1$ act on X by rotations about the z-axis. Clearly at each point except the vertex (which is the only singular point of X) there are slices in the usual sense and they are roughly of the same nature as the slices for the action of S^1 on $\mathbb{R}$ (fig. 1). At

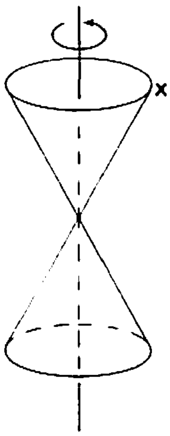

Fig. 3.

the vertex (0, 0, 0), the point set $S_{(0,0,0)}(\rho) := \{(x, y, z) \in X \mid z^2 \le \rho\}$ seems to satisfy most of the conditions of definition 1.1. The only problems are that $S_{(0,0,0)}(\rho)$ is not a submanifold at the vertex (no point set could be), and the map in (S3) is not a diffeomorphism at the vertex (no map could be). The generalization of the definition of slice we shall use calls sets like $S_{(0,0,0)}(\rho)$ slices despite these problems.

The key to the generalized definition of slice is the generalized definition of submanifold. Let X be a stratified manifold with strata X_i. A subset $S \subset X$ is called a *submanifold* of X if for each of these strata, $S \cap X_i$ is a submanifold of X_i in the usual sense. If a group G acts on X and leaves each stratum invariant, one finds that (with the usual Ebin–Palais reservations about function spaces) the original definition of a slice makes sense.

The set in fig. 3 is clearly a stratified manifold: the cone is the union of the vertex point $\{0, 0, 0\} = 0$ and the remainder $X\backslash 0$. The sets 0 and $X\backslash 0$ are both manifolds and the vertex point lies in the boundary of $X\backslash 0$. Each of these manifolds is therefore a stratum. Note that both are invariant under the action of S^1. Hence the above definition applies, and we see that $S_{(0,0,0)}(\rho)$ is indeed a slice in this generalized sense.

The set $\mathscr{E}(V)$ is also a stratified manifold (as shown by Fischer, Marsden and Moncrief [1980] and Arms, Marsden and Moncrief [1981, 1982]). Its strata consist of spacetimes with conjugate isometry groups. Clearly $\mathscr{D}(V)$ leaves each stratum invariant and so again, the generalized definition of slice can be used.

While the example of S^1 acting on the cone provides a useful analog to keep in mind when dealing with $\mathscr{D}(V)$ acting on $\mathscr{E}(V)$, it is misleading in one way. In the cone example, X/S^1 is a smooth manifold, diffeomorphic to the real line $\mathbb{R}$. In this case, the singularity in X and the singularity in performing the quotient have accidently fit together to produce a smooth manifold. Generally, however, X/G is not a smooth manifold. Another example which bears a close resemblance to $\mathscr{E}$ is to let X be the space of zero angular momentum for a particle in $\mathbb{R}^3$, i.e.

$$X = \{(x, p) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \times p = 0\}$$

which is a cone over the compact 3-manifold $M = \{(x, p) \in S^5 \mid x \times p = 0\}$.† Let G = SO(3), the rotation group of $\mathbb{R}^3$ acting simultaneously on x and p. The only singular point is the origin and we have a slice

†M is topologically $S^2 \times S^1/n$ where n is the equivalence relation $(x, y) \sim (-x, -y)$.

at each point. The quotient space X/G is, however, not a manifold but is a stratified set with two strata of dimension 2 and 0 respectively.

The last example brings in a new feature of X/G, namely the fact that each stratum is an (even dimensional) symplectic manifold. This property is also shared by $\mathscr{G} = \tilde{\mathscr{E}}/\mathscr{D}$ as we describe in section 8.

2. The dynamics of general relativity and the space of solutions of Einstein's equations

According to general relativity, a mathematical model for the physical universe is a spacetime (V, g, ψ) where V is a (connected, Hausdorff) 4-manifold, g is a Lorentz metric on V, and ψ is a collection of other fields on V (e.g., electromagnetism, Yang–Mills, fluid, elastic, etc.). Here, we shall consider only vacuum spacetimes (V, g), in which ψ is turned off. Let us label by $\mathscr{L}^s(V)$ the collection of all Lorentz metrics on V which lie in H^s on every nice compact subset C of V. (We call C "nice" if it is a compact four manifold with C^∞ boundary.) We endow $\mathscr{L}^s(V)$ with the compact-open H^s topology; i.e. $g_n \to g$ when $g_n|C \to g|C$ in $H^s(C)$ for every nice compact set C.

In classical (i.e., nonquantum) physics we are usually interested not in all of $\mathscr{L}^s(V)$ but only in the sub-class which satisfy Einstein's (vacuum) equations $\mathrm{Ric}(g) = 0$ i.e. $R_{\mu\nu} = 0$. We call this sub-class $\mathscr{E}^s(V)$.

Just as in the Ebin–Palais theorem, we let $\mathscr{D}^{s'}(V)$ denote the set of all $H^{s'}$ class diffeomorphisms of V to itself (that is, of class $H^{s'}$ when restricted to nice compact sets C, again with the compact-open H^s topology). Then $\mathscr{D}^{s'}(V)$ acts on $\mathscr{L}^s(V)$ (and therefore on $\mathscr{E}^s(V)$) by pullback. Since isometric spacetimes (i.e. spacetimes related by a diffeomorphism) are presumed in general relativity to be physically equivalent, we would like to be able to quotient $\mathscr{E}^s(V)$ by $\mathscr{D}^{s'}(V)$. A slice theorem for $\mathscr{D}^{s'}(V)$ acting on $\mathscr{E}^s(V)$ is an important step in that direction.

Superficially, this may appear to be simply a special case of the Ebin–Palais theorem. However, even if we were to restrict to the (physically bizarre) case of compact V, the Lorentz signature of the metrics contained in $\mathscr{L}^s(V)$ still prohibits a direct application of the Ebin–Palais techniques. The most immediate problem is that the elliptic Laplacian operator, which is used in obtaining $(T_x\mathcal{O}_x)^\perp$ in the Ebin–Palais proof, is replaced by the hyperbolic d'Alembertian, and no one (yet) knows how to *directly* find $(T_x\mathcal{O}_x)^\perp$ using this replacement. Other problems come up as well.

Thus, one needs a different approach, and that is what we will now begin to set-up. This approach does not work for all of $\mathscr{L}^s(V)$; it uses Einstein's equations and is therefore restricted to $\mathscr{E}^s$. It is further restricted as follows. Let V be a fixed four manifold and Σ a 3-manifold. (Σ is presumed, in this paper, to be *compact*, although as noted in the introduction, a generalization to noncompact Σ might also work if suitable "asymptotic flatness" conditions are added.)

(1) (V, g) must be globally hyperbolic with spatial topology Σ; i.e. there exists a spacelike Cauchy surface diffeomorphic to Σ; and

(2) (V, g) must contain a spacelike hypersurface of constant mean curvature (necessarily diffeomorphic to Σ by Budic et al. [1978]).

These two restrictions are quite strong, but many believe them to be physically reasonable (c.f., the discussions of "cosmic censorship" by Penrose [1979]). We use $\bar{\mathscr{E}}^s = \bar{\mathscr{E}}^s(\Sigma, V)$ to denote the Lorentz metrics in $\mathscr{E}^s$ which satisfy the first restriction, while $\tilde{\mathscr{E}}^s = \tilde{\mathscr{E}}^s(\Sigma, V)$ is used to denote those satisfying the second one as well.

Next, we discuss some of the general relativistic machinery which is used in proving the slice theorem. Most of this apparatus concerns the $(3+1)$ dynamical treatment of spacetime and Einstein's equations. We remind the reader (for the last time) that while the (3+1) formalism is used in the proof, the result itself is a covariant one.

The (3+1)-analysis of spacetime physics translates the physical fields from tensors covariant on the 4-manifold V into time-dependent tensors covariant on the 3-manifold Σ. This translation is done using embeddings $i:\Sigma\to V$. Given a spacetime (V, g), if one chooses an embedding i_0, then one locally translates the spacetime metric g into the intrinsic metric on Σ

$$\gamma := i_0^* g, \tag{2.1}$$

i.e. $\gamma(v, w) = g(Ti_0 \cdot v, Ti_0 \cdot w)$ where $v, w \in T_\Sigma$; and the second fundamental form k on Σ

$$k(v, w) := g(Ti_0 \cdot v, \nabla_Z Ti_0 \cdot w). \tag{2.2}$$

[Here, ∇ is the Levi-Civita connection built from g; v, w are vectors tangent to Σ; and Z is the unit forward-pointing normal to $i_0(\Sigma)$.] A one-parameter family of embeddings $i_t:\Sigma\to V$ gives us the time-dependent set $(\gamma(t), k(t))$ to represent g.

To avoid confusion when different metrics and embeddings are under consideration, we shall use the notation $(\gamma(g, i), k(g, i))$ to indicate the intrinsic metric γ and the 2nd fundamental form k which is induced by an embedding i on the spacetime (V, g).

Define $\mathrm{Emb}^{s+1}(\Sigma, V, g)$ to be the set of embeddings $i:\Sigma\to V$ of Sobolev class H^{s+1} such that $i(\Sigma)$ is everywhere spacelike. One finds by standard techniques (see Palais [1968] and Ebin–Marsden [1970]) that $\mathrm{Emb}^{s+1}(\Sigma, V, g)$ is a C^∞ Hilbert manifold, with the tangent space at the embedding i given by $T_i\mathrm{Emb}^{s+1}(\Sigma, V, g) = \{N:\Sigma\to TV \mid N \text{ is class } H^s \text{ and covers } i \text{ (i.e. } N(p)\in T_{i(p)}V \text{ for } p\in\Sigma)\}$.

Each $N\in T_i\mathrm{Emb}^{s+1}(\Sigma, V, g)$ determines an H^s vector field $N_\parallel$ on Σ called the *shift* and an H^{s+1} real valued function $N_\perp$ on Σ called the *lapse*, as follows. Decompose N into its tangential and normal components to $i(\Sigma)$. Then use Ti to "pull these back" to Σ as follows: For $p\in\Sigma$, $N_\parallel(p)\in T_p\Sigma$ and $N_\perp(p)\in\mathbb{R}$ are such that

$$N(p) = T_p i \cdot N_\parallel(p) + N_\perp(p)\, Z(i(p))$$

is the tangent-normal decomposition of $N(p)$. A *slicing* of (V, g) is a curve $i:\mathbb{R}\to\mathrm{Emb}^{s+1}(\Sigma, V, g)$. We will write i_t for its value at $t\in\mathbb{R}$. If one starts at some point $i_0\in\mathrm{Emb}^{s+1}(\Sigma, V, g)$ and if one specifies a rule for choosing $N_\parallel$ and $N_\perp$ as functions of t, then one determines a slicing i_t, at least for some t-interval about $t=0$. If the surfaces $i_t(\Sigma)$ are non-intersecting (this holds if $N_\perp>0$) they define a *foliation* of V. We shall reserve the word "slicing" for the embeddings i_t and the word "foliation" for their images: i.e. a slicing consists of *maps*, while a foliation consists of *surfaces*. The distinction will be important throughout this work.

The space of all pairs (γ, k) of Sobolev classes (H^s, H^{s-1}) $(s>\frac{3}{2})$ also forms a manifold. This manifold will be denoted $T\mathcal{M}^s(\Sigma)$ and is essentially the tangent bundle of the space $\mathcal{M}^s(\Sigma)$ of all Riemannian metrics γ on Σ.

It is also convenient to introduce the *conjugate momentum* π by

$$\pi := -(k - \gamma\, \mathrm{tr}k)^{\#}\mu(\gamma) \tag{2.3}$$

where $\mu(\gamma) = \sqrt{\det\gamma}\, d^3x$ is the volume form of γ, and # denotes that the indices are raised, so π is a symmetric 2-contravariant (indices up) tensor density. In other words, $\pi^{ab} = -(k^{ab} - \gamma^{ab}\kappa)(\det \gamma)^{1/2}$ where $\kappa := k^a{}_a$ is the mean curvature of i. As above we will write $\pi(g, i)$ to indicate that π depends on the spacetime metric g and the embedding i.

The set of pairs (γ, π) of class $(\mathrm{H}^s, \mathrm{H}^{s-1})$ forms an open set in the Banach space of H^s, H^{s-1} symmetric covariant tensors and symmetric contravariant tensor densities. This set will be denoted $\mathrm{T}^*\mathcal{M}^s(\Sigma)$ and is thought of as the cotangent bundle of $\mathcal{M}^s(\Sigma)$.

Now $\mathrm{T}^*\mathcal{M}^s(\Sigma)$ carries a canonical symplectic structure, defined in the same way as for any cotangent bundle, but with the pairing between covectors π and vectors k being given by

$$\langle \pi, k \rangle = \int_\Sigma \pi \cdot k\,,$$

where $\pi \cdot k$ is the contraction of π and k (a density on Σ). The symplectic form Ω at (g, π) is then given by

$$\Omega((h_1, \omega_1), (h_2, \omega_2)) = \langle \omega_2, h_1 \rangle - \langle \omega_1, h_2 \rangle \tag{2.4}$$

(cf. Abraham and Marsden [1978] pp. 178–9). The (weak) symplectic manifold $\mathrm{T}^*\mathcal{M}^s(\Sigma)$ is an appropriate phase space for the dynamical analysis of spacetimes. A path $(\gamma(t), \pi(t))$ in $\mathrm{T}^*\mathcal{M}^s(\Sigma)$ and a path of shift and lapse functions $N_\parallel(t)$, $N_\perp(t)$ determines a spacetime (V, g) with a slicing whose lapse and shift are $N_\perp$ and $N_\parallel$. Conversely, a spacetime with a given slicing determines a unique path $(\gamma(t), \pi(t))$ in $\mathrm{T}^*\mathcal{M}^s(\Sigma)$ and a path $N_\parallel(t)$, $N_\perp(t)$.

A useful picture for the relationship between $\mathscr{L}^s(\mathrm{V})$, $\mathrm{T}^*\mathcal{M}^s(\Sigma)$, and $\mathrm{Emb}^{s+1}(\Sigma, \mathrm{V}, g)$ is given by considering the following space of pairs (g, i):

$$\mathscr{H}^s(\Sigma, \mathrm{V}) = \{(g, i) | g \in \mathscr{L}^s(\mathrm{V}), i \in \mathrm{Emb}^{s+1}(\Sigma, \mathrm{V}, g)\}\,.$$

This space is a bundle over $\mathscr{L}^s(\mathrm{V})$ with projection $\Pi_{\mathscr{L}} : \mathscr{H}^s(\Sigma, \mathrm{V}) \to \mathscr{L}^s(\mathrm{V})$ given by $\Pi_{\mathscr{L}}(g, i) = g$. The fibre $\Pi_{\mathscr{L}}^{-1}(g)$ is obviously $\mathrm{Emb}^{s+1}(\Sigma, \mathrm{V}, g)$. More interestingly, $\mathscr{H}^s(\Sigma, \mathrm{V})$ is also a bundle over $\mathrm{T}^*\mathcal{M}^s(\Sigma)$. Here the projection $\Pi_{\mathrm{T}^*\mathcal{M}} : \mathscr{H}^s(\Sigma, \mathrm{V}) \to \mathrm{T}^*\mathcal{M}^s(\Sigma)$ is given by mapping (g, i) to the data $\gamma(g, i)$, $\pi(g, i)$ which is induced by the embedding $i : \Sigma \to \mathrm{V}$ and the Lorentz metric g. The fibre $\Pi_{\mathrm{T}^*\mathcal{M}}^{-1}(\gamma_1, \pi_1)$ at some point (γ_1, π_1) in $\mathrm{T}^*\mathcal{M}^s(\Sigma)$ consists of pairs (g, i) such that $\gamma(g, i) = \gamma_1$ and $\pi(g, i) = \pi_1$.

Let us now consider what happens when we impose Einstein's equations. Firstly, Einstein's equations place a restriction upon the allowable data sets $(\gamma, \pi) \in \mathrm{T}^*\mathcal{M}^s$ which may occur in a spacetime (V, g) with $g \in \mathscr{E}^s(\Sigma, \mathrm{V})$. Specifically, for *any* hypersurface $i \in \mathrm{Emb}^{s+1}(\Sigma, \mathrm{V}, g)$, $\gamma = \gamma(g, i)$ and $\pi = \pi(g, i)$ must satisfy the constraint equations

$$\Phi(\gamma, \pi) = 0 \tag{2.5}$$

on Σ, where

$$\Phi(\gamma, \pi) := \begin{pmatrix} \mathscr{H}(\gamma, \pi) \\ \mathscr{J}(\gamma, \pi) \end{pmatrix} := \begin{pmatrix} -\frac{1}{2}(\mathrm{tr}\, \pi)^2 + \pi^m{}_n \pi^n{}_m - (\det \gamma)^{1/2} R \\ 2\nabla_a \pi^{ab} \end{pmatrix}. \tag{2.6}$$

The set of all $(\mathrm{H}^s, \mathrm{H}^{s-1})$ class pairs $(\gamma, \pi) \in \mathrm{T}^*\mathcal{M}^s(\Sigma)$ which satisfy (2.5) is called the *constraint set* and is denoted $\mathscr{C}^s(\Sigma)$. We shall discuss the structure of this important set below.

In addition to imposing constraints on allowable Cauchy data (γ, π), the Einstein equations determine how such data may evolve in a sliced spacetime (V, g). That is, if (V, g) is a chosen globally hyperbolic spacetime (i.e., a point in $\bar{\mathscr{E}}^s(\Sigma, V)$) and if i is a chosen slicing of (V, g) (i.e., a path in $\mathrm{Emb}^s(\Sigma, V, g)$), then the parametrized set $(\gamma(t), \pi(t)) = (\gamma(g, i_t), \pi(g, i_t))$ (which is a path in $T^*\mathscr{M}^s(\Sigma)$) must satisfy the Hamiltonian evolution equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\gamma_{ab}\\ \pi^{cd}\end{pmatrix} = -\mathbb{J}\cdot D\Phi(\gamma, \pi)^*\begin{pmatrix}N_\perp\\ N_\parallel\end{pmatrix}$$
$$= \begin{bmatrix} 2N_\perp(\det\gamma)^{-1/2}[\pi_{ab} - \frac{1}{2}\gamma_{ab}\operatorname{tr}\pi] + (\mathscr{L}_{N_\parallel}\gamma)_{ab} \\ -N_\perp(\det\gamma)^{-1/2}[(\det\gamma)(R^{cd} - \frac{1}{2}\gamma^{cd}R) + \pi^{cd}\operatorname{tr}\pi - 2\pi^c{}_m\pi^{md} \\ \quad + \frac{1}{2}\gamma^{cd}(\pi^n{}_m\pi^m{}_n - \frac{1}{2}\operatorname{tr}\pi^2) + (\det\gamma)^{1/2}(\nabla^c\nabla^d N_\perp - \gamma^{cd}\nabla^2 N_\perp) + (\mathscr{L}_{N_\parallel}\pi)^{cd}\end{bmatrix} \tag{2.7}$$

where $\mathbb{J} = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$ is the matrix representation of the symplectic form Ω on $T^*\mathscr{M}^s(\Sigma)$, and $D\Phi^*$ is the adjoint of the differential of the map Φ.

As well as describing the time evolution of the fields in any specified slicing of a known spacetime, the evolution equations (2.7) can be used to construct spacetimes from chosen initial conditions. The extent to which this program can be successful, and the extent to which the spacetimes it produces are unique, is reviewed and summarized in proposition 2.3, which is preceded by a pair of useful definitions.

2.1. **Definition** (*Development*). *A spacetime* (V, g) *is a* ***development*** *of a set of initial data* $(\gamma_0, \pi_0) \in \mathscr{C}^s(\Sigma)$ *if*

(a) (V, g) *is a globally hyperbolic spacetime satisfying Einstein's equations, and*

(b) *there exists an embedding* $i_0 \in \mathrm{Emb}^{s+1}(\Sigma, V, g)$ *such that* $(\gamma_0, \pi_0) = (\gamma(g, i_0), \pi(g, i_0))$.

2.2. **Definition** (*Extension*). *A spacetime* (V, g) *is an* ***extension*** *of another spacetime* (V', g') *if there exists a diffeomorphism* η *of* V' *into* V *which maps* g' *into* g *on* $\eta(V')$.

2.3. **Proposition** (*Existence and Uniqueness for the Einstein Cauchy Problem*). *Assume that* $s > \frac{3}{2} + 1$. *Then*

(a) *For any choice of* $(\gamma_0, \pi_0) \in \mathscr{C}^s(\Sigma)$, *there exists an* H^s *spacetime* (V, g) *which is a development of* (γ_0, π_0).

(b) *If* (V, g) *and* (V', g') *are both developments of* (γ_0, π_0), *then there exists a third development* (V'', g'') *such that both* (V, g) *and* (V', g') *are extensions of* (V'', g'').

(c) *There exists a globally hyperbolic* H^s *spacetime* (V, g) *which is a development of* (γ_0, π_0) *and which is an extension of every other development of* (γ_0, π_0). *This "maximal development" of* (γ_0, π_0) *is unique up to isometry.*

Remarks. 1. There may exist spacetimes which further extend (V, g), but these must either violate Einstein's equations or they must fail to be globally hyperbolic.

2. Unless we state otherwise, we shall henceforth demand that the spacetimes contained in $\bar{\mathscr{E}}^s(\Sigma, V)$ admit no globally hyperbolic extensions.

3. Uniqueness in 2.3(c) means explicitly, the following. Let (V, g) be an H^s maximal development of (γ_0, π_0) for the embedding i_0 and let $(\bar{V}, \bar{g})$ be another maximal development of (γ_0, π_0) for the embedding $\bar{i}_0$. Then there is an H^{s+1} diffeomorphism $\eta: V \to \bar{V}$ such that $\eta^*\bar{g} = g$ and $\bar{i}_0 = \eta \circ i_0$.

We refer to Hughes, Kato and Marsden [1977] (or Fischer and Marsden [1979]) for the proof of the first two parts of the above proposition; this reference together with Choquet-Bruhat and Geroch [1969] or Hawking and Ellis [1973] contains the proof of the last part.

At least formally, every development can be obtained by integrating the evolution equations (2.7) with a suitable lapse and shift. It should be clear from these equations how different spacetime developments can be evolved from the same initial data (γ_0, π_0): Everything in (2.7) is fixed by the choice of (γ_0, π_0) except for $N_\perp$ and $N_\|$. These are free. Different choices of $N_\perp(t)$ and $N_\|(t)$ lead to different developments and correspond to the non-uniqueness up to isometry.

Proposition 2.3 is useful for studying the structure of $\bar{\mathscr{E}}^s(\Sigma, V)$. To see this, let us return to the space $\mathscr{H}^s(\Sigma, V)$ described earlier as a bundle over $\mathscr{L}^s(V)$ (with fibre $\Pi_{\mathscr{L}}^{-1}(g) = \mathrm{Emb}^{s+1}(\Sigma, V, g)$) and as a bundle over $T^*\mathscr{M}^s(\Sigma)$ with fibre $\Pi_{T^*\mathscr{M}}^{-1}(\gamma, \pi) = \{(g, i) | \gamma(g, i) = \gamma, \pi(g, i) = \pi\}$. If we now restrict $\mathscr{H}^s(\Sigma, V)$ to solutions of Einstein's equations by defining

$$\mathscr{H}^s_{\mathscr{E}}(\Sigma, V) = \{(g, i) | g \in \mathscr{E}^s(\Sigma, V), i \in \mathrm{Emb}^{s+1}(\Sigma, V, g)\},$$

then we have a bundle over $\mathscr{E}^s(\Sigma, V)$ and over $\mathscr{C}^s(\Sigma)$. The bundle structure for $\Pi_{\mathscr{E}} : \mathscr{H}^s_{\mathscr{E}}(\Sigma, V) \to \mathscr{E}^s(\Sigma, V)$ is much like that for $\Pi_{\mathscr{L}} : \mathscr{H}^s(\Sigma, V) \to \mathscr{L}^s(V)$. However the bundle structure of $\Pi_{\mathscr{C}} : \mathscr{H}^s_{\mathscr{E}}(\Sigma, V) \to \mathscr{C}^s(\Sigma)$ is much different from that of $\Pi_{T^*\mathscr{M}} : \mathscr{H}^s(\Sigma, V) \to T^*\mathscr{M}^s(\Sigma)$ because the Cauchy uniqueness results of proposition 2.3 considerably reduces the size of the fibres. Indeed, $\Pi_{\mathscr{C}}^{-1}(\gamma, \pi)$ can be identified with the orbit $\mathcal{O}_g \subset \mathscr{E}^s(V)$ of g under the action of the diffeomorphism group $\mathscr{D}^{s+1}(V)$, for *any* metric g such that (V, g) is a development of (γ, π). Further, choosing a point $(g_0, i_0) \in \mathscr{H}^s_{\mathscr{E}}(\Sigma, V)$, one can define a projection map $p_{i_0} : \mathscr{E}^s(\Sigma, v) \to \mathscr{C}^s(\Sigma) : g \mapsto (\gamma(g, i_0), \pi(g, i_0))$ which defines a bundle structure. The fibres $p_{i_0}^{-1}(\gamma_1, \pi_1)$ of this bundle can be identified with all diffeomorphisms of V which leave the points of $i_0(\Sigma) \subset V$ invariant. Thus, $\mathscr{E}^s(\Sigma, V)$ becomes a principal bundle over $\mathscr{C}^s(\Sigma)$. This bundle structure is useful when relating the properties of $\bar{\mathscr{E}}^s(\Sigma, V)$ to those of $\mathscr{C}^s(\Sigma)$.

The most striking of these properties is that $\mathscr{C}^s(\Sigma)$, and therefore $\bar{\mathscr{E}}^s(\Sigma, V)$, is not a manifold. One discovers this by studying linearization stability of the constraint function Φ. Since $\mathscr{C}^s(\Sigma) = \Phi^{-1}(0)$, the set of points $\mathscr{B}^s(\Sigma) := \{(\gamma, \pi) \in T^*\mathscr{M}^s(\Sigma) | \Phi(\gamma, \pi) = 0$ and Φ is not linearization stable$\}$ is exactly the set at which $\mathscr{C}^s(\Sigma)$ is not a manifold. It is not known what the structure of $\mathscr{B}^s(\Sigma) \subset \mathscr{C}^s(\Sigma)$ is. However, the subset $\tilde{\mathscr{B}}^s(\Sigma) := \mathscr{B}^s(\Sigma) \cap \tilde{\mathscr{C}}^s(\Sigma)$, where $\tilde{\mathscr{C}}^s(\Sigma)$ is the subset of $\mathscr{C}^s(\Sigma)$ with constant mean curvature (equivalently, with $\kappa = \mathrm{tr}\, k = \mathrm{const}$ on Σ) is known exactly. We state the results in the following proposition. The statement is slightly informal for expository reasons, but will suffice for our present needs.

2.4. **Proposition** (*Conical Singularities in* $\tilde{\mathscr{C}}^s(\Sigma)$). *The set* $\tilde{\mathscr{C}}^s(\Sigma)$ *is a manifold everywhere except at the points* $\tilde{\mathscr{B}}^s(\Sigma)$ *of linearization instability.* $\tilde{\mathscr{B}}^s(\Sigma)$ *consists of all those sets of data* (γ, π) *which generate spacetime developments with nonzero global Killing vector fields. In the neighborhood of a point* $(\gamma_0, \pi_0) \in \tilde{\mathscr{B}}^s(\Sigma)$, *the set* $\tilde{\mathscr{C}}^s(\Sigma)$ *has a conical structure* (*whose generators can be determined by certain second order conditions of Taub*).

See Arms, Marsden and Moncrief [1982] for the proof and for a more technically precise statement. The set $\tilde{\mathscr{C}}^s(\Sigma)$ is of "measure zero" (is nowhere dense) in $\mathscr{C}^s(\Sigma)$ and may therefore seem unimportant in studying the structure of $\bar{\mathscr{E}}^s(\Sigma, V)$. However, $\tilde{\mathscr{C}}(\Sigma)$ is quite important as is seen by the following argument. Recall the set $\bar{\mathscr{E}}^s(\Sigma, V)$ which consists of all globally hyperbolic spacetime solutions of Einstein's equations which contain a compact Cauchy surface of constant mean curvature. Clearly each spacetime in $\bar{\mathscr{E}}^s(\Sigma, V)$ is a development of data in $\tilde{\mathscr{C}}^s(\Sigma)$. It is also clear that $\bar{\mathscr{E}}^s(\Sigma, V)$ is an invariant subset of the action of $\mathscr{D}^{s+1}(V)$ on $\tilde{\bar{\mathscr{E}}}^s(\Sigma, V)$. Hence we may restrict the local bundle structure $p_{i_0}: \bar{\mathscr{E}}^s(\Sigma, V) \to \mathscr{C}^s(\Sigma)$ and obtain $\tilde{p}_{i_0}: \tilde{\bar{\mathscr{E}}}^s(\Sigma, V) \to \tilde{\mathscr{C}}^s(\Sigma)$. It follows that $\tilde{\bar{\mathscr{E}}}^s(\Sigma, V)$ has the differentiable structure of $\mathscr{C}^s(\Sigma)$ – it is a manifold with conical singular points. $\tilde{\bar{\mathscr{E}}}^s(V)$ is not the same as $\bar{\mathscr{E}}^s(V)$; however we have the following, assuming as usual that Σ is compact without boundary.

2.5. **Proposition.** $\tilde{\bar{\mathscr{E}}}^s(\Sigma, V)$ *is an open subset of* $\bar{\mathscr{E}}^s(\Sigma, V)$.

This proposition simply says that if a spacetime $(V, g) \in \tilde{\bar{\mathscr{E}}}^s(\Sigma, V)$ contains a constant mean curvature hypersurface, then so must all small perturbations of (V, g). The result is a direct consequence of the perturbation theory of CMC hypersurfaces appearing in Choquet-Bruhat, Fischer and Marsden [1979]. The basic techniques used will be reviewed in the next section.

It follows from proposition 2.5 that $\bar{\mathscr{E}}^s(\Sigma, V)$ inherits the conical singularity structure of $\tilde{\mathscr{E}}^s(\Sigma)$. It is possible that $\bar{\mathscr{E}}^s(\Sigma, V)$ contains other singularities as well. However, not much is known about the regions $\bar{\mathscr{E}}^s(\Sigma, V) \backslash \tilde{\bar{\mathscr{E}}}^s(\Sigma, V)$ in which these other singularities can occur. Indeed, no one knows whether or not $\bar{\mathscr{E}}^s(\Sigma, V) \backslash \tilde{\bar{\mathscr{E}}}^s(\Sigma, V)$ is non-empty. There is a well-known conjecture on this point.

2.6. **Conjecture.** $\bar{\mathscr{E}}^s(\Sigma, V) = \tilde{\bar{\mathscr{E}}}^s(\Sigma, V)$; i.e. *every vacuum globally hyperbolic spacetime solution of Einstein's equations with compact Cauchy surfaces contains at least one spacelike, constant mean curvature hypersurface.*

With the validity of this conjecture as yet far from certain, we shall avoid this problem by working exclusively with $\tilde{\bar{\mathscr{E}}}^s(\Sigma, V)$, whose structure is known.

We conclude this section with some additional concepts and terminology that will be useful in subsequent sections.

Let (V, g) be a spacetime contained in $\tilde{\bar{\mathscr{E}}}^s(\Sigma, V)$ and let i_t be a slicing of (V, g); so $i_t: \Sigma \to V$ is a spacelike embedding for each value of t. Let η be a diffeomorphism of V. While in general there need be no correlation between i_t and η, we will be concerned with situations in which there is such a correlation as spelled out in the following definitions:

2.7. **Definition** (*Foliation-Compatible Diffeomorphism*). *If there exists a monotonic function* $f: \mathbb{R} \to \mathbb{R}$ *and a parametrized set of* (*spatial*) *diffeomorphisms* $\xi_t = \xi(t): \Sigma \to \Sigma$ *such that*

$$\eta \circ i_t = i_{f(t)} \circ \xi(t) , \tag{2.8}$$

then η *is called* ***compatible*** *with the foliation* $\{i_t(\Sigma)\}$. *If* $f(t) = t + \tau$ *for some constant* τ, *then* η *is called* ***rigidly compatible*** *with the foliation* $\{i_t(\Sigma)\}$.

2.8. **Definition** (*Foliation-Preserving Diffeomorphism*). *If there exists a parameterized set of spatial diffeomorphism* $\xi(t): \Sigma \to \Sigma$ *such that*

$$\eta \circ i_t = i_t \circ \xi(t) \tag{2.9}$$

then we say η ***preserves*** *the foliation* $\{i_t(\Sigma)\}$.

Obviously these conditions are increasingly restrictive on η; i.e. foliation-preservation implies rigid foliation-compatibility (with $\tau = 0$), which in turn implies foliation-compatibility (with $f(t) = t$).

We are interested in these conditions for two reasons. Firstly, if η is compatible with a foliation then we can compare the Cauchy data of $\eta^* g$ with that of g directly since (2.8) implies that

$$(\gamma(\eta^* g, i_t), \pi(\eta^* g, i_t)) = (\xi^*(t)\, \gamma(g, i_{f(t)}), \xi^*(t)\, \pi(g, i_{f(t)}))\,. \tag{2.10}$$

The analogous result for η rigidly compatible with i_t is obtained by replacing $f(t)$ with $t + \tau$ in (2.10); for a foliation-preserving diffeomorphism, replace $f(t)$ by t. Secondly, as is discussed in the next section, if i_t is a constant mean curvature or a maximal slicing and if η is an isometry of g, then η necessarily satisfies either 2.7 or 2.8.

3. Constant mean curvature foliations

As noted in the introduction, constant mean curvature ("CMC") hypersurfaces are crucial to the proof of our slice theorem. This section reviews and extends some of their properties.

Let $g \in \tilde{\mathscr{E}}^s(\Sigma, \mathrm{V})$, i.e., (V, g) is a globally hyperbolic spacetime which contains at least one compact spacelike CMC hypersurface $i_0(\Sigma)$. We would like to have more – enough to completely foliate (V, g). In a local sense, this can often be done.

3.1. **Proposition** (*Local Existence of CMC Foliations*). *Let* (V, g) *be a nonflat spacetime metric contained in* $\tilde{\mathscr{E}}^s(\Sigma, \mathrm{V})$, *and let* $\mathrm{S}_0 = i_0(\Sigma)$ *be a CMC hypersurface in* (V, g). *Then there exists an open neighborhood* $\mathrm{U} \subset \mathrm{V}$ *containing* $i_0(\Sigma)$ *and there exists a CMC foliation* $\{i_t(\Sigma)\}$ *of* (U, g) *where* i_t *is a path in* $\mathrm{Emb}^{s+1}(\Sigma, \mathrm{V}, g)$ *such that every hypersurface* $i_t(\Sigma)$ *is a CMC hypersurface. Furthermore the spatial constant* $\operatorname{tr} k(t) = \kappa(t)$ *is a monotonic function of* t.

Proof (based on Choquet-Bruhat, Fischer and Marsden [1979]; an outline of this idea of proof was first given by York [1972]). Let V be a neighborhood of S_0 obtained using Gaussian normal coordinates. Let $\mathscr{F}^{s+1}$ denote the set of all H^{s+1} real valued functions f on S_0 with $s \geq 3$. Let S_f be the hypersurface that is the graph of f over S_0 in Gaussian normal coordinates. If f is sufficiently small, S_f is well defined.

Define $P: \mathscr{F}^{s+1} \to \mathscr{F}^{s-1}$; $P(f)$ = mean curvature of S_f. Then P is a smooth mapping with derivative at zero given by

$$\mathrm{d}P(0) \cdot f = -(k_0 \cdot k_0 + \Delta) f\,,$$

which is a variant of the standard Raychaudurhi equation, where k_0 is the second fundamental form of S_0 and $\Delta = -\nabla^2$ is the positive Laplacian. (See Choquet-Bruhat, Fischer and Marsden [1979] and note that our k differs in sign from theirs.) Let us first assume $k_0 \neq 0$, i.e. $k_0 \cdot k_0 > 0$, so that from elliptic theory (see, for instance Hormander [1969] Ch. X), $\mathrm{d}P(0)$ is an isomorphism. Thus P is a local diffeomorphism by the implicit function theorem. The equation $P(f) = \kappa$ is then uniquely solvable for f as a function of κ. The result now readily follows in case $k_0 \neq 0$.

Finally we must dispense with the case $k_0 = 0$. Indeed in this case (V, g) must be flat, for $\mathrm{Ric}(g) = 0$ and $k_0 = 0$ implies that $\gamma_0 = \gamma(g, i_0)$ has zero Ricci curvature and hence is flat (as S_0 is three dimensional). This follows directly from the Gauss–Codazzi equations (see for instance, Fischer and Marsden [1972] p. 563). Therefore, by uniqueness for the Cauchy problem, g is flat. ■

Remarks. For a flat spacetime, one still can prove that there is local foliation by CMC hypersurfaces. The mean curvature $\kappa = \operatorname{tr} k$ of these foliations is not necessarily monotonic, however, if the initial surface is maximal (which may or may not be the case in a flat spacetime). We discuss flat spacetimes further below.

Note that the argument used to prove 3.1 also shows that if we allow g to vary, then $P(f, g) = \kappa$ can be uniquely solved for f as a function of g and κ. This is how the openness of $\mathscr{E}$ in $\bar{\mathscr{E}}$ is proved near nonflat spacetimes. For perturbations of flat spacetimes, one needs to let κ "float" and solve $P(f) = \kappa$ modulo constants; technically, one uses the fact that P is transversal to the set of constant functions.

The implication of proposition 3.1 is that if one considers "small enough" developments of data in $\mathscr{C}^s(\Sigma)$, one obtains spacetimes in $\bar{\mathscr{E}}^s(\Sigma, V)$ which admit constant mean curvature foliations. Unfortunately, it is not known whether *every* spacetime in $\bar{\mathscr{E}}^s(\Sigma, V)$ (particularly those which are maximal globally hyperbolic developments) admits a CMC foliation. There are no known counter-examples, so we may conjecture, following Marsden and Tipler [1980] (and everybody else) that:

3.2. **Conjecture** (*Global CMC Foliations of Spacetimes in* $\bar{\mathscr{E}}^s(\Sigma, V)$). *Every spacetime in* $\bar{\mathscr{E}}(\Sigma, V)$ *admits a global foliation by smooth, spacelike constant mean curvature hypersurfaces.*

The strongest and cleanest version of our slice theorem depends upon the validity of this conjecture, which we henceforth refer to as the *CMC conjecture.* We still obtain a slice theorem for a localized version of $\bar{\mathscr{E}}^s(\Sigma, V)$ if conjecture 3.2 is false, however (see section 8 below).

Whether or not conjecture 3.2 is true, one can prove a global uniqueness result for CMC hypersurfaces:

3.3. **Proposition** (*Uniqueness of CMC Hypersurfaces in Nonflat Spacetimes*). *Let g be a spacetime metric in $\bar{\mathscr{E}}^s(\Sigma, V)$ which is not (everywhere) flat. Suppose that $i_1, i_2 \in \mathrm{Emb}^{s+1}(\Sigma, V, g)$ are two embeddings with the same constant mean curvature $\kappa = \mathrm{tr}\, k$. Then $i_1(\Sigma) = i_2(\Sigma)$.*

Local uniqueness follows from the proof of 3.1. The proof of global uniqueness is given in Marsden and Tipler [1980] (see Theorem 1.B), using a technique developed by Brill and Flaherty [1976, 1978].

Note that proposition 3.3 does *not* assert that $i_1 = i_2$; rather it concludes only that the *image* of i_1 *equals* that of i_2. An equivalent way of stating this result is to say that there exists a diffeomorphism of Σ – let us call it $\xi \in \mathscr{D}^{s+1}(\Sigma)$ – such that $i_1 = i_2 \circ \xi$.

If we combine this result with proposition 3.1, we get (locally) for any given nonflat (vacuum) spacetime $g \in \bar{\mathscr{E}}^s(\Sigma, V)$ a *unique* constant mean curvature foliation. If we assume conjecture 3.2, then the unique CMC foliation becomes global. That is, the CMC surfaces themselves are unique, and every pair of slicings i and j, which give them are related by a surface-compatible diffeomorphism $\eta \in \mathscr{D}^{s+1}(V)$. If we further stipulate that $\kappa(t) = \mathrm{tr}(k(g, i_t)) = \mathrm{tr}(k(g, j_t))$ then it follows that the diffeomorphism η which relates i_t and j_t must be *foliation preserving* (in the sense of definition 2.8).

To nail down a slicing which gives a CMC foliation for a nonflat spacetime as far as possible, it will often be convenient to assume that the mean curvature $\kappa(g, i) = \mathrm{tr}\, k(g, i)$ satisfies

$$\kappa(g, i_t) = t\,;$$

i.e. that the slicing is parametrized by κ itself.

The uniqueness of CMC foliations (up to purely spatial diffeomorphisms) makes them ideal for splitting the analysis of the action of $\mathscr{D}^{s+1}(V)$ on $\bar{\mathscr{E}}^s(\Sigma, V)$ into temporal and spatial pieces. We need such a split not only for single spacetimes, but for neighborhoods in $\bar{\mathscr{E}}^s(\Sigma, V)$ as well. The following perturbation result will be useful. The first part will assume the CMC conjecture.

3.4. **Proposition.** (a) *Let* $g_0 \in \bar{\mathscr{E}}^s(\Sigma, V)$ *be a nonflat (vacuum) spacetime with a CMC foliation* $\{i_t^0(\Sigma)\}$ *with mean curvature* $\kappa_0(t) = \kappa(g_0, i_t^0) = t$. *There is a neighborhood* U *of* g_0 *in* $\bar{\mathscr{E}}^s(\Sigma, V)$ *such that for each* $g \in U$, *there is a diffeomorphism* $\eta \in \mathscr{D}^{s+1}$ *(close to the identity on a compact set) of which* $i_t = i_t^0 \circ \eta$ *is a CMC foliation for* (V, g) *with* $\kappa(g, i_t) = t$.

(b) *Let* $g_\infty \in \bar{\mathscr{E}}^s(\Sigma, V)$ be a nonflat (vacuum) spacetime with a CMC foliation $\{i_t^\infty(\Sigma)\}$ with mean curvature $\kappa_\infty(t) := \kappa(g_\infty, i_t) = t$. Let $g_n \in \bar{\mathscr{E}}^s(\Sigma, V)$, $g_n \to g_\infty$ *and let* g_n *have a CMC foliation* $\{i_t^n(\Sigma)\}$ *with* $\kappa_n(t) := \kappa(g_n, i_t^n) = t$. *Then*

(i) *there exists* $\xi_{t,n} \in \mathscr{D}^{s+1}(\Sigma)$ *such that*

$$i_t^n \circ \xi_{t,n} \to i_t^\infty$$

and (ii) $(\xi_{t,n}^* \gamma_t^n, \xi_{t,n}^* \pi_t^n) \to (\gamma_t^\infty, \pi_t^\infty)$ *in* $H^s \times H^{s-1}$ *where* $\gamma_t^n = \gamma(g_n, i_t^n)$ *and* $\pi_t^n = \pi(g_n, i_t^n)$.

Proof. (a) A CMC foliation defines an isometry ϕ of (V, g) with $(\Sigma \times \mathbb{R}, \phi_* g)$ where $\Sigma \times \{t\}$ is a CMC surface. For g close to g_0, g has a CMC foliation close to that of g_0 as in proposition 3.1 and in the subsequent remark. The map η is uniquely defined by $i_t = i_t^0 \circ \eta$.

(b) For n large, we know from 3.3 and the proof of 3.1 that g_n has a globally unique CMC hypersurface S_t^n near $i_t(\Sigma)$ with mean curvature t. Moreover, that proof shows that $S_t^n \to i_t(\Sigma)$ as $n \to \infty$ in the following sense: If we represent S_t^n as the graph of f_n in Gaussian normal coordinates, then $f_n \to 0$ in H^{s+1}. If we let $\xi_{t,n}$ be such that $i_t^n \circ \xi_{t,n}$ is i_t composed with the graph map of f_n in Gaussian normal coordinates, then (i) follows. Part (ii) follows directly from (i). ■

Isometries of a metric g play two crucial roles in our slice theorem. First of all, they comprise the isotropy group at g for the action of the diffeomorphism group. Secondly it is at points with isometries that the space of solutions is singular. It is thus appropriate to examine the effect of isometries on CMC foliations.

3.5. **Proposition** (*Isometries of Nonflat Spacetimes and CMC Foliations*). *Let* (V, g) *be a nonflat spacetime in* $\bar{\mathscr{E}}^s(\Sigma, V)$, *and let* i_t *be a constant mean curvature foliation of* (V, g). *If* η *is an isometry of* (V, g), *then* η *preserves the foliation* i_t *(in the sense of definition* 2.8*). In other words, the isometry* η *specifies a unique family of spatial diffeomorphisms* $\xi_t : \Sigma \to \Sigma$ *such that the Cauchy data of* g *induced by* i_t *satisfies*

$$(\gamma(g, i_t), \pi(g, i_t)) = (\xi_t^* \gamma(g, i_t), \xi_t^* \pi(g, i_t)) .$$

Proof. Since η is an isometry it maps a CMC hypersurface to another with the same mean curvature κ. By uniqueness, the two hypersurfaces are coincident. Thus η induces a diffeomorphism of $i_t(\Sigma)$ and hence defines ξ_t by

$$\eta \circ i_t = i_t \circ \xi_t .$$

Since $\eta^* g = g$, the last formula follows from the general fact that

$$\gamma(\eta^* g, i_t) = \xi_t^* \gamma(g, j_t) ,$$

where $j_t \circ \xi_t = \eta \circ i_t$, with a similar formula for π. ■

Most of our discussion thus far excludes spacetimes which are flat. There are of course flat spacetimes in $\bar{\mathscr{E}}^s(\Sigma, V)$ (at least for some choices of Σ, such as the 3-torus, or such as one of the *schraubungen* of Hanzsche and Wendt [1935]) and they are important in our slice theorem, so we discuss them and their foliations now.

An obvious consequence of proposition 3.5 is that a nonflat spacetime (with a compact Cauchy surface) cannot contain a timelike Killing vector field. This does *not*, however, imply that every flat spacetime in $\bar{\mathscr{E}}^s(V)$ *does* contain a timelike Killing vector field. Indeed, there are standard counter-examples (see Marsden and Tipler [1980] p. 134).

We shall classify flat spacetimes into two types: Type I – those which contain a CMC hypersurface with $\kappa \neq 0$ – and Type II – those containing a maximal hypersurface (with $\kappa = 0$), but no CMC hypersurface with $\kappa \neq 0$. The next proposition shows that these classes are quite different with regard to Killing fields:

3.6. **Proposition.** *Let* (V, g) *be a flat spacetime with a CMC hypersurface* S *which is a compact Cauchy surface.*

(a) *If* (V, g) *has a timelike Killing field then* (V, g) *is of Type II. In fact* S *must be maximal, be flat and be a moment of time symmetry.*

(b) *If* (V, g) *is Type II then* (V, g) *admits a timelike Killing field.*

Proof. (a) Using the Killing field, X, we can propagate S by the flow of X to produce a family of maximal hypersurfaces S(t) with the same volume. The second variation formula

$$\frac{d^2}{dt^2}(\text{Vol}\, S(t))|_{t=0} = -\int_S \{X_\perp^2 k \cdot k + (\nabla X_\perp)^2\}\, d^3V$$

implies $k = 0$ and $X_\perp$ is constant.

Thus S has zero extrinsic curvature and so by the Gauss–Codazzi equations, S is flat. As is well known, S is covered by Euclidean space $\mathbb{R}^3$; i.e., it is $\mathbb{R}^3$ with identifications (Wolf [1977] p. 42). Thus, (V, g) is isometric to the product $S \times \mathbb{R}$, which is covered by Minkowski space.

(b) We know from the local foliation theorem (the flat version of 3.1; see remarks following that theorem) that S is part of a foliation {S(t)} of CMC hypersurfaces. Since (V, g) is Type II, S(t) are all maximal. The first variation formula shows S(t) all have the same volume and as in (a) the second variation formula shows $k = 0$ for each of them. Thus S is flat with $k = 0$, and so (V, g) is isometric to the product $S \times \mathbb{R}$; it therefore has Killing fields. ■

For spacetimes which are of Type I, one can show that propositions 3.1, 3.3, 3.4 and 3.5 are still valid [just replace "nonflat" by "$\kappa \neq 0$" in the proofs of these propositions]. Thus, as far as CMC hypersurfaces and foliations are concerned, this class behaves like nonflat spacetimes. We shall, therefore, treat them as such. Thus *for the remainder of the paper, by "flat" we shall mean "Type II flat"; i.e., by proposition* 3.6, *"flat with a timelike Killing field".*

The key result for the "flat" spacetimes, as far as foliations are concerned, is the following:

3.7. **Proposition.** *Every flat spacetime in* $\bar{\mathscr{E}}^s(\Sigma, V)$ *admits a global foliation by maximal* (i.e., $\kappa = 0$) *hypersurfaces. The surfaces of this foliation are unique in the following sense. Any pair of parameterized embeddings* i_t *and* $j_{t'}$ *which induce maximal foliations are related by* $t' = f(t)$ *for* $f: \mathbb{R} \to \mathbb{R}$ *some monotonic* *function, and*

$$i_t = j_{f(t)} \circ \xi_t$$

for $\xi_t \in \mathscr{D}^{s+1}(\Sigma)$.

Proof. As in 3.6, (V, g) is isometric to $S \times \mathbb{R}$ where S is a flat 3-manifold covered by $\mathbb{R}^3$. The surfaces $S \times \{t\}$ obviously form a global maximal foliation of (V, g). If S is maximal, then it lifts to a maximal hypersurface in Minkowski space, which by Calabi [1970] is a flat hyperplane (see also Cheng and Yau [1976]). But the only such planes which project down to spatially compact surfaces are $t = \text{constant}$ where t is the standard time coordinate. The results now follows. ■

There is too much arbitrariness in the time coordinatization of maximal foliations to use them in constructing slices near flat spacetimes. We therefore define "proper maximal foliations" which do the job.

3.8. **Definition** (*Proper Maximal Foliation*). *A* ***proper maximal foliation*** *of a flat spacetime* $(\Sigma \times \mathbb{R}, g)$ *is a maximal foliation* $\{i_t(\Sigma)\}$ such that the *proper time between* $i_{t_1}(\Sigma)$ *and* $i_{t_2}(\Sigma)$ *is exactly* $|t_1 - t_2|$.

3.9. **Proposition** (*Existence and Uniqueness of Proper Maximal Foliations*). *Let* (V, g) *be a flat spacetime. Then there exists a proper maximal foliation of* (V, g). *If* i_t *and* j_t *are a pair of proper maximal foliations of* (V, g), *then there exists a constant* τ *and a set of spatial diffeomorphisms* $\xi_t \in \mathscr{D}^{s+1}(\Sigma)$ *such that*

$$i_{t+\tau} = j_t \circ \xi_t \,.$$

Proof. The maximal foliation we constructed in the proof of 3.7 is proper. The uniqueness assertion is clear. ■

As in 3.5, we are especially concerned about isometries. For flat spacetimes we must deal with "screw motions".

3.10. **Proposition** (*Isometries of Flat Spacetimes and Maximal Foliations*). *Let* (V, g) *be a flat spacetime in* $\tilde{\mathscr{E}}^s(V, \Sigma)$ *and let* i_t *be a maximal foliation of* (V, g).

(a) *If* η *is an isometry of* (V, g), *then* η *is compatible with the foliation* i_τ (*in the sense of definition 2.8*).

(b) *If* i_t *is a proper maximal foliation, then* η *is rigidly compatible with it* (*i.e.* η *is a "screw motion"*). *Therefore, in the latter case, if we fix* i_t *then each isometry* η *specifies a constant* τ *and a unique parametrized set of spatial diffeomorphisms* ξ_t *such that the Cauchy data of* g *induced by* i_t *satisfies*

$$(\gamma(g, i_t), \pi(g, i_t)) = (\xi_t^* \gamma(g, i_{t+\tau}), \xi_t^*(g, i_{t+\tau})).$$

Proof. Part (a) follows from uniqueness in proposition 3.7. Part (b) follows directly from the fact that $\eta \circ i_t$ is a maximal proper foliation (since η is an isometry) and from uniqueness in proposition 3.9. ■

We now want a result analogous to proposition 3.4 for flat spacetimes. For the first part of 3.4, our result here is basically the same. For the second part, we need a small modification, which should not be surprising in view of the nature of the "uniqueness" of maximal foliations of flat spacetimes as described above. Again for convenience we assume the CMC conjecture.

3.11. **Proposition** (*Maximal Foliations and Perturbations for Flat Spacetimes*).

(a) *Let* (V, g_0) *be a flat spacetime contained in* $\bar{\mathscr{E}}^s(\Sigma, V)$. *Then there exists a neighborhood* $U \subset \bar{\mathscr{E}}^s(\Sigma, V)$ *such that for each* $g \in U$, *there exists a diffeomorphism* $\eta \in \mathscr{D}^{s+1}(V)$ *satisfying* $i_t^0 \circ \eta = i_t$, *where* $i_t^{(0)}$ *is a maximal foliation of* (V, g_0) *and* i_t *is a CMC foliation of* (V, g).

(b) *Let* (V, g_n) *be a sequence of (flat or nonflat) spacetimes in* $\bar{\mathscr{E}}^s(\Sigma, V)$ *which converges to a limit spacetime* (V, g_∞) *which is flat. Let* $i_t^{(\infty)}$ *denote a maximal foliation for* (V, g_∞) *and let* i_t^n *denote a CMC foliation for* (V, g^n). *Then there exists a reparametrization* $t' = f(t)$ *such that for every value of* t, *there exists a sequence* $\xi_{t,n} \in \mathscr{D}^{s+1}(\Sigma)$ *such that*

(i) $\lim_{n\to\infty} i_t^n \circ \xi_{t,n} = i_{f(t)}^\infty$

and

(ii) $(\xi_{t,n}^* \gamma_t^n, \xi_{t,n}^* \pi_t^n) \to (\gamma_{f(t)}^\infty, \pi_{f(t)}^\infty)$ in $H^s \times H^{s-1}$ *where* $\gamma_t^n = \gamma(g_n, i_t^n)$ *and* $\pi_t^n = \pi(g_n, i_t^n)$.

Proof. Arguments similar to those used in proving proposition 3.1 show that any spacetime close to a flat one (V, g_0) has a CMC foliation (that might be maximal) close to the maximal one for (V, g_0).

Part (a) follows as in 3.4(a). Part (b) follows as in 3.4(b) using the uniqueness result 3.7. ■

Note that if we replace "maximal foliation" by "proper maximal foliation" and replace "$f(t)$" by "$t + \tau$" (for some constant τ), then 3.11(b) (as modified) still applies.

4. Statement of the slice theorem

We shall present two statements of the slice theorem for the diffeomorphism group acting on the set of spacetime solutions of Einstein's equations. In this section, we state the version which depends upon the validity of the CMC conjecture. After proving this version of the slice theorem we state (in section 7) the messier version which does not require the CMC conjecture.

4.1. **Theorem** (*Slice Theorem* [*assuming the CMC conjecture*]). *Assume* $s > 2.5$ $(=\frac{1}{2}n + 1)$. *Let the group* $\mathscr{D}^{s+1}(V)$ *act upon the (stratified) manifold* $\bar{\mathscr{E}}^s(\Sigma, V)$ *in the usual pullback sense. Let* $r \geq 1$ *and* $g_0 \in \bar{\mathscr{E}}^{s+r}(\Sigma, V)$. *Then we have the following*:

(a) *The orbit* $\mathcal{O}_{g_0}$ *is a closed,* C^r, *embedded submanifold of* $\bar{\mathscr{E}}^s(\Sigma, V)$.

(b) *There exists a submanifold* $S_{g_0} \subset \bar{\mathscr{E}}^s(\Sigma, V)$ *containing* g_0 *such that*

(S1) *if* $g \in S_{g_0}$ *and if* $\eta \in I_{g_0}$, *then* $\eta^* g \in S_{g_0}$;

(S2) *if* $\eta \in \mathscr{D}^{s+1}(V)$, *if* $g \in S_{g_0}$, *and if* $\eta^* g \in S_{g_0}$, *then* $\eta \in I_{g_0}$; *and*

(S3) *there is a local cross-section* $\mu : \mathscr{D}^{s+1}(V)/I_{g_0} \to \mathscr{D}^{s+1}(V)$ *defined in a neighborhood* U *of the identity coset such that the map* $F: U \times S_{g_0} \to \bar{\mathscr{E}}^s(\Sigma, V)$, *defined by* $F(u, g) = \mu(u)^* g$, *is a homeomorphism onto a neighborhood* V *of* g_0.

Part (b) of this theorem of course simply says that there is a slice at each $g_0 \in \bar{\mathscr{E}}^{s+r}(\Sigma, V)$. Statements (S1)–(S3) are formally the same as those given in the original definition 1.1. The only difference is the context. Here we do not have a C^∞ Lie group acting smoothly on a C^∞ manifold. Rather we have a topological group acting on a stratified manifold with each stratum invariant. It is in this context that "(embedded) submanifold" is understood, as was explained in section 1. Note that there is not a slice at *every* $g_0 \in \bar{\mathscr{E}}^s(\mathbb{R} \times \Sigma)$ for the action of $\mathscr{D}^{s+1}(V)$. One needs $g_0 \in \bar{\mathscr{E}}^{s+r}(\Sigma, V)$ so that the $\mathcal{O}_{g_0}$ is a C^r

embedded submanifold ($r \geq 1$ is necessary for the implicit function theorem, which is used in the proof). This situation is the same as in the Ebin–Palais theorem (see section 1 and Ebin [1970]).

One can use the regularity argument of Ebin [1970] to let $s \to \infty$ and thereby obtain a slice theorem for the C^∞ diffeomorphisms acting on C^∞ solutions of Einstein's equations (solutions, that is, which are globally hyperbolic and contain a compact CMC hypersurface).

The proof of theorem 4.1 will occupy the next two sections.

5. Proof that the orbits are closed submanifolds

In this section we prove that for any $g_0 \in \tilde{\mathscr{E}}^{s+r}(\Sigma, V)$, the oribit $\mathcal{O}_{g_0}$ is a closed *embedded* C^r submanifold of $\tilde{\mathscr{E}}^s(\Sigma, V)$. To deal with this we shall regard $\mathcal{O}_{g_0}$ as the image of the map

$$\psi_{g_0} : \mathscr{D}^{s+1}(V) \to \mathscr{L}^s(\Sigma, V)$$

defined by $\psi_{g_0}(\eta) = \eta^*(g_0)$ (the standard pull-back map). Actually the image of ψ_{g_0} lies in the stratum of $\tilde{\mathscr{E}}^s(\Sigma, V)$ which passes through g_0; this is a C^∞ Hilbert manifold (from Fischer, Marsden and Moncrief [1980]). This will be understood to be the target space for ψ_{g_0}.

The proof proceeds in three steps:

(A) Use the implicit function to show that image (ψ_{g_0}) is an immersed submanifold of $\mathscr{L}^s(\Sigma, V)$.

(B) Show that image (ψ_{g_0}) is closed in $\tilde{\mathscr{E}}^s(\Sigma, V)$.

(C) Combine (A) and (B) together with a standard topological argument to show that image (ψ_{g_0}) is an embedded submanifold of $\tilde{\mathscr{E}}^s(\Sigma, V)$.

(A) For our purposes here, the most useful version of the implicit mapping theorem is the following:

5.1. **Lemma.** *Let* Q *and* P *be Hilbert manifolds and let* $f: Q \to P$ *be a* C^r *map* ($r \geq 1$) *satisfying two conditions*: (i) *the kernel of* Tf *defines a* C^r *sub-bundle of* TQ; (ii) *at each* $x \in Q$, *the image of* T$f(x)$ *is closed in* $T_{f(x)}P$. *Then* $f(Q)$ *is a* C^r *locally immersed submanifold of* P.

The proof proceeds in a standard way; see Abraham and Marsden [1978] p. 51.

To apply lemma 5.1 to our situation, we first restrict all our mappings to a nice compact set $C \subset V$ so that our spaces become Hilbert spaces or manifolds; this will be assumed implicitly in what follows. Secondly, the results concerning Sobolev space mappings discussed in the appendix guarantee that ψ_{g_0} is a C^r map. Next we consider the kernel of $T\psi_{g_0}$. This is the collection of Killing fields of the spacetime (V, g_0), made into a right invariant subset of $T\mathscr{D}^{s+1}(V)$. The arguments in appendix A of Ebin and Marsden [1970], show that this is a C^r sub-bundle of $T(\mathscr{D}^{s+1}(V))$. Hence condition (i) of the lemma is satisfied. To check the other condition, we consider the derivative of ψ_{g_0} at the identity $e \in \mathscr{D}^{s+1}(V)$; for any H^{s+1} vector field X on $C \subset V$ [i.e., for $X \in Te(\mathscr{D}^{s+1}(V))$], one gets

$$T\psi_{g_0}(e) \cdot X = (g_0, \mathscr{L}_X g_0)$$

where $\mathscr{L}_X$ is the Lie derivative operator. If we can show that the range of the mapping $X \mapsto \mathscr{L}_X g_0$ is closed, then condition (ii) of lemma 5.1 follows from this. We cannot use elliptic theory directly at this point since g_0 has Lorentz signature so $X \mapsto \mathscr{L}_X g_0$ is not elliptic and, moreover, our nice compact set C has a temporal boundary. Instead we proceed as follows.

In a slicing, one finds that the perpendicular and parallel projections on a hypersurface S of the Lie derivative are

$$(\mathscr{L}_X g)_{\|\|} = -2X_\perp k + \mathscr{L}_{X_\|}\gamma$$

$$(\mathscr{L}_X g)_{\perp\perp} = -2N_\| \cdot \nabla X_\perp + \frac{2}{N_\perp} X_\| \cdot \nabla N_\perp + 2N_\perp \frac{\partial X_\perp}{\partial t}$$

$$(\mathscr{L}_X g)_{\|\perp} = -N_\perp \nabla\left(\frac{X_\perp}{N_\perp}\right) + \frac{1}{N_\perp}\gamma \frac{\partial X_\|}{\partial t} + \frac{1}{N_\perp} \cdot \mathscr{L}_{X_\|} N_\| .$$

Picking a slicing for which $N_\| = 0$ for convenience, one sees that this system is elliptic in the variables $X_\|$, $X_\perp$, $\dot{X}_\|$, $\dot{X}_\perp$ (elliptic in the sense of systems; cf. Hormander [1966]). Suppose $h_n = \mathscr{L}_{X_n} g_0$ lies in the range of $\mathscr{L}_{(\cdot)} g_0$ and $h_n \to h$ in H^s. We need to show that $h = \mathscr{L}_X g_0$ for some X. The range of the projection onto S is closed by ellipticity on S, so we can find $X_\perp$, $X_\|$, $\dot{X}_\perp$, $\dot{X}_\|$ in (H^{s+1}, H^s) such that $h_{\|\|}$, $h_{\perp\perp}$ and $h_{\|\perp}$ are given by the right-hand sides of the preceding equations. However X itself is determined by this Cauchy data and the hyperbolic evolution equation $\delta(\mathscr{L}_X g_0) = \delta h$, where δ is the divergence with respect to g_0; cf. Fischer, Marsden and Moncrief [1980] p. 160. Thus, there is an $X \in H^{s+1}$ such that $\mathscr{L}_X g_0 = h$.

This completes the verification of lemma 5.1 and hence of step (A) of the proof.

(B) To show that $\mathcal{O}_{g_0}$ is closed in $\bar{\mathscr{E}}^s(\Sigma, V)$, we consider a sequence $\eta_n \in \mathscr{D}^{s+1}(V)$ for which $\eta_n^* g_0$ has a limit $g_\infty \in \bar{\mathscr{E}}^s(\Sigma, V)$; i.e.

$$\lim_{n\to\infty} n_n^* g_0 = g_\infty , \quad \text{in } H^s .$$

It suffices to prove that there exists some $\eta_\infty \in \mathscr{D}^{s+1}(V)$ such that $g_\infty = \eta_\infty^* g_0$. In fact, we shall prove more, which is needed for step (C), namely that η_∞ is the limit of a subsequence of η_n.

We shall do this using constant mean curvature foliations, so we consider two cases. First, we assume that g_0 is *not* flat. Since diffeomorphisms cannot map flat spacetimes to spacetimes with curvature, it follows that for each n, $\eta_n^* g_0$ is not flat as well. We also note that g_∞ is not flat, since, for example, integrals of scalar functions of the curvature tensor of g_0 over Cauchy surfaces are preserved by the action of the diffeomorphisms η_n, and therefore convergence of $\eta_n^* g_0$ to g_∞ in the Sobolev norm implies that the corresponding curvature invariants of g_∞ cannot vanish.

We can now apply proposition 3.4. If we denote $\eta_n^* g_0$ by g_n, it tells us that there exist CMC foliations i_t^n of g_n and i_t^∞ of g_∞ such that the intrinsic metrics $\gamma_n(t) = \gamma(g_n, i_t^n)$, $\gamma_\infty(t) = \gamma(g_\infty, i_t^\infty)$ induced by these foliations satisfy (for each value of t)

$$\lim_{n\to\infty} \xi_n(t)^* \gamma_n(t) = \gamma_\infty(t) \tag{5.1}$$

and $\kappa(t) = \kappa(g_n, i_t^n) = t$, where $\xi_n(t)$ is a sequence of diffeomorphisms of Σ (i.e., $\xi_n(t) \in \mathscr{D}^{s+1}(\Sigma)$). By definition,

$$\gamma_n(t) = i_t^{n*} g_n = i_t^{n*}(\eta_n^* g_0) .$$

Recall that if i_t and j_t are CMC slicings for g and η^*g with $\kappa(g, i_t) = t = \kappa(\eta^*g, j_t)$ then there is a diffeomorphism $\zeta(t)$: $\Sigma \to \Sigma$ such that $\eta \circ j_t = i_t \circ \zeta(t)$. (We could, therefore, redefine our slicings to arrange ζ = identity, if desired.) Thus, there exists a sequence $\zeta_n(t) \in \mathscr{D}^{s+1}(\Sigma)$ such that

$$\gamma_n(t) = \zeta_n(t)^* \gamma_0(t) \tag{5.2}$$

for each value of t. Combining equations (5.1) and (5.2), we have

$$\lim_{n\to\infty} \bar{\eta}_n(t)^* \gamma_0(t) = \gamma_\infty(t) \tag{5.3}$$

where $\bar{\eta}_n(t) := \xi_n(t) \circ \zeta_n(t)$ is again a sequence in $\mathscr{D}^{s+1}(\Sigma)$.

The results of Ebin and Palais, when applied to equation (5.3) imply that for each value of t, the sequence $\bar{\eta}_n(t)$ has a convergent subsequence. (See Ebin [1970] proof of Prop. 6.13, p. 29.) Hence there exists some $\bar{\eta}_\infty(t) \in \mathscr{D}^{s+1}(\Sigma)$ such that

$$\bar{\eta}_\infty(t)^* \gamma_0(t) = \gamma_\infty(t)\,. \tag{5.4}$$

If $\kappa(g_0, i_t^0) = t$ and $\kappa(g, i_t^\infty) = t$ and $\bar{\eta}_\infty(t)$ is such that (5.4) holds, then the condition $\eta_\infty \circ i_t^\infty = i_t^0 \circ \bar{\eta}_\infty(t)$ defines a unique diffeomorphism $\eta_\infty \in \mathscr{D}^{s+1}(\mathrm{V})$ such that η_∞ is a limit of a subsequence of η_n and so $\eta_\infty^* g_0 = g_\infty$.

Thus we have proved that $\mathcal{O}_{g_0}$ is closed for nonflat g_0.

Let us now presume that g_0 is flat. It follows that the metrics $g_n := \eta_n^* g_0$ are also flat (all n), as is the limit metric g_∞. [We argue that g_∞ is flat using the vanishing of the full curvature tensor Riem (g_n) for all g_n and the identity Riem $(\eta^*g) = \eta^*$ Riem(g).] Thus we can apply proposition 3.11. Since the time translation constant τ needed to obtain equation (5.1) for these flat spaces can in the end be absorbed into g_∞, the argument used above goes through for the flat as well as the nonflat case, thus proving that for *any* spacetime (V, g), the orbit $\mathcal{O}_g$ is closed in $\tilde{\mathscr{E}}^s(\Sigma, \mathrm{V})$.

(C) So far we have shown that the orbits $\mathcal{O}_g$ are closed immersed manifolds. Now we wish to show that they are honest submanifolds. This follows from a "standard" lemma.

5.2. **Lemma.** *Let the hypotheses of lemma 5.1 hold and assume furthermore that for each sequence $x_n \in \mathrm{Q}$ such that $f(x_n) = y_n$ converges in* P, *there is a convergent subsequence in* Q. *Then $f(Q) \subset \mathrm{P}$ is a submanifold.*

Proof (see fig. 4). Let $x_0 \in \mathrm{Q}$ and let V_0 be a neighborhood of x_0 such that $f(\mathrm{V}_0)$ is a submanifold of P (this is guaranteed by 5.1). Suppose U_0 is a neighborhood of $f(x_0)$ which provides a submanifold chart for $f(x_0)$. Let $\mathrm{V}_n \searrow \{x_0\}$ and $\mathrm{U}_n \searrow \{f(x_0)\}$ be decreasing sequences of neighborhoods such that $f(\mathrm{V}_0) \cap \mathrm{U}_n = f(\mathrm{V}_n)$. It suffices to show that $f(\mathrm{V}_n) = \mathrm{U}_n \cap f(\mathrm{Q})$ for large n. If this were not the case there would

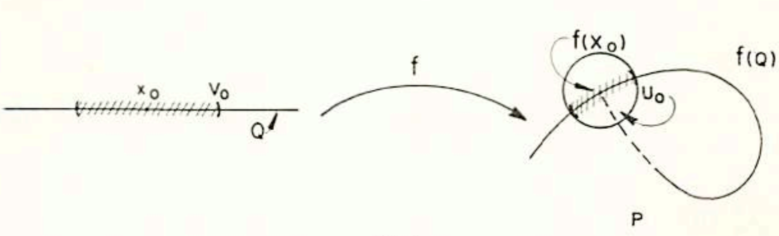

Fig. 4.

be a sequence $y_n = f(x_n) \in U_n$ such that $y_n \notin f(V_n)$ and hence $y_n \notin f(V_0)$. But $y_n \to f(x_0)$ and so by hypothesis there is a subsequence $\hat{x}_n$ of x_n converging to, say, x_∞. Now $f(x_0) = f(x_\infty)$ and the proof of lemma 5.1 shows that $f(V_0) \supset f(V_\infty)$ for a sufficiently small neighborhood V_∞ of x_∞.† But $y_n = f(\hat{x}_n) \in f(V_\infty)$ for large n so $\hat{y}_n \in f(V_0)$ for large n, which is a contradiction. ■

This now completes the proof that the orbits $\mathcal{O}_g$ are closed (embedded) submanifolds of $\tilde{\mathscr{E}}^s(\Sigma, V)$. Note that each orbit lies in a single stratum of $\tilde{\mathscr{E}}^s(\Sigma, V)$ and so is a manifold in the usual sense, without singularities.

6. Construction of the slice

To prove that a slice S_{g_0} exists for every spacetime (V, g_0) with $g_0 \in \tilde{\mathscr{E}}^{s+r}(\Sigma, V)$, we rely upon

(1) the preferred 3+1 decompositions of spacetimes which are provided by constant mean curvature foliations of spacetimes and

(2) a slice theorem (due to Fischer, Marsden and Moncrief [1980]) for diffeomorphisms of Σ acting on the space of Cauchy data.

Roughly speaking, S_{g_0} shall consist of those spacetime metrics g near g_0 such that for each value of κ, the data $(\gamma(\kappa), \pi(\kappa))$ on a CMC slicing corresponding to g lies on the Fischer–Marsden–Moncrief (FMM) slice $S_{(\gamma_0(t), \pi_0(t))}$ of $(\gamma_0(t), \pi_0(t))$ corresponding to g_0. In this section, we first make this construction of S_{g_0} precise, and then we prove that S_{g_0} is indeed a slice for the action of $\mathscr{D}^{s+1}(V)$ on $\tilde{\mathscr{E}}^s(\Sigma, V)$. We remind the reader that, in this section, we presume that the CMC conjecture holds.

We have already discussed (in section 3) the uniqueness of CMC (and proper maximal) foliations, including the key results (propositions 3.4 and 3.11) which permit us to compare nearby spacetimes surface-by-surface. The Fischer–Marsden–Moncrief slice theorem, which we now state, is a useful tool in making these comparisons.

6.1. **Lemma** (*F-M-M Slice Theorem for $\mathscr{D}^{s+1}(\Sigma)$ Acting on $\tilde{\mathscr{C}}^s(\Sigma)$*). *Let the group $\mathscr{D}^{s+1}(\Sigma)$ act upon the (stratified) manifold $\tilde{\mathscr{C}}^s(\Sigma)$ in the usual pullback sense: $\xi(\gamma, \pi) = (\xi^*\gamma, \xi^*\pi)$. Take $(\gamma_0, \pi_0) \in \tilde{\mathscr{C}}^{s+r}(\Sigma)$. Then we have the following:*

(a) *The orbit $\mathcal{O}_{(\gamma_0, \pi_0)}$ is a closed C^r embedded submanifold of $\tilde{\mathscr{C}}^s(\Sigma)$ in the sense of stratified manifolds.*

(b) *There exists a slice $S_{(\gamma_0, \pi_0)}$ (in the sense of theorem 4.1).*

Proof. This lemma is proved in section 5 of Fischer, Marsden and Moncrief [1980] (see lemma 5.1 and theorem 5.5). However, since we rely so heavily upon it in the proof of our slice theorem we sketch here how $S_{(\gamma_0, \pi_0)}$ is constructed.

It is most convenient to work within the space $T^*\mathcal{M}^s(\Sigma)$ and treat $\tilde{\mathscr{C}}^s(\Sigma)$ as the zero set of the Einstein constraint map Φ (see eq. (2.6)). Then, since $T^*\mathcal{M}^s(\Sigma)$ is an open subset of a Banach space, one can use the "affine method" to construct a slice as follows:

(a) Pick a Riemannian metric $\mathscr{D}_{(\gamma,\pi)}(\ ,\ )$ on $T^*\mathcal{M}^s(\Sigma)$ which is covariant under the action of $\mathscr{D}^{s+1}(\Sigma)$; a useful example is

$$\mathscr{G}_{(\gamma,\pi)}((h, \omega), (\bar{h}, \bar{\omega})) := \int_\Sigma (h_{ab}\bar{h}_{cd}\gamma^{ac}\gamma^{bd} + \omega^{ab}\bar{\omega}^{cd}\gamma_{ac}\gamma_{bd})\,\mu(\gamma)$$

where $(h, \omega), (\bar{h}, \bar{\omega}) \in T(T^*\mathcal{M}^s(\Sigma))$ and where $\mu(\gamma)$ is the volume element of γ on Σ.

† In our application Q is a group and the passage from x_0 to x_∞ can be effected simply by a group translation.

(b) Use $\mathscr{G}_{(\gamma,\pi)}$ to obtain an orthogonal decomposition of the tangent space $T_{(\gamma_0,\pi_0)}(T^*\mathscr{M})$ into two pieces:

$$T_{(\gamma_0,\pi_0)}(T^*\mathscr{M}^s) = T_{(\gamma_0,\pi_0)}(\mathscr{O}_{(\gamma_0,\pi_0)}) \oplus T^{\perp}_{(\gamma_0,\pi_0)}(\mathscr{O}_{(\gamma_0,\pi_0)})\,. \tag{6.1}$$

The first piece equals the range of $X \mapsto \mathbb{J} \circ D\Phi(\gamma_0, \pi_0)^*(0, X)$ and is tangent to the orbit $\mathscr{O}_{(\gamma_0,\pi_0)}$ generated by $\mathscr{D}^{s+1}(\Sigma)$; the second piece is the orthogonal complement to $T_{(\gamma_0,\pi_0)}(\mathscr{O}_{(\gamma_0,\pi_0)})$. This decomposition relies upon the ellipticity of the operator $D\Phi(\gamma_0, \pi_0)^*$ (where * is the $\mathscr{G}$-adjoint of the operator $D\Phi_{(\gamma_0,\pi_0)}$) and upon the Fredholm alternative theorem in $T_{(\gamma_0,\pi_0)}(T^*\mathscr{M}^s)$ (see Fischer and Marsden [1975]).

The decomposition (6.1) is related to Moncrief's [1975] decomposition which reflects the action of $\mathscr{D}^{s+1}(V)$ (rather than $\mathscr{D}^{s+1}(\Sigma)$) on the space of Cauchy data. This states that $T_{(\gamma_0,\pi_0)}(T^*\mathscr{M}^s) = \text{range}(\mathbb{J} \circ D\Phi(\gamma_0, \pi_0)^*) \oplus \text{Ker}(D\Phi(\gamma_0, \pi_0) \circ \mathbb{J}) \cap \text{Ker}(D\Phi(\gamma_0, \pi_0)) \oplus \text{range}(D\Phi(\gamma_0, \pi_0)^*)$. Here the second summand is essentially the "TT" perturbation data of Arnowitt, Deser and Misner [1962] and will parametrize the slice we ultimately want.

(c) Construct the ρ-ball, $N_\rho(\gamma_0, \pi_0)$, in $T^{\perp}_{(\gamma_0,\pi_0)}(\mathscr{O}_{(\gamma_0,\pi_0)})$ using a Sobolev norm $\|\cdot\|^s_{\gamma_0}$ which involves γ_0-compatible covariant derivatives rather than partial derivatives. $N_\rho(\gamma_0, \pi_0)$ defines a disc in $T^*\mathscr{M}^s(\Sigma)$ as well as in the tangent space because of the linearity of $T^*\mathscr{M}^s(\Sigma)$. [Note that the topology defined by the "γ_0-covariant" Sobolev norm $\|\ \|^s_{\gamma_0}$ is compatible with that defined by the original Sobolev norm $\|\ \|^s$ (which uses chart-dependent partial derivatives). The latter is not covariant under the $\mathscr{D}^{s+1}(\Sigma)$ action, however, and it therefore cannot be used to define a slice.]

(d) For sufficiently small ρ, $N_\rho(\gamma_0, \pi_0)$ determines a slice in $T^*\mathscr{M}^s$. This slice intersects the strata of $\mathscr{C}^s(\Sigma)$ in manifolds (see Fischer, Marsden and Moncrief [1980]) so it defines a slice in $\mathscr{C}^s(\Sigma)$.

To verify that indeed $S_{(\gamma_0,\pi_0)} = N_\rho(\gamma_0, \pi_0)$ is a slice in $T^*\mathscr{M}^s$ one needs to check the three properties: The first one follows from the covariance of both Φ and $\mathscr{G}$ under the action of $\mathscr{D}^{s+1}(\Sigma)$. This covariance implies that Φ and $\mathscr{G}$ are both *invariant* under the action of $I_{(\gamma_0,\pi_0)}$. Therefore if $(\gamma_1, \pi_1) \in S_{(\gamma_0,\pi_0)}$ and if $\xi \in I_{(\gamma_0,\pi_0)}$ then $\xi^*(\gamma_1, \pi_1) \in S_{\xi^*(\gamma_0,\pi_0)} = S_{(\gamma_0,\pi_0)}$; so $\xi^* S_{(\gamma_0,\pi_0)} = S_{(\gamma_0,\pi_0)}$.

To check the other two properties, let us assume first that (γ_0, π_0) has no Killing vector fields. Then $\tilde{\mathscr{C}}^s(\Sigma)$ is a C^∞ manifold in a neighborhood of $\mathscr{O}_{(\gamma_0,\pi_0)}$, and we find that (by construction), the set

$$\bigcup_{\xi \in \mathscr{D}^{s+1}(\Sigma)} S_{\xi^*(\gamma_0,\pi_0)} = \bigcup_{\xi \in D^{s+1}(\Sigma)} \xi^* S_{(\gamma_0,\pi_0)}$$

(which we shall call $\mathscr{B}_{(\gamma_0,\pi_0)}$) is a C^∞ bundle over $\mathscr{O}_{(\gamma_0,\pi_0)}$. The decomposition (6.1) allows us to use the implicit function theorem and therefore (for sufficiently small ρ) obtain $\mathscr{B}_{(\gamma_0,\pi_0)}$ as a tubular neighborhood of $\mathscr{O}_{(\gamma_0,\pi_0)}$ foliated by the submanifolds $\phi^* S_{(\gamma_0,\pi_0)}$. Then the properties (S2) and (S3) become obvious.

If (γ_0, π_0) has Killing vector fields (i.e., $I_{(\gamma_0,\pi_0)}$ is not trivial), we must be a bit more careful. $\tilde{\mathscr{C}}^s(\Sigma)$ is not a manifold near such (γ_0, π_0) and therefore $\mathscr{B}_{(\gamma_0,\pi_0)}$ cannot be a bundle in the usual sense. However, since $\tilde{\mathscr{C}}^s(\Sigma)$ is a stratified manifold and since $\mathscr{O}_{(\gamma_0,\pi_0)}$ respects the strata, we still obtain $\mathscr{B}_{(\gamma_0,\pi_0)}$ as a tubular (stratified) neighborhood of $\mathscr{O}_{(\gamma_0,\pi_0)}$ which is foliated (in the sense of definition 3.1) by the $\xi^* S_{(\gamma_0,\pi_0)}$. Again the properties (S2) and (S3) become obvious. ■

Using the slices constructed in lemma 6.1, we now build slices for $\mathscr{D}^{s+1}(V)$ acting on $\tilde{\mathscr{E}}^s(\Sigma, V)$ as follows:

6.2. **Construction** (*Slice Construction for Nonflat Spacetimes*). *Assume* $g_0 \in \tilde{\mathscr{E}}^{s+r}(\Sigma, V)$ *is not flat. Pick a*

CMC slicing $i_t^0 \in \mathrm{Emb}^{s+1}(\Sigma, V)$ such that for each value of t, i_t^0 induces a set of Cauchy data $(\gamma_0(t), \pi_0(t)) = (\gamma(g_0, i_t^0), \pi(g_0, i_t^0)) \in \tilde{\mathscr{C}}^{s+r}(\Sigma)$ with mean curvature $\kappa_0(t) = \kappa(g_0, i_t^0) = t$. Define

$$\begin{aligned} S_{g_0} = \{g \in \tilde{\mathscr{E}}^s(\Sigma, V) | i_t^0 \text{ is a CMC slicing with induced data } (\gamma(t), \pi(t)) \\ = (\gamma(g, i_t^0), \pi(g, i_t^0)) \text{ satisfying } \kappa(g, i_t^0) = t \text{ and } (\gamma(t), \pi(t) \in S_{(\gamma_0(t), \pi_0(t))} \text{ for all } t\}. \end{aligned} \tag{6.2}$$

We claim that S_{g_0} is a slice. To show this, we verify the three properties (S1)–(S3) one-by-one:

(S1) Consider some $g \in S_{g_0}$. We have to show that if $\eta \in I_{g_0}$, then $\eta^* g \in S_{g_0}$. Since $\eta^* g_0 = g_0$ (by definition of I_{g_0}), the uniqueness of CMC foliations guarantees that η leaves the surfaces $i_t^0(\Sigma) \subset V$ invariant; that is, $\eta(i_t^0(\Sigma)) = i_t^0(\Sigma)$. Hence the action of η on V is just that of a parametrized set of spatial diffeomorphisms $\eta(t) \in \mathscr{D}^{s+1}(\Sigma)$ on the CMC surfaces i.e. $i_t^0 \circ \eta(t) = \eta \circ i_t^0$. It follows that i_t^0 is a CMC foliation for $\eta^* g_0$, with $\kappa(\eta^* g_0, i_t^0) = t$. If we can show that for each value of t,

$$\eta(t)^*(\gamma(t), \pi(t)) \in S_{(\gamma_0(t), \pi_0(t))}, \tag{6.3}$$

then we're done with (S1). But $\eta^* g_0 = g_0$ implies that $\eta(t)^*(\gamma_0(t), \pi_0(t)) = (\gamma_0(t), \pi_0(t))$. Therefore, since $\eta(t)^* S_{(\gamma_0(t), \pi_0(t))} = S_{(\gamma_0(t), \pi_0(t))}$ (by lemma 6.1) we have (6.3) satisfied. This proves that $\eta^* S_{g_0} = S_{g_0}$.

(S2) Suppose there exists some $g \in \tilde{\mathscr{E}}^s(\Sigma, V)$ and some $\xi \in \mathscr{D}^{s+1}(V)$ such that both g and $\xi^* g$ are contained in S_{g_0}. Then we must show that $\xi \in I_{g_0}$. By definition of S_{g_0}, we know that if $g \in S_{g_0}$ and $\xi^* g \in S_{g_0}$, then i_t^0 must be a CMC foliation of both g and $\xi^* g$ with $\kappa(g, i_t^0) = \kappa(\xi^* g, i_t^0) = t$. It follows from proposition 3.3 that the diffeomorphism ξ leaves the surfaces $i_t^0(\Sigma)$ invariant and so induces a parametrized set of spatial diffeomorphisms $\xi_t = \xi(t) \in \mathscr{D}^{s+1}(\Sigma)$. For each t, we have both $(\gamma(t), \pi(t))$ and $\xi_t^*(\gamma(t), \pi(t))$ contained in $S_{(\gamma_0(t), \pi_0(t))}$. It follows from lemma 6.1 that $\xi_t \in I_{(\gamma_0(t), \pi_0(t))}$ for all t. But if η preserves the foliation, and if it leaves the Cauchy data of g_0 invariant on every surface $i_t(\Sigma)$, then it must be an isometry of g_0.

(S3) Clearly if we make *any* choice of cross-section $\mu : U \to \mathscr{D}^{s+1}(V)$ for U a neighborhood of the identity coset in $\mathscr{D}^{s+1}(V)/I_{g_0}$, we obtain a well-defined map $F : U \times S_{g_0} \to \tilde{\mathscr{E}}^s(\Sigma, V)$ given by $F(u, g) = \mu(u)^* g$. To show that there exists a μ for which F is a *homeomorphism* onto a neighborhood $V \subset \tilde{\mathscr{E}}^s(\Sigma, V)$, it is sufficient to show the following: For all g_1 contained in some neighborhood W of g_0, there exists a diffeomorphism $\eta \in \mathscr{D}^{s+1}(V)$ for which $\eta^* g \in S_{g_0}$; furthermore this diffeomorphism η must be unique modulo elements of I_{g_0}, and modulo elements of I_{g_0} it must lie in a neighborhood of the identity. We proceed to verify this now.

We know from proposition 3.4 that there exists a W such that if i_t^0 is a CMC slicing of g_0 and if i_t^1 is a CMC foliation of $g_1 \in W$, then there exists a diffeomorphism $\bar{\eta}$ such that $\bar{\eta} \circ i_t^1 = i_t^0$. Proposition 3.3 guarantees that for g_0 not flat, $\bar{\eta}$ is unique up to a parametrized set $\xi(t)$ of diffeomorphisms of the surfaces $i_t^0(\Sigma)$.

Let $(\gamma_1(t), \pi_1(t)) = (\gamma(g_1, i_t^1), \pi(g_1, i_t^1))$ be the data induced by a chosen g_1. Generally $(\gamma_1(t), \pi_1(t)) \notin S_{(\gamma_0(t), \pi_0(t))}$. However, by lemma 6.1, if (for fixed t) the data $(\gamma_1(t), \pi_1(t))$ lies in a sufficiently small neighborhood $U(t)$ of $(\gamma_0(t), \pi_0(t))$, then there exists a diffeomorphism $\zeta(t) \in \mathscr{D}^{s+1}(\Sigma)$ such that $\zeta(t)^*(\gamma_1(t), \pi_1(t))$ does lie in $S_{(\gamma_0(t), \pi_0(t))}$; and this $\zeta(t)$ is unique up to elements of $I_{(\gamma_0(t), \pi_0(t))}$. Now $(\gamma_1(t), \pi_1(t))$ generally does not lie in such a $U(t)$. Proposition 3.4, however, guarantees that we can choose $\bar{\eta}$ (using the freedom in $\xi(t)$) so that it does. Combining $\bar{\eta}$ and $\zeta(t)$, we get a diffeomorphism η for which $\eta^* g_1 \in S_{g_0}$, with η unique up to elements of V_{g_0}. That this η lies in a neighborhood of the identity (modulo elements of V_{g_0}) follows from our proof that $\mathcal{O}_{g_0}$ is a closed embedded submanifold of $\tilde{\mathscr{E}}^s(\Sigma, V)$.

Finally, that S_{g_0} is a manifold (in the stratified sense) may be proved by the transversality techniques of Fischer, Marsden and Moncrief [1980]; see especially lemma 7.3, p. 187.

Thus we have verified that we have a slice at nonflat spacetimes.

To complete the proof of our slice theorem 4.1, we need a prescription for constructing a slice S_{g_0} for a flat spacetime (V, g_0). This prescription, which we present in construction 6.3 (below), is much like that used for nonflat spacetimes (construction 6.2) in that g can be in S_{g_0} only if the Cauchy data on each CMC (or maximal) surface of g lies in the FMM slice of the Cauchy data on some maximal surface of g_0. There is an important difference here, however. While in construction 6.2 (for nonflat g_0) we require that $\kappa(g, i_t^0) = \kappa(g_0, i_t^0) = t$ and thereby rigidly tie the CMC surfaces of (V, g) to those of (V, g_0) before checking that $(\gamma(t), \pi(t)) \in S_{(\gamma_0(t), \pi_0(t))}$, here we let the surface-pairing float to some extent.

It should be clear that something like this allowance for floating is needed when g_0 is flat. Firstly, since the maximal surfaces of a flat spacetime don't have a preferred parametrization (see proposition 3.7), there is no unambiguous way to pair up the surfaces. Secondly, since the isotropy group I_{g_0} for a flat spacetime g_0 can be larger than that of any nonflat spacetime, the slice for a flat spacetime must also be larger; construction of S_{g_0} with floating surface-pairing accomplishes this. But the floating can't be totally free. The extra dimension of S_{g_0} for g_0 flat should match the increase in size of the isotropy group I_{g_0} (see section 1). To control the floating, we need proper maximal foliations. Using these, we construct a slice S_{g_0} near g_0 flat as follows:

6.3. **Construction** (*Slice Construction for Flat Spacetimes*). *Let* (V, g_0) *be a flat spacetime in* $\tilde{\mathscr{E}}^{s+r}(\Sigma, V)$.

Let i_t^0 *be a proper maximal slicing for* (V, g_0). *Define* $S_{g_0} = \{g \in \tilde{\mathscr{E}}^s(\Sigma, V) | i_t^0$ *is a CMC slicing for* (V, g) *and there is a constant* T *such that*

(a) *if* g *is flat,* i_t^0 *is a proper maximal slicing for* g;

(b) *if* g *is not flat,* $\kappa(g, i_t^0) = t + T$;

(c) $\gamma(t) = \gamma(g, i_t^0)$ *and* $\pi(t) = \pi(g, i_t^0)$ *satisfy* $(\gamma(t), \pi(t)) \in S_{(\gamma_0(t+T), \pi_0(t+T))}\}$. (Note that $\pi_0(t) = 0$.)

As before, we claim that S_{g_0} is a slice at g_0. First of all, S_{g_0} is a sub-manifold in the stratified sense, as in Fischer, Marsden and Moncrief [1980] p. 180. Next we verify (S1)–(S3).

(S1) Let $g \in S_{g_0}$ and $\eta \in I_{g_0}$. Then η is a screw motion by proposition 3.10, so

$$\eta \circ i_t^0 = i_{t+T} \circ \xi(t)$$

for a constant T and $\xi(t)$ an isometry of γ_0, so $\xi(t) \in I_{(\gamma_0, \pi_0)} \subset \mathscr{D}^{s+1}(\Sigma)$. It is clear from this and the transformation laws for Cauchy data that (a), (b) and (c) hold for $\eta^* g$.

(S2) Now suppose $g \in S_{g_0}$ and $\eta^* g \in S_{g_0}$. Suppose for example that g is flat. Then by uniqueness of proper maximal slicings (proposition 3.9), $\eta \circ i_t^0 = i_{t+T}^0 \circ \xi(t)$ for $\xi(t) \in \mathscr{D}^s(\Sigma)$. But then

$$(\gamma(t), \pi(t)) \in S_{(\gamma_0(t+T), \pi_0(t+T))}$$

and

$$\xi(t)^*(\gamma(t), \pi(t)) \in S_{(\gamma_0(t+T), \pi_0(t+T))} .$$

Hence by property (S2) for the $\mathscr{D}^{s+1}(\Sigma)$ slice, $\xi(t) \in I_{(\gamma_0(t+T), \pi_0(t+T))}$ i.e. $\xi(t)$ is an isometry for $\gamma_0(t+T)$. Thus η is a screw motion, so $\eta \in S_{g_0}$.

The proof for nonflat g is similar.

(S3) This proof proceeds in a manner analogous to (S3) above for g_0 nonflat, this time using instead the perturbation result for proper maximal foliations of flat spacetimes (see proposition 3.11). Note that if we were to attempt to use maximal foliations rather than proper maximal foliations in constructing S_{g_0}, property (S3) would fail.

Thus the properties of a slice are verified in all cases.

7. A localized slice construction

The construction we have used in sections 5 and 6 to prove the slice theorem 4.1 for $\mathscr{D}^{s+1}(V)$ acting upon $\bar{\mathscr{E}}^s(\Sigma, V)$ depends upon the validity of the CMC conjecture 3.2. What can be done if this conjecture – which says that every spacetime $\bar{\mathscr{E}}^s(\Sigma, V)$ has a global CMC foliation – is invalid? We can no longer prove theorem 4.1 using our CMC foliation procedure (although the theorem may still be true). We can, however, still prove a modified slice theorem.

Roughly speaking, the chief modification we make is to restrict attention to portions of spacetimes which are finite in temporal extent (yet still globally hyperbolic). As will be seen, the appropriate collections of such "truncated spacetimes" form stratified manifolds similar to $\bar{\mathscr{E}}^s(\Sigma, V)$ itself. Thus, once we have proven a slice theorem for the action of appropriate diffeomorphism groups on these collections, then all of the corollaries of the slice theorem (stratified symplectic structure, local maxima of spacetime symmetries, etc.) follow. The importance to gravitational physics and cosmology is not as immediate for these results as for those which follow from the slice theorem 4.1. However, as will become clear after we have described the modified theorem, these results are still useful.

To motivate the collections of spacetimes which appear in the modified slice theorem, let us consider an arbitrary (nonflat) spacetime (V, g_0) contained in $\bar{\mathscr{E}}^s(\Sigma, V)$. By definition of $\bar{\mathscr{E}}^s(\Sigma, V)$, (V, g_0) contains at least one CMC hypersurface. By proposition 3.1, this hypersurface can be extended into a local CMC foliation, which we label by i^0_κ, with κ taking on all values in $[\kappa_1, \kappa_2]$ for some pair of real numbers κ_1 and κ_2, $\kappa_1 < \kappa_2$. (We adopt our usual convention that $\kappa(g_0, i^0_\kappa) = \kappa$.)

Now let us consider spacetimes which are near (V, g_0) in $\bar{\mathscr{E}}^s(\Sigma, V)$. By proposition 2.5, there exists a neighborhood $\mathscr{U} \subset \bar{\mathscr{E}}^s(\Sigma, V)$ containing (V, g_0) such that for every $(V, g) \in \mathscr{U}$, there is a local CMC foliation i_κ with values of κ covering the interval $[\kappa_1, \kappa_2]$. Thus, even if conjecture 3.2 is false, one finds that for any given (nonflat) spacetime (V, g_0) in $\bar{\mathscr{E}}^s(\Sigma, V)$, there is an open neighborhood $\mathscr{U}$ of spacetimes (with $(V, g_0) \in \mathscr{U}$) such that all spacetimes in $\mathscr{U}$ have portions with corresponding CMC foliations.

One might be tempted to try to use these corresponding partial CMC foliations to construct a slice (via construction 5.2) about (V, g_0) in $\bar{\mathscr{E}}^s(\Sigma, V)$. Clearly such an attempt is doomed since the CMC foliations leave portions of the spacetimes uncovered. However, if we chop off the spacetimes to the future of $i_{\kappa_2}(\Sigma)$ and to the past of $i_{\kappa_1}(\Sigma)$, then the CMC foliations do cover what remains of the spacetimes, and a slice construction can be carried out. The collection of these truncated spacetimes we shall call $\bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$, where $V' \subset V$ is a four-manifold with boundary, that boundary consisting of two disjoint diffeomorphic copies of Σ. It is defined explicitly as follows: $\bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$ is the set of all globally hyperbolic H^s-class solutions of Einstein's (vacuum) equations on V' for which there exists a CMC foliation $i_\kappa : \Sigma \to V'$ satisfying thee conditions: (a) κ runs from κ_1 to κ_2; (b) $\kappa(g, i_\kappa) = \kappa$; and (c) the union of the images of i_κ, for $\kappa \in [\kappa_1, \kappa_2]$, equals V'.

We emphasize that the spacetimes contained in $\bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$ are generally *not* maximal developments (i.e., they generally have globally hyperbolic extensions). We also emphasize that $\bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$ is *not* a subspace of $\bar{\mathscr{E}}^s(\Sigma, V)$. Rather, one may view it as a quotient space of a subspace of $\bar{\mathscr{E}}^s(\Sigma, V)$, with the quotient being by means of the action of the group of diffeomorphisms of V which are the identity on $V' \subset V$.

To show that $\bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$ has the structure of a stratified manifold, it is best to think of the elements of $\bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$ not as spacetimes from $\bar{\mathscr{E}}^s(\Sigma, V)$ which have been truncated, but rather as (non-maximal) spacetime developments of Cauchy data in $\bar{\mathscr{C}}^s_{[\kappa_1,\kappa_2]}(\Sigma)$, the set of all ($H^s$ class) CMC Cauchy data with $\kappa_1 \le \kappa \le \kappa_2$. Now $\bar{\mathscr{C}}^s_{[\kappa_1,\kappa_2]}(\Sigma)$ is open in $\bar{\mathscr{C}}^s(\Sigma)$, so $\bar{\mathscr{C}}^s_{[\kappa_1,\kappa_2]}(\Sigma)$ has the conical stratified structure of $\bar{\mathscr{C}}^s(\Sigma)$. Just as $\bar{\mathscr{E}}^s(\Sigma, V)$ is a principal bundle over $\bar{\mathscr{C}}^s(\Sigma)$, $\bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$ is a principal bundle over $\bar{\mathscr{C}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$, the fibre of which can be identified with diffeomorphisms of V′ which leave some fixed Cauchy surface $i_0(\Sigma)$ pointwise invariant. Therefore $\bar{\mathscr{E}}^s_{[\kappa_0,\kappa_1]}(\Sigma, V)$ inherits the structure of $\bar{\mathscr{C}}^s_{[\kappa_1,\kappa_2]}(\Sigma)$; it is a stratified manifold which looks very much like $\bar{\mathscr{E}}^s(\Sigma, V)$, the space which is considered in the slice theorem 4.1.

If we are to set up and prove a slice theorem for $\bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V)$, we of course need a group which acts on this space, and not surprisingly the appropriate group is $\mathscr{D}^{s+1}(V')$ with the usual action via pullback. One might think that the artificial boundedness in temporal extent of V′ would make $\mathscr{D}^{s+1}(V')$ significantly different from the group of diffeomorphisms of a manifold V which has no boundary, and perhaps lead to difficulties in proving a slice theorem. However, as shown in Ebin and Marsden [1970], $\mathscr{D}^{s+1}(V')$ is a C^∞-manifold and is a topological group. We are thus led to the following.

7.1. **Theorem** (*Localized Slice Theorem – Nonflat Case*). *Let κ_1 and κ_2, be any*† *pair of real numbers, and assume $s > 2.5$. Let the group $\mathscr{D}^{s+1}(V')$ act on the (stratified) manifold $\bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$ in the usual pullback sense. Let $r > 1$ and $g_0 \in \bar{\mathscr{E}}^{s+r}_{[\kappa_1,\kappa_2]}(\Sigma, V')$. Then we have the following*:

(a) *The orbit $\mathcal{O}_{g_0}$ is a closed $\mathscr{C}^r$ embedded submanifold of $\bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$.*

(b) *There exists a submanifold $S_{g_0} \subset \bar{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$ containing g_0 which satisfies the three conditions (S1)–(S3) of Theorem 4.1; i.e., S_{g_0} is a slice.*

The proof of this result is essentially identical to that of theorem 4.1 (without the complications involving flat spacetimes with timelike Killing fields) so we need not detail it.

The reason the proof of theorem 7.1 can avoid the complications associated with flat spacetimes with time-like Killing fields is simply that $\bar{\mathscr{E}}_{[\kappa_1,\kappa_2]}$ doesn't contain any – see proposition 3.6. So, to obtain a comprehensive set of slice theorems to replace theorem 4.1 in all cases if in fact conjecture 3.2 is false, we seek a set of spacetimes analogous to $\mathscr{E}_{[\kappa_1,\kappa_2]}(\Sigma, V')$ which does contain the flat ones.

Although every flat spacetime (with a timelike Killing vector field) admits a global foliation by maximal hypersurfaces, there is nothing which is known to preclude every neighborhood of a given flat spacetime (V, g_0) from containing at least one nonflat spacetime (V, g) which has no global CMC foliation. Thus, if we wish to use our slice constructions (6.2) and (6.3), we again must cut V down to a fixed manifold V′. (If we were perhaps to use different manifolds V′ for the different spacetimes in the collection, then we couldn't find one group of diffeomorphisms which acts on the collection.)

For flat spacetimes, the appropriate way to limit (V, g) is via proper time. One might be tempted to continue to use intervals of κ to limit the nonflat spacetimes in the collection. However, it is not clear that a set which limits proper time extension for some of its elements and limits κ time for others can be fit together nicely into a stratified manifold of some sort (especially since, for sufficiently small perturbations of flat space, d(tr k)/d (proper time) can be arbitrarily small). Hence we use proper time to limit the nonflat spacetimes as well as the flat ones. We are led to define the following: $\bar{\mathscr{E}}^s_T(\Sigma, V')$ is

†The topology of Σ may place restrictions on permissible choices of κ_1 and κ_2. For example, if $\Sigma = T^3$ then κ_1 and κ_2 must have the same sign; see Schoen and Yau [1979].

the set of all globally hyperbolic H^s-class solutions of Einstein's (vacuum) equations on V′such that

(a) if $(V', g_F) \in \mathscr{E}^s(\Sigma, V')$ is flat (with a timelike Killing field), then there exists a proper maximal foliation i_t of (V', g_F) with t running from 0 to T and with the union of the images of i_t $(0 \le t \le T)$ equalizing V′; and

(b) if $(V', g) \in \bar{\mathscr{E}}^s(\Sigma, V')$ is *not* flat, then there exists a CMC foliation i_t of (V', g), the union of whose image equals V′, and further the proper distance between the initial and final CMC hypersurfaces of (V', g) is T. Note that in this definition of $\bar{\mathscr{E}}^s_T(\Sigma, V')$, we demand that a nonflat spacetime (V', g) possess a CMC foliation, but we do not in any way restrict the range of values $\kappa = \operatorname{tr} k$ may take.

It is not clear what the global structure of $\bar{\mathscr{E}}^s_T(\Sigma, V')$ is. However, in a sufficiently small neighborhood of a flat spacetime, $\bar{\mathscr{E}}^s_T(\Sigma, V')$ has the usual stratified manifold structure. One can prove this by combining (1) continuity and differentiability results for the Cauchy problem on compact regions of spacetime (see section 7.6 of Hawking and Ellis [1973] and Fischer and Marsden [1979]); (2) proposition 3.11 which guarantees that for all sufficiently small perturbations of a flat spacetime in a compact region, one has a CMC foliation; and (3) results of Fischer, Marsden and Moncrief [1980] on the structure of the constraint set.

With at least open subsets of $\bar{\mathscr{E}}^s_T(\Sigma, V')$ having the usual structure, we can consider the action of $\mathscr{D}^{s+1}(V')$ on $\bar{\mathscr{E}}^s_T(\Sigma, V')$ and prove a slice theorem. Before stating this theorem, however, we note an important consequence of the proper time boundedness of the spacetimes in $\bar{\mathscr{E}}^s_T(\Sigma, V')$: there can be no timelike isometry. This fact is reflected in the construction of the slices.

7.2. **Theorem** (*Localized Slice Theorem–Flat Case*). *Let* $T > 0$, *and let* $\bar{\mathscr{E}}^s_{T,\mathrm{Flat}}(\Sigma, V')$ *be the set of all flat spacetimes in* $\bar{\mathscr{E}}^s_T(\Sigma, V')$ *for any real value of T. Then there exists a neighborhood* $\mathscr{U}$ *of* $\bar{\mathscr{E}}^s_{T,\mathrm{Flat}}(\Sigma, V')$ *in* $\bar{\mathscr{E}}^s_T(\Sigma, V)$ *which is invariant under the action of the group* $\tilde{\mathscr{D}}^{s+1}(V')$, *and is such that for* $s > 2.5$, $r > 1$ *and* $g_0 \in \mathscr{U}$, *we have the following*:

(a) *The orbit* $\mathcal{O}_{g_0}$ *is a closed,* C^r *embedded submanifold of* $\mathscr{U}$.

(b) *There exists a submanifold* $S_{g_0} \subset \bar{\mathscr{E}}^s_T(\Sigma, V')$ *which satisfies the three conditions (S1)–(S3) of theorem 4.1 so that* S_{g_0} *is a slice.*

Proof. Again, the proof is very similar to that of theorem 4.1, so we can leave out most of the details. We do note some important modifications used in constructing the slices, however. These deal with how one matches up maximal or CMC surfaces for g_0 and those for $g \in S_{g_0}$. Let us first consider g_0 flat. One chooses a proper maximal slicing i_t for g_0, specifying that t run from 0 to T. For a flat g to be in S_{g_0}, we demand that i_t be a proper maximal slicing with $t \in [0, T]$, while for g nonflat, i_t is to be CMC with

$$\operatorname{tr} k(g, i_T) = \kappa_1(g) + (1 - t/T)(\kappa_2(g) - \kappa_1(g)). \tag{7.1}$$

Here $\kappa_1(g)$ is the smallest value of tr k taken by a CMC surface in V′ and $\kappa_2(g)$ is the largest value.

We now consider a spacetime $(g_0, V') \in \bar{\mathscr{E}}^s_T(\Sigma, V')$ which is not flat. One chooses a CMC slicing for g_0 and reparametrizes it so that t runs from 0 to T and with $\operatorname{tr} k(g_0, i_t) = \kappa_1(g_0) + (1 - t/T)(\kappa_2(g_0) - \kappa_1(g_0))$. Then for g flat to be in S_{g_0}, i_t must be a proper maximal foliation, while for g nonflat, i_t must be a CMC foliation satisfying equation (7.1). One verifies that with these modified pairings of surfaces, the steps of the proof as outlined in sections 6–7 still go through. Note that the slice construction for flat g_0 does not involve a "floating" mean curvature, as is required in construction 6.3. This result reflects the loss of the timelike isometry due to truncation. ■

Taken together, theorems 7.1 and 7.2 permit one to take any spacetime for $\bar{\mathscr{E}}^s(\Sigma, V)$, and then, after restricting it and all its neighboring spacetimes to regions V′ which have CMC foliations, to find a local slice for the action of the diffeomorphism group on these truncated spacetimes.

Does this result illuminate the study of the space of gravitational degrees of freedom? If one wants to focus on physical systems of finite temporal extent, then the answer is clearly yes. Even if one only considers the maximal spacetime developments to be important, the result is still useful however. For, as noted above, one can think of the sets $\tilde{\mathscr{E}}^s_{[\kappa_1,\kappa_2]}(\Sigma, V')$ and $\tilde{\mathscr{E}}^s_{T,\mathrm{Flat}}(\Sigma, V')$ as subsets of $\tilde{\mathscr{E}}^s(\Sigma, V)$ with the diffeomorphisms that are the identity inside V′ divided out. The modified slice theorem thus tells us nothing about the action of diffeomorphism in the complementary region V\V′, but it does tell us that relative to the action of the diffeomorphisms in the CMC region V′, the space of solutions does behave nicely.

Of course if, as we believe, conjecture 3.2 is true, then there is no need for theorems 7.1 and 7.2; the more powerful theorem 4.2 can then be used.

8. The stratification by symplectic manifolds

In Fischer [1970] and Bourguignon [1975] it is shown that $\mathscr{M}/\mathscr{D}$ is stratified by orbit type. This result uses the slice theorem in a crucial way. The same methods together with our slice theorem show that $\tilde{\mathscr{E}}(\Sigma, V)/\mathscr{D}(V)$ is stratified. (Recall that $\tilde{\mathscr{E}}(\Sigma, V)$ has quadratic singularities.) However a new phenomena occurs here: *each stratum is a symplectic manifold.*

Near generic points with no symmetry the quotient space is a symplectic manifold. This is closely related to the reduction results of Marsden and Weinstein [1974]. For the other strata, we shall invoke the techniques of Arms, Marsden and Moncrief [1981].

Throughout this section we shall assume the CMC conjecture to avoid complexities in the formulation. Also it will be convenient to work with the regularized theorems (i.e., $s \to \infty$).

8.1. **Theorem.** *The set $\tilde{\mathscr{E}}(\Sigma, V)/\mathscr{D}(V)$ is a stratified manifold with each stratum being symplectic. The generic stratum, consisting of (equivalence classes of) solutions of Einstein's equations with no isometries, is open and dense.*

Proof. Let us deal with the generic stratum first. Thus, let $\mathscr{E}_{\mathrm{gen}}(\Sigma, V) \subset \tilde{\mathscr{E}}(\Sigma, V)$ be the subset of solutions of Einstein's equations which have no isometries. From the structure theory for $\tilde{\mathscr{E}}$ (see section 2), we know that $\mathscr{E}_{\mathrm{gen}}$ is a smooth manifold which is open in $\tilde{\mathscr{E}}$. We shall postpone the proof that $\mathscr{E}_{\mathrm{gen}}/\mathscr{D}$ is dense until the next section (see corollary 9.3).

By construction, $\mathscr{E}_{\mathrm{gen}}/\mathscr{D}$ is well defined. By our slice theorem, $\mathscr{E}_{\mathrm{gen}}/\mathscr{D}$ is a smooth manifold with local coordinate charts given by the slices. To show that $\mathscr{E}_{\mathrm{gen}}/\mathscr{D}$ is symplectic, we rely upon the fact that, by construction, the tangent space to the slice is just the TT component of the Moncrief decomposition (see lemma 6.1). But, as shown in Fischer and Marsden [1979a], this TT component is naturally symplectic, so $\mathscr{E}_{\mathrm{gen}}/\mathscr{D}$, the generic stratum, is as well. The form so defined is closed, as in Marsden and Weinstein [1974], and so $\mathscr{E}_{\mathrm{gen}}/\mathscr{D}$ is symplectic.

We now turn to the other strata (i.e., those containing spacetimes with symmetries). Based on arguments such as those in Bourguignon [1975] and Fischer [1970], our slice theorems together with the structure theory for $\tilde{\mathscr{E}}$ guarantees that $\tilde{\mathscr{E}}/\mathscr{D}$ is stratified. Let $g_0 \in \tilde{\mathscr{E}}$ have a non-trivial isometry group and let $N_{g_0} := \{g \in \tilde{\mathscr{E}} \mid$ the isometry groups of g and g_0 are conjugate$\}$ i.e. N_{g_0} consists of solutions of the same orbit type as g_0.* It is shown in Fischer, Marsden and Moncrief [1980] that N_{g_0} is a smooth

*Here it is important to work with the full symmetry groups including discrete symmetries. The difference results in a branched covering.

manifold (without singularities). It is also obvious that $\mathscr{D}$ leaves N_{g_0} invariant. Thus the quotient space

$$\mathcal{N}_{g_0} = N_{g_0}/\mathscr{D}$$

is a smooth manifold and is locally parametrized by the slice within N_{g_0}, i.e. by

$$N_{g_0} \cap S_{g_0}$$

which, as we have remarked before, is a smooth manifold. Lemma 18 of Arms, Marsden and Moncrief [1980] guarantees that $\mathcal{N}_{g_0}$ is symplectic.

Now $\mathcal{N}_{g_0}$ is the stratum in $\bar{\mathscr{E}}/\mathscr{D}$ through (the equivalence class of) g_0. Thus $\bar{\mathscr{E}}/\mathscr{D}$ is the union of its strata, each of which is a smooth symplectic manifold. ■

The space $\mathscr{G} = \bar{\mathscr{E}}/\mathscr{D}$ is the space of "true degrees of freedom", i.e., the space of solutions of Einstein's equations modulo gauge transformations. We have just shown that for V spatially *compact*, $\mathscr{G}$ is a stratified symplectic manifold. Note, however, that in this spatially compact case, *all* the dynamics has been factored out; i.e., each point of $\mathscr{G}$ represents a class of equivalent four dimensional solutions.

For other classes of spacetimes [e.g., the asymptotically flat ones, with or without radiation] $\mathscr{G}$ would presumably have some dynamics remaining. It is reasonable therefore to carry out a similar reduction program for such spacetimes. The work of Cantor [1978, 1981], of Ashtekar and Magnon-Ashtekar (e.g. [1978]), of Regge and Teitelboim [1974] and of Stumbles [1981] might prove useful in such a program.

9. Parametrization of the slices by York data

In the early 1970's, York developed a program for solving the gravitational constraint equations (2.6) using certain tensor decompositions. One goal of York's program has been to provide a representation of the gravitational degrees of freedom. Since the slice S_{g_0} represents these degrees of freedom, at least in a neighborhood of the spacetime (V, g_0), it is interesting to compare our construction of slices with York's free data. We shall show that the two approaches are consistent, and in particular, that York's free data provides a practical parametrization of S_{g_0}.

For our purposes, it is most convenient to think of York's program in terms of a map

$$\mathscr{Y}_\tau : T^*\mathcal{M}^s(\Sigma) \to \bar{\mathscr{C}}^s(\Sigma) \subset T^*\mathcal{M}^s(\Sigma) \tag{9.1}$$

which, for a given $(\bar{\gamma}, \bar{\pi}) \in T^*\mathcal{M}^s(\Sigma)$, produces $(\gamma, \pi) \in \bar{\mathscr{C}}^s(\Sigma)$ via the following steps: (1) Extract from $\bar{\pi}^{ab}$ the piece $\bar{\sigma}^{ab}$ which is transverse and traceless with respect to $\bar{\gamma}_{ab}$; i.e.

$$\bar{\nabla}_a \bar{\sigma}^{ab} = 0\,, \qquad \bar{\gamma}_{ab}\bar{\sigma}^{ab} = 0\,. \tag{9.2}$$

(Here $\bar{\nabla}_a$ is the covariant derivative for the metric $\bar{\gamma}_{ab}$.) As shown by York [1974], for any given $(\bar{\gamma}_{ab}, \bar{\pi}^{cd}) \in T^*\mathcal{M}^s(\Sigma)$ (with $s \geq 2$), this $\bar{\sigma}^{ab}$ is unique. (2) Solve the equation

$$\bar{\nabla}^2 \phi = \frac{-1}{8 \det \bar{\gamma}} (\bar{\sigma}_{cb}\bar{\sigma}^{cd}) \phi^{-7} + \frac{\bar{R}}{8}\phi + \frac{1}{12}\tau^2 \phi^5 \tag{9.3}$$

for the scalar function ϕ (here $\bar{R}$ is the scalar curvature for $\bar{\gamma}_{ab}$, and τ is a real number which labels the map $\mathcal{Y}_\tau$). As shown by O'Murchadha and York [1974], solutions to this equation exist and are unique for an open dense subset† of $T^*\mathcal{M}^s(\Sigma)$. (3) Define

$$\left.\begin{aligned} &\gamma_{ab} = \phi^4 \bar{\gamma}_{ab} \\ \text{and} \quad & \\ &\pi^{cd} = \phi^{-4}\bar{\sigma}^{cd} + \tfrac{2}{3}\phi^2 \bar{\gamma}^{cd} \det \bar{\gamma}\tau . \end{aligned}\right\} \tag{9.4}$$

One easily verifies that (γ, π) satisfy the constraints and have mean curvature τ. Since the Sobolev class is preserved by this construction and since τ is constant on Σ, one has $(\gamma_{ab}, \pi^{cd}) \in \bar{\mathscr{C}}^s(\Sigma)$.

Since $\mathcal{Y}_\tau$ maps $T^*\mathcal{M}^s(\Sigma)$ into a space (namely $\bar{\mathscr{C}}^s(\Sigma)$) which is smaller than itself, we should expect that there are a number of invariances of this map. Indeed, one easily verifies that $\mathcal{Y}_\tau$ is invariant under the action of the conformal group $\Theta^s(\Sigma)$, whose elements θ are positive definite scalar functions in $H^s(\Sigma)$ which act on $T^*\mathcal{M}^s(\Sigma)$ via

$$\theta : (\bar{\gamma}_{ab}, \bar{\pi}^{cd}) \mapsto (\theta\bar{\gamma}_{ab}, \theta^{-1}\bar{\pi}^{cd}) . \tag{9.5}$$

The York map $\mathcal{Y}_\tau$ is also invariant under the addition of arbitrary trace and longitudinal pieces to $\bar{\pi}^{cd}$.

In addition to these invariances, $\mathcal{Y}_\tau$ also has an important covariance. In particular, it follows directly from the three step definition given above that $\mathcal{Y}_\tau$ commutes with the action of the spatial diffeomorphism group $\mathscr{D}^s(\Sigma)$; that is,

$$\mathcal{Y}_\tau(\eta^*\bar{\gamma}_{ab}, \eta^*\bar{\pi}^{cd}) = \eta^*\mathcal{Y}_\tau(\bar{\gamma}_{ab}, \bar{\pi}^{cd}) = (\eta^*\gamma_{ab}, \eta^*\pi^{cd}) \tag{9.6}$$

for any $\eta \in \mathscr{D}^s(\Sigma)$.

To get the York free data, one must divide these invariances and covariances out of $T^*\mathcal{M}^s(\Sigma)$. Actually, in view of the way $\mathcal{Y}_\tau$ works, what one really wants is to restrict to the subspace $T^*\mathcal{M}^s(\Sigma)_{TT}$ consisting of all $(\bar{\gamma}, \bar{\pi}) \in T^*\mathcal{M}^s(\Sigma)$ satisfying the transverse traceless conditions (9.2), and then divide out by the combined action of $\Theta^s(\Sigma)$ and $\mathscr{D}^s(\Sigma)$. The resulting quotient space

$$Y^s(\Sigma) := T^*\mathcal{M}^s(\Sigma)_{TT}/\mathscr{D}^s(\Sigma) \otimes \theta^s(\Sigma) \tag{9.7}$$

has been studied by Fischer and Marsden [1977]. It is the Marsden–Weinstein reduced space for the action of the "conformal rescaling-diffeomorphism group" (which is just the semi-direct product $\mathscr{D}^s(\Sigma) \times \Theta^s(\Sigma)$) on $T^*\mathcal{M}^s(\Sigma)$ (see theorem 3.2 of Fischer–Marsden [1977]). Combining the results of Fischer and Marsden [1977]‡ with those of Arms, Marsden and Moncrief [1981] and of Fischer, Marsden and Moncrief [1980], we obtain the following:

9.1. **Proposition** (*Structure of* $Y^s(\Sigma)$).

(a) *At every point* $(\bar{\gamma}_0, \bar{\pi}_0) \in T^*\mathcal{M}^{s+r}(\Sigma)_{TT}$ *(for* $s > 1.5$*) there exists a* C^r *slice* $S_{(\bar{\gamma}_0, \bar{\pi}_0)}$ *for the action of* $\mathscr{D}^{s+1}(\Sigma) \otimes \Theta^s(\Sigma)$ *on* $T^*\mathcal{M}^s(\Sigma)_{TT}$.

(b) *The set* $Y^s(\Sigma)$ *is a stratified manifold, with each stratum being (weakly) symplectic. The elements of any given stratum all have mutually conjugate conformal isotropy groups.*

The actual construction of the slice $S_{(\bar{\gamma}_0, \bar{\pi}_0)}$ used in proving proposition 9.1 is, not surprisingly, much

† The points of $T^*\mathcal{M}^s(\Sigma)$ for which (9.3) cannot be solved are those with $\bar{\sigma}^{ab} = 0$, and with $\bar{R} \geq 0$ for all metrics $\bar{\gamma}_{ab}$ conformally related to $\bar{\gamma}_{ab}$. (See Isenberg [1979] chapter III.)

‡ As was pointed out by Tony Tromba, the proof of the slice theorem in Fischer and Marsden [1977] is not complete. We believe it can be patched directly. Alternatively, our slice theorem can probably be used to construct one for the York data by means of the York map.

like that of the slice $S_{(\gamma_0,\pi_0)}$ for the diffeomorphism group acting alone on $T^*\mathcal{M}^s(\Sigma)$ or on its subset $\tilde{\mathscr{C}}^s(\Sigma)$ (see lemma 6.1). We also note that the stratification structure of $Y^s(\Sigma)$, like that of $\tilde{\mathscr{C}}^s(\Sigma)$, is conical (i.e., quadratic) in form.

Since the map $\mathscr{Y}_\tau : T^*\mathcal{M}^s(\Sigma) \to T^*\mathcal{M}^s(\Sigma)$ is invariant under the action of $\Theta^s(\Sigma)$ and since it commutes with the action of $\mathscr{D}^s(\Sigma)$, it follows that there exists a unique ("reduced") map

$$\hat{\mathscr{Y}}_\tau : Y^s(\Sigma) \to T^*\mathcal{M}^s(\Sigma)/\mathscr{D}^s(\Sigma)\,. \tag{9.8}$$

In the remainder of this section, we shall show that $\hat{\mathscr{Y}}_\tau$ establishes a local isomorphism between the York free data $Y^s(\Sigma)$ and the gravitational degrees of freedom as represented by the slices S_{g_0} defined in section 6 (and section 7). The main substance of this isomorphism is contained in the following result:

9.2. **Proposition** (*York Free Data and the Gravitational Degrees of Freedom*). *The map $\hat{\mathscr{Y}}_\tau$ is a homeomorphism from (an open, dense subset of) $Y^s(\Sigma)$ onto the set $\tilde{\mathscr{C}}^s_\tau(\Sigma)/\mathscr{D}^s(\Sigma)$ consisting of those (diffeomorphism equivalence classes of) data in $T^*\mathcal{M}^s(\Sigma)/\mathscr{D}^s(\Sigma)$ which satisfy the Einstein vacuum constraint* (2.6) *and which have constant mean curvature τ.*

Proof. It follows from its definition (in terms of $\mathscr{Y}_\tau$) that $\hat{\mathscr{Y}}_\tau$ maps an open dense subset of $Y^s(\Sigma)$ into $\tilde{\mathscr{C}}^s_\tau(\Sigma)/\mathscr{D}^s(\Sigma)$. To verify that $\hat{\mathscr{Y}}_\tau$ is surjective onto $\tilde{\mathscr{C}}^s_\tau(\Sigma)/\mathscr{D}^s(\Sigma)$, consider an arbitrary point $[(\gamma_0, \pi_0)] \in \tilde{\mathscr{C}}^s(\Sigma)/\mathscr{D}^s(\Sigma)$. (Here, the notation "$[(\gamma_0, \pi_0)]$" refers to "the diffeomorphism equivalence class containing (γ_0, π_0)".) If we subtract $\operatorname{tr}\pi_0 = -2\sqrt{\det\gamma_0}\,\tau$ from π_0, then the resulting tensor μ_0 is necessarily transverse and traceless (recall the second constraint equation in (2.6)), and so $[[(\bar{\gamma}_0, \bar{\sigma}_0)]] :=$ (conformal equivalence class of $[(\gamma_0, \mu_0)]$) specifies a point in $Y^s(\Sigma)$. (Here, the notation "$[[(\bar{\gamma}_0, \bar{\sigma}_0)]]$" refers to "the conformal and diffeomorphism equivalence class containing (γ_0, σ_0)".) But $\hat{\mathscr{Y}}_\tau$ clearly maps $[[(\bar{\gamma}_0, \bar{\sigma}_0)]]$ to $[(\gamma_0, \pi_0)]$ and so $\hat{\mathscr{Y}}_\tau$ is surjective.

To show that $\hat{\mathscr{Y}}_\tau$ is injective, consider a pair of distinct elements $[[(\bar{\gamma}_1, \bar{\sigma}_1)]]$ and $[[(\bar{\gamma}_2, \bar{\sigma}_2)]]$ in $Y^s(\Sigma)$. By definition of $Y^s(\Sigma)$, $[[(\bar{\gamma}_1, \bar{\sigma}_1)]]$ and $[[(\bar{\gamma}_2, \bar{\sigma}_2)]]$ are not conformally related. However, it follows from eq. (9.4) that the image $[(\gamma_1, \pi_1)] := \mathscr{Y}_\tau[[(\bar{\gamma}_1, \bar{\sigma}_1)]]$ must (apart from the trace of π_1) be conformally related to $[(\bar{\gamma}_1, \bar{\sigma}_1)]$, while $[(\gamma_2, \pi_2)] := \hat{\mathscr{Y}}_\tau[[(\bar{\gamma}_2, \bar{\sigma}_2)]]$ must (apart from the trace of π_2) be conformally related to $[(\bar{\gamma}_2, \bar{\sigma}_2)]$. Hence $[(\gamma_2, \pi_2)] \neq [(\gamma_1, \pi_1)]$, which proves injectivity. ■

9.3. **Corollary.** *The generic stratum $\mathscr{E}_{\text{gen}}/\mathscr{D} \subset \tilde{\mathscr{E}}/\mathscr{D}$* (see proposition 8.1) *is dense.*

Proof. It suffices to show that for $(\gamma_0, \pi_0) \in \tilde{\mathscr{C}}^s_\tau(\Sigma)$ there is a nearby $(\gamma, \pi) \in \tilde{\mathscr{C}}^s_\tau(\Sigma)$ with no isometries. It suffices to assume $\tau \neq 0$ and thereby deal with just spacelike isometries, since the case $\tau = 0$ can always be slightly perturbed to the case $\tau \neq 0$ (see section 3). Now let $[[(\bar{\gamma}_0, \bar{\pi})]]$ map to $[(\gamma_0, \pi_0)]$ by the York map. By Ebin [1970] p. 35 there is a $\bar{\gamma}$ near $\bar{\gamma}_0$ with no isometries. By the York procedure we can then find $\bar{\pi}$ near $\bar{\pi}_0$ which is TT relative to $\bar{\gamma}$. Then, since the York map is continuous (see Choquet-Bruhat [1976] p. 229), the image of $[[(\bar{\gamma}, \bar{\pi})]]$ gives us a solution $[(\gamma, \pi)]$ which cannot have any isometries by covariance of the York map. ■

We now want to use proposition 9.2 to relate $Y^s(\Sigma)$ to the space of gravitational degrees of freedom, as locally represented by our slices. Consider a spacetime $(V, g_0) \in \tilde{\mathscr{E}}^s(V)$, and for convenience assume that it is not flat. The slice S_{g_0}, recall, is constructed by including all nearby spacetimes (V, g) which, on all corresponding CMC surfaces (as specified by some foliation i_τ) have data $(\gamma(\tau), \pi(\tau))$ lying in the Fischer–Marsden–Moncrief slice $S_{(\gamma_0(\tau), \pi_0(\tau))}$ (see lemma 6.1). An alternative, but equivalent, characterization of S_{g_0} focuses on a single CMC surface: say $\tau = 5$. One can then take S_{g_0} to include all nearby spacetimes (V, g) such that $(\gamma(5), \pi(5)) \in S_{(\gamma_0(5), \pi_0(5))}$ and such that the future and past of the CMC surface $i_{\tau=5}$ are built using the Einstein evolution equation (2.7) with lapse $N_\perp$ determined by the CMC evolution condition and shift $N_\| = 0$. From this point of view, the FMM slice $S_{(\gamma_0(5),\pi_0(5))}$ parametrizes the slice S_{g_0}. But $S_{(\gamma_0(5),\pi_0(5))}$ is exactly the slice which parametrizes $\tilde{\mathscr{C}}^s_5(\Sigma)/\mathscr{D}^s(\Sigma)$; and, as shown in proposition 9.2, $\tilde{\mathscr{C}}^s_5(\Sigma)/\mathscr{D}^s(\Sigma)$ is isomorphic to $Y^s(\Sigma)$ via the York map $\hat{\mathscr{y}}_5$. Hence we have an isomorphism between the York free data $Y^s(\Sigma)$ and the gravitational degrees of freedom.

This isomorphism is clearly not unique; indeed, we have a different one for every appropriate value of τ.

A pleasant feature of the York map $\hat{\mathscr{y}}_\tau$ is that it is symplectic. Thus the York program not only faithfully parametrizes the space of degrees of freedom but it is consistent with the symplectic structure. This result is somewhat tricky since $\mathscr{y}_\tau$ itself is not symplectic. Details of the proof that $\hat{\mathscr{y}}_\tau$ is symplectic will appear in another publication. The isomorphism is still quite useful, since the York data is freely specifiable, while the points $\tilde{\mathscr{C}}^s(\Sigma)$ are not so easily specified. Thus $Y^s(\Sigma)$ provides a practical parametrization of the slice S_{g_0} and hence a practical local parametrization of the gravitational degree of freedom.

Conclusion

The focus of this paper is theorem 4.1, the slice theorem for the group of diffeomorphisms $\mathscr{D}$ acting upon $\tilde{\mathscr{E}}$, the set of spatially compact, globally hyperbolic spacetime solutions of the vacuum Einstein's equations which contain at least one constant mean curvature hypersurface. Using this theorem, we can show that the space of gravitational degrees of freedom (for this class of spacetimes) is a stratified symplectic manifold. We can also show that every spacetime in this class, in a sense, is a local maximum in terms of isometries. The spacetimes with no isometries are found to be open and dense in this class.

It is interesting to note how important the role of CMC hypersurfaces and foliations is in proving the slice theorem. For the spacetimes with global CMC foliations, the proof of the slice theorem is reduced by that preferred foliation into a familiar elliptic problem which can readily be treated. Spacetimes without at least one CMC hypersurface (if such spacetimes exist) can't be handled at all; spacetimes with only partial CMC foliations (again, it is not known if they exist) can only be handled within their CMC foliated portions. So the evidence accumulates that CMC hypersurfaces are, both mathematically and physically, extremely important tools for the study of spacetimes.

We have commented earlier in the paper on the various directions in which further work might proceed. One direction would be to consider spacetimes containing other fields besides gravity (or gravitational fields satisfying some field equations other than those of Einstein). Since a slice theorem is known for pure Yang–Mills, it should now be straightforward to find one for Einstein–Yang–Mills. More challenging would be to examine supergravity. (Bao, Isenberg, Marsden and Yasskin are presently developing a dynamical formulation of supergravity which is consistent with the point-of-view of this paper.)

Another direction which might lead to interesting results is to attempt to prove a slice theorem for asymptotically flat spacetimes. As noted earlier, most of the technology is available. The results could prove to be quite interesting since the quotient space – i.e., the gravitational degrees of freedom – might have dynamics remaining (which is not so for the spatially compact spacetimes). The amount of dynamics should depend upon the group of diffeomorphisms considered: One might have Poincaré dynamics for certain choices of $\mathscr{D}$ and BMS dynamics for others. These issues are currently under study.

Appendix: On the function spaces used in the slice theorem

The proof of our slice theorem for the action of the diffeomorphism group $\mathscr{D}$ upon the set of solutions to Einstein's equations $\mathscr{E}$ depends upon such things as the implicit function theorem and the smoothness of the group action. To use these, we must make sure that $\mathscr{D}$ and $\mathscr{E}$ (and related spaces such as the set of solutions to the Einstein constraint equations, $\mathscr{C}$) are sufficiently nice sets. As discussed in the paper, we (following Ebin [1970] and Palais [1970]) use Sobolev spaces to do this. In the appendix, we briefly review some of the results concerning Sobolev spaces which make them effective for proving the slice theorem. More details may be found in Adams [1975], Palais [1965, 1968], Ebin and Marsden [1970] and in Marsden and Hughes [1982].

A1. *Definition of Sobolev spaces*

Let V be a compact C^∞ manifold, and let $C^\infty(V)$ be the set of real-valued C^∞ functions on V. For s and p a pair of positive integers, the Sobolev norm on $f \in C^\infty(V)$ is defined as

$$\|f\|_{s,p} := \sum_{0 \le \alpha \le s} \|D^\alpha f\|_{L_p(V)} \tag{A1}$$

where $D^\alpha f$ is the αth derivative on f, and where

$$\|g\|_{L_p(V)} := \left(\int_V |g|^p \, d\mu \right)^{1/p} \tag{A2}$$

is the L_p norm on $C^\infty(V)$. The Sobolev space $W^{s,p}(V)$ is then defined to be the (Cauchy) completion of $C^\infty(V)$ with respect to $\| \; \|_{s,p}$.

The generalization of this definition to $\mathbb{R}^n$-valued functions on V, and further to sections of a bundle $B \to V$ should be clear (presuming the existence of an inner product on the fibres of B). We shall discuss certain important examples of such generalizations below.

A2. *Properties of Sobolev spaces*

(a) *Vector spaces*: For all values of s and p, $W^{s,p}(V)$ is a Banach space (an infinite dimensional vector space over the reals, with the norm $\| \; \|_{s,p}$). For $p = 2$, $W^{s,2}(V)$ is a Hilbert space, since the $\| \; \|_{s,2}$ norm defines an inner product

$$\langle\langle f, g \rangle\rangle_s := \sum_{0 \le \alpha \le s} \left[\int_V (D^\alpha f \, D^\alpha g) \right]^{1/2}. \tag{A3}$$

The notation $H^s(V) := W^{s,2}(V)$ is used for this special case.

Since a Banach space is automatically a smooth (C^∞) manifold, all of the Sobolev spaces are smooth manifolds.

(b) *Differentiability*: If $s > n/p + k$, where $n := \dim(V)$ and k is a positive integer (or zero) *then* $W^{s,p}(V)$ is continuously and compactly imbedded in the function space $C^k(V)$.

This result (the "Morrey–Sobolev–Rellich embedding theorem") tells one how to choose s and p in order to guarantee that the functions contained in $W^{s,p}(V)$ have a desired degree of differentiability in the classical sense. Note that $\lim_{s\to\infty} W^{s,p}(V) = C^{\infty}(V)$, a result used in the Ebin "regularity" argument.

(c) *Embedding*: If $s > s'$, then $W^{s,p}(V)$ is compactly embedded in $W^{s',p}(V)$.

(d) *Multiplication*: If $s' \geq s$ and $s' > n/p$, then for $f \in W^{s,p}(V)$ and $g \in W^{s',p}(V)$, one has

$$fg \in W^{s,p}\,. \tag{A4}$$

Thus one may think of function multiplication "×" as a bilinear map on Sobolev spaces:

$$\times : W^{s,p}(V) \oplus W^{s',p}(V) \to W^{s,p}(V)\,. \tag{A5}$$

Note that the map "×" is C^{∞} on the C^{∞} manifolds $W^{s,p}(V)$ and $W^{s',p}(V)$ since it is linear on these linear spaces. Not surprisingly, the image of this map is the "larger" Sobolev space; the one with the weaker norm.

A3. *Open sets in Sobolev spaces*

Consider the set of all strictly positive functions in $W^{s,p}(V)$. Clearly this set (which we shall call $W^{s,p}_{(+)}(V)$) is not a Sobolev space since it fails to be closed under addition. If, however, we find that $W^{s,p}_{(+)}(V)$ is an open set in a Sobolev space, then many of the properties discussed above will still hold.

Elementary properties of continuous functions guarantee that $C^k_+(V)$ is open in $C^k(V)$. This is not generally true for $W^{s,p}_{(+)}(V)$ in $W^{s,p}(V)$, since the latter is a completion space. However if $s > n/p$ then $W^{s,p}(V)$ is continuously imbedded in $C^0(V)$. Then

$$W^{s,p}_{(+)}(V) = W^{s,p}(V) \cap C^0_{(+)}(V)\,, \tag{A6}$$

from which it follows that $W^{s,p}_{(+)}(V)$ is an open subset of $W^{s,p}(V)$.

As noted, an open subset of a Sobolev space is generally not a vector space. However, the differentiability, the embedding, and the multiplication properties discussed above all hold (in appropriate form) for open subsets of Sobolev spaces. As well, such a set is clearly a C^{∞} manifold.

The discussion so far has been general. Now we examine the specific examples of Sobolev spaces with which we work in this paper: the space of metrics on V, and the space of diffeomorphisms of V. We shall also discuss the action induced by the space of diffeomorphisms on the space of metrics.

A4. *The space of metrics on* V

Let $S^{s,p}_2(V)$ be the Sobolev space consisting of rank-2 tensors with components contained in $W^{s,p}(V)$. [As noted above, such a space is well-defined, independent of the choice of coordinate chart.] Define $\mathcal{M}^{s,p}(V)$ to be the subset of $S^{s,p}_2(V)$ whose elements are nondegenerate everywhere on V and have the proper signature. $\mathcal{M}^{s,p}(V)$ is "the Sobolev space" of metrics. It is not a Sobolev space; however for $s > n/p$, one can use an argument similar to that of section A3 to show that $\mathcal{M}^{s,p}(V)$ is open in $S^{s,p}_2(V)$. It therefore has all the properties of section A2 except that it is not a vector space.

A5. *The group of diffeomorphisms of* V

Here we define the collection of diffeomorphisms of V as open subsets of Sobolev spaces, we discuss (in terms of Sobolev spaces) the composition of diffeomorphisms with scalar functions, and finally we consider the diffeomorphisms as a topological group.

(a) *The space of diffeomorphisms of* V: A map $\eta: V \to V$ is locally (in a coordinate chart of V) $\mathbb{R}^n$-valued and hence we can define a Sobolev space $W^{s,p}(V, V)$ of such maps. One can check that this space is chart-independent if $s > n/p$ and hence well-defined. The diffeomorphisms are the invertible elements of $W^{s,p}(V, V)$. Since invertibility is not a linear condition, one can't construct a Sobolev space of diffeomorphisms. One therefore again looks for conditions sufficient to guarantee that they make up an open subset of $W^{s,p}(V, V)$. Now, the continuous diffeomorphisms aren't open in the set of continuous maps $C^0(V, V)$. However in $C^1(V, V)$, the inverse function theorem is available and one can therefore verify openness. So, using the usual intersection arguments, one finds that for $s > n/p + 1$, one has the set of diffeomorphisms $\mathscr{D}^{s,p}(V)$ open in the Sobolev space $W^{s,p}(V, V)$.

(b) *Composing scalar functions with diffeomorphisms*: Consider a map on Sobolev open sets defined as follows:

$$F: \mathscr{D}^{s',p}(V) \oplus W^{s'',p}(V) \to W^{s,p}(V)$$
$$F(\eta, f) = f \circ \eta . \tag{A7}$$

Using the chain rule, one easily proves that one can choose $s = \min\{s', s''\}$.

Now consider two maps related to the composition map F:

$$F_\eta: W^{s,p}(V) \to W^{s,p}(V) \tag{A8}$$
$$F_\eta(f) := f \circ \eta \qquad (\text{for fixed } \eta \in \mathscr{D}^{s',p}(V), \text{ with } s' \geq s + 1)$$
$$\mathcal{O}_f: \mathscr{D}^{s',p}(V) \to W^{s,p}(V) \tag{A9}$$
$$\mathcal{O}_f(\eta) := f \circ \eta \qquad (\text{for fixed } f \in W^{s'',p} \text{ with } s' \geq s \text{ and } s'' \geq s) .$$

The first of these, F_η, is the pullback map. It is linear [since $F_\eta(f + g) = (f + g) \circ \eta = f \circ \eta + g \circ \eta = F_\eta(f) + F_\eta(g)$]. Since one easily verifies that it is also C^0, one has that F_η is a smooth map.

The second map, $\mathcal{O}_f$, is the orbit map; its image on $W^{s,p}(V)$ is the orbit of $\mathscr{D}^{s',p}(V)$ through f. $\mathcal{O}_f$ is not linear and so smoothness is not automatic. One does, however, find that if $f \in W^{s+k,p}(V)$, then $\mathcal{O}_f: \mathscr{D}^{s',p}(V) \to W^{s,p}(V)$ is a C^k map as long as $s' > s$.

(c) $\mathscr{D}^{s,p}(V)$ *as a topological group*: The group structure for $\mathscr{D}^{s,p}(V)$ is based upon using composition for the group multiplication. There is a left and right multiplication: For $\eta \in \mathscr{D}^{s,p}(V)$, one has

$$\rho_\eta: \mathscr{D}^{s,p}(V) \to \mathscr{D}^{s,p}(V): \mu \mapsto \mu \circ \eta \tag{A10}$$

and

$$\lambda_\eta: \mathscr{D}^{s,p}(V) \to \mathscr{D}^{s,p}(V): \mu \mapsto \eta \circ \mu . \tag{A11}$$

To ensure that $\mathscr{D}^{s,p}(V)$ is a group, one requires that $s > n/p + 1$ so that an inverse exists. One can use the results on composition from section A5.b to show that while right multiplication is smooth, left multiplication is only C^0. Hence $\mathscr{D}^{s,p}(V)$ is a topological group and not a differentiable Lie group.

In general, there is no well-defined exponential map for a topological group. For $\mathscr{D}^{s,p}(V)$, however, there is. Specifically, Ebin and Marsden [1970] show the following: Let $T_e\mathscr{D}^{s,p}(V)$, the "tangent space of

$\mathcal{D}^{s,p}(V)$ at the identity diffeomorphism e", consist of all $W^{s,p}(V)$ class vector fields on V. Then for every $v \in T_e\mathcal{D}^{s,p}(V)$, there is a unique flow η_t which is a one-parameter subgroup of $\mathcal{D}^{s,p}(V)$. The curve η_t is C^1, and the map

$$\mathrm{Exp}: T_e\mathcal{D}^{s,p}(V) \to \mathcal{D}^{s,p}(V): v \to \eta_1 \tag{A12}$$

is continuous.

A6. $\mathcal{D}^{s',p}(V)$ *acting on* $\mathcal{M}^{s,p}(V)$

The slice theorem focuses on the action of $\mathcal{D}^{s',p}(V)$ on $\mathcal{M}^{s,p}(V)$, as defined by the composition map

$$F: \mathcal{D}^{s',p}(V) \oplus \mathcal{M}^{s,p}(V) \to \mathcal{M}^{s,p}(V)$$
$$F(\eta, g) = g \circ \eta (D\eta)(D\eta). \tag{A13}$$

We now summarize the conditions which must be imposed upon s', s and p for the proof of the slice theorem to work:

(i) $s > n/p$ so that $\mathcal{M}^{s,p}(V)$ is an open subset of a Sobolev space.

(ii) $s' > n/p + 1$ so that $\mathcal{D}^{s',p}(V)$ is an open subset of a Sobolev space and so that $\mathcal{D}^{s',p}(V)$ is a topological group.

(iii) $s' > s + 1$ so that F takes values in $\mathcal{M}^{s,p}(V)$.

(iv) $s = s' + k$ (k a positive integer) so that the orbit map $\mathcal{O}_g: \mathcal{D}^{s',p}(V) \to \mathcal{M}^{s,p}(V)$ is C^k (and therefore permits use of the implicit function theorem).

The first three of these conditions are compatible. The last is not. Thus we cannot obtain a slice through *every* $g \in \mathcal{M}^{s,p}(V)$. We can, however, impose conditions (i), (ii) and (iii), and then obtain a slice through every $g \in \mathcal{M}^{s+k+1,p}(V) \subset \mathcal{M}^{s,p}(V)$.

Index of symbols

Here, we list some of the symbols used in the paper (in roughly alphabetical order). For each one, we give a brief definition and we note the page on which it first appears. The list does not include symbols which appear only locally (e.g., only in the course of proving one of the theorems).

- $\mathcal{B}^s(\Sigma)$ The set of data $(\gamma, \pi) \in T^*\mathcal{M}^s(\Sigma)$ which satisfy the constraints $\Phi(\gamma, \pi) = 0$, and at which Φ is *not* linearization stable (p. 192).
- $\tilde{\mathcal{B}}^s(\Sigma)$ The subset of $\mathcal{B}^s(\Sigma)$ with constant mean curvature (p. 192).
- $C^r(X)$ The set of functions on a manifold X whose r^{th} order derivatives are continuous (similarly for $C^\infty(X)$).
- $\mathcal{C}^s(\Sigma)$ The set of data $(\gamma, \pi) \in T^*\mathcal{M}^s(\Sigma)$ which satisfy the constraints $\Phi(\gamma, \pi) = 0$ (p. 190).
- $\tilde{\mathcal{C}}^s(\Sigma)$ The subset of $\mathcal{C}^s(\Sigma)$ with constant mean curvature (p. 192).
- $\tilde{\mathcal{C}}^s_{[\kappa_1, \kappa_2]}(\Sigma)$ The subset of $\tilde{\mathcal{C}}^s(\Sigma)$ with constant mean curvature κ such that $\kappa_1 \le \kappa \le \kappa_2$ (p. 208).
- $\mathcal{D}^s(\Sigma)$ The set of H^s-Sobolev class diffeomorphisms of the compact manifold Σ (p. 185).

$\mathscr{D}^s(V)$ The set diffeomorphisms of the spacetime manifold V which are H^s-Sobolev class when restricted to open subsets of V with compact closure (p. 188).

$\mathscr{D}^s(V')$ The set of H^s-Sobolev class diffeomorphisms of the manifold-with-boundary V' (p. 208).

∇ The Levi–Civita connection (metric-compactible and torsion-free) associated with a spacetime metric g (p. 189).

$D\Phi^*$ The adjoint of the derivative of Φ; it generates the dynamical evolution of the fields (γ, π) (p. 191).

$\mathscr{E}(V)$ The set of all spacetime metrics g which are solutions of Einstein's equations on V (p. 181).

$\bar{\mathscr{E}}(V)$ The subset of $\mathscr{E}(V)$ which contain at least one constant mean curvature surface (p. 181).

$\mathscr{E}^s(V)$ The subset of $\mathscr{E}(V)$ which are Sobolev class H^s (p. 188).

$\bar{\mathscr{E}}^s(\Sigma, V)$ The subset of $\mathscr{E}^s(V)$ which contain a Cauchy surface which is an imbedding of a manifold Σ into V (p. 188).

$\bar{\bar{\mathscr{E}}}^s(\Sigma, V)$ The subset of $\bar{\mathscr{E}}^s(\Sigma, V)$ for which there exists a Cauchy surface with constant mean curvature (p. 188).

$\bar{\mathscr{E}}^s_{[\kappa_1, \kappa_2]}(\Sigma, V')$ The set of globally hyperbolic H^s class solutions of Einstein's equations on V' which are foliated by constant mean curvature surfaces with κ running from κ_1 to κ_2 (p. 207).

$\bar{\mathscr{E}}^s_T(\Sigma, V')$ The set of globally hyperbolic H^s class solutions of Einstein's equations on V' which are foliated by CMC surfaces, with a proper distance T separating the first and last Cauchy surfaces (p. 208).

$\bar{\mathscr{E}}^s_{T,\mathrm{Flat}}(\Sigma, V')$ The subset of $\bar{\mathscr{E}}^s_T(\Sigma, V')$ which are flat (with a time-like Killing vector field) (p. 209).

$\mathscr{E}_{\mathrm{gen}}(\Sigma, V)$ The globally hyperbolic solutions of Einstein's equations which are generic, i.e. have no symmetries (p. 210).

$\mathrm{Emb}^s(\Sigma, V, G)$ The H^s-class embeddings of Σ in V which are spacelike with respect to g (p. 189).

η A diffeomorphism (of Σ or V) (p. 185).

$\Phi(\gamma, \pi)$ The Einstein initial data constraints (p. 190).

ϕ The conformal factor obtained by solving the constraints via the York program (p. 211).

g The spacetime metric (p. 188).

$\mathscr{G}$ The space of degrees of freedom of the gravitational field (p. 181).

$\mathscr{G}_{(\gamma,\pi)}(\,,\,)$ A Riemannian inner product defined on $T^*\mathscr{M}^s(\Sigma)$ (p. 203).

$\gamma(g, i)$ The Riemannian metric induced on Σ by $i : \Sigma \to V$ in a spacetime (V, g) (p. 189).

H^s The Sobelev space of index s (p. 185).

$\mathscr{H}(\gamma, \pi)$ The super Hamiltonian constraint (p. 190).

I_x The isotropy subgroup of the point $x \in X$ for a group G acting on X (p. 183).

I_g The isotropy subgroup of the metric $g \in \mathscr{E}(V)$ for the action of $\mathscr{D}(V)$ on $\mathscr{E}(V)$ (p. 199).

i An imbedding (usually of Σ into V) (p. 189).

i_t A parametrized set of imbeddings (p. 189).

$\mathscr{J}(\gamma, \pi)$ The supermomentum constraint (p. 190).

$\mathbb{J}$ The symplectic matrix $\begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$ (p. 191).

$k(g, i)$	The second fundamental form induced on Σ by $i: \Sigma \to V$ in a spacetime (V, g) (p. 189).
$\kappa(g, i)$	The trace of $k(g, i)$; the mean curvature of $i(\Sigma) \subset V$ (p. 194).
$\mathscr{K}^s(\Sigma, V)$	The space of pairs (g, i) where g is an H^s class Lorentz metric on V, and i is an H^{s+1} class imbedding of Σ into V (p. 190).
$\mathscr{K}^s_{\mathscr{E}}(\Sigma, V)$	The subspace of $\mathscr{K}^s(\Sigma, V)$ which satisfies Einstein's equations.
$\mathscr{L}^s(V)$	The set of Lorentz signature metrics on V, which are H^s-class on compact subsets of V (p. 188).
$\mathscr{L}_{N_\parallel}$	The Lie derivative along a vector field $N_\parallel$ (p. 201).
$\mathscr{M}^s(\Sigma)$	The set of H^s-class Riemannian metrics on a compact manifold Σ (p. 185).
N	An element of $T_i\mathrm{Emb}^{s+1}(\Sigma, V, g)$ which projects into the lapse $N_\perp$ and the shift $N_\parallel$ (p. 189).
$N_\rho(\gamma, \pi)$	A neighborhood of $(\gamma, \pi) \in T^*\mathscr{M}^s(\Sigma)$ with radius ρ with respect to $\mathscr{G}_{(\gamma,\pi)}(\,,\,)$ (p. 204).
$N_{g_0}(V)$	The space of metrics in $\tilde{\mathscr{E}}(V)$ whose isometry groups are conjugate to that of the spacetime metric g_0 (p. 210).
$\mathscr{N}_{g_0}(V)$	The quotient of $N_{g_0}(V)$ with respect to the diffeomorphism group $\mathscr{D}(V)$ (p. 211).
$\mathscr{O}_x$	The orbit through $x \in X$ of a group G acting on a space X (p. 183).
$\mathscr{O}_{g_0}$	The orbit through g_0 of the diffeomorphism group acting on the space of metrics (p. 200).
$\Omega(\,,\,)$	The symplectic form on $T^*\mathscr{M}^s(\Sigma)$ (p. 190).
p_{i_0}	The bundle projection map from $\mathscr{E}^s(\Sigma, V)$ to $\mathscr{C}^s(\Sigma)$ defined by $p_{i_0}(g) = (\gamma(g, i_0), \pi(g, i_0))$ for some given imbedding i_0 (p. 192).
π	The momentum conjugate to γ (p. 189).
$\Pi_{\mathscr{L}}$	The bundle projection map from $\mathscr{K}^s(\Sigma, V)$ to $\mathscr{L}^s(V)$ (p. 190).
$\Pi_{T^*\mathscr{M}}$	The bundle projection map from $\mathscr{K}^s(\Sigma, V)$ to $T^*\mathscr{M}^s(\Sigma)$ (p. 190).
ψ_{g_0}	A map from $\mathscr{D}^{s+1}(V)$ to $\mathscr{L}^s(\Sigma, V)$ defined by $\psi_{g_0}(\eta) = \eta^* g_0$. The image of ψ_{g_0} is the orbit $\mathscr{O}_{g_0}$ (p. 200).
$\mathrm{Ric}(g)$	The Ricci tensor associated to a metric g (via the Levi–Civita connection) (p. 188).
$\mathrm{Riem}(g)$	The Riemann tensor associated to a metric g (via the Levi–Civita connection) (p. 202).
S_x	A slice through $x \in X$ for the action of a group G on X (p. 183).
S_g	A slice through $g \in \tilde{\mathscr{E}}^{s+r}(\Sigma, V)$ for the action of $\mathscr{D}^{s+1}(V)$ on $\tilde{\mathscr{E}}^s(\Sigma, V)$ (p. 199).
$S_{(\gamma,\pi)}$	The Fischer–Marsden–Moncrief slice through $(\gamma, \pi) \in T^*\mathscr{M}^{s+r}(\Sigma)$ for the action of $\mathscr{D}^{s+1}(\Sigma)$ on $T^*\mathscr{M}^s(\Sigma)$ (p. 203).
$S_{(\bar\gamma,\bar\pi)}$	A slice through $(\bar\gamma, \bar\pi) \in T^*\mathscr{M}^{s+r}(\Sigma)_{TT}$ for the action of $\mathscr{D}^{s+1}(\Sigma) \otimes \Theta^s(\Sigma)$ on $T^*\mathscr{M}^s(\Sigma)_{TT}$ (p. 212).
$\bar\sigma^{ab}$	The transverse-traceless piece of $\bar\pi^{ab}$ (p. 211).
Σ	A compact 3-manifold (p. 185).
$T_x\mathscr{O}_x$	The tangent space to the orbit $\mathscr{O}_x \subset X$ at $x \in X$ (p. 184).
$(T_x\mathscr{O}_x)^\perp$	The orthogonal complement to $T_x\mathscr{O}_x$ (p. 184).
$T^*\mathscr{M}^s(\Sigma)$	The set of pairs (symmetric covariant tensors, symmetric contravariant tensor densities) of (H^s, H^{s-1})-class; the cotangent space to $\mathscr{M}^s(\Sigma)$ (p. 190).
$T^*\mathscr{M}^s(\Sigma)_{TT}$	The subset of elements $(\gamma_{ab}, \sigma^{cd}) \in T^*\mathscr{M}(\Sigma)$ which satisfy $\gamma_{ab}\sigma^{ab} = 0$ and $\nabla_a\sigma^{ac} = 0$ (p. 212).
Ti	The tangent of the imbedding map i (p. 189).

$T_i\mathrm{Emb}^s(\Sigma, V, g)$ The tangent space to $i \in \mathrm{Emb}^s(\Sigma, V, g)$ (p. 189).
$T_p i \cdot N_\parallel(p)$ The push forward of $N_\parallel(p)$ along $i: \Sigma \to V$ (p. 189).
$\Theta^s(\Sigma)$ The space of conformal mappings of H^s-class on $(\gamma, \pi) \in T^*\mathcal{M}^s$ defined by $(\gamma, \pi) \to (\theta\gamma, \theta^{-1}\pi)$ for θ an H^s class positive definite function (p. 212).
V The spacetime manifold (p. 188).
(V, g) A spacetime (p. 188).
V' A spacetime manifold with boundary given by a pair of 3 manifolds diffeomorphic to some Σ (p. 207).
X A manifold (p. 183).
ξ A diffeomorphism (usually of Σ) (p. 185).
$Y^s(\Sigma)$ The quotient $T^*\mathcal{M}^s(\Sigma)_{TT}/\mathcal{D}^s(\Sigma) \oplus \Theta^s(\Sigma)$; the space of (reduced) York data (p. 212).
$\mathcal{Y}_\tau$ The York map from $T^*\mathcal{M}^s(\Sigma)$ to $\bar{\mathcal{C}}^s(\Sigma)$ (p. 211).
$\hat{\mathcal{Y}}_\tau$ The reduced York map from $Y^s(\Sigma)$ to $T^*\mathcal{M}^s(\Sigma)/\mathcal{D}^s(\Sigma)$ (p. 213).
$Z(i)$ A vector field normal to the surface $i(\Sigma)$ in a spacetime (V, g) (p. 189).

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